Mixed principal eigenvalues in dimension one

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1. Problem

- Consider one dimensional diffusion process:

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad x \in (0, D), \quad D \leqslant \infty,$$

where a and b are Borel measurable, $a > 0$ on $(0, D)$.

Boundary conditions: Neumman ('N'); Dirichlet ('D').
Four cases: ND DN NN DD

- Eigenequation:

$$Lg = -\lambda g, \qquad \text{'+'} \qquad \text{boundary conditions.}$$

g is called the eigenfunction corresponding to the eigenvalue λ.

1. Problem

- Let $C(x) = \int_0^x b/a$. Define two important measures:

  Speed measure: $\mu(\mathrm{d}x) = e^{C(x)}/a(x)\mathrm{d}x$,

  Scale measure: $\nu(\mathrm{d}x) = e^{-C(x)}\mathrm{d}x$.

- ND case. Let $\lambda_0^{(ND)}$ be the minimal solution to the following eigenequation:

$$Lg = -\lambda g, \qquad g'(0) = 0, \quad g(D) = 0 \text{ if } D < \infty.$$

- $\lambda_0^{(ND)}$ has a classical variational formula:

$$\lambda_0^{(ND)} = \inf\{D(f) : \mu(f^2) = 1, f \in \mathscr{C}_K[0, D], f(D) = 0 \text{ if } D < \infty\},$$

where $D(f) = \int_0^D af'^2 \mathrm{d}\mu$,
$\mathscr{C}_K[0, D] = \{f : f \in \mathscr{C}^1(0, D) \cap \mathscr{C}[0, D], \text{ supp}(f) \text{ is compact}\}$.

1. Problem

$$\mathscr{C}[0,D] = \{f : f \text{ is continuous on } [0,D]\},$$
$$\mathscr{C}^k(0,D) = \{f : f \text{ has continuous derivatives of order } k \text{ on } (0,D)\}, k \geqslant 1$$

We focus on

How to estimate on $\lambda_0^{(ND)}$ and $\lambda_0^{(DN)}$?

Consider $\|P_t f\| \leqslant \|f\| e^{-\varepsilon t}$. Then $\lambda_0 = \varepsilon_{\max}$, i.e.,$\lambda_0$ is the maximal decay(stability) speed of diffusion process.

2. Background

There are a great number of publications for the background and motivation of the study on the principal eigenvalue or the Poincaré-type inequalities. Now we recall some results of our group as follows.

In 1997, a variational formula for the lower bound of spectral gap of an elliptic operator was presented by M.F. Chen and F.Y. Wang. The eigenfunction g is monotone. (NN)

In 1999, Chen obtained again some dual variational formulas for the lower and upper bounds of eigenvalue by analytic method and proved that the eigenfunction g in the DN case is increasing. (NN and DN)

In 2000, Chen obtained variational formulas and basic estimates for eigenvalue in the DN case. (DN)

2. Background

In 2002, Chen, Y.H. Zhang and X.L. Zhao obtained variational formulas and explicit estimates for the processes with DD boundaries including continuous and discrete cases. (DD)

In 2003, using the variational formulas, M.F. Chen obtained approximating procedure for one dimensional diffusion process with NN boundary. (NN)

In 2010, Chen obtained variational formulas and approximating procedure for birth-death process.(Four cases)

Our work is a continuation of the above paper (in the continuous situation).

3(1). Our work —Analysis

Another form of the operator: $L = \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}\nu}$.

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\nu} g(\beta) - \frac{\mathrm{d}}{\mathrm{d}\nu} g(\alpha) = -\lambda_0 \int_\alpha^\beta g \mathrm{d}\mu, \quad \alpha < \beta.$$

$\Downarrow$ by $g'(0) = 0$

$$\frac{\mathrm{d}}{\mathrm{d}\nu} g(x) = -\lambda_0 \int_0^x g \mathrm{d}\mu \Rightarrow I(g)(x) := -\frac{e^{-C(x)}}{g'(x)} \int_0^x g \mathrm{d}\mu = \lambda_0^{-1}$$

$\Downarrow$

$$g(x) - g(D) = \lambda_0 \int_x^D \nu(\mathrm{d}s) \int_0^s g \mathrm{d}\mu.$$

$\Downarrow$ if $g(D) = 0$

$$g(x) = \lambda_0 \int_x^D \nu(\mathrm{d}s) \int_0^s g \mathrm{d}\mu \Rightarrow I\!I(g)(x) := \frac{1}{g(x)} \int_x^D \nu(\mathrm{d}s) \int_0^s g \mathrm{d}\mu = \lambda_0^{-1}.$$

3(1). Our work—Analysis

- If we 'know' the eigenfunction g, then $\lambda_0 = I(g)(x)^{-1} = I\!I(g)(x)^{-1}$. Moreover, ($h = g'/g$)

$$\lambda_0 = -Lg(x)/g(x) = -(ah^2 + bh + ah')(x) =: R(h)(x);$$

and also ($h = g'$)

$$\lambda_0 = -(Lg)'(x)/g'(x) = -\left(ah' + bh\right)'/h(x) =: \overline{R}(h)(x).$$

- However, it is very hard to know the explicit expression of eigenfunction in general case.
- The property of eigenfunction: either positive or negative; monotone; $g(D) := \lim_{x\to D} g(x) = 0$ (Essential)
  Lemmas 4, 5, 6 consist of the basis of the test functions used in the following definitions of $\mathscr{F}_{\#}$ and $\mathscr{H}$ $\rightarrow$ how to choose the test function.

3(2). Our work—Notations

$$\mathscr{F}_I = \left\{f : f \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D],\ f|_{(0,D)} > 0,\ \text{and}\ f'|_{(0,D)} < 0\right\},$$

$$\mathscr{F}_{II} = \{f : f \in \mathscr{C}[0,D] \text{ and } f|_{(0,D)} > 0\},$$

$$\mathscr{H} = \Big\{h : h \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D],\ h(0) = 0,\ h|_{(0,D)} < 0 \text{ if } \nu(0,D) < \infty,$$
$$\text{and } h|_{(0,D)} \leqslant 0 \text{ if } \nu(0,D) = \infty\Big\}.$$

$$\widetilde{\mathscr{F}}_I = \Big\{f : f \in \mathscr{C}^1(x_0,x_1) \cap \mathscr{C}[x_0,x_1],\ f'|_{(x_0,x_1)} < 0 \text{ for some } x_0, x_1 \in [0,D)$$
$$\text{with } x_0 < x_1 \text{ and } f = f(\cdot \vee x_0)\mathbf{1}_{[0,x_1)}\Big\},$$

$$\widetilde{\mathscr{F}}_{II} = \left\{f : \exists x_0 \in (0,D) \text{ such that } f = f\mathbf{1}_{[0,x_0)} \text{ and } f \in \mathscr{C}[0,x_0]\right\},$$

$$\widetilde{\mathscr{H}} = \Big\{h : \exists x_0 \in (0,D) \text{ such that } h \in \mathscr{C}^1(0,x_0) \cap \mathscr{C}[0,x_0],\ h|_{(0,x_0)} < 0,$$
$$h(0) = 0,\ h|_{[x_0,D]} = 0, \text{ and } \sup_{(0,x_0)} \left(ah^2 + bh + ah'\right) < 0\Big\}.$$

3(2). Our work—Notations

Hypothesis

$$e^C/a \text{ and } e^{-C} \text{ are locally integrable.} \tag{1}$$

- single integral form

$$I(f)(x) = -\frac{e^{-C(x)}}{f'(x)} \int_0^x f\,\mathrm{d}\mu, \quad x \in (0, D);$$

- double integral form

$$I\!I(f)(x) = \frac{1}{f(x)} \int_{(x,D)\cap \mathrm{supp}(f)} \nu(\mathrm{d}s) \int_0^s f\,\mathrm{d}\mu, \quad x \in \mathrm{supp}(f);$$

- differential forms

$$R(h)(x) = -\big(ah^2 + bh + ah'\big)(x), \quad x \in (0, D);$$
$$\overline{R}(h)(x) = -\big(ah' + bh\big)'(x)/h(x), \quad x \in (0, D).$$

3(3). Main results: (I)

Theorem 1

(1)Single integral form:

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{x\in E} I(f)(x)^{-1}=\lambda_0=\sup_{f\in\mathscr{F}_I}\inf_{x\in E} I(f)(x)^{-1}.$$

(2) Double integral form:

$$\inf_{f\in\widetilde{\mathscr{F}}_{I\!I}}\sup_{x\in \text{supp(f)}} I\!I(f)(x)^{-1}=\lambda_0=\sup_{f\in\mathscr{F}_{I\!I}}\inf_{x\in E} I\!I(f)(x)^{-1},$$

Moreover, if $a, b\in\mathscr{C}[0,D]$, then we additionally have a (3) differential form:

$$\inf_{h\in\widetilde{\mathscr{H}}}\sup_{x\in E} R(h)(x)=\lambda_0=\sup_{h\in\mathscr{H}}\inf_{x\in E} R(h).$$

When $\exists$ a solution (λ_0, g) for the eigeneq., the sup. can be attained.

3(3). Main results: (I)

Proposition 1

$$\lambda_0 = \inf\left\{D(f) : \mu\left(f^2\right) = 1,\, f \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D] \text{ and } f(D) = 0\right\}$$

$$=: \tilde{\lambda}_0, \text{ where } f(D) = \lim_{x\to D} f(x) \text{ in the case of } D = \infty.$$

$$\inf_{f\in\widetilde{\mathscr{F}}_I'} \sup_{x\in(0,D)} I(f)(x)^{-1} = \lambda_0 = \sup_{f\in\mathscr{F}_I} \inf_{x\in(0,D)} I\!I(f)(x)^{-1},$$

$$\inf_{f\in\widetilde{\mathscr{F}}_{I\!I}\bigcup\widetilde{\mathscr{F}}_{I\!I}'} \sup_{x\in\text{supp}(f)} I\!I(f)(x)^{-1} = \lambda_0 = \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{x\in\text{supp}(f)} I\!I(f)(x)^{-1},$$

where

$$\widetilde{\mathscr{F}_I'} = \{f : \exists x_0 \in (0,D) \text{ such that } f = f\mathbf{1}_{[0,x_0)},\, f \in \mathscr{C}^1(0,x_0) \cap \mathscr{C}[0,x_0], \text{ and } f'|_{(0,x_0)} < 0\},$$

$$\widetilde{\mathscr{F}_{I\!I}'} = \{f : f > 0,\, f \in \mathscr{C}[0,D], \text{ and } fI\!I(f) \in L^2(\mu)\}.$$

Besides, the supremum over $\{f \in \mathscr{F}_I\}$ can be attained.

3(3). Main results: (I)

$$\lambda_0 \geqslant \tilde{\lambda}_0 \geqslant \sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I(f)(x)^{-1}$$
$$= \sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I\!I(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{I\!I}}\inf_{x\in(0,D)} I\!I(f)(x)^{-1}$$
$$\geqslant \sup_{h\in\mathscr{H}}\inf_{x\in(0,D)} R(h)(x) \geqslant \lambda_0 \text{ provided that } \lambda_0 = \tilde{\lambda}_0.$$

$$\lambda_0 \leqslant \inf_{f\in\widetilde{\mathscr{F}}_{I\!I}\bigcup\widetilde{\mathscr{F}}'_{I\!I}}\sup_{x\in\text{supp}(f)} I\!I(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I\!I}}\sup_{x\in\text{supp}(f)} I\!I(f)(x)^{-1}$$
$$= \inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{x\in\text{supp}(f)} I\!I(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{x\in(0,D)} I(f)(x)^{-1}$$
$$= \inf_{f\in\widetilde{\mathscr{F}}'_I}\sup_{x\in(0,D)} I(f)(x)^{-1}(\leqslant \tilde{\lambda}_0 \Rightarrow \lambda_0 = \tilde{\lambda}_0)$$
$$\leqslant \inf_{h\in\widetilde{\mathscr{H}}}\sup_{x\in(0,D)} R(h)(x) \leqslant \lambda_0.$$

3(3). Main results: (I)

Sketch of proof for the lower estimates:

$$\lambda_0 \geqslant \tilde{\lambda}_0 \geqslant \sup_{f\in\mathscr{F}_{II}} \inf_{x\in(0,D)} II(f)(x)^{-1}.$$

(a) $\lambda_0 \geqslant \tilde{\lambda}_0$ by definitions.

(b) Let g be a test function of $\tilde{\lambda}_0$. Then for every h with $h|_{(0,D)} > 0$, by Cauchy-Schwarz's inequality and Fubini's Theorem,

$$1 = \mu(g^2) \leqslant D(g) \sup_{t\in(0,D)} \frac{1}{h(t)} \int_0^t \frac{e^{C(x)}}{a(x)} \mathrm{d}x \int_x^D \frac{h(s)}{e^{C(s)}} \mathrm{d}s =: D(g) \sup_{t\in(0,D)} H(t).$$

For $f \in \mathscr{F}_{II}$, we specify $h(t) = \int_0^t a(s)^{-1} e^{C(s)} f(s) \mathrm{d}s$. Then by Cauchy's mean value theorem, $\sup_{t\in(0,D)} H(t) \leqslant \sup_{x\in(0,D)} II(f)(x) \Rightarrow$

$$\inf_{x\in(0,D)} II(f)(x)^{-1} \leqslant \inf_{t\in(0,D)} H(t)^{-1} \leqslant D(g) \Rightarrow \cdots \leqslant \tilde{\lambda}_0.$$

3(3). Main results: (I)

$$\sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I(f)(x)^{-1} = \sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I\!I(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{I\!I}}\inf_{x\in(0,D)} I\!I(f)(x)^{-1}.$$

(c) For $f \in \mathscr{F}_I$, by using Cauchy's mean value theorem, we have

$$\sup_{x\in(0,D)} I\!I(f)(x) \leqslant \sup_{t\in(0,D)} -\frac{e^{-C(t)}}{f'(t)}\int_0^t f\,\mathrm{d}\mu = \sup_{x\in(0,D)} I(f)(x).$$

Since $\mathscr{F}_I \subset \mathscr{F}_{I\!I}$,

$$\sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I(f)(x)^{-1} \leqslant \sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I\!I(f)(x)^{-1} \leqslant \sup_{f\in\mathscr{F}_{I\!I}}\inf_{x\in(0,D)} I\!I(f)(x)^{-1}$$

$$? \leqslant \sup_{f\in\mathscr{F}_I}\inf_{x\in(0,D)} I(f)(x)^{-1}$$

3(3). Main results: (I)

(c)(continued) Let $f \in \mathscr{F}_{I\!I}$. Put $g(x) = f I\!I(f)(x)$. Then $g \in \mathscr{F}_I$ and $I(g)(s)^{-1} \geqslant \inf_{x\in(0,D)} I\!I(f)(x)^{-1} \Rightarrow$

$$\inf_{s\in(0,D)} I(g)(s)^{-1} \geqslant \inf_{x\in(0,D)} I\!I(f)(x)^{-1} \Rightarrow \cdots .$$

In the following, we pre-assume that $\lambda_0 = \tilde{\lambda}_0$.
A different way to prove the equalities:
By the property of eigenfunction g, we have seen that $\lambda_0 = I(g)(x)^{-1}$ for $x \in (0, D)$ and $g \in \mathscr{F}_I \Rightarrow$

$$\lambda_0 = \inf_{x\in(0,D)} I(g)(x)^{-1} \leqslant \sup_{f\in\mathscr{F}_I} \inf_{x\in(0,D)} I(f)(x)^{-1} \leqslant \cdots \leqslant \lambda_0.$$

(Lemmas 4, 5)

3(3). Main results: (I)

Now we assume that $a, b \in \mathscr{C}[0, D]$.

$$\sup_{f\in\mathscr{F}_{I\!I}} \inf_{x\in(0,D)} I\!I(f)(x)^{-1} \geqslant \sup_{h\in\mathscr{H}} \inf_{x\in(0,D)} R(h)(x).$$

(d) For each $h \in \mathscr{H}$ with $h = g'/g$, we have $R(h) = -Lg/g$.
If $R(h) > 0$ for a positive g with $g'(0) = 0$ and $h = g'/g$, then g must be strictly decreasing.
Let $f = gR(h)$. Then $Lg = -f$ and $f \in \mathscr{F}_{I\!I}$. Since $Lg = -f$ and $g'(0) = 0 \Rightarrow g(x) \geqslant f(x)I\!I(f)(x) \Rightarrow R(h)(x) \leqslant I\!I(f)(x)^{-1}$ for $x \in (0, D) \Rightarrow$

$$\inf_{x\in(0,D)} R(h)(x) \leqslant \inf_{x\in(0,D)} I\!I(f)(x)^{-1} \leqslant \sup_{f\in\mathscr{F}_{I\!I}} \inf_{x\in(0,D)} I\!I(f)(x)^{-1}.$$

3(3). Main results: (I)

$$\sup_{h\in\mathscr{H}} \inf_{x\in(0,D)} R(h)(x) \geqslant \lambda_0.$$

(e) Since $a,\ b \in \mathscr{C}[0,D]$, by the property of eigenfunction(Lemmas 1, 4, 5) , there exists an eigenfunction g such that $Lg = -\lambda_0 g$,

$$g'(0) = 0,\ g|_{(0,D)} > 0,\ g'|_{(0,D)} < 0 \text{ and } g \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D].$$

Let $h = g'/g$. Then $h \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D]$, $h(0) = 0$, $h \in \mathscr{H}$, and

$$R(h)(x) = -\frac{Lg(x)}{g(x)} = \lambda_0 \qquad \text{for } x \in (0,D).$$

3(3). Main results: (II)

Proposition 2

Suppose that $a, b \in \mathscr{C}^1(0,D) \cap \mathscr{C}[0,D]$ and $a > 0$ on $(0,D)$. Set

$$\overline{\mathscr{H}} = \left\{h : h(0) = 0,\ h \in \mathscr{C}^2(0,D) \cap \mathscr{C}[0,D],\ \text{and } h|_{(0,D)} < 0\right\}.$$

Then (1) we have $\sup_{h\in\overline{\mathscr{H}}} \inf_{x\in(0,D)} \overline{R}(h)(x) \geqslant \lambda_0$ and the equality sign holds once $\mu(0,D) = \infty$.
(2) In general, we have

$$\lambda_0 = \sup_{h\in\mathscr{H}_*} \inf_{x\in(0,D)} \overline{R}(h)(x),$$

where $\mathscr{H}_* = \left\{h : h(0) = 0,\ h \in \mathscr{C}^2(0,D) \cap \mathscr{C}[0,D], h|_{(0,D)} < 0, h' < -a^{-1}bh \text{ on } (0,D)\right\}$. Moreover, the supremum can be attained.

This differential form can be used as a 'bridge' between ND and DN.

3(3). Main results: (III)

Applying Theorem 1(1) to the test function $\nu(x, D)^\gamma$ with $\gamma = 1/2$, 1:

Theorem 2(Criterion and basic estimates)

$$\lambda_0 > 0 \Leftrightarrow \delta := \sup_{x \in (0,D)} \mu(0,x)\,\nu(x,D) < \infty.$$

More precisely, we have

$$(4\delta)^{-1} \leqslant \lambda_0 \leqslant \delta^{-1}.$$

In particular, when $D = \infty$, we have $\lambda_0 = 0$ if $\nu(0,D) = \infty$, and $\lambda_0 > 0$ if $\int_0^\infty \mu(0,x)\,\nu(\mathrm{d}x) < \infty$.

3(3). Main results: (III)

Applying Theorem 1(2), repeatedly with $f = f_n$:

Theorem 3(Approximating procedure)

Assume that $\delta < \infty$. Set $\varphi(x) = \nu(x, D)$.
(1) Let $f_1 = \sqrt{\varphi}$, $f_n = f_{n-1} I\!I(f_{n-1})$, and $\delta_n = \sup_{x\in(0,D)} I\!I(f_n)(x)$. Then δ_n is decreasing in n and

$$\lambda_0 \geqslant \delta_n^{-1} \geqslant (4\delta)^{-1}, \quad n \geqslant 1.$$

(2) For fixed $x_0,\ x_1 \in [0, D)$ with $x_0 < x_1$, define

$$f_1^{x_0,x_1} = \nu(\cdot \vee x_0,\ x_1)\,\mathbf{1}_{[0,x_1)},\ f_n^{x_0,x_1} = \left(f_{n-1}^{x_0,x_1} I\!I\left(f_{n-1}^{x_0,x_1}\right)\right)(\cdot \vee x_0)\,\mathbf{1}_{[0,x_1)},\ n \geqslant 1,$$

and let $\delta_n' = \sup_{x_0,x_1:x_0<x_1} \inf_{x<x_1} I\!I({f_n}^{x_0,x_1})(x)$. Then δ_n' is increasing in n and

$$\delta^{-1} \geqslant {\delta_n'}^{-1} \geqslant \lambda_0, \quad n \geqslant 1.$$

3(3). Main results: (III)

Theorem 3(continued)

(3) Define

$$\bar{\delta}_n = \sup_{x_0, x_1 : x_0 < x_1} \frac{\|f_n^{x_0,x_1}\|}{D(f_n^{x_0,x_1})}, \qquad n \geqslant 1.$$

Then $\bar{\delta}_n^{-1} \geqslant \lambda_0$, $\bar{\delta}_{n+1} \geqslant \delta_n'$ $(n \geqslant 1)$, and $\bar{\delta}_1 = \delta_1'$.

Corollary (Improved estimates)

$$\delta^{-1} \geqslant {\delta_1'}^{-1} \geqslant \lambda_0 \geqslant \delta_1^{-1} \geqslant (4\delta)^{-1},$$

$$\delta_1 = \sup_{x \in (0,D)} \left(\sqrt{\varphi(x)} \int_0^x \sqrt{\varphi}\, \mathrm{d}\mu + \frac{1}{\sqrt{\varphi(x)}} \int_x^D \varphi^{3/2} \mathrm{d}\mu \right),$$

$$\delta_1' = \sup_{x \in (0,D)} \left(\mu(0,x)\, \varphi(x) + \frac{1}{\varphi(x)} \int_x^D \varphi^2\, \mathrm{d}\mu \right) \in [\delta, 2\delta].$$

3(4). From ND to DN

For the DN case, the similar results can be given by the parallel method.
$\rightarrow$ 'parallel results'

Duality relation: $\hat{\nu} := \mu;\qquad \hat{\mu} := \nu;$

$$L = \frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}\nu};\qquad \text{dual with}\qquad \hat{L} = \frac{\mathrm{d}}{\mathrm{d}\hat{\mu}}\frac{\mathrm{d}}{\mathrm{d}\hat{\nu}}.$$

Then $\lambda_0(L) = \lambda_0(\hat{L})$ and $\delta = \hat{\delta}$, where $\lambda_0(\hat{L})$ is the eigenvalue of the corresponding diffusion process with DN boundaries.

The identity is a combination of the formula in differential form $\overline{R}$ (Proposition 2) in ND case and the one in differential form R in DN case. $\rightarrow$ 'dual results'

3(4). From ND to DN

‘parallel results’ $\neq$ ‘dual results’

The variational formulas and then the approximating procedure in DN case by parallel method are different from those deduced by the dual approach. It is interesting that in the discrete situation, the approximating procedure given by the parallel method is often less powerful than those in terms of duality. Similar phenomenon happens in the continuous situation with $D < \infty$.

M.F. Chen

General estimate of the first eigenvalue on manifolds, Front. Math. China 2011, 6(6): 1025-1043.

3(5). The property of eigenfunction

- The properties of eigenfunction in DN case were shown by M.F. Chen in 1999.
- The properties of eigenfunction in ND case are not so easy! The method for the DN case does not work now!

When $D < \infty$, one may simply reverse the variable to obtain one from the other of the ND- and DN- cases. But when $D = \infty$, these two cases are certainly different since the Dirichlet boundary at 0 is touchable but not the one at ∞.

Six lemmas, except Lemma 2, are mainly devoted to describe the eigenfunction of λ_0 in ND case. The lemmas are essential in our study.

3(5). The property of eigenfunction

Let $\mathscr{A}[\alpha, \beta]$ be the set of all absolutely continuous functions on $[\alpha, \beta]$.

Lemma 1 (Existence) Zettl A. Sturm-Liouville Theory, AMS, 2005

(1) $\exists$ uniquely a non-zero function $g \in \mathscr{C}^1[0, D]$ such that

$$g' \in \{f \in \mathscr{A}[\alpha, \beta] : 0 \leqslant \alpha < \beta < D\} \quad \text{and} \quad Lg = -\lambda g, \ a.e..$$

(2) Suppose additionally $a, b \in \mathscr{C}[0, D]$, then $g \in \mathscr{C}^2[0, D]$ and $Lg = -\lambda g$ on $[0, D]$.

We call the function g given in Lemma 1(1) a.e. eigenfunction of λ.

Lemma 2

Define $\lambda_* = \inf\{D(f) : f \in \mathscr{A}[0, D], \ \|f\| = 1, \ \text{and} \ f(D) = 0\}$. Then $\tilde{\lambda}_0 = \lambda_*$.

$$\tilde{\lambda}_0 := \inf\left\{D(f) : \mu(f^2) = 1, \ f \in \mathscr{C}^1(0, D) \cap \mathscr{C}[0, D] \text{ and } f(D) = 0\right\}.$$

3(5). The property of eigenfunction

Lemma 3

$\lambda_0^{(0,p_n)} \downarrow\downarrow \lambda_0^{(0,D)} = \tilde{\lambda}_0$ as $p_n \uparrow\uparrow D$. Moreover, for $p \in (0, D)$, we have $\lambda_1^{(0,p)} > \lambda_0^{(0,p)}$, where $\lambda_1^{(0,p)}$ is the principal NN-eigenvalue.

Lemma 4(Monotonicity)

Let g be a non-zero a.e. eigenfunction of $\tilde{\lambda}_0 > 0$. Then g is strictly monotone.

Sketch of proof:

1. Since $\tilde{\lambda}_0 > 0$, $g \neq$ cons. It suffices to show that $g'|_{(0,D)} \neq 0$.
2. If $\exists p \in (0, D)$ such that $g'(p) = 0$, then $\tilde{\lambda}_0 \geqslant \lambda_1^{(0,p)}$.
3. From the above arguments, we have

$$\tilde{\lambda}_0 \geqslant \lambda_1^{(0,p)} > \lambda_0^{(0,p)} > \lambda_0^{(0,D)} = \tilde{\lambda}_0.$$

This is a contradiction.

3(5). The property of eigenfunction

Lemma 5

The a.e. eigenfunction g of $\tilde{\lambda}_0$ is either positive or negative everywhere.

Sketch of proof:
1. If $\tilde{\lambda}_0 = 0$, then $g = const$. and so the assertion is obvious.
2. Let $\tilde{\lambda}_0 > 0$. Assume that $g'|_{(0,D)} < 0$ and $g(0) > 0$. We need only to prove that $g \neq 0$ on $(0, D)$.
3. If otherwise $g(p) = 0$ for some $p \in (0, D)$, then, since $\lambda_0^{(0,p)}$ is the minimal ND-eigenvalue on $(0, p)$, the eigenequation restricted to $(0, p)$ shows that

$$\tilde{\lambda}_0 \geqslant \lambda_0^{(0,p)} > \lambda_0^{(0,D)} = \tilde{\lambda}_0,$$

which is a contradiction.

3(5). The property of eigenfunction

Lemma 6 (Vanishing property)

Let $D = \infty$. If $\lambda_0 > 0$, then its a.e. eigenfunction g satisfies $g(\infty) = 0$.

Sketch of proof: Since $\lambda_0 > 0$ and $Lg = -\lambda_0 g$, we have

$$\int_0^D e^{-C(x)}\mathrm{d}x \int_0^x g(t)\mu(\mathrm{d}t) = \frac{g(0) - g(\infty)}{\lambda_0} < \infty.$$

Let $f = g - g(\infty)$. If $g(\infty) > 0$, then $f \in \mathscr{F}_{I\!I}$. By the lower variational formula and Cauchy's mean value theorem,

$$\lambda_0^{-1} \leqslant \inf_{f \in \mathscr{F}_{I\!I}} \sup_{x\in(0,D)} I\!I(f)(x) \leqslant \sup_{x\in(0,D)} I\!I(f)(x)$$

$$= \sup_{x\in(0,D)} \int_x^D e^{-C(x)}\mathrm{d}x \int_0^x f(t)\mu(\mathrm{d}t) \Big/ \lambda_0 \int_x^D e^{-C(x)}\mathrm{d}x \int_0^x g(t)\mu(\mathrm{d}t)$$

$$\leqslant \sup_{x\in(0,D)} \frac{f(x)}{\lambda_0 g(x)} = \sup_{x\in(0,D)} \frac{1}{\lambda_0}\left(1 - \frac{g(\infty)}{g(x)}\right) \leqslant \frac{1}{\lambda_0}\left(1 - \frac{g(\infty)}{g(0)}\right) < \lambda_0^{-1}.$$

4. Other related research

- J.H. Shao and Y.H. Mao studied principal eigenvalue of birth-death processes on trees by taking advantage of the related results on half line. (Variational formulas, DN)
- Ma Y.T. discussed the regularity, recurrence and ergodic criterions, $\cdots$ for birth-death process on trees by constructing two corresponding birth-death process on Z_+.
- weighted Hardy inequality:

$$\mu(|f|^q) \leqslant C\nu(|f|^p).$$

A. Kufner, L.-E. Persson

Weighted Inequalities of Hardy type, World Scientific, 2003.

H.Y. Jin and Y.H. Mao obtained some estimates on the best constant of weighted Hardy inequality in L^p space using the similar method for estimates on λ_0.(DD, continuous case)

4. Other related research

- Especially, let $p = q = 2, \mu(x) = e^{C(x)}/a(x)$, $\nu(x) = e^{C(x)}$. Then $C = \lambda_0^{-1}$, i.e., the estimates on mixed eigenvalue is just the estimates on the best constant of a special kind of Hardy inequality.
- We now study the weighted Hardy inequality and mixed eigenvalues of p-Laplacian operator in discrete and continuous cases.

5. Main references

- M.F. Chen, F.Y. Wang , *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. 349:3 1997, 1239-1267.
- M.F. Chen, *Anaylitic proof of dual variational formula for the first eigenvalue in dimension one*, Sci. Sin. (A) 1999, 42:8, 805-815.
- M.F. Chen, *Variational formulas and approximation theorems for the first eigenvalue in dimension one*, Sci. Sin. (A) 2001, 31:1, 28-36 (Chinese Edition); 44:4 (English Edition), 409-418.
- M.F. Chen, *Eigenvalues, Inequalities, and Ergodic Theory*, New York, Springer, 2005.
- M.F. Chen, *Speed of stability for birth-death process*, Front. Math. China 5(3):379-515 2010

5. Main references

- M.F. Chen, *Basic esimates and stability rate for one-dimentional diffusion*, Chapter 6 in ”Probability Approximations and Beyond”, Lecture notes in statistics 205, 2012, 75-99.
- H.Y. Jin, Y.H. Mao, *Estimation of the optimal constants in the L^p-Poincaré inequalities on the half line*, Acta Math. Sin. (Chinese vertion) 2012, Vol.55, No.1.
- Y.T. Ma, *Birth and death process on trees*, Sci. Sin. (A), 2010,Vol.53, No.1:10.
- J.H. Shao, Y.H. Mao, *Eiatimation of the Dirichlet Eigenvalue of birth-delth Process on trees*. Acta Math. Sin. (Chinese vertion) 2007, Vol.50, No.3.
- M.F. Chen, L.D. Wang, Y.H. Zhang, *Mixed principal eigenvalues in dimension one*, Front. Math. China 2012+.

Thank you for your attention!