Separation cutoff for skip-free upwardly chains

Yu-Hui Zhang

Beijing Normal Univesity

(June 19, 2015, Moscow State Lemonosov University)

This talk is based on the joint work with Yong-Hua Mao & Chi Zhang.

Outline

1. Background

Cutoff is referred to a family of ergodic Markov chains shows a sharp transition when converging to their stationary distributions. Let $P^{(n)}(t)$ be the distribution of an finite ergodic Markov chains $X_t^{(n)}$ at time t, its stationary distribution is $\pi^{(n)}$. Then for any "distance" D between probability measures,

$$\lim_{t\to\infty} D(P^{(n)}(t), \pi^{(n)}) = 0$$

holds for every n.
However involving n, it may happen that

$$\lim_{n\to\infty} D(P^{(n)}(ct_n), \pi^{(n)}) = \begin{cases} 0 & \text{for } c > 1; \\ 1 & \text{for } c < 1. \end{cases}$$

1. Background

This is called (D-)cutoff phenomenon by Persi Diaconis (1996). The “distance” D usually can be chosen to be

- separation (Diaconis & Saloff-Coste (2006))
- total variance (Ding et al(2010))
- $\max -L^2$ distance (Chen & Saloff-Coste(2010))

For two probability measures μ and ν, the separation is defined as

$$\text{sep}(\mu, \nu) = \max_i (1 - \mu_i/\nu_i).$$

Aldous & Diaconis (1987). separation is not a distance.

However separation is easily handled and powerful in the following sense. For a finite Markov chain, let $P(t)$ be the distribution at t, there exists a so-called fastest strong stationary time (FSST) τ such that

$$\text{sep}(P(t), \pi) = \mathbb{P}[\tau > t], \; t \geq 0.$$

($X_\tau \sim^{\text{d}} \pi$, X_τ and τ are independent.)

Aldous & Diaconis (1987) for the definitions and properties for FSST and Fill(1991) for the existence of FSST.

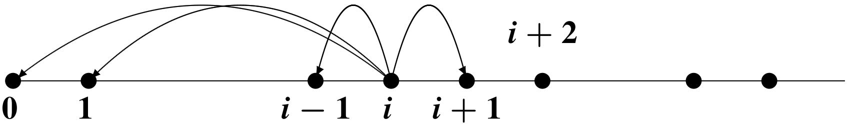
Diaconis & Saloff-Coste(2006) first gave the criteria for the continuous time or discrete time birth and death chains, these criteria involves all the eigenvalues of each chain.

Mao & Z.(2014) give the explicit criteria, which depend only on the one-step transition probability in the discrete time case, and on the birth and death rates in the continuous time case.

2. Our main results

In this talk, we study the separation cutoff for the skip-free upwardly chains (or single birth processes). Birth and death chains are special cases of the skip-free chains.

$Q = (q_{ij} : i,j \geq 0)$: $q_{i,i+1} > 0, \ q_{i,i+j} = 0, \ i \geq 0, j \geq 2$.



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let us first recall some basics for the skip-free upwardly chains. On the finite state space $\{0, 1, \ldots, N\}$, let $Q = (q_{ij})$ be the generator of an irreducible and conservative upward skip-free chain. That is, for $0 \le i < N$ and $j > i+1$, $q_{i,i+1} > 0, q_{ij} = 0$; and for $0 \le i \le N, q_i := -q_{ii} = \sum_{j \ne i} q_{ij} < \infty$. For $0 \le k < i \le N$, define $q_i^{(k)} = \sum_{j=0}^{k} q_{ij}$, and

$$F_i^{(i)} = 1, \quad F_i^{(k)} = \sum_{j=k+1}^{i} \frac{F_i^{(j)} q_j^{(k)}}{q_{j,j+1}}.$$

Then define

$$m_i = \sum_{k=0}^{i} \frac{F_i^{(k)}}{q_{k,k+1}}, \quad 0 \le i \le N.$$

Z.(2013) gives the stationary distribution as follows

$$\pi_i = \frac{F_N^{(i)}}{q_{i,i+1} m_N}, \quad 0 \le i \le N.$$

2. Our main results

Define

$$T = \sum_{i=0}^{N} \pi_i \sum_{j=0}^{i-1} m_j, \quad S = \sum_{i=0}^{N} \pi_i \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{F_j^{(k)}}{q_{k,k+1}} \sum_{\ell=0}^{k-1} m_\ell.$$

Theorem

For each n, assume that $X_t^{(n)}$ is an upward skip-free chain on $\{0, 1, \ldots, N_n\}$, started at 0 and with the stochastically monotone time-reversal. Define $T^{(n)}$ and $S^{(n)}$ with N replaced by N_n. Then there exists the separation cutoff with $t_n = T^{(n)}$ if and only if

$$\lim_{n\to\infty} \frac{S^{(n)}}{[T^{(n)}]^2} = \frac{1}{2}.$$

2. Our main results

The following corollary gives a useful sufficient condition for separation cutoff.

Corollary

For each $n = 1, 2, \ldots$, let $Q^{(n)} = (q_{ij}^{(n)})$ be the generator for a skip-free upwardly chains on $\{0, 1, \ldots, N_n\}$, where $q_{i,i+1}^{(n)} = 1$ for $1 \le i < N_n$, and $q_{ki}^{(n)} \le q_{k,i+1}^{(n)}$ for $k > i + 1$. If there is $C > 0$ and $\beta > 1$ such that $F_i^{(j)} \sim C\beta^{i-j}$ as $i - j \to \infty$, then the separation cutoff occurs.

$$m_i \sim \beta^{i+1}/(\beta - 1), \quad \pi_i = F_N^{(i)}/m_N \sim (\beta - 1)\beta^{-i-1}.$$

$$T \sim \frac{N}{\beta - 1}, \quad S \sim \frac{N^2}{2(\beta - 1)^2}.$$

2. Our main results

Next we would like to present some examples. We will focus on the restricted chains of a skip-free upwardly chain X_t with generator $Q = (q_{ij})$ on $E = \{0, 1, 2, \dots\}$.
For an increasing sequence $\{N_n\}$ with limit ∞ as $n \to \infty$, we define a sequence of ergodic chains $\{X_t^{(n)}\}$ as below. For each $X_t^{(n)}$, let $Q^{(n)} = (q_{ij}^{(n)})$ be its generator satisfying $q_{ij}^{(n)} = q_{ij}$ $(0 \le i \le N_n, 0 \le j \le N_n)$ except for $q_{N_n,N_n}^{(n)} = -\sum_{j=0}^{N_n-1} q_{N_n,j}$. Then $X_t^{(n)}$ is a restricted chain of X_t on $\{0, 1, \dots, N_n\}$.
We say that X_t exhibits the separation cutoff if it occurs for the family of restricted chains $\{X_t^{(n)}\}$. Obviously, the choice of the increasing sequence $\{N_n\}$ going to infinity has no impact on the occurrence of the separation cutoff for X_t.

Examples

(1) Let $q_{01} = q_{10} = q_{12} = 1$ and $q_{i,i+1} = q_{i,i-1} = q_{i,i-2} = 1$, $q_{ii} = -3$ $(i \geq 2)$. Then the skip-free chain exhibits the separation cutoff.

$$F_i^{(j)} \sim C(\sqrt{2} + 1)^{i-j}.$$

(2) Let $q_{i,i+1} = 1$ $(i \geq 0)$ and $q_{ij} = 1/i$ $(0 \leq j < i)$. Then the chain exhibits the separation cutoff.

$$F_i^{(j)} \sim 2^{i-j-1}.$$

(3) Let $q_{i,i+1} = 1$, $q_{i0} = p^i$ $(i \geq 0)$, and $q_{ij} = (1-p)p^{i-j-1}$ $(1 \leq j < i)$. Then the chain exhibits the separation cutoff if $0 < p \leq (\sqrt{5} - 1)/2$.

$$F_i^{(j)} = (1 + p)^{i-j-1}.$$

3. Finite skip-free upwardly chains

Define the time-reversal $\widetilde{Q} = (\tilde{q}_{ij})$ of Q as

$$\tilde{q}_{ij} = \frac{\pi_j}{\pi_i} q_{ji}.$$

Proposition

The time-reversal chain of Q is stochastically monotone if and only if

$$\sum_{k\geq j} \frac{F_N^{(k)} q_{ki}}{q_{k,k+1}} \leq \frac{q_{i+1,i+2} F_N^{(i)}}{q_{i,i+1} F_N^{(i+1)}} \sum_{k\geq j} \frac{F_N^{(k)} q_{k,i+1}}{q_{k,k+1}}, \quad i+1<j\leq N.$$

Corollary

Assume that $q_{i,i+1} \equiv 1 (i \geq 0)$. If $F_N^{(i)} \geq F_N^{(i+1)}$ for all $i < N-1$ and $q_{ki} \leq q_{k,i+1}$ for all $k > i+1$, then the time-reversal chain is stochastically monotone.

3. Finite skip-free upwardly chains

Under the assumption that the time-reversal chain is stochastically monotone, Fill(2009) obtained the following theorem

Theorem A

For an ergodic continuous-time skip-free upwardly chain in the state space $\{0, \ldots, N\}$ started at 0 and with stochastically monotone time-reversal, let Q be the generator. Then the fastest strong stationary time τ has the distribution

$$\mathbb{E}e^{-\lambda\tau} = \prod_{\nu=1}^{N} \frac{\lambda_\nu}{\lambda + \lambda_\nu}, \quad \lambda > 0,$$

where $\lambda_1, \ldots, \lambda_N$ are the nonzero eigenvalues of $-Q$.

Define the hitting time of state k as $\tau_k = \inf\{t \geq 0 : X_t = k\}$.

Lemma

For the ergodic continuous time upward skip-free chain X_t on $\{0, 1, \ldots, N\}$, for $0 \leq i,j < N$, let

$$\phi_{iN}(\lambda) = 0,\ \phi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}_i[X_t = j, t < \tau_N]dt$$

$$\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}_i[X_t = j]dt, \quad 0 \leq i,j \leq N.$$

It holds that for $0 \leq i,j \leq N$

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{\xi_i(\lambda)\eta_j(\lambda)}{\lambda \sum_{j=0}^N \eta_j(\lambda)},$$

where $\xi_i(\lambda) = 1 - \lambda \sum_{k=0}^{N-1} \phi_{ik}(\lambda), \eta_j(\lambda) = \pi_j - \lambda \sum_{k=0}^{N-1} \pi_k \phi_{kj}(\lambda)$.

4. A general condition for separation cutoff

To use Theorem A(Fill) to derive the criterion, we need the following result. This result was originated in Diaconis & Saloff-Coste (2006) and completed recently in Mao & Z.(2014).

Proposition B

For each n, let $\tau^{(n)}$ be a FSST of the ergodic Markov chains $X_t^{(n)}$. Assume that there is $C < \infty$ such that

$$\mathbb{E}(\tau^{(n)})^3 \le C(\mathbb{E}\tau^{(n)})^3 \text{ for all } n.$$

Then there exists the separation cutoff with $t_n = \mathbb{E}\tau^{(n)}$ if and only if

$$\frac{(\mathbb{E}\tau^{(n)})^2}{\mathrm{Var}(\tau^{(n)})} \to \infty, \text{ or equivalently, } \frac{(\mathbb{E}\tau^{(n)})^2}{\mathbb{E}(\tau^{(n)})^2} \to 1 \text{ as } n \to \infty.$$

4. A general condition for separation cutoff

We remark that the integrability condition is natural for Markov chains. For example, if $X_t^{(n)}$ is a family of upward skip-free chains started at $\mathbf{0}$ and with stochastically monotone time-reversals, then we can easily get from Theorem A that

$$\mathbb{E}(\tau^{(n)})^k \leq k!(\mathbb{E}\tau^{(n)})^k, \quad k = 1, 2, \ldots.$$

5. Proof on explicit criterion

We will obtain the explicit criterion for the separation cutoff of skip-free upwardly chains by deriving the explicit expressions of $\mathbb{E}\tau^{(n)}$ and $\mathbb{E}(\tau^{(n)})^2$ in Proposition B.
Firstly, we study the distribution of the FSST in the following theorem.

Theorem

Assume that X_t is an ergodic skip-free upwardly chain on $\{0, 1, \ldots, N\}$, started at 0 and with the stochastically monotone time-reversal. Let τ be a FSST and $P_i(t) = \mathbb{P}_0[X_t = i]$ for $0 \le i \le N$. Then
(1) $\mathbb{P}[\tau > t] = 1 - P_N(t)/\pi_N$;
(2)

$$\mathbb{E}e^{-\lambda\tau} = \frac{\lambda}{\pi_N}\int_0^\infty e^{-\lambda t} P_N(t)dt, \ \lambda \ge 0.$$

5. Proof on explicit criterion

Let $p_{ij}(t) = \mathbb{P}_i[X_t = j]$ and X_t^* be the time-reversal chain of X_t. Then

$$p_{ij}^*(t) := \mathbb{P}_i[X_t^* = j] = \frac{\pi_j p_{ji}(t)}{\pi_i}.$$

Since X_t^* is stochastically monotone, we have

$$p_{0N}(t)/\pi_N = p_{N0}^*(t)/\pi_0 = \min_i p_{i0}^*(t)/\pi_0 = \min_i p_{0i}(t)/\pi_i.$$

Thus,

$$1 - P_N(t)/\pi_N = \max_i (1 - p_{0i}(t)/\pi_i) = \mathbb{P}[\tau > t].$$

5. Proof on explicit criterion

To derive the explicit formulas for the moments of the FSST, we need the boundary theory below, which establishes a relationship between the FSST and the hitting times.

Theorem

For the chain X_t defined in the theorem above, the Laplace transform of the FSST can be expressed as

$$\mathbb{E}e^{-\lambda\tau} = \left(\pi_0 + \sum_{k=1}^{N} \pi_k \left(\mathbb{E}_0 e^{-\lambda\tau_k}\right)^{-1}\right)^{-1}, \ \lambda \geq 0.$$

5. Proof on explicit criterion

Now we can deduce the explicit criteria of the separation cutoff. For the chain X_t in the theorem above, we can obtain the explicit expressions for the moments of the FSST τ from those of the hitting times $\{\tau_k\}$.
In fact, by taking derivatives twice, we get

$$\mathbb{E}\tau = \sum_{i=0}^{N} \pi_i \mathbb{E}_0 \tau_i, \ \mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 + \sum_{i=0}^{N} \pi_i \mathbb{E}_0 \tau_i^2 - 2\sum_{i=0}^{N} \pi_i (\mathbb{E}_0 \tau_i)^2.$$

By Z.(2013),

$$\mathbb{E}_0 \tau_i = \sum_{k=0}^{i-1} m_k, \ \mathbb{E}_0 \tau_i^2 = 2(\mathbb{E}_0 \tau_i)^2 - 2\sum_{k=0}^{i-1} \sum_{\ell=0}^{k} \frac{F_k^{(\ell)}}{q_{\ell,\ell+1}} \mathbb{E}_0 \tau_\ell.$$

5. Proof on explicit criterion

We can easily get that

$$\mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 - 2\sum_{i=0}^{N} \pi_i \sum_{j=0}^{i-1} \sum_{k=0}^{j} \frac{F_j^{(k)}}{q_{k,k+1}} \mathbb{E}_0 \tau_k.$$

Then we have

$$\frac{\mathbb{E}\tau^2}{(\mathbb{E}\tau)^2} = 2\left(1 - \frac{S}{T^2}\right).$$

Thus our criterion follows from Proposition B.

6. References

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Thank you for your attention!

Homepage: http://math.bnu.edu.cn/~zhangyh/