Explicit criteria on separation cutoff for birth and death chains

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This talk is based on the joint work with Yong-Hua Mao and Chi Zhang.

Outline

1. Background

- What is separation cutoff?

Cutoff is referred to a family of ergodic Markov chains shows a sharp transition when converging to their stationary distributions. Let $P^{(n)}(t)$ be the distribution of an finite ergodic Markov chains X_t at time t, its stationary distribution is $\pi^{(n)}$. Then

$$\lim_{t\to\infty} D(P^{(n)}(t), \pi^{(n)}) = 0$$

holds for each n.
However involving n, it may happen that

$$\lim_{n\to\infty} D(P^{(n)}(ct_n), \pi^{(n)}) = \begin{cases} 0 & \text{for } c > 1; \\ 1 & \text{for } c < 1. \end{cases}$$

1. Background

This is called (D-)cutoff phenomenon by Persi Diaconis (1996). The "distance" D usually can be chosen to be

- separation (Diaconis & Saloff-Coste (2006))
- total variance (Ding et al(2010))
- max $-L^2$ distance (Chen & Saloff-Coste(2010))

1. Background

For two probability measures μ and ν, the separation is defined as

$$\mathrm{sep}(\mu,\nu)=\max_i(1-\mu_i/\nu_i).$$

Aldous & Diaconis (1987). separation is not a distance.
For a finite Markov chain, let $P(t)$ be the distribution at t, there exists a so-called fastest strong stationary time (FSST) τ such that

$$\mathrm{sep}(P(t),\pi)=\mathbb{P}[\tau>t],\ t\geq 0.$$

Aldous & Diaconis (1987), Fill(1991).
($X_\tau \sim^{\mathrm{d}} \pi$, X_τ and τ are independent.)

Diaconis & Saloff-Coste(2006) first gave the criteria for birth and death chains, these criteria involves all the eigenvalues of each chain.

1. Background

We give the explicit criteria, which depend only on the one-step transition probability in the discrete time case, and on the birth and death rates in the continuous time case.

Since the birth and death chain is symmetric, its eigenvalues are real. But for a general skip-free chains, eigenvalues may be complex.

Fill(2009) obtained the distribution of the FSST for upward skip-free chains (single birth chain). So we will study the separation cutoff for the upward skip-free chains.

To get the FSST, the upward skip-free chain is assumed to have the stochastically monotone time-reversal.

2. A general condition for separation cutoff

Proposition(Diaconis & Saloff-Coste(2006))

Let $\tau^{(n)}$ be FSST's for a family of Markov chains. Assume that there is $C < \infty$ such that

$$\mathbb{E}(\tau^{(n)})^3 \leq C(\mathbb{E}\tau^{(n)})^3 \text{ for all } n.$$

Then there is a separation cutoff with $t_n = \mathbb{E}\tau^{(n)}$ iff

$$\frac{(\mathbb{E}\tau^{(n)})^2}{\mathrm{Var}(\tau^{(n)})} \to \infty, \text{ or, equivalently } \frac{(\mathbb{E}\tau^{(n)})^2}{\mathbb{E}(\tau^{(n)})^2} \to 1.$$

Let $\xi^{(n)} = \tau^{(n)}/\mathbb{E}\tau^{(n)}$. Separation cutoff implies that $\xi^{(n)}$ converges to 1 in probability. $\lim_{n\to\infty} \mathbb{E}|\xi^{(n)} - 1|^2 = 0$.

$$\lim_{n\to\infty} \frac{(\mathbb{E}\tau^{(n)})^2}{\mathbb{E}(\tau^{(n)})^2} = \lim_{n\to\infty} \frac{(\mathbb{E}\xi^{(n)})^2}{\mathbb{E}(\xi^{(n)})^2} = 1.$$

2. A general condition for separation cutoff

The following Chebyshev type inequality (Feller 1971). For a non-negative ξ with mean $\mathbb{E}\xi$ and variance $\mathrm{Var}(\xi)$, we have for any $\epsilon > 0$,

$$\mathbb{P}[\xi \geq \mathbb{E}\xi + c\sqrt{\mathrm{Var}(\xi)}] \leq \frac{1}{1+c^2}, \quad \mathbb{P}[\xi \leq \mathbb{E}\xi - c\sqrt{\mathrm{Var}(\xi)}] \leq \frac{1}{1+c^2}.$$

Apply this Chebyshev inequality to the sequence of fastest strong stationary times $\tau^{(n)}$ to get that

$$\mathbb{P}[\tau^{(n)} \geq (1+\epsilon)\mathbb{E}\tau^{(n)}] = \mathbb{P}[\tau^{(n)} \geq \mathbb{E}\tau^{(n)} + \frac{\epsilon\mathbb{E}\tau^{(n)}}{\sqrt{\mathrm{Var}(\tau^{(n)})}}\sqrt{\mathrm{Var}(\tau^{(n)})}]$$

$$\leq \frac{1}{1+\epsilon^2\frac{[\mathbb{E}\tau^{(n)}]^2}{\mathrm{Var}(\tau^{(n)})}} \to 0$$

and similarly

$$\mathbb{P}[\tau^{(n)} \leq (1-\epsilon)\mathbb{E}\tau^{(n)}] \to 0.$$

3. FSST of birth-death chains

Theorem(Diaconis & Fill(1990))

For an ergodic continuous-time birth-death chain on the state space $\{0, \ldots, N\}$ started at 0, let Q be the generator. Then there exists a fastest strong stationary time (FSST) τ such that

$$\text{sep}(P(t), \pi) = \mathbb{P}[\tau > t]$$

and the distribution of τ is given in Laplace transformation:

$$\mathbb{E}e^{-\lambda\tau} = \prod_{\nu=1}^{N} \frac{\lambda_\nu}{\lambda + \lambda_\nu}, \ \lambda \geq 0,$$

where $\lambda_1 < \cdots < \lambda_N$ are positive eigenvalues of the matrix $-Q$.

Let $X_t^{(n)}$ be a sequence of birth and death process on $E_n = \{0, 1, \ldots, m_n\}$ with generator matrix $Q^{(n)}$. Let $\pi^{(n)}$ be the corresponding stationary distribution and $P^{(n)}(t)$ be the distribution $X_t^{(n)}$ starting from state 0. Let $0 = \lambda_0^{(n)} < \lambda_1^{(n)} < \cdots < \lambda_{m_n}^{(n)}$ be the eigenvalues of $-Q^{(n)}$.

$$\mathbb{E}\tau^{(n)} = T^{(n)} = \sum_{\nu=1}^{m_n} \frac{1}{\lambda_\nu^{(n)}}, \ \mathrm{Var}(\tau^{(n)}) = \sum_{\nu=1}^{m_n} \left(\frac{1}{\lambda_\nu^{(n)}}\right)^2.$$

$$\left(\frac{1}{\lambda_1^{(n)}}\right)^2 \le \mathrm{Var}(\tau^{(n)}) \le \frac{T^{(n)}}{\lambda_1^{(n)}},$$

$$\Longrightarrow \lambda_1^{(n)} T^{(n)} \le \frac{(\mathbb{E}\tau^{(n)})^2}{\mathrm{Var}(\tau^{(n)})} \le \left[\lambda_1^{(n)} T^{(n)}\right]^2.$$

Theorem(Diaconis & Saloff-Coste(2006))

There is a separation cutoff with $t_n = T^{(n)}$ iff $\lim_{n\to\infty} \lambda_1^{(n)} T^{(n)} = \infty$.

4. Explicit criteria for BD chains -Eigenvalues

The formula for $T^{(n)}$ comes from the so-called eigentime identity, appearing first in Aldous & Fill(2003). Let $\lambda_\nu^{(n)}, \nu \geq 1$ be the non-zero eigenvalues of $-Q^{(n)}$, then it holds that

$$\sum_{\nu=1}^{m_n} \frac{1}{\lambda_\nu^{(n)}} = \sum_{i,j=0}^{m_n} \pi_i^{(n)} \pi_j^{(n)} \mathbb{E}_i \tau_j^{(n)} = \sum_{j=0}^{m_n} \pi_j^{(n)} \mathbb{E}_i \tau_j^{(n)} \quad \text{for all} \quad i.$$

According to Mao(2004), for birth and death chains, we have

$$T^{(n)} = \sum_{\nu=1}^{m_n} \frac{1}{\lambda_\nu^{(n)}} = \sum_{i=0}^{m_n-1} \frac{1}{\pi_i^{(n)} b_i^{(n)}} \sum_{j=i+1}^{m_n} \pi_j^{(n)} \sum_{j=0}^{i} \pi_j^{(n)}.$$

4. Explicit criteria for BD chains -Eigenvalues

M.F. Chen(2010) gives an elegant estimate of spectral gap for ergodic birth and death chains. Let

$$\kappa^{(n)} = \min_{0\le\ell<m\le m_n} \left[\left(\sum_{i=0}^{\ell}\pi_i^{(n)}\right)^{-1} + \left(\sum_{i=m}^{m_n}\pi_i^{(n)}\right)^{-1}\right]\left(\sum_{i=\ell}^{m-1}\frac{1}{\pi_i^{(n)}b_i^{(n)}}\right)^{-1}.$$

Then

$$\frac{1}{4}\kappa^{(n)} \le \lambda_1^{(n)} \le \kappa^{(n)}.$$

Theorem 1

Separation cutoff occurs iff

$$\lim_{n\to\infty}\kappa^{(n)}T^{(n)} = \infty.$$

4. Explicit criteria for BD chains -Duality

Fill(1992) constructed a dual absorbing birth-death process, whose absorption time τ^* has the same distribution as τ the FSST.
Set for $0 \le i \le m, H_i = \sum_{j \le i} \pi_j$ and for $i \le m-1$

$$a_i^* = \frac{H_{i-1}}{H_i} b_i, \quad b_i^* = \frac{H_{i+1}}{H_i} a_{i+1}, \quad a_m^* = b_m^* = 0.$$

Let

$$\pi_i^* = b_0 H_i^2 / (\pi_i b_i).$$

Let $a_i^* (1 \le i \le m), b_i^* (0 \le i \le m)$ be the birth and death rates for a dual process $X^*(t)$. So m is an absorbing state for $X^*(t)$. Assume both $X(t)$ and $X^*(t)$ start at 0. Then the absorption time (hitting time to m) of $X^*(t)$ has the same distribution as the FSST τ of $X(t)$.

$$Q^* \Lambda = \Lambda Q, \ \Lambda = (\Lambda_{ij}), \Lambda_{ij} = \frac{H_j}{H_i} \mathbf{1}_{[j \le i]}.$$

$$\mathbb{E}\tau = \mathbb{E}\tau^* = \sum_{i=0}^{m-1} \frac{1}{\pi_i^* b_i^*} \sum_{j=0}^{i} \pi_j^*,$$

$$\mathbb{E}\tau^2 = \mathbb{E}(\tau^*)^2 = 2(\mathbb{E}\tau)^2 - 2\sum_{i=0}^{m-1} \frac{1}{\pi_i^* b_i^*} \sum_{j=0}^{i} \pi_j^* \sum_{k=0}^{j-1} \frac{1}{\pi_k^* b_k^*} \sum_{l=0}^{j} \pi_l^*,$$

Denote

$$S = (\mathbb{E}\tau)^2 - \frac{1}{2}\mathbb{E}\tau^2 = \sum_{i=0}^{m-1} \frac{1}{\pi_i^* b_i^*} \sum_{j=0}^{i} \pi_j^* \sum_{k=0}^{j-1} \frac{1}{\pi_k^* b_k^*} \sum_{l=0}^{j} \pi_l^*.$$

$$\mathbb{E}\tau = \sum_{j=0}^{m-1} \frac{1}{\pi_j b_j} H_j(1 - H_j) = T.$$

$$S = \sum_{j=0}^{m-1} \frac{1 - H_j}{\pi_j b_j} \sum_{k=0}^{j-1} \frac{H_k(H_j - H_k)}{\pi_k b_k}.$$

Since

$$\frac{\mathbb{E}\tau^2}{(\mathbb{E}\tau)^2} = 2 - 2\frac{S}{(\mathbb{E}\tau)^2} = 2 - 2\frac{S}{T^2},$$

Theorem 2

Separation cut-off occurs iff

$$\lim_{n\to\infty} \frac{S^{(n)}}{[T^{(n)}]^2} = \frac{1}{2}.$$

4. Explicit criteria for BD chains -Stochastic monotonicity

We will obtain the distribution of FSST for the birth and death process via the halting state. This method can be used to deal with general stochastically monotone Markov processes.

Levin et al(2008): $i \in E$ is called a halting state for the FSST τ if for all $t \geq 0$

$$\mathbb{P}[\tau > t] = 1 - P_i(t)/\pi_i = \text{sep}(P(t), \pi).$$

Since

$$\max_j (1 - P_j(t)/\pi_j) = \text{sep}(P(t), \pi),$$

i is the halting state if and only if for $t \geq 0$,

$$P_i(t)/\pi_i = \min_j P_j(t)/\pi_j.$$

Lemma

For the continuous time birth and death process chain on state space $E = \{0, 1, \cdots, m\}$, starting at 0, state m is a halting state for the FSST τ.

$$\frac{P_m(t)}{\pi_m} = \frac{p_{0m}(t)}{\pi_m} = \frac{p_{m0}(t)}{\pi_0} = \min_i \frac{p_{i0}(t)}{\pi_0} = \min_i \frac{p_{0i}(t)}{\pi_i} = \min_i \frac{P_i(t)}{\pi_i}.$$

Theorem

Let τ be the FSST for a continuous time birth and death process on $\{0, 1, \ldots, m\}$, starting at 0. Define $\psi_m(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}[X_t = m]dt$. Then

$$\mathbb{E}e^{-\lambda\tau} = \frac{\lambda\psi_m(\lambda)}{\pi_m}, \quad \lambda \geq 0.$$

Let τ_k be the hitting time of state k.

Lemma(Y. Gong, Y.H. Mao & C. Zhang(2012))

For the continuous time birth death process X_t, let

$$\phi_{im}(\lambda) = 0,\ \phi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}[X_t = j, t < \tau_m | X_0 = i]dt$$

for $0 \le i, j < m$ and

$$\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}[X_t = j | X_0 = i]dt$$

for $0 \le i, j \le m$. It holds that for $0 \le i, j \le m$

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{\xi_i(\lambda)\pi_j\xi_j(\lambda)}{\lambda\sum_{i=0}^m \pi_i\xi_i(\lambda)},$$

where $\xi_i(\lambda) = \mathbb{E}[e^{-\lambda\tau_m} | X_0 = i]$.

Theorem

For the continuous-time birth and death process starting at 0,

$$\mathbb{E}e^{-\lambda\tau} = \left(\pi_0 + \sum_{k=1}^{n} \pi_k \left(\mathbb{E}e^{-\lambda\tau_k}\right)^{-1}\right)^{-1}, \ \lambda \geq 0.$$

$$\mathbb{E}\tau = \sum_{i=0}^{m} \pi_i \mathbb{E}\tau_i, \ \mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 + \sum_{i=0}^{m} \pi_i \mathbb{E}\tau_i^2 - 2\sum_{i=0}^{m} \pi_i (\mathbb{E}\tau_i)^2.$$

$$\mathbb{E}\tau_i = \sum_{j=0}^{i-1} \frac{1}{\pi_j b_j} \sum_{k=0}^{j} \pi_k, \ \mathbb{E}\tau_i^2 = 2(\mathbb{E}\tau_i)^2 - 2\sum_{j=0}^{i-1} \frac{1}{\pi_j b_j} \sum_{k=0}^{j} \pi_k \mathbb{E}\tau_k.$$

$$\mathbb{E}\tau = \sum_{i=0}^{m} \frac{1}{\pi_i b_i} \sum_{j=0}^{i} \pi_j \sum_{j=i+1}^{m} \pi_j, \mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 - 2\sum_{i=0}^{m} \pi_i \sum_{j=0}^{i-1} \frac{1}{\pi_j b_j} \sum_{k=0}^{j} \pi_k \mathbb{E}\tau_k.$$

Let

$$S = (\mathbb{E}\tau)^2 - \frac{1}{2}\mathbb{E}\tau^2 = \sum_{i=0}^{m} \pi_i \sum_{j=0}^{i-1} \frac{1}{\pi_j b_j} \sum_{k=0}^{j} \pi_k \sum_{u=0}^{k-1} \frac{1}{\pi_u b_u} \sum_{v=0}^{u} \pi_v.$$

and exchange the first summation and the second one, then exchange the third one and the forth one.

5. Applications -Restricted chains

Let $a_i (i \geq 1), b_i (i \geq 0)$ be the death and birth rates respectively for the continuous time birth and death processes on $\{0, 1, \ldots\}$. For each $n = 1, 2, \ldots$, a restricted process $X_t^{(n)}$ on $\{0, 1, \ldots, n\}$ is referred to a birth and death process with birth rates $b_i^{(n)} = b_i$ for $0 \leq i \leq n-1$ and $b_n^{(n)} = 0$, and death rates $a_i^{(n)} = a_i$ for $1 \leq i \leq n$. Then $X_t^{(n)}$ is ergodic with reflecting boundary at n.

Corollary 1

Assume that X_t is exponential ergodic. Then $X_t^{(n)}$ has separation cutoff if and only if X_t is not strong ergodic.

Proof Let λ_1 be the spectral gap for X_t and $\lambda_1 > 0$ by exponential ergodicity. Let $\lambda_1^{(n)}$ as before. Then $\lim_{n\to\infty} \lambda_1^{(n)} = \lambda_1$.

$$\sum_{i=0}^{\infty} \frac{1}{\pi_i b_i} \sum_{j=i+1}^{\infty} \pi_j \geq \lim_{n\to\infty} T^{(n)} = T = \sum_{i=0}^{\infty} \frac{1}{\pi_i b_i} \sum_{j=0}^{i} \pi_j \sum_{j=i+1}^{\infty} \pi_j$$
$$\geq \pi_0 \sum_{i=0}^{\infty} \frac{1}{\pi_i b_i} \sum_{j=i+1}^{\infty} \pi_j.$$

But the birth and death process is not strongly ergodic if and only if

$$\sum_{i=0}^{\infty} \frac{1}{\pi_i b_i} \sum_{j=i+1}^{\infty} \pi_j = \infty.$$

Therefore, the non-strong ergodicity is equivalent to $\lim_{n\to\infty} \lambda_1^{(n)} T^{(n)} = \infty$. Diaconis & Saloff-Coste(2006).

Example 1

Let $a_i = a, b_i = b$.
(1) If $a > b > 0$, then $\lambda_1 = (\sqrt{a} - \sqrt{b})^2$. Thus this process is exponentially ergodic, but not strongly ergodic, so that by Corollary 1 there is separation cutoff for the restricted chains.
(2) If $b \geq a > 0$, the process is transient. In this case, we cannot apply Corollary 1. Instead, we apply Theorem 2. For $\beta := b/a > 1$, we have

$$Z = \frac{\beta^{m+1} - 1}{\beta - 1}, \ \pi_i = \frac{\beta^i}{Z};$$

A direct and tedious calculation implies the separation cutoff occurs.
(3) For $a = b$, we have $\pi_i = 1/m$. Again an easy calculation shows that there is no separation cutoff.

Example 2

Let $a_i = ai^\gamma, b_i = b$ with $\gamma \geq 0, a, b > 0$. Then there is separation cutoff for the restricted chains if and only if $\gamma \in [0, 1]$.

5. Applications -Metropolis algorithm

For each $m = 1, 2, \ldots,$ a Metropolis chain on $\{0, 1, \ldots, m\}$ is formulated as follows: for $1 \leq i \leq m$, $a_i = 1/2$ when $\pi_{i-1}/\pi_i \geq 1$, $a_i = \pi_{i-1}/\pi_i$ when $\pi_{i-1}/\pi_i < 1$; for $0 \leq i \leq m-1$, $b_i = 1/2$ when $\pi_{i+1}/\pi_i \geq 1$, $b_i = \pi_{i+1}/\pi_i$ when $\pi_{i+1}/\pi_i < 1$.

Corollary 2

Let $\pi_i = (i+1)^d/Z$ with $d \in \mathbb{R}$, where Z makes π a probability measure. For a family of Metropolis chains on $\{0, 1, \ldots, m\}$ defined as above, there is no separation cutoff.

Corollary 3

Let $\pi_i = \beta^i/Z$ with $\beta > 0$, where Z makes π a probability measure. For a family of Metropolis chains on $\{0, 1, \ldots, m\}$ defined as above, there is separation cutoff except $\beta = 1$.

6. FSST of single birth chains

Since the birth and death chain is symmetric, its eigenvalues are real. But for a general skip-free chains, eigenvalues may be complex.

Theorem(Fill(2009))

For an ergodic continuous-time single birth chain on the state space $\{0, \ldots, N\}$ started at 0 with stochastically monotone time-reversal, let Q be the generator. Then a fastest strong stationary time τ has the distribution

$$\mathbb{E}e^{-\lambda\tau} = \prod_{\nu=1}^{N} \frac{\lambda_\nu}{\lambda + \lambda_\nu}, \ \lambda \geq 0,$$

7. Explicit criteria for SB chains -Duality

Set for $0 \le i \le m, H_i = \sum_{j\le i} \pi_j$ and for $i \le m-1$

$$\tilde{q}_{ij} = \frac{\pi_j q_{ji}}{\pi_i},$$

$$q^*_{ij} = \frac{H_j}{H_i} \sum_{k\le i} (\tilde{q}_{jk} - \tilde{q}_{j+1,k}),\ i \le m-1; q^*_{mm} = 0.$$

Let Q^* be a single birth Q-matrix with an absorbing state m. Assume both $X(t)$ and $X^*(t)$ start at 0. Then the absorption time (hitting time to m) of $X^*(t)$ has the same distribution as the FSST τ of $X(t)$.

$$Q^*\Lambda = \Lambda Q,\ \Lambda = (\Lambda_{ij}), \Lambda_{ij} = \frac{H_j}{H_i}\mathbf{1}_{[j\le i]}.$$

8. Explicit criteria for SB chains -Stochastic monotonicity

Theorem

Assume the single birth chain on the state space $E = \{0, 1, \ldots, N\}$, started at 0, has the stochastically monotone time-reversal. Let τ be a FSST and $P(t) = (P_i(t))$ be the distribution of X_t. Then

(1) $\mathbb{P}[\tau > t] = 1 - P_N(t)/\pi_N$;

(2)

$$\mathbb{E}e^{-\lambda\tau} = \frac{\lambda}{\pi_N}\int_0^\infty e^{-\lambda t}P_N(t)dt, \ \lambda \geq 0.$$

$$p_{0N}(t)/\pi_N = p^*_{N0}(t)/\pi_0 = \min_i p^*_{i0}(t)/\pi_0 = \min_i p_{0i}(t)/\pi_i.$$

$$1 - P_N(t)/\pi_N = \max_j (1 - P_j(t)/\pi_j) = \text{sep}(P(t), \pi) = \mathbb{P}[\tau > t].$$

Let τ_k be the hitting time of state k.

Lemma

For the continuous time single birth process X_t on $\{0, 1, \ldots, N\}$, let

$$\phi_{iN}(\lambda) = 0,\ \phi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}_i[X_t = j, t < \tau_N]dt$$

for $0 \le i, j < N$ and

$$\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}\mathbb{P}_i[X_t = j]dt$$

for $0 \le i, j \le N$. It holds that for $0 \le i, j \le N$

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{\xi_i(\lambda)\eta_j(\lambda)}{\lambda \sum_{j=0}^N \eta_j(\lambda)},$$

where $\xi_i(\lambda) = 1 - \lambda \sum_{k=0}^{N-1} \phi_{ik}(\lambda), \eta_j(\lambda) = \pi_j - \lambda \sum_{k=0}^{N-1} \pi_k \phi_{kj}(\lambda)$.

Theorem

For the continuous-time single birth process with the stochastically monotone time-reversal, started at 0,

$$\mathbb{E}e^{-\lambda\tau} = \left(\pi_0 + \sum_{k=1}^{n} \pi_k \left(\mathbb{E}_0 e^{-\lambda\tau_k}\right)^{-1}\right)^{-1}, \ \lambda \geq 0.$$

$$\mathbb{E}\tau = \sum_{i=0}^{m} \pi_i \mathbb{E}\tau_i, \ \mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 + \sum_{i=0}^{m} \pi_i \mathbb{E}_0\tau_i^2 - 2\sum_{i=0}^{m} \pi_i (\mathbb{E}_0\tau_i)^2.$$

For $0 \le k < i \le N$, define $q_i^{(k)} = \sum_{j=0}^{k} q_{ij}$, and

$$F_i^{(i)} = 1,\ F_i^{(k)} = \sum_{j=k+1}^{i} \frac{F_i^{(j)} q_j^{(k)}}{q_{j,j+1}}.$$

$$m_i = \sum_{k=0}^{i} \frac{F_i^{(k)}}{q_{k,k+1}},\ 0 \le i \le N.$$

Here we set $q_{N,N+1} = 1$ for saving notations and convenience. By induction, it is easy to check that

$$F_i^{(k)} = \frac{1}{q_{i,i+1}} \sum_{j=k}^{i-1} q_i^{(j)} F_j^{(k)},\ 0 \le k < i \le N;$$

and

$$m_i = \frac{1}{q_{i,i+1}} \left(1 + \sum_{0 \le j \le i-1} q_i^{(j)} m_j \right),\ 0 \le i \le N.$$

Theorem

For $j < i$, we have

$$\mathbb{E}_j\tau_i = \sum_{k=j}^{i-1} m_k,\ \mathbb{E}_j\tau_i^2 = 2\sum_{k=j}^{i-1}\sum_{\ell=0}^{k}\frac{F_k^{(\ell)}}{q_{\ell,\ell+1}}\mathbb{E}_\ell\tau_i.$$

$$\mathbb{E}_0\tau_i = \sum_{k=0}^{i-1} m_k,\ \mathbb{E}_0\tau_i^2 = 2(\mathbb{E}_0\tau_i)^2 - 2\sum_{k=0}^{i-1}\sum_{\ell=0}^{k}\frac{F_k^{(\ell)}}{q_{\ell,\ell+1}}\mathbb{E}_0\tau_\ell.$$

For $i < j \le N$, we have

$$\mathbb{E}_j\tau_i = \sum_{k=i}^{j-1}\left(\frac{m_N}{F_N^{(i)}}F_k^{(i)} - m_k\right),$$

$$\mathbb{E}_j\tau_i^2 = 2\sum_{k=i}^{j-1}\left(\frac{F_k^{(i)}}{F_N^{(i)}}\sum_{\ell=0}^{N}\frac{F_N^{(\ell)}}{q_{\ell,\ell+1}}\mathbb{E}_\ell\tau_i - \sum_{\ell=0}^{k}\frac{F_k^{(\ell)}}{q_{\ell,\ell+1}}\mathbb{E}_\ell\tau_i\right).$$

$$\pi_i = \frac{F_N^{(i)}}{q_{i,i+1} m_N}, \ 0 \le i \le N.$$

Proposition

The time-reversal $\tilde{Q}$ is stochastically monotone iff

$$\sum_{k \ge j} \frac{F_N^{(k)} q_{ki}}{q_{k,k+1}} \le \frac{q_{i+1,i+2} F_N^{(i)}}{q_{i,i+1} F_N^{(i+1)}} \sum_{k \ge j} \frac{F_N^{(k)} q_{k,i+1}}{q_{k,k+1}}, \ i+1 < j \le N.$$

Corollary

Assume that $q_{i,i+1} \equiv 1$. If $F_N^{(i)} \ge F_N^{(i+1)}$ for all $i < N-1$ and $q_{ki} \le q_{k,i+1}$ for all $k > i+1$, then the time-reversal is stochastically monotone.

$$\mathbb{E}\tau^2 = 2(\mathbb{E}\tau)^2 - 2\sum_{i=0}^{N}\pi_i \sum_{j=0}^{i-1}\sum_{k=0}^{j}\frac{F_j^{(k)}}{q_{k,k+1}}\mathbb{E}_0\tau_k.$$

Let

$$T = \mathbb{E}\tau = \sum_{i=0}^{N}\pi_i \sum_{j=0}^{i-1} m_j,\ S = \sum_{i=0}^{N}\pi_i \sum_{j=0}^{i-1}\sum_{k=0}^{j}\frac{F_j^{(k)}}{q_{k,k+1}}\mathbb{E}_0\tau_k.$$

We have

$$\frac{\mathbb{E}\tau^2}{(\mathbb{E}\tau)^2} = 2 - 2\frac{S}{(\mathbb{E}\tau)^2} = 2 - 2\frac{S}{T^2}.$$

Theorem

For each n, assume the skip-free chain on $\{0,1,\ldots,N_n\}$ started at 0, with the stochastically monotone time-reversal and $T^{(n)}$ and $S^{(n)}$ defined similarly (with N_n in the place of N). Then there is a separation cutoff with $t_n = T^{(n)}$ iff

$$\lim_{n\to\infty} \frac{S^{(n)}}{[T^{(n)}]^2} = \frac{1}{2}.$$

Corollary

For each $n = 1,2,\ldots$, let $Q^{(n)} = (q_{ij}^{(n)})$ be the generator for a single birth chains on $\{0,1,\ldots,N_n\}$ stared at 0. Assume that $q_{i,i+1}^{(n)} = 1$ for $1 \le i \le N_n - 1$ and $q_{ki}^{(n)} \le q_{k,i+1}^{(n)}$ for $k > i+1$. If there is $C > 0$ and $\beta > 1$ such that $F_i^{(j)} \sim C\beta^{i-j}$ as $i - j \to \infty$, then there is a separation cutoff.

Examples

(1) Let $q_{i,i+1} = q_{i,i-1} = q_{i,i-2} = 1$. Then the chain exhibits the separation cutoff.

$$F_i^{(j)} \sim C(\sqrt{2}+1)^{i-j}.$$

(2) Let $q_{i,i+1} = 1, q_{ij} = 1/i$ for $0 \le j < i$. Then the chain exhibits the separation cutoff.

$$F_i^{(j)} \sim 2^{i-j-1}.$$

(3) Let $q_{i,i+1} = 1, q_{ij} = (1-p)p^{i-j-1}(1 \le j < i), q_{i0} = p^i$ for $i \ge 0$. Then the chain exhibits the separation cutoff if $0 < p \le (\sqrt{5}-1)/2$.

$$F_i^{(j)} = (1+p)^{i-j-1}.$$

Thank you for your attention!

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