Single Birth Processes

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Contents

Background

- State space $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$; Markov process $X(t)$ with transition probability matrix $P(t) = (p_{ij}(t))$.
- Transition rate Q-matrix $Q = (q_{ij})$:

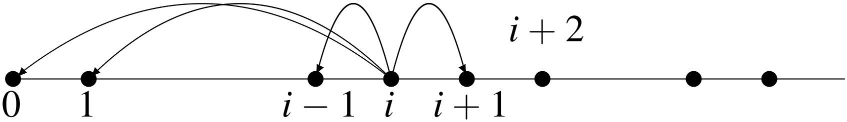
$$\lim_{t \to 0} \frac{p_{ij}(t) - \delta_{ij}}{t} = q_{ij}.$$

- Totally stable: $q_i := -q_{ii} < \infty$ for all $i \in \mathbb{Z}_+$.
- Conservative: $q_i = \sum_{j \neq i} q_{ij}$ for all $i \in \mathbb{Z}_+$.
- Regular: totally stable, conservative, determine uniquely one process.

Background

Single birth Q-matrix $Q = (q_{ij})$:

$$q_{i,i+1} > 0, \quad q_{ij} = 0 \quad \text{for all} \quad j > i+1, i \in \mathbb{Z}_+.$$



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background

- Single birth process(S.J. Yan & M.F. Chen, 1986).
- Upwardly skip-free processes (W.J. Anderson, 1991).
- Population Theory: birth and death processes with catastrophes.

$$
\begin{aligned}
i \to\; & i+1 && \text{at rate} && b_i \\
\to\; & i-1 && && a_i + c_i f_{i,i-1} \\
\to\; & i-2 && && c_i f_{i,i-2} \\
\to\; & \cdots && && \cdots \\
\to\; & 0 && && c_i f_{i0}
\end{aligned}
$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

Background

P. J. Brockwell, J. Gani, S. I. Resnick and A. G. Pakes et al (1982-1986), B. Cairns & P. Pollett (2004) for certain birth, death and catastrophes f_{ij}, such as geometric, uniform, binomial.

People concern over extinction problem, such as extinction probability and extinction times.

Key tool: generating function of Q-resolvent.

We won't talk about these work in detail today. Let's turn to the classical problems: uniqueness, recurrence, $\cdots$.

Uniqueness

For a totally stable, conservative single birth Q-matrix. Under what condition does it determine uniquely a single birth process?

Define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ for all $0 \leqslant k < n$ $(k, n \in \mathbb{Z}_+)$ and

$$m_0 = \frac{1}{q_{01}}, \quad m_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\right), \quad n \geq 1.$$

Theorem (J.K. Zhang(1984), Yan & Chen(1986))

The process is unique iff $R := \sum_{n=0}^{\infty} m_n = \infty.$

Uniqueness

Theorem (Feller(1957), Reuter(1957))

For a totally stable, conservative Q-matrix, the process is unique iff zero-exit.

Zero-exit: for some (equivalently, for all) $\lambda > 0$, the solution $x = (x_i)$ to the equation

$$x_i = \sum_{j \neq i} \frac{q_{ij}}{\lambda + q_i} x_j, \qquad 0 \le x_i \le 1, \ i \in \mathbb{Z}_+$$

is trivial.

Uniqueness

In the non-conservative cases, what happens for zero-exit? Define the non-conservative quantity at i by

$$c_i := q_i - \sum_{j \neq i} q_{ij}.$$

Theorem (L.D. Wang & Zhang(2012))

Given a totally stable single birth Q-matrix $Q = (q_{ij})$. *Then zero-exit iff* $\widehat{R} := \sum_{n=0}^{\infty} \hat{m}_n = \infty$, *where*

$$\hat{m}_n = \frac{1}{q_{n,n+1}}\left(1 + c_n + \sum_{k=0}^{n-1} q_n^{(k)} \hat{m}_k\right), \qquad n \in \mathbb{Z}_+.$$

Uniqueness

Sketch of the proof:

- Construct a conservative single birth Q-matrix $\widehat{Q}$ on $\{-1, 0, 1, \cdots\}$ with absorbing state -1:

$$\hat{q}_{i,-1} = c_i, \quad \hat{q}_{ij} = q_{ij}, \quad i,j \in \mathbb{Z}_+.$$

- Q is zero-exit if and only if so is $\widehat{Q}$.
- By Chen(1999), $\widehat{Q}$ is zero-exit iff

$$m_n = \frac{1}{q_{n,n+1}}\left(1+c_n+\sum_{k=0}^{n-1}(c_n+q_n^{(k)})m_k\right), \qquad n \in \mathbb{Z}_+.$$

- Prove that the two series $\sum_{n=0}^{\infty} \hat{m}_n$ and $\sum_{n=0}^{\infty} m_n$ converge or diverge simultaneously.

Uniqueness

The criteria on zero-exit are unified for conservative and non-conservative cases.

Note that the process satisfying the backward equation is unique iff zero-exit.

Corollary

Given a totally stable single birth Q-matrix $Q = (q_{ij})$. *Then the process satisfying the backward equation is unique iff* $\widehat{R} := \sum_{n=0}^{\infty} \hat{m}_n = \infty$.

Recurrence

Define

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n.$$

Theorem (Yan & Chen(1986))

For a regular and irreducible single birth Q-matrix, the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Key: recurrence iff the equation

$$x_i = \sum_{k \neq 0,i} \frac{q_{ik}}{q_i} x_k, \quad 0 \leqslant x_i \leqslant 1, i \in \mathbb{Z}_+$$

has only zero solution.

Ergodicity

Ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$. Define

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\right),$$

Theorem (Yan & Chen(1986))

For a regular and irreducible single birth Q-matrix, the process is ergodic iff

$$d := \sup_{k\geq 0}\left(\sum_{n=0}^{k} d_n\right)\Big/\left(\sum_{n=0}^{k} F_n^{(0)}\right) < \infty.$$

Key: ergodicity iff the following equation has a finite non-negative solution:

$$x_i \geqslant \sum_{j\neq i} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \quad i \geqslant 1.$$

Strong ergodicity

Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$.

Theorem (Zhang(2001))

For a regular and irreducible single birth Q-matrix, the process is srongly ergodic iff

$$S := \sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty.$$

Key: strong ergodicity iff the following equation has a bounded nonnegative solution:

$$x_i \geqslant \sum_{j\neq i} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \quad i \geqslant 1.$$

Strong ergodicity

- H.J. Zhang, X. Lin and Z.T. Hou (2000) prove the criterion for the birth-death process.
- Tweedie(1981) used the condition "$S < \infty$" as the sufficient one for the exponential ergodicity of the birth-death process.

Moments of the first hitting time

Fix $i_0 \in \mathbb{Z}_+$. Consider the first hitting time of i_0: $\tau_i = \inf\{t > 0 : X(t) = i_0\}$. Define

$$m_0^{(n)} = \frac{\mathbf{E}_0 \tau_{i_0}^{n-1}}{q_{01}}, \quad m_i^{(n)} = \frac{1}{q_{i,i+1}} \left(\mathbf{E}_i \tau_{i_0}^{n-1} + \sum_{k=0}^{i-1} q_i^{(k)} m_k^{(n)} \right)$$

for $i \geqslant 1$, where in convention $\mathbf{E}_{i_0} \tau_{i_0}^0 = 1$. Set

$$c_k^{(n)} = \sup_{i \geqslant k} \frac{\sum_{j=k}^i m_j^{(n)}}{\sum_{j=k}^i F_j^{(k)}}, \quad k \in \mathbb{Z}_+.$$

When $n = 1$, we omit the supscript "(1)" for m and c.

Moments of the first hitting time

Theorem (Zhang(2012))

Given a totally stable and conservative single birth Q-matrix.

$$\mathbf{E}_i\tau_{i_0}^n = n\sum_{j=i}^{i_0-1} m_j^{(n)}, \quad i < i_0;$$

$$\mathbf{E}_i\tau_{i_0}^n = n\sum_{j=i_0}^{i-1} \left(F_j^{(i_0)} c_{i_0}^{(n)} - m_j^{(n)}\right), \quad i \geqslant i_0 + 1.$$

In particular, $\mathbf{E}_{i_0-1}\tau_{i_0}^n = n m_{i_0-1}^{(n)}$, $\mathbf{E}_{i_0+1}\tau_{i_0}^n = n\left(c_{i_0}^{(n)} - m_{i_0}^{(n)}\right)$.

Moments of the first hitting time

Key: $(\mathbf{E}_i\tau_{i_0}^n)$ is the minimal non-negative solution of the following equation:

$$x_{i_0} = 0, \quad x_i = \sum_{k\neq i} \frac{q_{ik}}{q_i} x_k + \frac{n\mathbf{E}_i\tau_{i_0}^{n-1}}{q_i}, \quad i \neq i_0.$$

Remark

$$d = \mathbf{E}_1\tau_0, \quad m = \mathbf{E}_0\tau_\infty, \quad S = \sup_{i\geqslant 1} \mathbf{E}_i\tau_0.$$

Approaches

The above problems (uniqueness, recurrence, ergodicity, strong ergodicity, moments) are connected with one linear equation with non-negative coefficients:

$$x_i = \sum_{j \neq ?} a_{ij} x_j + b_i.$$

By the property of single birth, we know that the equation has only one (single) parameter. The explicit criteria or representations can be expected to obtain.

So we need only patience!

Stationary distribution

Consider the return time: $\sigma_j = \inf\{t \geq$ the first jumping time: $X(t) = j\}$.

Theorem (Zhang(2004))

Given a totally stable and conservative single birth Q-matrix. Then

$$\mathbf{E}_i\sigma_i = \frac{q_{i,i+1}c_i}{q_i}, \qquad i \in \mathbb{Z}_+.$$

In the ergodic case, the stationary distribution

$$\pi_i = \frac{1}{q_{i,i+1}c_i}, \quad i \in \mathbb{Z}_+.$$

Finite space

Consider totally stable and conservative single birth Q-matrix on the finite space $\{0, 1, 2, \cdots, N\}$. We use the former notations and in convention $q_{N,N+1} = 1$.

Theorem (Zhang(2012))

$$\mathbf{E}_i \tau_{i_0}^n = n \sum_{j=i}^{i_0-1} m_j^{(n)}, \qquad i < i_0;$$

$$\mathbf{E}_i \tau_{i_0}^n = n \sum_{j=i_0}^{i-1} \left(F_j^{(i_0)} \frac{m_N^{(n)}}{F_N^{(i_0)}} - m_j^{(n)} \right), \qquad i_0 < i \leqslant N.$$

Finite space

Theorem (Zhang(2012))

$$\pi_i = \frac{F_N^{(i)}}{q_{i,i+1} m_N}, \quad 0 \leqslant i \leqslant N.$$

$$m_N = \sum_{i=0}^{N} \frac{F_N^{(i)}}{q_{i,i+1}}; \quad m_k = \sum_{i=0}^{k} \frac{F_k^{(i)}}{q_{i,i+1}}.$$

ℓ-ergodicity

Definition

For a positive integer ℓ, the recurrent chain $P(t)$ is said to be ℓ-ergodic if $\mathbf{E}_j\sigma_j^\ell < \infty$ for some (and hence for all) $j \in E$.

1-ergodic = positive recurrent (ergodic).
0-ergodic = null recurrent.
Discrete time:

- J.G. Kemeny, J.L. Snell & A.W. Knapp(1976)
- Y.H. Mao(2003); Z.T. Hou & Y.Y. Liu(2003).

Continuous time:

- P. Coolen-Schrijner & E.A. van Doorn(2002)
- Y.H. Mao (2004)

ℓ-ergodicity

- ℓ-ergodicity provides an algebraic convergence rate: $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \to \infty$.
- Key: The single birth process is ℓ-ergodic iff the equation

$$x_i \geqslant \sum_{j \neq i} \frac{q_{ij}}{q_i} x_j + \frac{\ell \mathbf{E}_i \sigma_0^{\ell-1}}{q_i}, \quad i \geqslant 1$$

has a finite nonnegative solution.

ℓ-ergodicity

Define

$$d_0^{(\ell)} = 0, \quad d_i^{(\ell)} = \frac{1}{q_{i,i+1}}\left(\mathbf{E}_i\sigma_0^{\ell-1} + \sum_{k=0}^{i-1} q_i^{(k)} d_k^{(\ell)}\right), \quad i \geq 1.$$

$$d^{(\ell)} := \sup_{i\geq 0} \frac{\sum_{j=0}^{i} d_j^{(\ell)}}{\sum_{j=0}^{i} F_j^{(0)}}.$$

Theorem (Zhang(2010))

Given a regular and irreducible single birth Q-matrix. Then the process is ℓ-ergodic iff $d^{(\ell)} < \infty$ for $\ell \geq 1$.

ℓ-ergodicity

Remark

$$\ell d^{(\ell)} = \mathbf{E}_1 \sigma_0^\ell.$$

By the property of single birth,

$$\mathbf{E}_0 \sigma_0^\ell < \infty \Leftrightarrow \mathbf{E}_1 \sigma_0^\ell < \infty \Leftrightarrow d^{(\ell)} < \infty.$$

Exponential ergodicity

Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t)-\pi_j|=0.$

Theorem (Mao & Zhang (2004))

Given a regular and irreducible single birth Q-matrix. If

$$\inf_i q_i > 0 \text{ and } \delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty,$$

then the process is exponentially ergodic.

Exponential ergodicity

Key: the single birth process is exp. erg. iff for some $\lambda > 0$ with $\lambda < q_i$ for all i, the following equation

$$x_i \geqslant \sum_{j \neq i} \frac{q_{ij}}{q_i - \lambda} x_j + \frac{1}{q_i - \lambda}, \quad i \geqslant 1$$

has a finite nonnegative solution.

Note that now the equation has two (double) parameters. So it is not easy to construct the finite nonnegative solution. But we finish the task eventually under our conditions: the constructed solution has the form of double summations!

Exponential ergodicity

Unfortunately, in general, the condition may be not necessary for exp. erg. (except for birth-death processes and the next example).

Example

Given $q_{01} > 0$ and $q_{i,i+1} = i^{\gamma}, q_{i0} = i^{\gamma-1}$ for all $i \geqslant 1$, where the constant $\gamma \in (1,2)$; $q_{ij} = 0$ for other $j \neq i$. Then the single birth process is strongly ergodic (so exp. erg.) but $\delta = \infty$.

Essentially, the condition implies that the single process is controlled by the following birth-death one:

$$a_i = \frac{q_{i,i+1} F_i^{(0)}}{F^{(0)}}, \quad i \geqslant 1; \qquad b_i = q_{i,i+1}, \quad i \geqslant 0.$$

Exponential ergodicity

$$\mathbf{E}_i\tau_0^n \leqslant \mathbf{E}_i\bar{\tau}_0^n, \quad n \geqslant 0 \Rightarrow \mathbf{E}_i e^{\lambda\tau_0} \leqslant \mathbf{E}_i e^{\lambda\bar{\tau}_0}.$$

So $\delta < \infty \Leftrightarrow$ the birth-death process is exp. erg. $\Rightarrow$ so is the single birth one.

Theorem (Chen(2000))

For a regular birth-death Q-matrix (a_i, b_i). The process is exponentially ergodic iff $\delta < \infty$.

Two facts: $\lambda_1 > 0 \Longleftrightarrow \delta < \infty$ and $\lambda_1 =$ exp. conv. rate.

Another proof is presented by constructing the solution in Mao & Zhang(2004).

Exponential ergodicity

Example (Zhang(2010))

Given a regular and irreducible single birth Q-matrix satisfying $\inf_i q_i > 0$ *and*

$$q_{i+1,j} = p_i q_{ij}, \quad 0 \leqslant j \leqslant i-1,$$

where $0 \leqslant p_i \leqslant c < 1 (i \geqslant 1)$ *for some constant c. There the process is exponentially ergodic iff* $\delta < \infty$.

Conjecture: the criterion should be

$$\tilde{\delta} = \sup_{i \geqslant 0} \sum_{j=0}^{i} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

Uniform decay

Assume that the (minimal) single birth process is transient, that is, the Green matrix $G = (g_{ij})$ satisfies $g_{ij} = \int_0^\infty p_{ij}(t)dt < \infty$ so that $\lim_{t\to\infty} p_{ij}(t) = 0$.

Theorem (Mao(2006))

$$tr(G) = R := \sum_{n=0}^{\infty} m_n.$$

$$\mathrm{tr}(G) = \sum_{j=0}^{\infty} g_{jj} = \sum_{j=0}^{\infty} \frac{1}{q_j \mathbf{P}_j(\sigma_j = \infty)} = R.$$

Uniform decay

Uniform decay rate:

$$\beta = \sup\{\varepsilon > 0 : \exists C < \infty \text{s.t.} ||P_t||_{\infty\to\infty} \leqslant Ce^{-\varepsilon t}, \forall t \geqslant 0\}.$$

Theorem (Mao(2006))

For a single birth processes, $\beta \geqslant R^{-1}$.

$$\begin{aligned}
\beta &\geqslant (\sup_i \mathbf{E}_i \eta)^{-1} \quad \text{(the first leap time)} \\
&= (\sup_i \mathbf{E}_i \tau_\infty)^{-1} \quad \text{(the first time arriving } \infty) \\
&= (\mathbf{E}_0 \tau_\infty)^{-1} \quad \text{(the single birth property)} \\
&= R^{-1}.
\end{aligned}$$

Hitting time distribution

Consider an irreducible single birth process on $\{0, 1, \cdots, N\}$ with absorbing state N.
Define $T_{0,N} = \inf\{t \geqslant 0 : X(t) = N | X(0) = 0\}$.

Theorem (Brown & Shao(1987), J.A. Fill(2009))

Let $\lambda_1, \cdots, \lambda_N$ be nonzero eigenvalue of $-Q$. Then

$$\mathbf{E}e^{-sT_{0,N}} = \prod_{j=1}^{N} \frac{\lambda_j}{s + \lambda_j}.$$

In particular, if the λ_j's are real, $T_{0,N}$ is distributed as the sum of N independent exponential random variables with rates $\{\lambda_1, \cdots, \lambda_N\}$.

SST distribution

A strong stationary time (SST): a stopping time τ for $X(t)$ such that $X(\tau)$ has the distribution π and is independent of τ.

Theorem (J.A. Fill(2009))

Consider an ergodic single birth process on $\{0, 1, \cdots, N\}$ with stochastically monotone time-reversal. Let $\lambda_1, \cdots, \lambda_N$ be nonzero eigenvalue of $-Q$. Then the fastest SST τ has Laplace transform:

$$\mathbf{E}e^{-s\tau} = \prod_{j=1}^{N} \frac{\lambda_j}{s+\lambda_j}.$$

Hitting times and SST distribution

How to deal with single birth process on $\mathbb{Z}_+$? Our colleague and students are doing this work.

For birth-death process on $\mathbb{Z}_+$, the corresponding results has been obtained by Y. Gong, Mao and C. Zhang recently.

Mao will give you this story in detail.

Thank you for your attention!