Unified treatment for some problems of single birth processes

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Outline

1. Poisson equation

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t))$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0$, $\sum_j p_{ij}(t) \leq 1$.

  (2) Chapman-Kolmogorov eq.: $p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s)$.

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$, $i, j \in \mathbb{Z}_+$.

  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij}$, $i, j \in \mathbb{Z}_+$, i.e. Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

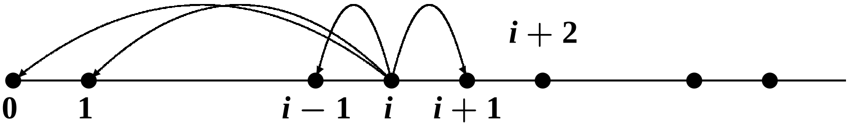
$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty$, $i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij}$, $i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

1. Poisson equation

Single birth process: $Q = (q_{ij})$ satisfies

$$q_{i,i+1} > 0,\ q_{i,i+j} = 0,\ i \geq 0, j > 1.$$



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Poisson equation

1. Upwardly skip-free processes, population proc. (W.J. Anderson, 1991);
2. Generalized birth and death proc. (J.K. Zhang, 1984);
3. Birth and death proc. with catastrophes; birth, death and catastrophe proc. (Brockwell, Gani, Resnick and Pakes et al, 1982-1986; B. Cairns & P. Pollett, 2004);
4. Single birth proc. (S.J. Yan & M.F. Chen, 1986)

1. Poisson equation

For a totally stable and conservative single birth Q-matrix, given a function c, define

$$\Omega g = Qg + cg,$$

Clearly, if $c \le 0$, then Ω is an operator corresponding to a single birth process with killing rates $(-c_i)$. Define

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad n > i \ge 0,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \le k < n.$$

Once $c_i \equiv 0$, we omit the superscript $\widetilde{\ }$.

Theorem 1[M.F. Chen & Z.(2014)]

Given a totally stable and conservative single birth Q-matrix $Q = (q_{ij})$ and functions c and f, the solution g to the Poisson equation

$$\Omega g = f$$

has the following representation:

$$g_n = g_0 + \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)} (f_j - c_j g_0)}{q_{j,j+1}}, \quad n \ge 0.$$

In particular, the harmonic function of Ω can be represented as

$$g_n = g_0 \left(1 - \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \ge 0.$$

Proof For each $i \geq 0$, we have

$$
\begin{aligned}
(\Omega g)_i &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \le j \le i-1} q_{ij} \sum_{k=j}^{i-1} (g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \le j \le i-1} \sum_{j=0}^{k} q_{ij} (g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \le j \le i-1} \left(\sum_{j=0}^{k} q_{ij} - c_i \right) (g_{k+1} - g_k) + c_i g_0 \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \le j \le i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + c_i g_0.
\end{aligned}
$$

$$
g_{i+1} - g_i = \frac{1}{q_{i,i+1}} \left(\sum_{0 \le k \le i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + f_i - c_i g_0 \right).
$$

$$g_{i+1} - g_i = \sum_{j=0}^{i} \frac{\widetilde{F}_i^{(j)} (f_j - c_j g_0)}{q_{j,j+1}}, \qquad i \geq 0.$$

Proposition[M.F. Chen & Z.(2014)]

For given f, the sequence (h_n) defined successively by

$$h_n = \frac{1}{q_{n,n+1}} \Bigg(f_n + \sum_{i \leq k \leq n-1} \tilde{q}_n^{(k)} h_k \Bigg), \qquad n \geq i$$

has a unified expression as follows

$$h_n = \sum_{k=i}^{n} \frac{\widetilde{F}_n^{(k)} f_k}{q_{k,k+1}}, \qquad n \geq i.$$

- Uniqueness. J.K. Zhang(1984), Yan & Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P.J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988).
- Ergodicity. Yan & Chen(1986).
- Strong ergodicity. Z.(2001).
- Polynomial moments and stationary dist. Z.(2003, 2004, 2013)
- Exponential moments. Z.(2003)
- Laplace transform. P. J. Brockwell(1986), W.J. Anderson(1991)

2. Uniqueness

Q-process: $P'(t)|_{t=0} = Q$.
Backward Kolmogorov equation: $P'(t) = QP(t)$.
Forward Kolmogorov equation: $P'(t) = P(t)Q$.

$$p_{ij}(t) = \sum_{k \neq i} \int_0^t q_{ik} e^{-q_i(t-s)} p_{kj}(s) ds + \delta_{ij} e^{-q_i t},$$

$$p_{ij}(t) = \sum_{k \neq j} \int_0^t p_{ik}(s) q_{kj} e^{-q_j(t-s)} ds + \delta_{ij} e^{-q_j t},$$

- Q-process exist? unique?

Theorem

Let $Q = (q_{ij})$ be totally stable and conservative. Every Q-process satisfies the backward Kolmogorov equation.

2. Uniqueness

Existence Theorem

Given a totally stable and conservative Q-matrix, there always exist a Q-process. In details, the minimal solution is a Q-process. Moreover, BE and FE have the same minimal solution.

Laplace transform: $p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t) dt$.

$$p_{ij}(\lambda) = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i},$$

$$p_{ij}(\lambda) = \sum_{k \neq j} p_{ik}(\lambda) \frac{q_{kj}}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_i}.$$

Denote the minimal process by $P^{\text{min}}(\lambda)$. Consider $P(\lambda) - P^{\text{min}}(\lambda)$:

2. Uniqueness

$$u_i = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} u_k, \quad 0 \leq u_i \leq 1, i \in \mathbb{Z}_+.$$

Uniqueness Theorem[Feller(1957), Reuter(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$, the Q-process is unique iff the equation has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

2. Uniqueness

Define

$$\widetilde{m}_0 = \frac{1}{q_{01}}, \quad \widetilde{m}_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \widetilde{m}_k\right), \quad n \geq 1.$$

$$\widetilde{m}_n = \sum_{k=0}^{n} \frac{\widetilde{F}_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Uniqueness[J.K. Zhang(1984), Yan & Chen(1986)]

Given a totally stable and conservative single birth Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

Sketch of proof. unique iff the solution (u_i) to the equation

$$(\lambda + q_i)u_i = \sum_{j\neq i} q_{ij}u_j, \qquad i \geq 0; \qquad u_0 = 1$$

is unbounded for some (equivalently for all) $\lambda > 0$. Rewrite it as

$$\Omega u = Qu - \lambda u = 0; \qquad u_0 = 1.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i \equiv 0$, we obtain:

$$u_n = 1 + \lambda \sum_{0\leq k\leq n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0\leq k\leq n-1} \widetilde{m}_k, \qquad n \geq 0.$$

Clearly, u_n is increasing in n and then is unbounded iff $\sum_n \widetilde{m}_n = \infty$. Thus, it remains to show that $\sum_n \widetilde{m}_n = \infty$ iff $\sum_n m_n = \infty$.

3. Recurrence, return/extinction probability

- $P(t)$ is recurrent if for each $h > 0$, $P(h)$ is recurrent. Equivalently, $\int_0^\infty p_{ii}(t)dt = \infty$.

embedding chain (Π_{ij}): $\Pi_{ij} = I_{\{q_i \neq 0\}}(1-\delta_{ij})\frac{q_{ij}}{q_i} + I_{\{q_i=0\}}\delta_{ij}$.

Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$,

$$\int_0^\infty p_{ij}^{\text{min}}(t)dt = \sum_{n=0}^\infty \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if Q is irreducible and regular, then $P(t)$ is recurrent iff so is its embedding chain.

regular: totally stable, conservative and Q-process is unique.

3. Recurrence, return/extinction probability

Theorem

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \leq x_i \leq 1, \; i \geq 0$$

has only a trivial solution.

For embedding chain (Π_{ij}), $(f_{i,j_0} = \mathbb{P}_i(\sigma_{j_0} < \infty) : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_i = \sum_{k \neq j_0} \Pi_{ik} x_k + \Pi_{i,j_0}, \; i \in \mathbb{Z}_+.$$

(Π_{ij}) is recurrent iff $f_{i,j_0} \equiv 1$.

3. Recurrence, return/extinction probability

Recurrence Theorem[Yan & Chen(1986)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Sketch of proof. recurrent iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad 0 \leq x_i \leq 1, \qquad i \geq 0$$

has only zero solution, where $\Pi_{ik} = (1 - \delta_{ik}) q_{ik} / q_i$. It is equivalent to that the solution to the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad i \geq 0; \qquad x_0 = 1$$

is unbounded.

3. Recurrence, return/extinction probability

(contin.)
Rewrite the previous equation as

$$(Qx)_0 = 0, \qquad (Qx)_i = q_{i0}, \qquad i \geq 1; \qquad x_0 = 1.$$

Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})$, we obtain the unique solution as follows

$$x_0 = 1, \quad x_n = 1 + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly, (x_n) is unbounded iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$. In other words, the equation has only a trivial solution iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$.

3. Recurrence, return/extinction probability

Define $\sigma_0 = \inf\{t \geq \text{the first jumping time: } X_t = 0\}$.

Proposition

Let the single birth Q-matrix $Q = (q_{ij})$ be non-explosive and irreducible. Then the return/extinction probability is as follows:

$$\mathbb{P}_0(\sigma_0 < \infty) = \frac{\sum_{k=1}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \mathbb{P}_n(\sigma_0 < \infty) = \frac{\sum_{k=n}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, n \geq 1.$$

Furthermore, $\mathbb{P}_n(\sigma_0 < \infty) = 1$ for all $n \geq 0$ iff $\mathbb{P}_0(\sigma_0 < \infty) = 1$, equivalently iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

P. J. Brockwell(1986), W. J. Anderson(1991)

Sketch of proof. $(\mathbb{P}_i(\sigma_0 < \infty) : i \in E)$ is the minimal nonnegative solution to the equation

$$x_i = \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i}(1 - \delta_{i0}), \qquad i \in E.$$

Then, we have $(Qx)_i = q_{i0}(1 - \delta_{i0})(x_0 - 1), i \geq 0$. Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$, we get

$$x_n = x_0 \left(1 + \sum_{1 \leq k \leq n-1} F_k^{(0)}\right) - \sum_{1 \leq k \leq n-1} F_k^{(0)}, \quad n \geq 0.$$

Because $x_n > 0$, it follows that

$$x_0 \geq \sup_{n \geq 1} \frac{\sum_{k=1}^{n-1} F_k^{(0)}}{\sum_{k=0}^{n-1} F_k^{(0)}} = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}.$$

From here, we obtain the minimal nonnegative solution:

$$x_0^* = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}, \qquad x_n^* = 1 - \frac{\sum_{k=0}^{n-1} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad n \geq 1.$$

3. Recurrence, return/extinction probability

Consider the occupation time of the state j:

$$T_j := \int_0^\infty \mathbf{1}_{\{X(t)=j\}} \mathrm{d}\, t.$$

By the return probability and the embedding chain, we get

Proposition[P.S. Li & Z.(2016)]

Let the single birth Q-matrix $Q = (q_{ij})$ be regular and irreducible. Then the mean occupation time

$$\mathrm{E}_i T_j = \frac{1}{q_{j,j+1}} \sum_{n=i\vee j}^{\infty} F_n^{(j)}.$$

4. Laplace transform of return/extinction time

Take $c_i \equiv -\lambda < 0$. Define $\left(\widetilde{F}_k^{(i)}\right)$ and

$$\tilde{d}_0 = 0,\ \tilde{d}_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{d}_k\right),\ n \geq 1.$$

Equivalently, $\tilde{d}_n = \sum_{1 \leq j \leq n} \frac{\widetilde{F}_n^{(j)}}{q_{j,j+1}},\ n \geq 0.$

Proposition

Let the single birth process be recurrent. Then the Laplace transform of σ_0 is given by

$$\mathbb{E}_0 e^{-\lambda\sigma_0} = \frac{q_{01}(1-\lambda\tilde{d})}{q_{01}+\lambda}, \mathbb{E}_n e^{-\lambda\sigma_0} = 1-\lambda\sum_{k=0}^{n-1}\left(\widetilde{F}_k^{(0)}\tilde{d}-\tilde{d}_k\right), n \geq 1,$$

where

$$\tilde{d} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\tilde{d}_k}{\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)}} = \lim_{n\to\infty}\frac{\tilde{d}_n}{\widetilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

P. J. Brockwell(1986), W. J. Anderson(1991)

Sketch of proof. $(\mathbb{E}_i e^{-\lambda\sigma_0} : i \in E)$ is the minimal solution (x_i^*) of the following equation

$$x_i = \frac{1}{q_i+\lambda}\sum_{k\notin\{0,i\}} q_{ik}x_k + \frac{q_{i0}(1-\delta_{i0})}{q_i+\lambda}, \qquad x_i \geq 1, \qquad i\in E.$$

Then, we have

$$(Qx)_i - \lambda x_i = q_{i0}(x_0-1), \quad i\geq 1; \quad (Qx)_0 - \lambda x_0 = 0.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i = q_{i0}(1-\delta_{i0})(x_0-1)$ for all $i\geq 0$,

$$\begin{aligned} x_n &= x_0\left(1+\frac{\lambda}{q_{01}}\right)\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)} - \sum_{k=0}^{n-1}\left(\widetilde{F}_k^{(0)} - \lambda\tilde{d}_k\right) + 1 \\ &=: x_0\alpha_{n-1} - \beta_{n-1}, \quad n\geq 1. \end{aligned}$$

By the minimal nonnegative property, $x_0^* = \sup_{n\geq 1} \beta_n/\alpha_n$, then

$$x_0^* = \overline{\lim_{n\to\infty}} \frac{\beta_n}{\alpha_n}.$$

We can replace $\overline{\lim}_{n\to\infty}$ by $\lim_{n\to\infty}$. On the one hand, since $x_n \in (0,1]$,

$$\frac{\beta_n}{\alpha_n} < x_0 \leq \frac{\beta_n + 1}{\alpha_n}, \qquad n \geq 1.$$

On the other hand, we can prove that $\sum_k \widetilde{F}_k^{(0)} = \infty$ since $\sum_k F_k^{(0)} = \infty$ by the recurrent assumption. Therefore, we have

$$x_0^* = \frac{q_{01}}{q_{01}+\lambda} \lim_{n\to\infty} \left[1 - \lambda \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \widetilde{F}_k^{(0)}}\right] = \frac{q_{01}}{q_{01}+\lambda}\left[1 - \lambda\tilde{d}\,\right].$$

$$x_n^* = (1-\lambda\tilde{d}) \sum_{k=0}^{n-1} \widetilde{F}_k^{(0)} - \sum_{k=0}^{n-1} \left(\widetilde{F}_k^{(0)} - \lambda\tilde{d}_k\right) + 1$$
$$= 1 - \lambda \sum_{k=0}^{n-1} \left(\widetilde{F}_k^{(0)}\tilde{d} - \tilde{d}_k\right), \quad n \geq 1.$$

Example [uniform catastrophes]

Let $q_{i,i+1} = b\,i$, $i \geq 0$; $q_{ij} = a$, $j = 0, 1, \ldots, i-1$; and $q_{ij} = 0$ for other $j > i+1$, where a and b are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n \mathrm{e}^{-\lambda\tau_0} = \frac{a}{a+\lambda}, \qquad \lambda > 0,\ n \geq 1.$$

It is surprising that the distribution is independent of b and the starting point n. Redefine $q_{01} = 1$. Then the irreducible process is indeed strongly ergodic.

P. J. Brockwell et al(1982)

5. Summary

Roughly speaking, the unified treatment consists of the following three steps.

(a) Find out the Poisson equation corresponding to the problem .

(b) Apply Theorem 1 to get the solution to Poisson equation.

(c) Work out the answer for the problem using the solution obtained in (b).

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv 0$
Uniqueness	$c_i \equiv -\lambda < 0$	$f_i \equiv 0$
Recurrence	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})$
Ext./return probability	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv 0$	$f_i^{(\ell)}$
Exp. moment/ergod.	$c_i \equiv \lambda > 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Laplace transform	$c_i \equiv -\lambda < 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$

where $f_i^{(\ell)} = q_{ii_0}(1-\delta_{ii_0})g_{i_0} - \ell\mathbb{E}_i\sigma_{i_0}^{\ell-1}$.

Let the single death Q-matrix $Q = (q_{ij})$ be regular and irreducible. Assume that the corresponding process is ergodic. To get the stationary distribution of the process, solve

$$\pi Q = 0.$$

Proposition[L.D. Wang & Z.(2014)]

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n/\mu,\ n \geq 0$, where

$$\mu_0 = 1,\ \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} \bar{q}_k^{(n)} \mu_k,\ n \geq 1;$$

$$\mu = \sum_{n=0}^{\infty} \mu_n;\ \bar{q}_k^{(n)} = \sum_{j=n}^{\infty} q_{kj},\ n > k.$$

Transpose the matrix equation $\pi Q = \mathbf{0}$ to

$$Q^{\mathsf{T}} \pi^{\mathsf{T}} = \mathbf{0}^{\mathsf{T}}.$$

Then

$$Q^{\mathsf{T}} g =: \Omega g = Q^* g + c g = \mathbf{0}^{\mathsf{T}},$$

where Q^* is a totally stable and conservative single birth Q-matrix and

$$c_i = \sum_{j=0}^{i-1} q_{ji} + q_{i+1,i} - \sum_{j=i+1}^{\infty} q_{ij} - q_{i,i-1}, \; i \geq 0;$$

$$f_i \equiv 0.$$

Proposition[Z.(2016)]

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n/\mu,\ n \ge 0$, where

$$\mu_0 = 1,\ \mu_n = 1 - \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j+1,j}}, n \ge 1;\ \mu := \sum_{n=0}^{\infty} \mu_n;$$

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \frac{1}{q_{n+1,n}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \quad n > i \ge 0,$$

$$\tilde{q}_n^{(k)} = \sum_{j=0}^{k} q_{jn} - c_n, \quad n > k \ge 0.$$

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Thank you for your attention!

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