单生过程一些问题的统一处理

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Chen, Mu-Fa, Z. Unified representation of formulas for single birth processes, Frontiers of Mathematics in China, 2014, 9(4): 761-796.

Outline

1. Background
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1. Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t))$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0$, $\sum_j p_{ij}(t) \leq 1$.

  (2) Chapman-Kolmogorov eq.: $p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s)$.

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$, $i, j \in \mathbb{Z}_+$.

  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij}$, $i, j \in \mathbb{Z}_+$, i.e. Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

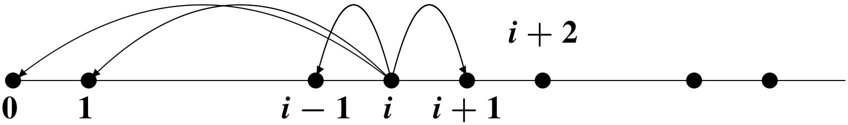
- totally stable: $q_i < \infty$, $i \in \mathbb{Z}_+$.

- conservative: $q_i = \sum_{j\neq i} q_{ij}$, $i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

1. Background

Single birth process: $Q = (q_{ij})$:

$$q_{i,i+1} > 0,\ q_{i,i+j} = 0,\ i \geq 0, j \geq 2.$$



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Background

1. The exit boundary of the process consists at most one single extremal point
2. Fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes
3. Usually the single birth process is non-symmetric (irreversible)
4. Applications in other fields

1. Background

1. Upwardly skip-free proc. (W.J. Anderson, 1991);
2. generalized birth and death processes (J.K. Zhang, 1984);
3. Population proc.: birth and death proc. with catastrophes; birth, death and catastrophe proc.

$$\begin{array}{lll} i \longrightarrow i+1 & \text{at rate} & b_i \\ \quad\longrightarrow i-1 & & a_i + c_i f_{i,i-1} \\ \quad\longrightarrow i-2 & & c_i f_{i,i-2} \\ \quad\longrightarrow \cdots & & \cdots \\ \quad\longrightarrow 0 & & c_i f_{i0} \end{array}$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

1. Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1-p)^{i-1-j}$ $(1 \leq j < i)$;$f_{i0} = (1-p)^{i-1}$;

uniform: $f_{ij} = 1/i$ $(0 \leq j < i)$;

binomial: $f_{ij} = \binom{i-1}{j} p^j (1-p)^{i-1-j}$ $(0 \leq j < i)$.

Aim: extinction times and probability of extinction.

Keys: generating function of Q-resolvent.

B. Cairns & P. Pollett (2004).

$$\begin{aligned} i \longrightarrow\ & i+1 && \text{at rate} && g_i b \\ \longrightarrow\ & i-1 && && g_i d_1 \\ \longrightarrow\ & \cdots && && \cdots \\ \longrightarrow\ & 1 && && g_i d_{i-1} \\ \longrightarrow\ & 0 && && g_i \sum_{k\geq i} d_k \end{aligned}$$

where $b + \sum_{k\geq 1} d_k = 1$.

1. Background

single birth process:
Yan, S.-J. and Chen, M.-F.(1986), Multidimensional Q-processes, Chinese Ann. Math., **7(B)**(1), 90-110.

Chen, M.-F.(2004), From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore.

陈木法, 毛永华(2007), 随机过程导论, 高等教育出版社, 北京.

Population processes:

Anderson, W. J.(1991), Continuous-Time Markov Chains, Springer-Verlag, New York.

2. Poisson equation

Given a function c, define

$$\Omega g = Qg + cg, \quad \text{where} \quad (Qg)_i = \sum_j q_{ij}(g_j - g_i).$$

Clearly, if $c \leq 0$, then Ω is an operator corresponding to a single birth process with killing rates $(-c_i)$. Define

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad n > i \geq 0,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \leq k < n.$$

Theorem 1

Given a single-birth Q-matrix $Q = (q_{ij})$ and functions c and f, the solution g to the Poisson equation

$$\Omega g = f$$

has the following representation:

$$g_n = g_0 + \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad n \ge 0.$$

In particular, the harmonic function of Ω can be represented as

$$g_n = g_0 \left(1 - \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \ge 0.$$

Proof For each $i \geq 0$, we have

$$
\begin{aligned}
(\Omega g)_i &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} q_{ij} \sum_{k=j}^{i-1} (g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \sum_{j=0}^{k} q_{ij}(g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \left(\sum_{j=0}^{k} q_{ij} - c_i \right)(g_{k+1} - g_k) + c_i g_0 \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + c_i g_0.
\end{aligned}
$$

$$
g_{i+1} - g_i = \frac{1}{q_{i,i+1}} \left(\sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + f_i - c_i g_0 \right).
$$

$$g_{i+1} - g_i = \sum_{j=0}^{i} \frac{\widetilde{F}_i^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \qquad i \geq 0.$$

Proposition 1

For given f, the sequence (h_n) defined successively by

$$h_n = \frac{1}{q_{n,n+1}}\left(f_n + \sum_{i\leq k\leq n-1} \tilde{q}_n^{(k)} h_k\right), \qquad n \geq i$$

has a unified expression as follows

$$h_n = \sum_{k=i}^{n} \frac{\widetilde{F}_n^{(k)} f_k}{q_{k,k+1}}, \qquad n \geq i.$$

In particular, $\widetilde{F}_i^{(i)} = 1, \widetilde{F}_n^{(i)} = \sum_{k=i+1}^{n} \frac{\widetilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, n \geq i+1.$

Proposition 2

The solution $(h_n)_{n\geq i}$ to the recursive equations

$$h_n = \frac{1}{\beta_n}\left(\sum_{i\leq k\leq n-1}\alpha_{nk}h_k + f_n\right), \qquad n\geq i$$

can be represented as

$$h_n = \sum_{i\leq k\leq n}\frac{\gamma_{nk}}{\beta_k}f_k, \qquad n\geq i,$$

where for each fixed i, $(\gamma_{ni})_{n\geq i}$ with $\gamma_{ii}=1$ is the solution to

$$\gamma_{ni} = \frac{1}{\beta_n}\sum_{i\leq k\leq n-1}\alpha_{nk}\gamma_{ki}, \qquad n>i.$$

Equivalently, $\gamma_{ii}=1, \gamma_{ni} = \sum_{i+1\leq k\leq n}\frac{\gamma_{nk}}{\beta_k}\alpha_{ki}, n\geq i+1.$

Proposition 3

For given real numbers $(\alpha_{nk})_{n-1\ge k\ge 0}$ and $(f_n)_{n\ge 0}$, the solution $(g_n)_{n\ge 0}$ to the recursive inhomogeneous equations

$$g_n = \sum_{0\le k\le n-1} \alpha_{nk} g_k + f_n, \qquad n \ge 0$$

can be represented as

$$g_n = \sum_{0\le k\le n} \gamma_{nk} f_k, \qquad n \ge 0,$$

where for fixed $k \ge 0$, $(\gamma_{nk})_{n\ge k}$ with $\gamma_{kk} = 1$ is the solution to the recursive equations

$$\gamma_{nk} = \sum_{k\le j\le n-1} \alpha_{nj}\gamma_{jk}, \qquad n > k.$$

3. Uniqueness

Q-process: $P'(t)|_{t=0} = Q$.
Backward Kolmogorov equation: $P'(t) = QP(t)$.
Forward Kolmogorov equation: $P'(t) = P(t)Q$.

$$p_{ij}(t) = \sum_{k \neq i} \int_0^t q_{ik} e^{-q_i(t-s)} p_{kj}(s) ds + \delta_{ij} e^{-q_i t},$$

$$p_{ij}(t) = \sum_{k \neq j} \int_0^t p_{ik}(s) q_{kj} e^{-q_j(t-s)} ds + \delta_{ij} e^{-q_j t},$$

- Q-process exist? unique?

Theorem

Let $Q = (q_{ij})$ be totally stable and conservative. Every Q-process satisfies the backward Kolmogorov equation.

3. Uniqueness

Existence Theorem

Given a totally stable and conservative Q-matrix, there always exist a Q-process. In details, the minimal solution is a Q-process. Moreover, BE and FE have the same minimal solution.

Laplace transform: $p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt$.

$$p_{ij}(\lambda) = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i},$$

$$p_{ij}(\lambda) = \sum_{k \neq j} p_{ik}(\lambda) \frac{q_{kj}}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_i}.$$

Denote the minimal process by $P^{\text{min}}(\lambda)$. Consider $P(\lambda) - P^{\text{min}}(\lambda)$:

3. Uniqueness

$$u_i = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} u_k, \quad 0 \le u_i \le 1, i \in \mathbb{Z}_+.$$

Uniqueness Theorem[Feller(1957), Reuter(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$, the Q-process is unique iff the equation has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

Define

$$\widetilde{m}_0 = \frac{1}{q_{01}}, \quad \widetilde{m}_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \widetilde{m}_k\right), \quad n \geq 1.$$

$$\widetilde{m}_n = \sum_{k=0}^{n} \frac{\widetilde{F}_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Uniqueness[J.K. Zhang(1984), S.J. Yan & M.F. Chen(1986)]

Given a totally stable and conservative single birth Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

Proof. unique iff the solution (u_i) to the equation

$$(\lambda + q_i)u_i = \sum_{j \neq i} q_{ij}u_j, \qquad i \geq 0; \qquad u_0 = 1$$

is unbounded for some (equivalently for all) $\lambda > 0$. Rewrite it as

$$\Omega u = Qu - \lambda u = 0; \qquad u_0 = 1.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i \equiv 0$, we obtain:

$$u_n = 1 + \lambda \sum_{0 \leq k \leq n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0 \leq k \leq n-1} \widetilde{m}_k, \qquad n \geq 0.$$

Clearly, u_n is increasing in n and then is unbounded iff $\sum_n \widetilde{m}_n = \infty$. Thus, it remains to show that $\sum_n \widetilde{m}_n = \infty$ iff $\sum_n m_n = \infty$.

4. Recurrence

- $P(t)$ is recurrent if for each $h > 0$, $P(h)$ is recurrent. Equivalently, $\int_0^\infty p_{ii}(t)dt = \infty$.

embedding chain (Π_{ij}): $\Pi_{ij} = I_{\{q_i \neq 0\}}(1 - \delta_{ij})\dfrac{q_{ij}}{q_i} + I_{\{q_i = 0\}}\delta_{ij}$.

Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$,

$$\int_0^\infty p_{ij}^{\min}(t)dt = \sum_{n=0}^{\infty} \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if Q is irreducible and regular, then $P(t)$ is recurrent iff so is its embedding chain.

regular: totally stable, conservative and Q-process is unique.

4. Recurrence

Theorem

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

has only a trivial solution.

For embedding chain (Π_{ij}), $(f_{i,j_0} = \mathbb{P}_i(\sigma_{j_0} < \infty) : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_i = \sum_{k \neq j_0} \Pi_{ik} x_k + \Pi_{i,j_0},\ i \in \mathbb{Z}_+.$$

(Π_{ij}) is recurrent iff $f_{i,j_0} \equiv 1$.

4. Recurrence: single birth processes

Recurrence Theorem[Yan & Chen(1986)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Proof. recurrent iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad 0 \leq x_i \leq 1, \qquad i \geq 0$$

has only zero solution, where $\Pi_{ik} = (1 - \delta_{ik}) q_{ik}/q_i$. It is equivalent to that the solution to the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad i \geq 0; \qquad x_0 = 1$$

is unbounded.

(contin.)
Rewrite the previous equation as

$$(Qx)_0 = 0, \qquad (Qx)_i = q_{i0}, \qquad i \geq 1; \qquad x_0 = 1.$$

Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})$, we obtain the unique solution as follows

$$x_0 = 1, \quad x_n = 1 + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly, (x_n) is unbounded iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$. In other words, the equation has only a trivial solution iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$.

5. Laplace transform of return/extinction time

Define

$$\tilde{d}_0 = 0, \quad \tilde{d}_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{d}_k\right), n \geq 1,$$

$$\tilde{d}_n = \sum_{1 \leq j \leq n} \frac{\widetilde{F}_n^{(j)}}{q_{j,j+1}}, \quad n \geq 0.$$

$$\sigma_0 = \inf\{t \geq \text{the first jumping time: } X_t = 0\}.$$

5. Laplace transform of return/extinction time

Proposition

Define $(\widetilde{F}_k^{(i)})$ and $(\tilde{d}_k)$ with $c_i \equiv -\lambda < 0$. Let the single birth process be recurrent. Then the Laplace transform of σ_0 is given by

$$\mathbb{E}_0 e^{-\lambda\sigma_0} = \frac{q_{01}(1-\lambda\tilde{d})}{q_{01}+\lambda}, \mathbb{E}_n e^{-\lambda\sigma_0} = 1-\lambda\sum_{k=0}^{n-1}\left(\widetilde{F}_k^{(0)}\tilde{d}-\tilde{d}_k\right), n \geq 1,$$

where

$$\tilde{d} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\tilde{d}_k}{\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)}} = \lim_{n\to\infty}\frac{\tilde{d}_n}{\widetilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

P. J. Brockwell(1986), W. J. Anderson(1991)

Proof. $(\mathbb{E}_i e^{-\lambda\sigma_0} : i \in E)$ is the minimal solution (x_i^*) of the following equation

$$x_i = \frac{1}{q_i+\lambda}\sum_{k\notin\{0,i\}} q_{ik}x_k + \frac{q_{i0}(1-\delta_{i0})}{q_i+\lambda}, \qquad x_i \geq 1, \qquad i \in E.$$

Then, we have

$$(Qx)_i - \lambda x_i = q_{i0}(x_0 - 1), \quad i \geq 1; \quad (Qx)_0 - \lambda x_0 = 0.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i = q_{i0}(1-\delta_{i0})(x_0-1)$ for all $i \geq 0$,

$$\begin{aligned} x_n &= x_0\left(1+\frac{\lambda}{q_{01}}\right)\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)} - \sum_{k=0}^{n-1}\left(\widetilde{F}_k^{(0)} - \lambda\tilde{d}_k\right) + 1 \\ &=: x_0\alpha_{n-1} - \beta_{n-1}, \quad n \geq 1. \end{aligned}$$

By the minimal nonnegative property, $x_0^* = \sup_{n\geq 1} \beta_n/\alpha_n$, then

$$x_0^* = \overline{\lim_{n\to\infty}} \frac{\beta_n}{\alpha_n}.$$

We can replace $\overline{\lim}_{n\to\infty}$ by $\lim_{n\to\infty}$. On the one hand, since $x_n \in (0,1]$,

$$\frac{\beta_n}{\alpha_n} < x_0 \leq \frac{\beta_n + 1}{\alpha_n}, \qquad n \geq 1.$$

On the other hand, we can prove that $\sum_k \widetilde{F}_k^{(0)} = \infty$ since $\sum_k F_k^{(0)} = \infty$ by the recurrent assumption. Therefore, we have

$$x_0^* = \frac{q_{01}}{q_{01}+\lambda} \lim_{n\to\infty} \left[1 - \lambda \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \widetilde{F}_k^{(0)}}\right] = \frac{q_{01}}{q_{01}+\lambda}\left[1 - \lambda\tilde{d}\,\right].$$

$$x_n^* = (1-\lambda\tilde{d}) \sum_{k=0}^{n-1} \widetilde{F}_k^{(0)} - \sum_{k=0}^{n-1} \left(\widetilde{F}_k^{(0)} - \lambda\tilde{d}_k\right) + 1$$

$$= 1 - \lambda \sum_{k=0}^{n-1} \left(\widetilde{F}_k^{(0)}\tilde{d} - \tilde{d}_k\right), \quad n \geq 1.$$

Example [uniform catastrophes]

Let $q_{i,i+1} = b\,i,\ i \geq 0$; $q_{ij} = a, j = 0, 1, \ldots, i-1$; and $q_{ij} = 0$ for other $j > i+1$, where a and b are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n e^{-\lambda\tau_0} = \frac{a}{a+\lambda}, \qquad \lambda > 0,\ n \geq 1.$$

It is surprising that the distribution is independent of b and the starting point n. Redefine $q_{01} = 1$. Then the irreducible process is indeed strongly ergodic.

P. J. Brockwell et al(1982)

6. Summary

Roughly speaking, the unified treatment consists of the following three steps.

(a) Find out the Poisson equation corresponding to the problem .

(b) Apply Theorem 1 to get the solution to Poisson equation.

(c) Work out a criterion for the problem using the solution obtained in (b).

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv 0$
Uniqueness	$c_i \equiv -\lambda < 0$	$f_i \equiv 0$
Recurrence	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})$
Ext./return probability	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv 0$	$f_i^{(\ell)}$
Exp. moment/ergod.	$c_i \equiv \lambda > 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Laplace transform	$c_i \equiv -\lambda < 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$

where $f_i^{(\ell)} = q_{ii_0}(1-\delta_{ii_0})g_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$.

- Uniqueness. J.K. Zhang(1984), Yan & Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P. J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988).
- Ergodicity. S.J. Yan & M.F. Chen(1986).
- Strong ergodicity. Z.(2001).
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- Exponential moments. Z.(2003)
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Thank you for your attention!

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