Mixed Eigenvalues of Discrete p-Laplacian

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This talk is based on the joint work with Mu-Fa Chen and Ling-Di Wang.

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1. Problem

- the discrete version of p-Laplacian eigenvalue problem:

$$\text{Eigeneq.:} \quad \Omega_p g(k) = -\lambda \mu_k |g_k|^{p-2} g_k, \; k \in E := \mathbb{Z}^+ \cap [0, N],$$

$$\begin{aligned}\Omega_p g(k) :=& \nu_{k-1}|g_{k-1} - g_k|^{p-2}(g_{k-1} - g_k) \\ & - \nu_k |g_k - g_{k+1}|^{p-2}(g_k - g_{k+1}).\end{aligned}$$

- ND boundary conditions ($N \leqslant \infty$)

$$0 \neq g_{-1} = g_0 \text{ and } g_{N+1} = 0 \text{ if } N < \infty.$$

Problem

How to estimate on the first eigenvalue $\lambda_p (1 < p < \infty)$?

2. Background

$p = 2$, DD, DN, ND and NN

- Chen M F. Speed of stability for birth-death process, Front. Math. China, 2010, 5(3): 379-516.
- Chen M F, Wang L D, Zhang Y H. Mixed principal eigenvalues in dimension one, Front. Math. China, 2013, 8(2): 37-343.

$p \neq 2$, NN and DD

- Chen M F. Bilateral Hardy-type inequalities, Acta Math. Sinica, 2013, 29(1): 1-32.

- the classical variational formula:

$$\lambda_p = \inf\left\{D_p(f) : \mu\left(|f|^p\right) = 1, f \in \mathscr{C}_K\right\},$$

where $\mathscr{C}_K$ is the set of functions having compact support and

$$D_p(f) := \sum_{k\in E} \nu_k |f_k - f_{k+1}|^p, \quad p \geqslant 1, \quad f \in \mathscr{C}_K.$$

- the weighted Hardy inequality:

$$\mu\left(|f|^p\right) \leqslant A D_p(f), \quad f \in \mathscr{C}_K$$

- the optimal constant $A = \lambda_p^{-1}$. This explains the relationship between the p-Laplacian eigenvalue and the Hardy's inequality.

3(1). Our work: Analysis & Notations

eigenequation + ND boundary conditions

$$\Longrightarrow \nu_j|g_j - g_{j+1}|^{p-2}(g_j - g_{j+1}) = \lambda_p \sum_{k=0}^{j} \mu_k |g_k|^{p-2} g_k.$$

$$\Downarrow \text{if } g \text{ monotone} \Longrightarrow I_i(g) := \frac{1}{\nu_i(g_i - g_{i+1})^{p-1}} \sum_{j=0}^{i} \mu_j g_j^{p-1} = \lambda_p^{-1}$$

$$g_n - g_{N+1} = \lambda_p^{p'-1} \sum_{j=n}^{N} \left(\frac{1}{\nu_j} \sum_{k=0}^{j} \mu_k |g_k|^{p-2} g_k \right)^{p'-1}$$

$$\Downarrow \text{if } g_{N+1} = 0 \Rightarrow I\!I_n(g) := \left\{ \frac{1}{g_n} \sum_{j=n}^{N} \left(\frac{1}{\nu_j} \sum_{k=0}^{j} \mu_k |g_k|^{p-2} g_k \right)^{p'-1} \right\}^{p-1} = \lambda_p^{-1}$$

- For the eigenfunction g of λ_p, then

$$\lambda_p = I_i(g)^{-1} = I\!I_i(g)^{-1}.$$

About the eigenfunction g

(1) g is either positive or negative.

(2) g is strictly monotone.

(3) $g_{N+1} := \lim_{i\to N+1} g_i = 0$ for $N = \infty$.

Based the properties of eigenfunction, the set of test functions is defined as follows:

- for lower bounds $\mathscr{F}_{I\!I} = \{f : f > 0\}$.
- for upper bounds

$$\widetilde{\mathscr{F}_{I\!I}} = \{f : f_i > 0 \text{ up to some } m \in [1, N+1) \text{ and then vanishes}\}.$$

- $$\inf_{f \in \widetilde{\mathscr{F}_{I\!I}}} \sup_{i \in \mathrm{supp}(f)} I\!I_i(f)^{-1} = \lambda_p = \sup_{f \in \mathscr{F}_{I\!I}} \inf_{i \in E} I\!I_i(f)^{-1}.$$

3(2). Variational formulas

Theorem 1 (Variational formulas)

(1)Single summation form:

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{i\in E} I_i(f)^{-1}=\lambda_p=\sup_{f\in\mathscr{F}_I}\inf_{i\in E} I_i(f)^{-1}.$$

(2) Double summation form:

$$\inf_{f\in\widetilde{\mathscr{F}}_{I\!I}}\sup_{i\in \text{supp(f)}} I\!I_i(f)^{-1}=\lambda_p=\sup_{f\in\mathscr{F}_{I\!I}}\inf_{i\in E} I\!I_i(f)^{-1},$$

(3) Difference form:

$$\inf_{w\in\widetilde{\mathscr{W}}}\sup_{x\in E} R_i(w)=\lambda_p=\sup_{w\in\mathscr{W}}\inf_{i\in E} R_i(w).$$

Moreover, the supremun on the right-hand side of the three above formulas can be attained.

$$\mathscr{F}_I = \{f : f > 0 \text{ and } f \text{ is strictly decreasing}\},$$

$$\widetilde{\mathscr{F}_I} = \{f : f \text{ is strictly decreasing on some } [n, m], 0 \leqslant n < m < N+1, \\ f_\cdot = f_{\cdot \vee n} \mathbb{1}_{\cdot \leqslant m}\},$$

$$\mathscr{W} = \left\{ w : w_i \in (0,1) \text{ if } \sum_{j \in E} \hat{\nu}_j < \infty \text{ and } w_i \in (0,1] \text{ if } \sum_{j \in E} \hat{\nu}_j = \infty \right\},$$

$$\widetilde{\mathscr{W}} = \Big\{ w : \exists\, m \in [1, N+1) \text{ such that } w_i > 0 \text{ up to } m-1, w_m = 0, \\ w_i < 1 - (\nu_{i-1}/\nu_i)^{p-1} \left(w_{i-1}^{-1} - 1 \right) \text{ for } i = 0, 1, \cdots, m \Big\}.$$

$$R_i(w) = \mu_i^{-1} \left[\nu_i (1 - w_i)^{p-1} - \nu_{i-1} \left(w_{i-1}^{-1} - 1 \right)^{p-1} \right] \quad \text{(difference form)}.$$

3(3). Basic estimates

Applications of variational formulas:

Proposition

Assume that $\lambda_p > 0$. Then, for the eigenfunction g of λ_p, $g_{N+1} := \lim_{i\to N+1} g_i = 0$ for $N = \infty$.

Define
$\hat{\nu}_j = \left(\nu_j\right)^{1-p'}$, $k(p) = pp'^{p-1}$, $\varphi_n = \hat{\nu}[n, N]^{p-1}$, $\sigma_p = \sup_{n\in E} \mu[0, n]\varphi_n$

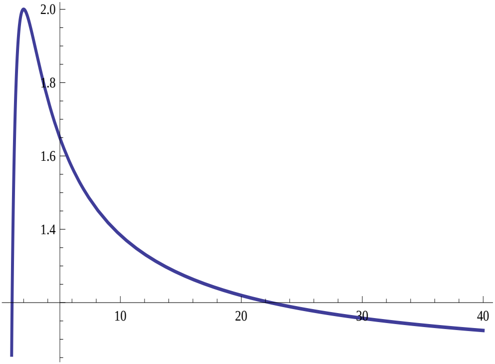
Theorem 2 (Basic estimates)

$$\lambda_p > 0 \Leftrightarrow \sigma_p < \infty.$$

More precisely,

$$(k(p)\sigma_p)^{-1} \leqslant \lambda_p \leqslant \sigma_p^{-1}.$$

- Kufner A. et al. 2007. The Hardy Inequality, Theorem 7, P.58; Mao Y H, Nash ineq., 2002.
- Kufner A, Persson L E. 2003, Weighted Hardy Inequality.



The function $p \to k(p)^{1/p}$ is unimodal with maximum 2 at $p = 2$.

3(4). Approximating procedure

Theorem 3 (Approximating procedure)

Assume that $\sigma_p < \infty$.

(1) Define $f_1 = \varphi^{1/p}$, $f_n = f_{n-1} I\!I(f_{n-1})^{p'-1}$, $\delta_n = \sup_{i\in E} I\!I_i(f_n)$. Then $\delta_n \downarrow$ and

$$\lambda_p \geqslant \delta_\infty^{-1} \geqslant \cdots \geqslant \delta_1^{-1} \geqslant (k(p)\sigma_p)^{-1}.$$

(2) Define $\delta_n' = \sup_{\ell,m:\ell<m} \min_{i\leqslant m} I\!I_i(f_n^{(\ell,m)})$. Then $\delta_n' \uparrow$ and

$$\sigma_p^{-1} \geqslant \delta_1'^{-1} \geqslant \cdots \geqslant \delta_\infty'^{-1} \geqslant \lambda_p.$$

(3) Define $\bar{\delta}_n = \sup_{\ell,m:\ell<m} \dfrac{\mu(|f_n^{(\ell,m)}|^p)}{D_p(f_n^{(\ell,m)})}$. Then $\bar{\delta}_n^{-1} \geqslant \lambda_p$, $\bar{\delta}_{n+1} \geqslant \delta_n'$.

Corollary (Improved estimates)

$$\sigma_p^{-1} \geqslant \delta_1'^{-1} (\text{or } \bar{\delta}_1^{-1}) \geqslant \lambda_p \geqslant \delta_1^{-1} \geqslant \big(k(p)\sigma_p\big)^{-1},$$

where

$$\delta_1 = \sup_{i\in E} \left[\frac{1}{\hat\nu[i,N]^{1/p'}} \sum_{j=i}^{N} \hat\nu_j \left(\sum_{k=0}^{j} \mu_k \hat\nu[k,N]^{(p-1)/p^*} \right)^{p'-1} \right]^{p-1},$$

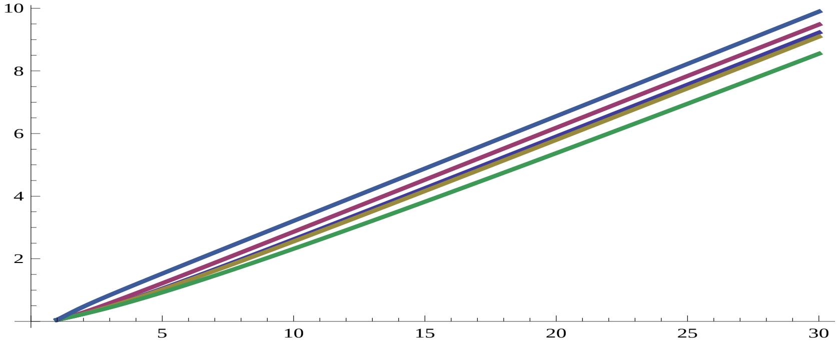
$$\delta_1' = \sup_{\ell\in E} \frac{1}{\hat\nu[\ell,N]^{p-1}} \left[\sum_{j=\ell}^{N} \hat\nu_j \left(\sum_{k=0}^{j} \mu_k \hat\nu[k\vee \ell,N]^{p-1} \right)^{p'-1} \right]^{p-1},$$

$$\bar\delta_1 = \sup_{m\in E} \frac{1}{\hat\nu[m,N]} \sum_{j=0}^{N} \mu_j \hat\nu[j\vee m,N]^p \in [\sigma_p, p\sigma_p].$$

For $1 < p \leqslant 2$, $\bar\delta_1 \leqslant \delta_1'$; for $p \geqslant 2$, $\bar\delta_1 \geqslant \delta_1'$.

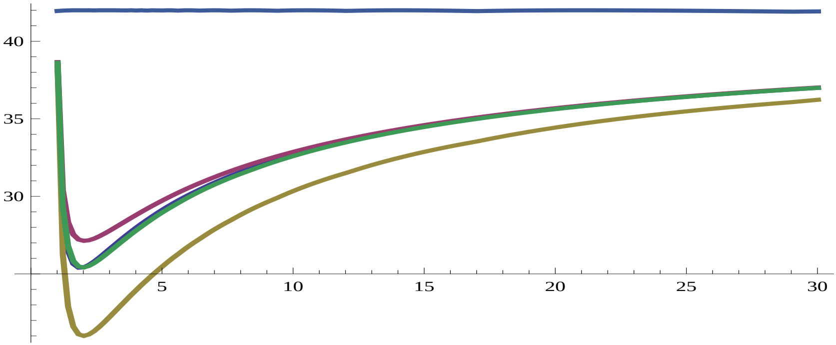
4. Examples

$E = \{0, 1, \cdots, 80\}$. Let $\mu_k = 20^k$, $\nu_k = 20^{k+1}$.



Let p vary over $(1.001, 30.001)$ avoiding the singularity at $p = 1$. The curves from top to bottom are $(k(p)\sigma_p)^{1/p}$, ${\delta_1}^{1/p}$, $\bar{\delta}_1^{1/p}$, ${\delta_1'}^{1/p}$ and $\sigma_p^{1/p}$ res. The lower bounds $\bar{\delta}_1^{1/p}$ and ${\delta_1'}^{1/p}$ of $\lambda_p^{-1/p}$ are nearly overlapped.

$E = \{0, 1, \cdots, 40\}$. Let $\mu_k = 1$, $\nu_k = 1$.



Let p vary over $(1.0175, 30.0175)$. The curves from top to bottom are $(k(p)\sigma_p)^{1/p}$, ${\delta_1}^{1/p}$, $\bar{\delta}_1^{1/p}$, ${\delta_1'}^{1/p}$ and $\sigma_p^{1/p}$. The exact λ_p is unknown except that $\lambda_p = \sin^2 \frac{\pi}{2(N+2)}$ when $p = 2$.

Thank you for your attention!