

# Unified treatment for some problems of single birth processes

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# Outline

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# 1. Background

- $Q$ -process: on  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ , suppose that sub-Markov transition probability matrix  $P(t) = (p_{ij}(t))$  satisfies
  - (1) Normal condition:  $p_{ij}(t) \geq 0$ ,  $\sum_j p_{ij}(t) \leq 1$ .
  - (2) Chapman-Kolmogorov eq.:  $p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s)$ .
  - (3) Jump condition:  $\lim_{t \rightarrow 0} p_{ij}(t) = \delta_{ij}$ ,  $i, j \in \mathbb{Z}_+$ .
  - (4)  $Q$ -condition:  $\lim_{t \rightarrow 0} (p_{ij}(t) - \delta_{ij})/t = q_{ij}$ ,  $i, j \in \mathbb{Z}_+$ , i.e.  $Q$ -matrix  $Q = (q_{ij})$  is the derivative matrix at time  $0$  of  $P(t)$ ,

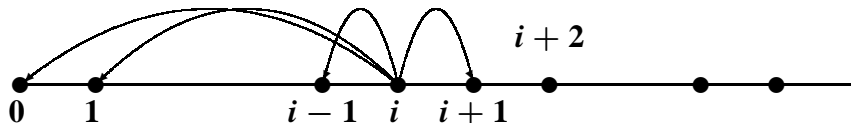
$$0 \leq q_{ij} < \infty, \quad i \neq j; \quad \sum_{j \neq i} q_{ij} \leq -q_{ii} =: q_i \leq \infty, \quad i \in \mathbb{Z}_+.$$

- totally stable:  $q_i < \infty$ ,  $i \in \mathbb{Z}_+$ .
- conservative:  $q_i = \sum_{j \neq i} q_{ij}$ ,  $i \in \mathbb{Z}_+$ , i.e.  $\sum_{j \in \mathbb{Z}_+} q_{ij} = 0$ .

# 1. Background

Single birth process:  $Q = (q_{ij})$  satisfies

$$q_{i,i+1} > 0, q_{i,i+j} = 0, i \geq 0, j > 1.$$



$$Q = \begin{pmatrix} - & + & \mathbf{0} & \mathbf{0} & \cdots \\ * & - & + & \mathbf{0} & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# 1. Background

- 1 The exit boundary of the process consists at most one single extremal point
- 2 Fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes
- 3 Usually the single birth process is non-symmetric (irreversible)
- 4 Applications in other fields

# 1. Background

- 1 Upwardly skip-free processes, population proc. (W.J. Anderson, 1991);
- 2 Generalized birth and death proc. (J.K. Zhang, 1984);
- 3 Birth and death proc. with catastrophes; birth, death and catastrophe proc. (Brockwell, Gani, Resnick and Pakes et al, 1982-1986; B. Cairns & P. Pollett, 2004);
- 4 Single birth proc. (S.J. Yan & M.F. Chen, 1986)

# 1. Background

$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } b_i \\ \rightarrow i - 1 & a_i + c_i f_{i,i-1} \\ \rightarrow i - 2 & c_i f_{i,i-2} \\ \rightarrow \dots & \dots \\ \rightarrow 0 & c_i f_{i0} \end{array}$$

where  $\sum_{j=0}^{i-1} f_{ij} = 1$ .

# 1. Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes  $f_{ij}$  ( $b_i = b + \lambda i$ ,  $a_i = \mathbf{0}$ ,  $c_i = ci$ ,  $i \geq \mathbf{0}$ ).

geometric:  $f_{ij} = p(1 - p)^{i-1-j}$  ( $1 \leq j < i$ );  $f_{i0} = (1 - p)^{i-1}$ ;

uniform:  $f_{ij} = 1/i$  ( $\mathbf{0} \leq j < i$ );

binomial:  $f_{ij} = \binom{i-1}{j} p^j (1 - p)^{i-1-j}$  ( $\mathbf{0} \leq j < i$ ).

Aim: extinction times and probability of extinction.

Keys: generating function of  $Q$ -resolvent.



## 2. Poisson equation

For a totally stable and conservative single birth  $Q$ -matrix, given a function  $c$ , define

$$\Omega g = Qg + cg,$$

Clearly, if  $c \leq \mathbf{0}$ , then  $\Omega$  is an operator corresponding to a single birth process with killing rates  $(-c_i)$ . Define

$$\tilde{F}_i^{(i)} = \mathbf{1}, \quad \tilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \tilde{F}_k^{(i)}, \quad n > i \geq \mathbf{0},$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^k q_{nj} - c_n, \quad \mathbf{0} \leq k < n.$$

Once  $c_i \equiv \mathbf{0}$ , we omit the superscript  $\sim$ .

## Theorem 1[M.F. Chen & Z.(2014)]

Given a totally stable and conservative single birth  $Q$ -matrix  $Q = (q_{ij})$  and functions  $c$  and  $f$ , the solution  $g$  to the Poisson equation

$$\Omega g = f$$

has the following representation:

$$g_n = g_0 + \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)} (f_j - c_j g_0)}{q_{j,j+1}}, \quad n \geq 0.$$

In particular, the harmonic function of  $\Omega$  can be represented as

$$g_n = g_0 \left( 1 - \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \geq 0.$$

*Proof* For each  $i \geq 0$ , we have

$$\begin{aligned}(\Omega \mathbf{g})_i &= q_{i,i+1}(\mathbf{g}_{i+1} - \mathbf{g}_i) - \sum_{0 \leq j \leq i-1} q_{ij} \sum_{k=j}^{i-1} (\mathbf{g}_{k+1} - \mathbf{g}_k) + c_i \mathbf{g}_i \\ &= q_{i,i+1}(\mathbf{g}_{i+1} - \mathbf{g}_i) - \sum_{0 \leq j \leq i-1} \sum_{j=0}^k q_{ij} (\mathbf{g}_{k+1} - \mathbf{g}_k) + c_i \mathbf{g}_i \\ &= q_{i,i+1}(\mathbf{g}_{i+1} - \mathbf{g}_i) - \sum_{0 \leq j \leq i-1} \left( \sum_{j=0}^k q_{ij} - c_i \right) (\mathbf{g}_{k+1} - \mathbf{g}_k) + c_i \mathbf{g}_0 \\ &= q_{i,i+1}(\mathbf{g}_{i+1} - \mathbf{g}_i) - \sum_{0 \leq j \leq i-1} \tilde{q}_i^{(k)} (\mathbf{g}_{k+1} - \mathbf{g}_k) + c_i \mathbf{g}_0.\end{aligned}$$

$$\mathbf{g}_{i+1} - \mathbf{g}_i = \frac{1}{q_{i,i+1}} \left( \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} (\mathbf{g}_{k+1} - \mathbf{g}_k) + f_i - c_i \mathbf{g}_0 \right).$$

$$g_{i+1} - g_i = \sum_{j=0}^i \frac{\tilde{F}_i^{(j)} (f_j - c_j g_0)}{q_{j,j+1}}, \quad i \geq 0.$$

### Proposition[M.F. Chen & Z.(2014)]

For given  $f$ , the sequence  $(h_n)$  defined successively by

$$h_n = \frac{1}{q_{n,n+1}} \left( f_n + \sum_{i \leq k \leq n-1} \tilde{q}_n^{(k)} h_k \right), \quad n \geq i$$

has a unified expression as follows

$$h_n = \sum_{k=i}^n \frac{\tilde{F}_n^{(k)} f_k}{q_{k,k+1}}, \quad n \geq i.$$

- Uniqueness. J.K. Zhang(1984), Yan & Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P.J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988).
- Ergodicity. Yan & Chen(1986).
- Strong ergodicity. Z.(2001).
- Polynomial moments and stationary dist. Z.(2003, 2004, 2013)
- Exponential moments. Z.(2003)
- Laplace transform. P. J. Brockwell(1986), W.J. Anderson(1991)

### 3. Uniqueness

$Q$ -process:  $P'(t)|_{t=0} = Q$ .

Backward Kolmogorov equation:  $P'(t) = QP(t)$ .

Forward Kolmogorov equation:  $P'(t) = P(t)Q$ .

$$p_{ij}(t) = \sum_{k \neq i} \int_0^t q_{ik} e^{-q_i(t-s)} p_{kj}(s) ds + \delta_{ij} e^{-q_i t},$$

$$p_{ij}(t) = \sum_{k \neq j} \int_0^t p_{ik}(s) q_{kj} e^{-q_j(t-s)} ds + \delta_{ij} e^{-q_j t},$$

- $Q$ -process exist? unique?

#### Theorem

Let  $Q = (q_{ij})$  be totally stable and conservative. Every  $Q$ -process satisfies the backward Kolmogorov equation.

### 3. Uniqueness

#### Existence Theorem

Given a totally stable and conservative  $Q$ -matrix, there always exist a  $Q$ -process. In details, the minimal solution is a  $Q$ -process. Moreover, BE and FE have the same minimal solution.

Laplace transform:  $p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t) dt.$

$$p_{ij}(\lambda) = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i},$$

$$p_{ij}(\lambda) = \sum_{k \neq j} p_{ik}(\lambda) \frac{q_{kj}}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_i}.$$

Denote the minimal process by  $P^{\min}(\lambda)$ . Consider  $P(\lambda) - P^{\min}(\lambda)$ :

### 3. Uniqueness

$$u_i = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} u_k, \quad \mathbf{0} \leq u_i \leq \mathbf{1}, i \in \mathbb{Z}_+.$$

#### Uniqueness Theorem[Feller(1957), Reuter(1957)]

For a given totally stable and conservative  $Q$ -matrix  $Q = (q_{ij})$ , the  $Q$ -process is unique iff the equation has only the trivial solution  $u_i \equiv \mathbf{0}$  for some (equivalently, for all)  $\lambda > \mathbf{0}$ .



### 3. Uniqueness

Define

$$\tilde{m}_0 = \frac{1}{q_{01}}, \quad \tilde{m}_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{m}_k \right), \quad n \geq 1.$$

$$\tilde{m}_n = \sum_{k=0}^n \frac{\tilde{F}_n^{(k)}}{q_{k,k+1}}, \quad n \geq 0.$$

**Uniqueness[J.K. Zhang(1984), Yan & Chen(1986)]**

Given a totally stable and conservative single birth  $Q$ -matrix  $Q = (q_{ij})$ , the process is unique (non-explosive) iff  $\sum_{n=0}^{\infty} m_n = \infty$ .

Sketch of proof. unique iff the solution  $(u_i)$  to the equation

$$(\lambda + q_i)u_i = \sum_{j \neq i} q_{ij}u_j, \quad i \geq 0; \quad u_0 = 1$$

is unbounded for some (equivalently for all)  $\lambda > 0$ . Rewrite it as

$$\Omega u = Qu - \lambda u = 0; \quad u_0 = 1.$$

Applying Theorem 1 to  $c_i \equiv -\lambda$  and  $f_i \equiv 0$ , we obtain:

$$u_n = 1 + \lambda \sum_{0 \leq k \leq n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0 \leq k \leq n-1} \tilde{m}_k, \quad n \geq 0.$$

Clearly,  $u_n$  is increasing in  $n$  and then is unbounded iff  $\sum_n \tilde{m}_n = \infty$ . Thus, it remains to show that  $\sum_n \tilde{m}_n = \infty$  iff  $\sum_n m_n = \infty$ .

## 4. Recurrence, return/extinction probability

- $P(t)$  is recurrent if for each  $h > 0$ ,  $P(h)$  is recurrent. Equivalently,  $\int_0^\infty p_{ii}(t) dt = \infty$ .

embedding chain  $(\Pi_{ij})$ :  $\Pi_{ij} = I_{\{q_i \neq 0\}}(1 - \delta_{ij}) \frac{q_{ij}}{q_i} + I_{\{q_i = 0\}} \delta_{ij}$ .

### Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative  $Q$ -matrix  $Q = (q_{ij})$ ,

$$\int_0^\infty p_{ij}^{\min}(t) dt = \sum_{n=0}^{\infty} \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if  $Q$  is irreducible and regular, then  $P(t)$  is recurrent iff so is its embedding chain.

regular: totally stable, conservative and  $Q$ -process is unique.

## 4. Recurrence, return/extinction probability

### Theorem

For a regular and irreducible  $Q = (q_{ij})$ ,  $P(t)$  is recurrent iff for some (equivalently, all)  $j_0$ ,

$$x_i = \sum_{j \neq j_0, i} \frac{q_{ij}}{q_i} x_j, \quad 0 \leq x_i \leq 1, i \geq 0$$

has only a trivial solution.

For embedding chain  $(\Pi_{ij})$ ,  $(f_{i,j_0} = \mathbb{P}_i(\sigma_{j_0} < \infty) : i \in \mathbb{Z}_+)$  is the minimal solution to the equation

$$x_i = \sum_{k \neq j_0} \Pi_{ik} x_k + \Pi_{i,j_0}, \quad i \in \mathbb{Z}_+.$$

$(\Pi_{ij})$  is recurrent iff  $f_{i,j_0} \equiv 1$ .

## 4. Recurrence, return/extinction probability

### Recurrence Theorem[Yan & Chen(1986)]

Assume the single birth  $Q$ -matrix  $Q = (q_{ij})$  is non-explosive and irreducible. Then the process is recurrent iff  $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$ .

Sketch of proof. recurrent iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \quad \mathbf{0} \leq x_i \leq \mathbf{1}, \quad i \geq \mathbf{0}$$

has only zero solution, where  $\Pi_{ik} = (\mathbf{1} - \delta_{ik})q_{ik}/q_i$ . It is equivalent to that the solution to the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \quad i \geq \mathbf{0}; \quad x_0 = \mathbf{1}$$

is unbounded.

## 4. Recurrence, return/extinction probability

(contin.)

Rewrite the previous equation as

$$(Qx)_0 = \mathbf{0}, \quad (Qx)_i = q_{i0}, \quad i \geq 1; \quad x_0 = \mathbf{1}.$$

Applying Theorem 1 to  $c_i \equiv \mathbf{0}$  and  $f_i = q_{i0}(1 - \delta_{i0})$ , we obtain the unique solution as follows

$$x_0 = \mathbf{1}, \quad x_n = \mathbf{1} + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly,  $(x_n)$  is unbounded iff  $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$ . In other words, the equation has only a trivial solution iff  $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$ .

## 4. Recurrence, return/extinction probability

Define  $\sigma_0 = \inf\{t \geq 0 : X_t = \mathbf{0}\}$ .

### Proposition

Let the single birth  $Q$ -matrix  $Q = (q_{ij})$  be non-explosive and irreducible. Then the return/extinction probability is as follows:

$$\mathbb{P}_0(\sigma_0 < \infty) = \frac{\sum_{k=1}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \mathbb{P}_n(\sigma_0 < \infty) = \frac{\sum_{k=n}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, n \geq 1.$$

Furthermore,  $\mathbb{P}_n(\sigma_0 < \infty) = 1$  for all  $n \geq 0$  iff  $\mathbb{P}_0(\sigma_0 < \infty) = 1$ , equivalently iff  $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$ .

P. J. Brockwell(1986), W. J. Anderson(1991)

Sketch of proof.  $(\mathbb{P}_i(\sigma_0 < \infty) : i \in E)$  is the minimal nonnegative solution to the equation

$$x_i = \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i} (1 - \delta_{i0}), \quad i \in E.$$

Then, we have  $(Qx)_i = q_{i0}(1 - \delta_{i0})(x_0 - 1), i \geq 0$ . Applying Theorem 1 to  $c_i \equiv 0$  and  $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$ , we get

$$x_n = x_0 \left( 1 + \sum_{1 \leq k \leq n-1} F_k^{(0)} \right) - \sum_{1 \leq k \leq n-1} F_k^{(0)}, \quad n \geq 0.$$

Because  $x_n > 0$ , it follows that

$$x_0 \geq \sup_{n \geq 1} \frac{\sum_{k=1}^{n-1} F_k^{(0)}}{\sum_{k=0}^{n-1} F_k^{(0)}} = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}.$$

From here, we obtain the minimal nonnegative solution:

$$x_0^* = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad x_n^* = 1 - \frac{\sum_{k=0}^{n-1} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad n \geq 1.$$



## 4. Recurrence, return/return/return probability

Consider the occupation time of the state  $j$ :

$$T_j := \int_0^\infty \mathbf{1}_{\{X(t)=j\}} dt.$$

By the return probability and the embedding chain, we get

### Proposition[P.S. Li & Z.(2016)]

Let the single birth  $Q$ -matrix  $Q = (q_{ij})$  be regular and irreducible. Then the mean occupation time

$$\mathbf{E}_i T_j = \frac{1}{q_{j,j+1}} \sum_{n=i \vee j}^{\infty} F_n^{(j)}.$$

## 5. Laplace transform of return/extinction time

Take  $c_i \equiv -\lambda < \mathbf{0}$ . Define  $(\tilde{F}_k^{(i)})$  and

$$\tilde{d}_0 = \mathbf{0}, \quad \tilde{d}_n = \frac{\mathbf{1}}{q_{n,n+1}} \left( \mathbf{1} + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{d}_k \right), \quad n \geq 1.$$

$$\text{Equivalently, } \tilde{d}_n = \sum_{1 \leq j \leq n} \frac{\tilde{F}_n^{(j)}}{q_{j,j+1}}, \quad n \geq 0.$$

## 5. Laplace transform of return/extinction time

### Proposition

Let the single birth process be recurrent. Then the Laplace transform of  $\sigma_0$  is given by

$$\mathbb{E}_0 e^{-\lambda \sigma_0} = \frac{q_{01}(1 - \lambda \tilde{d})}{q_{01} + \lambda}, \mathbb{E}_n e^{-\lambda \sigma_0} = 1 - \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k), n \geq 1,$$

where

$$\tilde{d} = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \tilde{F}_k^{(0)}} = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{\tilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

P. J. Brockwell(1986), W. J. Anderson(1991)

Sketch of proof.  $(\mathbb{E}_i e^{-\lambda \sigma_0} : i \in E)$  is the minimal solution  $(x_i^*)$  of the following equation

$$x_i = \frac{1}{q_i + \lambda} \sum_{k \notin \{0, i\}} q_{ik} x_k + \frac{q_{i0}(1 - \delta_{i0})}{q_i + \lambda}, \quad x_i \geq 1, \quad i \in E.$$

Then, we have

$$(Qx)_i - \lambda x_i = q_{i0}(x_0 - 1), \quad i \geq 1; \quad (Qx)_0 - \lambda x_0 = 0.$$

Applying Theorem 1 to  $c_i \equiv -\lambda$  and  $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$  for all  $i \geq 0$ ,

$$\begin{aligned} x_n &= x_0 \left( 1 + \frac{\lambda}{q_{01}} \right) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} - \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} - \lambda \tilde{d}_k) + 1 \\ &=: x_0 \alpha_{n-1} - \beta_{n-1}, \quad n \geq 1. \end{aligned}$$

By the minimal nonnegative property,  $x_0^* = \sup_{n \geq 1} \beta_n / \alpha_n$ , then

$$x_0^* = \overline{\lim}_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n}.$$

We can replace  $\overline{\lim}_{n \rightarrow \infty}$  by  $\lim_{n \rightarrow \infty}$ . On the one hand, since  $x_n \in (0, 1]$ ,

$$\frac{\beta_n}{\alpha_n} < x_0 \leq \frac{\beta_n + 1}{\alpha_n}, \quad n \geq 1.$$

On the other hand, we can prove that  $\sum_k \tilde{F}_k^{(0)} = \infty$  since  $\sum_k F_k^{(0)} = \infty$  by the recurrent assumption. Therefore, we have

$$x_0^* = \frac{q_{01}}{q_{01} + \lambda} \lim_{n \rightarrow \infty} \left[ 1 - \lambda \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \tilde{F}_k^{(0)}} \right] = \frac{q_{01}}{q_{01} + \lambda} [1 - \lambda \tilde{d}].$$

$$\begin{aligned} x_n^* &= (1 - \lambda \tilde{d}) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} - \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} - \lambda \tilde{d}_k) + 1 \\ &= 1 - \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k), \quad n \geq 1. \end{aligned}$$

## Example [uniform catastrophes]

Let  $q_{i,i+1} = b$ ,  $i \geq 0$ ;  $q_{ij} = a$ ,  $j = 0, 1, \dots, i - 1$ ; and  $q_{ij} = 0$  for other  $j > i + 1$ , where  $a$  and  $b$  are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n e^{-\lambda \tau_0} = \frac{a}{a + \lambda}, \quad \lambda > 0, \quad n \geq 1.$$

It is surprising that the distribution is independent of  $b$  and the starting point  $n$ . Redefine  $q_{01} = 1$ . Then the irreducible process is indeed strongly ergodic.

P. J. Brockwell et al(1982)

## 6. Summary

Roughly speaking, the unified treatment consists of the following three steps.

- (a) Find out the Poisson equation corresponding to the problem .
- (b) Apply Theorem 1 to get the solution to Poisson equation.
- (c) Work out the answer for the problem using the solution obtained in (b).

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv \mathbf{0}$
Uniqueness	$c_i \equiv -\lambda < \mathbf{0}$	$f_i \equiv \mathbf{0}$
Recurrence	$c_i \equiv \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})$
Ext./return probability	$c_i \equiv \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$
Ergodicity	$c_i \equiv \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv \mathbf{0}$	$f_i^{(\ell)}$
Exp. moment/ergod.	$c_i \equiv \lambda > \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$
Laplace transform	$c_i \equiv -\lambda < \mathbf{0}$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$

where  $f_i^{(\ell)} = q_{ii_0}(1 - \delta_{ii_0})g_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$ .



Let the single death  $Q$ -matrix  $Q = (q_{ij})$  be regular and irreducible. Assume that the corresponding process is ergodic. To get the stationary distribution of the process, solve

$$\pi Q = \mathbf{0}.$$

### Proposition[L.D. Wang & Z.(2014)]

The stationary distribution of the single death process is represented as follows:  $\pi_n = \mu_n / \mu$ ,  $n \geq 0$ , where

$$\mu_0 = 1, \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} \bar{q}_k^{(n)} \mu_k, n \geq 1;$$

$$\mu = \sum_{n=0}^{\infty} \mu_n; \bar{q}_k^{(n)} = \sum_{j=n}^{\infty} q_{kj}, n > k.$$

Transpose the matrix equation  $\pi Q = \mathbf{0}$  to

$$Q^T \pi^T = \mathbf{0}^T.$$

Then

$$Q^T \mathbf{g} =: \Omega \mathbf{g} = Q^* \mathbf{g} + \mathbf{c} \mathbf{g} = \mathbf{0}^T,$$

where  $Q^*$  is a totally stable and conservative single birth  $Q$ -matrix and

$$c_i = \sum_{j=0}^{i-1} q_{ji} + q_{i+1,i} - \sum_{j=i+1}^{\infty} q_{ij} - q_{i,i-1}, \quad i \geq 0;$$
$$f_i \equiv 0.$$

## Proposition[Z.(2016)]

The stationary distribution of the single death process is represented as follows:  $\pi_n = \mu_n / \mu$ ,  $n \geq 0$ , where

$$\mu_0 = 1, \quad \mu_n = 1 - \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)} c_j}{q_{j+1,j}}, \quad n \geq 1; \quad \mu := \sum_{n=0}^{\infty} \mu_n;$$

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \frac{1}{q_{n+1,n}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \tilde{F}_k^{(i)}, \quad n > i \geq 0,$$

$$\tilde{q}_n^{(k)} = \sum_{j=0}^k q_{jn} - c_n, \quad n > k \geq 0.$$

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***Thank you for your attention!***

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