Large positive solutions of semilinear elliptic equations with critical and supercritical growth

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Abstract

Existence and uniqueness of large positive solutions are obtained for some semilinear elliptic equations with critical and supercritical growth on general bounded smooth domains. It is shown that the large positive solution develops a boundary layer. The boundary derivative estimate of the large solution is also established. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

In this paper we study large positive solutions of the problem
\[-\Delta u = u^p - \epsilon u^q \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]
\((I_\epsilon)\)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) \((N \geq 3)\) with smooth boundary \(\partial \Omega\) and
\[q > p \geq p_N := (N + 2)/(N - 2), \quad \epsilon > 0.\]
Merle and Peletier [1] studied this problem for small values of \( \epsilon \) with \( \Omega \) being the unit ball. They showed that (I_\epsilon) has at least two positive radial solutions: one is the large solution \( \overline{u}_\epsilon(r) \), i.e., \( \epsilon \overline{u}_\epsilon^{q-p}(0) \to 1 \) as \( \epsilon \to 0 \); another one is the small solution \( u_\epsilon(r) \), i.e., \( \epsilon u_\epsilon^{q-p}(0) \to c^* \in (0, 1) \) as \( \epsilon \to 0 \). Moreover, \( \overline{u}_\epsilon(x) \to \infty \) as \( \epsilon \to 0 \) for every \( x \in \Omega \), while the small solution \( u_\epsilon \), 'concentrates' at the origin; i.e., \( u_\epsilon(0) \to \infty \) and \( u_\epsilon(x) \to 0 \) for \( x \neq 0 \), as \( \epsilon \to 0 \) (see also [2–9]). In [10], they showed that the asymptotic behaviour of \( u_\epsilon \) obtained in [1] holds for the small solution of (I_\epsilon), provided that \( \Omega \) is star-shaped. In this paper we are interested in the properties of large solutions of (I_\epsilon), which were not carefully studied by Merle and Peletier.

It is known from the Pohozaev identity [11] that the problem (I_0) will not have a positive solution if \( \Omega \) is a star-shaped domain. We will allow \( \Omega \) to be any bounded smooth domain and show that (I_\epsilon) has a unique large positive solution \( u_\epsilon \); i.e., \( \epsilon (\max_\Omega \overline{u}_\epsilon)^{q-p} \to 1 \) as \( \epsilon \to 0 \). Moreover, for any compact set \( K \subset \Omega \),

\[
\epsilon \overline{u}_\epsilon^{q-p}(x) \to 1 \quad \text{for } x \in K.
\]

This implies that the large positive solution \( \overline{u}_\epsilon \) develops a boundary layer. Meanwhile, we give the exact behaviour of \( \partial \overline{u}_\epsilon / \partial \nu(x) \) for any \( x \in \partial \Omega \) as \( \epsilon \to 0 \), where \( \nu(x) \) is the outward normal vector of \( \partial \Omega \) at \( x \). As a corollary of our results, we know that \( \overline{u}_\epsilon(r) \) obtained in [1] is the unique large positive solution of (I_\epsilon) for \( \epsilon \) sufficiently small.

Writing (I_\epsilon) in the form

\[
-\Delta w = \lambda \xi(w) \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega, \quad (P_\lambda)
\]

where \( w = \epsilon^{1/(q-p)}u \), \( \lambda = \epsilon^{-(p-1)/(q-p)} \) and \( \xi(s) = s^p - s^q \) for \( s > 0 \), we know that (P_\lambda) is a semilinear elliptic problem with a large parameter \( \lambda \). The nonlinearity \( \xi(s) \) satisfies that \( \xi(0) = \xi'(0) = 0 = \xi(1) \), \( \xi(s) > 0 \) for \( s \in (0, 1) \) and \( \xi(s) < 0 \) for \( s > 1 \). Problem (P_\lambda) with \( \xi'(0) > 0 \) has been studied by Dancer [12]. In this paper, we use the results obtained in [1] (where the conditions on \( p \) and \( q \) were used) to construct a family of subsolutions to (I_\epsilon). Using these subsolutions and Serrin’s sweeping principle, we obtain the asymptotic behaviour of any large positive solution of (I_\epsilon) as \( \epsilon \to 0 \). The uniqueness of large positive solutions of (I_\epsilon), for \( \epsilon \) sufficiently small, is then obtained by such asymptotic behaviour and blow up arguments.

In Section 2 we study a related problem to (I_\epsilon) in the unit ball, which is useful in the coming proofs. In Section 3 we obtain the asymptotic behaviour of any large positive solution of (I_\epsilon) as \( \epsilon \to 0 \). In Section 4 we obtain uniqueness of large positive solutions of (I_\epsilon) and in the final section we establish the boundary derivative estimate of the large positive solution.
2. Preliminaries

In this section we will study a related problem to \((I_\epsilon)\) in the unit ball \(B\) by arguments similar to those in [1]. The results obtained in this section will be useful in the construction of subsolutions of \((I_\epsilon)\) below. Since the nonlinearity of the related problem we discuss here is different from that of \((I_\epsilon)\), our study in this section is not a simple repeat of the study in [1].

Lemma 2.1. Let \(\alpha > 1\) be a fixed number and

\[
\mathcal{f}_\epsilon(s) = \begin{cases} 
-s^{\alpha}(1 - s)^{\alpha} & \text{for } 0 < s < 1, \\
\epsilon^{-1}(s - 1)^p - (s - 1)^q & \text{for } s \geq 1.
\end{cases}
\]

Then the problem

\[-\Delta u = \epsilon g(\epsilon)\mathcal{f}_\epsilon(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B\]  \hspace{1cm} (2.1)

has at least two positive (radial) solutions \(\overline{u}_\epsilon(r)\) and \(\underline{u}_\epsilon(r)\) for \(\epsilon\) sufficiently small such that \(\overline{u}_\epsilon(r) < 0, \underline{u}_\epsilon(r) < 0\) for \(r \in (0, 1)\) and

\[
\lim_{\epsilon \to 0} \epsilon (\overline{u}_\epsilon(0) - 1)^{q - p} = 1, \quad (2.2)
\]

\[
\lim_{\epsilon \to 0} \epsilon (\underline{u}_\epsilon(0) - 1)^{q - p} = c^*, \quad (2.3)
\]

where \(g \in C^0[0, 1]\) is an arbitrary increasing function of \(\epsilon\) with \(g(0) = 0\) (to be chosen below), \(c^*\) is the number as in [1]. Moreover,

\[
\lim_{\epsilon \to 0} \epsilon (\overline{u}_\epsilon(x) - 1)^{q - p} = 1 \quad \text{for } x \in B. \quad (2.4)
\]

Proof. It is known from [13] that any positive solution \(u_\epsilon\) of (2.1) is a radial solution and \(u_\epsilon'(r) < 0\) for \(0 < r \leq 1\). We first show that if \(u_\epsilon\) is a positive solution of (2.1), then \(\|u_\epsilon\|_\infty \to \infty\) as \(\epsilon \to 0\). In fact, we can write the equation in (2.1) in the form

\[-\Delta u = \epsilon g(\epsilon)(-\chi_{[0 < u < 1]}u^\alpha(1 - u)^\alpha
\]

\[+ \chi_{[u \geq 1]}(\epsilon^{-1}(u - 1)^p - (u - 1)^q)), \quad (2.5)
\]

where \(\chi_A(x)\) is the characteristic function on the set \(A\). Suppose that there exist a sequence \(\{\epsilon_n\}\) with \(\epsilon_n \to 0\) as \(n \to \infty\) and \(\{u_n\} \equiv \{u_{\epsilon_n}\}\) such that \(\|u_n\|_\infty \leq C\), then there are two cases: (i) \(\|u_n\|_\infty \to 0\) as \(n \to \infty\), (ii) \(\|u_n\|_\infty \not\to 0\) as \(n \to \infty\). (We can choose subsequences if necessary.) For the first case, we obtain a contradiction by dividing (2.5) by \(\|u_n\|_\infty\) and sending \(n\) to \(\infty\). For the second case, we easily derive a contradiction since the right-hand side of (2.5) tends to 0 as \(n \to \infty\).

In the following, we omit the subscript \(\epsilon\) on \(u_\epsilon\). Now we rescale the variables and write

\[y = [g(\epsilon)]^{1/2}y^{(p-1)/2}x, \quad v(y) = y^{-1}(u(x) - 1), \quad \gamma = \|u\|_\infty - 1.\]
This yields the following form of \( v \):

\[
-\Delta v = -\gamma^{-p} \epsilon \chi_{[-1<\gamma v<0]} (1 + \gamma v)^\alpha (-\gamma v)^\alpha
+ \chi_{[v \geq 0]} [v^p - \gamma^{q-p} \epsilon v^q].
\]

(2.6)

Notice that \( v(0) = 1, v'(0) = 0, \lim_{\epsilon \to 0} \gamma = \infty \). By theory of ordinary equations, we know that when \( \epsilon \) is sufficiently small and \([g(\epsilon)]^{1/2} \gamma^{(p-1)/2} \to \infty \) as \( \epsilon \to 0 \), the solution \( v(r) \sim w(r) \) on any bounded interval of \((0, \infty)\), where \( w \) is the solution of the problem

\[
w'' + \frac{N-1}{r} + w^p - c w^q = 0, \quad w(0) = 1, \quad w'(0) = 0,
\]

(2.7)

where \( c = \epsilon \gamma^{q-p} \). Define

\[
R(c) = \sup\{ r > 0: w(\cdot, c) > 0 \text{ on } (0, r) \}.
\]

It follows from Lemma 2.3 of [1] that there is a number \( c^* \in [0, 1] \) such that

\[
\lim_{c \uparrow c^*} R(c) = \infty \quad \text{and} \quad \lim_{c \downarrow c^*} R(c) = \infty,
\]

and \( R(c) \) is continuous on \((c^*, 1)\). Choosing \( g \) satisfies \( \epsilon^{-(p-1)/(q-p)} g(\epsilon) \to \infty \) as \( \epsilon \to 0 \), we know that

\[
[g(\epsilon)]^{1/2} \gamma^{(p-1)/2} \to \infty
\]

if \( \epsilon \gamma^{q-p} \to 1 \) or \( \epsilon \gamma^{q-p} \to c^* \) as \( \epsilon \to 0 \). Then if we define

\[
\widehat{R}(c) = \sup\{ r > 0: v(r, c) > -\gamma^{-1} \text{ on } (0, r) \}
\]

and

\[
R'(c) = \sup\{ r > 0: v(r, c) > 0 \text{ on } (0, r) \},
\]

we have \( R'(c) < \widehat{R}(c) \) and

\[
\lim_{c \uparrow c^*} \widehat{R}(c) = \infty \quad \text{and} \quad \lim_{c \downarrow c^*} \widehat{R}(c) = \infty,
\]

\[
\lim_{c \uparrow c^*} R'(c) = \infty \quad \text{and} \quad \lim_{c \downarrow c^*} R'(c) = \infty.
\]

We return to problem (2.1). We know that the function \( v(r, c) \) will correspond to a solution of (2.1) if \( c = \epsilon \gamma^{q-p} \). For \( c \to 1 \), we choose \( c \) satisfies

\[
\lim_{\epsilon \to 0} \epsilon^{-(p-1)/(2(q-p))} (1 - c^{1/(q-p)})
= \lim_{\epsilon \to 0} \epsilon^{-(p-1)/(2(q-p))} (1 - \epsilon^{-1/(q-p)} \gamma) = 1.
\]

In view of the properties of \( R'(c) \) and \( \widehat{R}(c) \), there exist for \( \epsilon \) sufficiently small, two solutions \( c^+(\epsilon) \) and \( c^-(\epsilon) \) such that

\[
c^+(\epsilon) \to 1, \quad c^-(\epsilon) \to c^* \quad \text{as } \epsilon \to 0
\]
and
\[ \hat{R}(c^\pm(\epsilon)) = \tilde{y}, \]
where \( \tilde{y} = [g(\epsilon)]^{1/2}y^{(p-1)/2} \). They correspond to two solutions \( \tilde{u}_\epsilon \) and \( u_\epsilon \) of (2.1) with
\[ \epsilon(\|\tilde{u}_\epsilon\|_\infty - 1)^{q-p} \to 1, \quad \epsilon(\|u_\epsilon\|_\infty - 1)^{q-p} \to c^* \quad \text{as } \epsilon \to 0. \quad (2.8) \]
Moreover,
\[ \tilde{u}_\epsilon(1) = 0, \quad u_\epsilon(1) = 0, \]
\[ \tilde{u}_\epsilon(R'(c^+(\epsilon))\tilde{y}^{-1}) = 1, \quad u_\epsilon(R'(c^-(\epsilon))\tilde{y}^{-1}) = 1. \]
By the second remark after Theorem C in [1] we know that
\[ y^{-1}(\tilde{u}_\epsilon - 1) \to 1 \quad \text{as } \epsilon \to 0 \text{ when } x \in B. \quad (2.9) \]
Therefore, \( R'(c^+(\epsilon))\tilde{y}^{-1} \to 1 \) as \( \epsilon \to 0 \). (2.4) can be obtained from (2.9). \( \square \)

**Proposition 2.2.** Let \( \alpha > 1 \) be a fixed number and
\[ f_\epsilon^*(s) = \begin{cases} 0 & \text{for } s \leq -1, \\ -(1+s)^\alpha|s|^\alpha & \text{for } -1 < s < 0, \\ \epsilon^{-1}s^p - s^q & \text{for } s \geq 0. \end{cases} \]
Then there exists \( \epsilon_0 > 0 \) sufficiently small such that for \( 0 < \epsilon < \epsilon_0 \), there is \( v_\epsilon \in C^1(\mathbb{R}^N) \), radially symmetric, which satisfies
\[ -\Delta v_\epsilon = \epsilon g(\epsilon)f_\epsilon^*(v_\epsilon) \quad \text{in } \mathbb{R}^N, \quad v_\epsilon(1) = -1. \quad (2.10) \]
Moreover, \( v_\epsilon'(r) < 0 \) for \( r > 0 \) and
\[ \lim_{\epsilon \to 0} \epsilon^{-(p-1)/(2(q-p))}[1 - \epsilon^{1/(q-p)} \max v_\epsilon] = 1. \]

**Proof.** Define \( \tilde{f}_\epsilon(s) = f_\epsilon^*(s - 1) \). Then \( \tilde{f}_\epsilon \) is the function \( f_\epsilon \) in Lemma 2.1 for \( s \geq 0 \).

Considering the problem
\[ -\Delta u = \epsilon g(\epsilon)\tilde{f}_\epsilon(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \quad (2.11) \]
we know from Lemma 2.1 that there exists \( \epsilon_0 > 0 \) such that for \( 0 < \epsilon < \epsilon_0 \), (2.11) has a positive radial solution \( \tilde{u}_\epsilon(r) \) satisfying that for any compact \( K \subset B \)
\[ \epsilon(\tilde{u}_\epsilon(x) - 1)^{q-p} \to 1 \quad \text{for } x \in K, \quad \text{as } \epsilon \to 0 \]
and
\[ y^{-1}(\tilde{u}_\epsilon(x) - 1) \to 1 \quad \text{for } x \in K, \quad \text{as } \epsilon \to 0, \]
when \( \epsilon^{-(p-1)/(q-p)} g(\epsilon) \to \infty. \)
Set
\[ v_\epsilon(r) = \begin{cases} \bar{u}_\epsilon(r) - 1, & \text{for } r \in [0, 1], \\ -1 - (N - 2)^{-1} (r^2 - N - 1) \bar{u}_\epsilon'(1), & \text{for } r \in (1, \infty). \end{cases} \]

We know that \( v_\epsilon(1) = -1 \). Let \( \beta_\epsilon \in (0, 1) \) be the first zero of \( v_\epsilon \). Then \( \beta_\epsilon \to 1 \) as \( \epsilon \to 0 \). One can easily verify that \( v_\epsilon \) is the required function. This completes the proof. \( \square \)

3. Existence and asymptotic behaviour of large solutions of \((I_\epsilon)\)

In this section we will first find a large positive solution of \((I_\epsilon)\) in general smooth domains \( \Omega \), then study its asymptotic behaviour as \( \epsilon \to 0 \). We call \( u_\epsilon \) a large positive solution of \((I_\epsilon)\) if \( u_\epsilon > 0 \) in \( \Omega \) satisfies \((I_\epsilon)\) and there exist \( x_0 \in \Omega \) (\( x_0 \) depends only upon \( u \)) and \( r > 0 \) independent of \( \epsilon \) such that
\[
\lim_{\epsilon \to 0} \inf_{B_r(x_0)} \epsilon u_\epsilon^{q-p}(x) > 0,
\]
where \( B_r(x_0) \) is a ball with center \( x_0 \) and radius \( r \).

**Theorem 3.1.** There exists \( \epsilon_1 > 0 \) such that for all \( 0 < \epsilon < \epsilon_1 \), \((I_\epsilon)\) possesses at least one large positive solution \( \bar{u}_\epsilon \) satisfying \( \max_{\Omega} \bar{u}_\epsilon < \epsilon^{-1/(q-p)} \) and \( \lim_{\epsilon \to 0} \epsilon \max_{\Omega} \bar{u}_\epsilon^{q-p} = 1 \). Moreover, any large positive solution \( \bar{u}_\epsilon \) of \((I_\epsilon)\) satisfies
\[ \epsilon \bar{u}_\epsilon^{q-p} \to 1 \quad \text{in } K, \quad \text{as } \epsilon \to 0, \]
where \( K \) is any compact set of \( \Omega \).

To prove Theorem 3.1, we need the following lemmas.

**Lemma 3.2.** Let \( v_\epsilon, \beta_\epsilon \in (0, 1), g(\epsilon) \) and \( f^*_\epsilon(s) \) be given in Proposition 2.2. Then for any \( y \in \Omega \) and
\[ 0 < \epsilon < \epsilon_y^*: = \min \{ \epsilon_0, g^{-1}(\beta_\epsilon^{-2}(\text{dist}(y, \partial \Omega))^2) \}, \]
\[ w(\epsilon, y)(x) := v_\epsilon \left( [g(\epsilon)]^{-1/2}(x - y) \right), \quad x \in \Omega, \]
is a subsolution of
\[
-\Delta u = \epsilon f^*_\epsilon(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \tag{3.1}
\]

**Proof.** The function \( w(\epsilon, y) \in C^1(\mathbb{R}^N) \) satisfies
\[ -\Delta w = \epsilon f^*_\epsilon(w) \quad \text{in } \mathbb{R}^N; \]
\[
\int_{\Omega} \left[ \nabla w \nabla \phi - \epsilon f_\epsilon^*(w) \phi \right] dx \leq 0
\]
for all \( \phi \in D^+(\Omega) \), where \( D^+ \) consists of all nonnegative functions in \( C_0^\infty(\Omega) \).
Since \( w(\epsilon, y) < 0 \) on \( \partial \Omega \) for \( \epsilon < g^{-1}(\beta_\epsilon^{-2} \text{dist}(y, \partial \Omega)^2) \). This implies that \( w(\epsilon, y) \) is a subsolution of (3.1).

**Lemma 3.3.** Let \( x^* \in \Omega \) and \( z_\epsilon(x) = w(\epsilon, x^*)(x) \), where \( w(\epsilon, x^*) \) is defined in Lemma 3.2. Then there exist \( \epsilon^{**} > 0 \) and \( c_\epsilon > 0 \) such that for \( 0 < \epsilon < \epsilon^{**} \), there exists at least one solution \( u_\epsilon \in [z_\epsilon, \epsilon^{-1/(q-p)}] \) of (3.1) such that
\[
u_\epsilon(x) \leq \min\{c_\epsilon[g(\epsilon)]^{-1/2} \text{dist}(x, \partial \Omega), \gamma_\epsilon \}
\]
for all \( x \in \Omega \), (3.2)
where \( \gamma_\epsilon = \nu_\epsilon(0) \), \( \nu_\epsilon \) is as in Lemma 3.2 (hence \( \epsilon \gamma_q^{q-p} \to 1 \) as \( \epsilon \to 0 \)).

**Proof.** It is clear that for a fixed \( \epsilon > 0 \) sufficiently small, \( \epsilon^{-1/(q-p)} \) is a supersolution of (3.1). It follows from Lemma 3.2 that \( z_\epsilon \) is a subsolution of (3.1) for \( 0 < \epsilon < \epsilon^{x*} \). Moreover, there exists \( M_\epsilon > 0 \) such that \( f_\epsilon^*(s) + M_\epsilon s \) is an increasing function for \( s \in (-\infty, \epsilon^{-1/(q-p)}) \). By the monotone method as in [14], we easily obtain the existence of a minimal solution \( u_1^\epsilon(x) \) and a maximal solution \( u_2^\epsilon(x) \) of (3.1) such that \( z_\epsilon \leq u_1^\epsilon \leq u_2^\epsilon \) on \( \Omega \) and
\[
\|u_1^\epsilon\|_{\infty} \leq \|u_2^\epsilon\|_{\infty} < \epsilon^{-1/(q-p)}.
\]
The last inequality can be obtained by the maximum principle.

Since \( \Omega \) satisfies a uniform interior sphere condition, there exists \( \eta_0 > 0 \) such that \( \Omega = \bigcup \{B(x, \eta): x \in \Omega_\eta\} \) for \( \eta \in (0, \eta_0] \), where \( \Omega_\eta = \{x \in \Omega: \text{dist}(x, \partial \Omega) > \eta\} \). Set
\[
\epsilon^{**} = \min\{\epsilon_{x*}, g^{-1}(\beta_\epsilon^{-2} \eta_0^2)\},
\]
\[
c_\epsilon = \inf\{ (\beta_\epsilon - r)^{-1} \nu_\epsilon(r): r \in [0, \beta_\epsilon) \}.
\]
Notice that \( c_\epsilon > 0 \) for \( 0 < \epsilon < \epsilon^{**} \), since \( \nu_\epsilon > 0 \) on \( [0, \beta_\epsilon) \) and \( \nu'_\epsilon(\beta_\epsilon) < 0 \). Moreover, there exists \( \zeta > 0 \) independent of \( \epsilon \) such that \( \epsilon^{1/(q-p)} \nu_\epsilon(r) \to 1 \) for \( r \in (0, 1) \) as \( \epsilon \to 0 \).

Let \( u_\epsilon \) be any solution of (3.1) with \( 0 < \epsilon < \epsilon^{**} \), \( u_\epsilon \in [z_\epsilon, \epsilon^{-1/(q-p)}] \). Since for \( 0 < \epsilon < \epsilon^{**} \), \( \Omega_{\beta_\epsilon[g(\epsilon)]^{1/2}} \) is arcwise connected and since \( w(\epsilon, y) \) is a subsolution for \( y \in \Omega_{\beta_\epsilon[g(\epsilon)]^{1/2}} \) with \( w(\epsilon, y) < 0 \) on \( \partial \Omega \), one finds by the sweeping principle of Serrin [12] that
\[
u_\epsilon(x) \leq w(\epsilon, y) \quad \text{in} \ \Omega \ \text{for all} \ y \in \Omega_{\beta_\epsilon[g(\epsilon)]^{1/2}}.
\]
Hence,
\[
u_\epsilon(x) \geq c_\epsilon[g(\epsilon)]^{-1/2} \text{dist}(x, \partial \Omega) \quad \text{for all} \ x \in \Omega \setminus \Omega_{\beta_\epsilon[g(\epsilon)]^{1/2}}, \quad (3.3)
\]
\[ u_\epsilon(x) \geq \gamma_\epsilon \quad \text{for all } x \in \Omega_{\beta_\epsilon [g(\epsilon)]^{1/2}}. \]  

(3.4)

This completes the proof. \[ \square \]

\textbf{Remark.} It easily follows from (3.3) and (3.4) that if \( u_\epsilon \in [z_\epsilon, \epsilon^{-1/(q-p)}] \) is a solution of (3.1), then \( u_\epsilon \) is a positive solution of (3.1) and thus it is a positive solution of \((I_\epsilon)\) since \( \epsilon f_\epsilon^*(s) = s^p - \epsilon s^q \) for \( s \geq 0 \). This also implies that there exists a positive solution of \((I_\epsilon)\) for \( 0 < \epsilon < \epsilon^{**} \).

\textbf{Lemma 3.4.} Let \( u_\epsilon \in [z_\epsilon, \epsilon^{-1/(q-p)}] \) be a solution of (3.1) for \( 0 < \epsilon < \epsilon^{**} \). Then for any compact set \( K \subset \Omega \),

\[ \epsilon u_\epsilon^{q-p}(x) \to 1 \quad \text{in } K \quad \text{as } \epsilon \to 0. \]

Moreover, for any sequence \( \{\epsilon_n\} \) satisfying \( \epsilon_n \to 0 \) as \( n \to \infty \), \( \{u_n\} \equiv \{u_{\epsilon_n}\} \), and \( \{x_n\} \equiv \{x_{\epsilon_n}\} \subset \Omega \) satisfying

\[ \text{dist}(x_n, \partial \Omega) > \text{dist}(x_{n+1}, \partial \Omega) \quad \forall n \]

and

\[ \epsilon_n^{-(p-1)/(2(q-p))} \text{dist}(x_n, \partial \Omega) \to \infty, \]

we have that

\[ \epsilon_n u_n^{q-p}(x_n) \to 1 \quad \text{as } n \to \infty. \]  

(3.5)

\textbf{Proof.} It follows from (3.4) that

\[ u_\epsilon \geq \gamma_\epsilon \quad \text{on } \Omega_{\beta_\epsilon [g(\epsilon)]^{1/2}}. \]

Since \( \beta_\epsilon \to 1 \) as \( \epsilon \to 0 \) and \( \epsilon \gamma_\epsilon^{q-p} \to 1 \) as \( \epsilon \to 0 \), we have that for any \( \delta > 0 \) and \( \epsilon < \epsilon^{**} \) sufficiently small,

\[ \epsilon \overline{\pi}_\epsilon^{q-p}(x) \geq 1 - \delta \quad \text{for } x \in \Omega_{\beta_\epsilon [g(\epsilon)]^{1/2}}. \]  

(3.6)

On the other hand, we also know that

\[ \overline{\pi}_\epsilon(x) \leq \epsilon^{-1/(q-p)} \quad \text{for } x \in \Omega. \]

This implies that

\[ \epsilon \overline{\pi}_\epsilon^{q-p}(x) \leq 1 \quad \text{for } x \in \Omega. \]  

(3.7)

It follows from (3.6) and (3.7) that for \( \epsilon \) sufficiently small,

\[ 1 - \delta \leq \epsilon \overline{\pi}_\epsilon^{q-p}(x) \leq 1 \quad \text{for } x \in \Omega_{\beta_\epsilon [g(\epsilon)]^{1/2}}. \]  

(3.8)

(3.8) implies that for any compact set \( K \subset \Omega \),

\[ \epsilon \overline{\pi}_\epsilon^{q-p}(x) \to 1 \quad \text{for } x \in K, \quad \text{as } \epsilon \to 0. \]  

(3.9)
Now we show (3.5). We can use contradiction argument to obtain this. Since
\[
\lim_{n \to \infty} \epsilon_n^{-(p-1)/(2(q-p))} \text{dist}(x_n, \partial \Omega) = \infty,
\]
we can choose an increasing function \( g \in C^0[0, 1] \) with \( g(0) = 0 \) such that
\[
\lim_{n \to \infty} [g(\epsilon_n)]^{-1/2} \text{dist}(x_n, \partial \Omega) = A \geq 4 \quad (A < \infty).
\]
This implies that
\[
\lim_{n \to \infty} \epsilon_n^{-(p-1)/(q-p)} g(\epsilon_n) = \infty.
\]
Now, (3.5) follows from (3.4) since \( x_n \in \Omega(\lambda_1/2)[g(\epsilon_n)]^{1/2} \subset \Omega_{\beta \epsilon_n} [g(\epsilon_n)]^{1/2} \) for \( n \) sufficiently large. This completes the proof.

Let \( \phi_1 \) be the eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of
\[-\Delta v = \lambda v \quad \text{in } B, \quad v = 0 \quad \text{on } \partial B,
\]
where \( B \) denotes the unit ball in \( \mathbb{R}^N \). Let \( \phi_1 \) be normalized so that \( \max \phi_1 = 1 \). It is well known that \( \phi_1 \) is radially symmetric and \( \phi_1(0) = 1 \).

**Lemma 3.5.** Let \( u \) satisfy \(-\Delta u = [u^p - \epsilon u^q] \) in an open set \( \Omega' \subset \Omega \) for any fixed \( \epsilon > 0 \) sufficiently small. Let \( \sigma > 0 \) be such that \( s^p - \epsilon s^q \geq \sigma(s - a) \) for \( s \in [a, b] \). Suppose that \( u(x) > a \) for \( x \in \Omega' \). If \( x_1 \in (\Omega'_{(\lambda_1/\sigma)}^{1/2}, \text{then } u(x_1) > b \).

**Proof.** Set \( \psi(x_1, t; x) = a + t \phi_1((\sigma / \lambda_1)^{1/2}(x - x_1)) \) for \( x \in \tilde{B} \) and \( t \in [0, b - a] \), where \( \tilde{B} = B(x_1, (\lambda_1/\sigma)^{1/2}) \). We claim that the set \( \{ \psi(x_1, t): t \in [0, b - a] \} \) is a family of subsolutions of the problem
\[-\Delta v = v^p - \epsilon v^q \quad \text{in } \tilde{B}, \quad v = u \quad \text{on } \partial \tilde{B},
\]
and the closure of \( \tilde{B} \) is contained in \( \Omega' \). By the sweeping out result, we obtain our conclusion.

**Lemma 3.6.** The conclusions of Lemma 3.4 hold for any large positive solution \( u_\epsilon \) of \((I_\epsilon)\) and \( \epsilon \) sufficiently small.

**Proof.** By the definition, we have that there exists \( 0 < \tilde{\beta} < 1 \) such that \( u_\epsilon(x) > \tilde{\beta} \epsilon^{-1/(q-p)} \) for \( x \in B_r(x_0) \) and \( \epsilon \) sufficiently small. On the other hand, since
\[
\epsilon^{-(p-1)/(2(q-p))} (1 - \epsilon^{1/(q-p)} \gamma_\epsilon) \to 1 \quad \text{as } \epsilon \to 0,
\]
we have that \( \gamma_\epsilon > \tilde{\beta} \epsilon^{-1/(q-p)} \) for \( \epsilon \) sufficiently small. Moreover,
\[
s^p - \epsilon s^q \geq [\tau_\epsilon^p - \gamma_\epsilon] \epsilon^{-p/(q-p)} \quad \text{for } s \in [\beta \epsilon^{-1/(q-p)}, \gamma_\epsilon],
\]
where \( \gamma_\epsilon = \epsilon^{-1/(q-p)} \tau_\epsilon. \) Therefore, \( \tau_\epsilon \to 1 \) as \( \epsilon \to 0 \) and
\[
\tau_\epsilon^p - \gamma_\epsilon = (q \gamma_\epsilon q^{-1} - p \gamma_\epsilon p^{-1})(1 - \tau_\epsilon),
\]
where \( \xi_{\epsilon} \in [\tau_{\epsilon}, 1] \). Thus,
\[
s^p - \epsilon s^q \geq \xi_{\epsilon} \left( s - \tilde{\beta} e^{-1/(q-p)} \right) \quad \text{for } s \in \left[ \beta e^{-1/(q-p)}, \gamma_{\epsilon} \right], \tag{3.13}
\]
where
\[
\xi_{\epsilon} := \frac{1}{2} (q - p) (1 - \tilde{\beta}) (1 - \tau_{\epsilon}) e^{-(p-1)/(q-p)}.
\]
We know from (3.11) \( e^{-(p-1)/(2(q-p))} (1 - \tau_{\epsilon}) \to 1 \); then \( \xi_{\epsilon} \to \infty \) as \( \epsilon \to 0 \).

It follows from Lemma 3.5 with \( a = \tilde{\beta} e^{-1/(q-p)} \), \( b = \gamma_{\epsilon} \) and \( \sigma = \xi_{\epsilon} \) that for \( \epsilon \) sufficiently small,
\[
u_{\epsilon}(x) > \gamma_{\epsilon} \quad \text{for } x \in B_{\rho_{\epsilon} [\xi_{\epsilon}^1]}(x_0) \subset B_{r_{\epsilon} [\lambda_{\epsilon}^1/\xi_{\epsilon}^1]}(x_0). \tag{3.14}
\]
Let \( w(\epsilon, x_0) \) be the function defined as in Lemma 3.3. Then \( w(\epsilon, x_0) \) is a subsolution of (3.1). We easily know that \( u_{\epsilon} \) is a supersolution of (3.1).

Observe that \( w(\epsilon, x_0) < u_{\epsilon} \) in \( \Omega \) for \( 0 < \epsilon < \epsilon^{**} := \min(\epsilon^{*}_{\epsilon x_0}, g^{-1}(\beta^{-2}\eta_0^{1/2})) \).

In fact, we know that \( w(\epsilon, x_0) \leq \gamma_{\epsilon} \) in \( B_{\rho_{\epsilon} [\xi_{\epsilon}^1]}(x_0) \) and \( w(\epsilon, x_0) \leq 0 \) in \( \Omega \setminus B_{\rho_{\epsilon} [\xi_{\epsilon}^1]}(x_0) \). Thus, this lemma can be obtained by arguments similar to those in the proof of Lemma 3.3. \( \square \)

**Proof of Theorem 3.1.** The first part of Theorem 3.1 can be obtained from the remark after the proof of Lemma 3.3. The second part of Theorem 3.1 can be obtained from Lemma 3.6. \( \square \)

Now we obtain the asymptotic behaviour of the large positive solutions of \( (I_{\epsilon}) \) as \( \epsilon \to 0 \). For convenience, we denote \( \xi(s) = s^p - s^q \).

We first consider the problem
\[
- y'' = y^p - y^q \quad \text{in } (0, \infty), \quad y(0) = 0, \quad y(\infty) = 1. \tag{3.15}
\]
By the first integral of the equation of (3.15), we easily see that (3.15) has a unique solution \( z_0(t) \) satisfying \( z_0 > 0, z'_0 > 0 \) in \( (0, \infty) \). In fact, we see that the unique solution \( \hat{y}(t) \) of the initial value problem
\[
- y'' = y^p - y^q \quad \text{in } (0, \infty),
\]
\[
y(0) = 0, \quad y'(0) = 2^{1/2} \left( \frac{1}{p+1} - \frac{1}{q+1} \right)^{1/2}
\]
satisfies (3.15). We explain a little here. By theory of ordinary differential equations, we know that \( \hat{y} \) increases up to a point \( t_0 > 0 \) where \( \hat{y}' = 0 \). Since the first integral of the equation in (3.15) implies
\[
G(\hat{y})(t) \equiv \frac{1}{2} |\hat{y}'(t)|^2 + \frac{|\hat{y}|^{p+1}(t)}{p+1} - \frac{|\hat{y}|^{q+1}(t)}{q+1} \equiv G_0
\]
for $t \in (0, \infty)$, where $G_0 = 1/(p + 1) - 1/(q + 1)$ and the function $\tilde{\xi}(s) = s^{p+1}/(p + 1) - s^{q+1}/(q + 1)$ has only one maximum point at $s = 1$ for $s \in (0, \infty)$, we can show that if such $t_0$ exists, $\hat{y}(t_0) = 1$. On the other hand,

$$|\hat{y}'|^2 = 2(\tilde{\xi}(1) - \tilde{\xi}(\hat{y})).$$

Thus,

$$\hat{y}(t) \int_0^1 (\tilde{\xi}(1) - \tilde{\xi}(s))^{-1/2} ds = 2^{1/2}t.$$

Since $\int_0^1 (\tilde{\xi}(1) - \tilde{\xi}(s))^{-1/2} ds = \infty$, we easily know that such $t_0 > 0$ cannot exist. Thus, $\hat{y}$ increases till to $\infty$ and $\lim_{t \to \infty} \hat{y}(t)$ exists. If this limit is $\infty$, we know that there exists $t_1 > 0$ such that $\hat{y}(t_1) = 1$ and thus $\hat{y}'(t_1) = 0$. This is impossible. The identity $G(\hat{y})(t) \equiv G_0$ implies that $\lim_{t \to \infty} \hat{y}(t) = 1$. If there were two solutions of (3.15), we easily know that these two solutions satisfy the same initial value problem as above. Thus they must be one function.

If $x \in \Omega$ and $x$ is near $\partial \Omega$, $x$ can be uniquely written in the form $x = s - tn(s)$ where $s \in \partial \Omega$, $n(s)$ denotes the outward normal vector to $\partial \Omega$ at $s$, and $t$ is small and positive. We will make frequent use of these coordinates. We denote $\eta_\epsilon(x) = z_0(\epsilon^{-(p-1)/(2(q-p))}t)$ if $x$ is near $\partial \Omega$ and $\eta_\epsilon(x) = 1$ otherwise.

**Theorem 3.7.** Given $\delta > 0$, there is $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$ and $u_\epsilon$ is a large positive solution of $(I_\epsilon)$. Then

$$(1 - \delta)\eta_\epsilon \leq \epsilon^{1/(q-p)}u_\epsilon \leq (1 + \delta)\eta_\epsilon \quad \text{in } \Omega.$$

**Proof.** Writing the equation in $(I_\epsilon)$ to the form

$$-\Delta(\epsilon^{1/(q-p)}u_\epsilon) = \epsilon^{-(p-1)/(q-p)}[(\epsilon^{1/(q-p)}u_\epsilon)^p - (\epsilon^{1/(q-p)}u_\epsilon)^q],$$

(3.16)

by Lemma 3.6, we only need to prove the result for points whose distance from $\partial \Omega$ is of order $\epsilon^{(p-1)/(2(q-p))}$. To prove this, we construct sub- and supersolutions. The main idea is similar to that in the proof of Theorem 2 of [12].

Near $\partial \Omega$, we use the $s, t$ coordinates. In these variables,

$$\Delta u = \frac{\partial^2 u}{\partial t^2} + b(s, t) \frac{\partial u}{\partial t} + \text{terms involving } s \text{ derivatives.}$$

If $\overline{\sigma} < z_0'(0)$ but is close, using the first integral of (3.15) we easily prove that the solution of (3.15) with $\tilde{z}(0) = 0, \tilde{z}'(0) = \overline{\sigma}$ first increases to a number near 1 but less than 1, and then decreases to zero. Hence there is $\tilde{t}$ near 1 and $\tilde{t} > 0$ such that $\tilde{z}(\tilde{t}) = \tilde{l}, \tilde{z}'(\tilde{t}) = 0$. Since $\tilde{z}'' = -\xi(\tilde{z}(\tilde{t})) \neq 0, \tilde{z}'(\tilde{t})$ changes sign at $\tilde{t}$. Hence if $\mu$ is close to 1 and $\tilde{\beta}$ is small, the solution $\tilde{z}$ of

$$-x'' - \tilde{\beta}x' = \mu \xi(x(t)), \quad x(0) = 0, \quad x'(0) = \overline{\sigma}$$

(3.17)
increases until \( \bar{r} \) where \( \mathcal{Z}'(\bar{r}) = 0 \) and \( \mathcal{Z}(\bar{r}) \) is close to 1 but less than 1. Define

\[
\lambda = \lambda(\epsilon) = e^{-(p-1)/(q-p)},
\]

\[
\tilde{\eta}_\lambda(x) = \begin{cases} 
\mathcal{Z}(\lambda^{1/2}t) & \text{if } x \text{ is close to } \partial \Omega \text{ and } 0 \leq t \leq \lambda^{-1/2}\bar{r}, \\
\mathcal{Z}(\bar{r}) & \text{otherwise,}
\end{cases}
\]

where \( x = s - tn(s) \) if \( x \) is near \( \partial \Omega \). (Thus \( \tilde{\eta}_\epsilon \) is constant except near \( \partial \Omega \).)

Suppose we can show that, if \( \epsilon \) is small and \( u_\epsilon \) is a large positive solution of \( (I_\epsilon) \), \( e^{1/(q-p)}u_\epsilon \geq \tilde{\eta}_\lambda \) in \( \Omega \), then Lemma 3.6 implies that \( e^{1/(q-p)}u_\epsilon \geq (1 - \delta) \eta_\epsilon \) for \( \epsilon \) small, since \( \mathcal{Z} \) is close to \( z_0 \) on compact intervals if \( \tilde{\alpha} \) is near \( z_0' \) (0), if \( \mu \) is near 1 and if \( \beta \) is small. This will prove half of Theorem 3.7.

By a simple calculation, \( \tilde{\eta}_{d\lambda} \) is a subsolution of \( (I_\epsilon) \) if

\[
d\mu \leq 1
\]

and

\[
b(s, t) \geq \beta(d\lambda)^{1/2}
\]

for \( 0 \leq t \leq \bar{r}(\lambda d)^{-1/2} \). Here we have used that \( \xi(s) \geq 0 \) on \((0, 1)\), \( \mathcal{Z}'(s) \geq 0 \) on \([0, \bar{r}]\) and that, to check that \( \tilde{\eta}_{d\lambda} \) is a subsolution, we need only to check the regions

\[
A = \{x : x \text{ is near } \partial \Omega, 0 \leq t < \bar{r}(e^{-(p-1)/(q-p)}d)^{-1/2}\} \text{ and } \Omega \setminus \overline{A}
\]

separately. Now, if \( \mu < 1 \), \( \beta < 0 \) and \( \bar{r} > 0 \), (3.18) and (3.19) hold for all \( d \in [\bar{r}, 1] \) and \( \epsilon \) small. Moreover, we can find an \( e = e(\epsilon) \in (0, 1) \) such that \( e^{1/(q-p)}u_\epsilon \geq \tilde{\eta}_{e(\epsilon)\lambda} \) for \( \epsilon \) small by Lemma 3.3 and Lemma 3.6. In fact, we know from Lemma 3.6 that for some \( \epsilon > 0 \) independent of \( \epsilon \),

\[
e^{1/(q-p)}u_\epsilon \geq \min \{\tilde{e}[g(\epsilon)]^{-1/2}\text{dist}(x, \partial \Omega), e^{1/(q-p)}g_\epsilon\} \in \Omega.
\]

Choosing \( e(\epsilon) = \ell[e^{(p-1)/(q-p)}g(\epsilon)]^{-1} \) with \( 0 < \ell < \tilde{\epsilon} \) sufficiently small, we easily know that \( e^{1/(q-p)}u_\epsilon \geq \tilde{\eta}_{e(\epsilon)\lambda} \) in \( \Omega \) for \( \epsilon \) sufficiently small (since \( \mathcal{Z}'(0) > 0 \) independent of \( \epsilon \) and \( e^{1/(q-p)}g_\epsilon \to 1 \) as \( \epsilon \to 0 \)). On the other hand, for any \( d \in [e(\epsilon), 1] \) (\( d \) may depend upon \( \epsilon \)), we easily know

\[
d\epsilon^{-(p-1)/(q-p)} \geq \ell[g(\epsilon)]^{-1} \to \infty
\]

and \( e(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Thus, (3.18) and (3.19) hold when \( \epsilon \) is small. Therefore, the sweeping principle implies that

\[
e^{1/(q-p)}u_\epsilon \geq \tilde{\eta}_{d\lambda} \quad \text{for } d \in [e(\epsilon), 1].
\]

Therefore,

\[
e^{1/(q-p)}u_\epsilon \geq \tilde{\eta}_\lambda \quad \text{in } \Omega.
\]

This is our claim.

To prove the estimate in the opposite direction, we use supersolutions. If \( \overline{\alpha}_1 > z_0'(0) \), it is easy to show from the first integral that the solution \( \tilde{z}_1 \) of (3.15) such that \( \tilde{z}_1(0) = 0 \), \( \tilde{z}_1'(0) = \overline{\alpha}_1 \), increases till it hits \( y = 1 \). Once again by
continuous dependence, the solution $\zeta_1$ of (3.17) such that $\zeta_1(0) = 0$, $\zeta'_1(0) = \tilde{\alpha}$ increases till it hits $y = 1$ at $t = \tilde{t}_1$ provided that $\mu$ is near 1 and $\tilde{\beta}$ is small. We define

$$
\lambda = \lambda(\epsilon) = \epsilon^{-(p-1)/(q-p)},
$$

$$
\tilde{\eta}_\lambda(x) = \begin{cases}
\zeta_1(\lambda^{1/2}t) & \text{if } 0 \leq t \leq \lambda^{-1/2}\tilde{t}_1, \\
1 & \text{otherwise.}
\end{cases}
$$

By an argument similar to before, $\tilde{\eta}_{d\lambda}$ is a supersolution if (3.18) and (3.19) are satisfied with the inequalities reversed. The difference is that $\partial\tilde{\eta}_\lambda/\partial n$ has a discontinuity when $t = \epsilon^{(p-1)/(2(q-p))}\tilde{t}_1$. However, the jump in the derivative is not a difficulty to obtain the sweeping out result as above. As before, we find that the conditions (3.18) and (3.19) are satisfied with inequalities reversed for all small $\epsilon$ if $\mu > 1$, $\tilde{\beta} > 0$ and $d \in [1, \tilde{\beta}]$ (where $\tilde{\beta} > 1$). Thus we can argue as before if we can choose $\epsilon > 1$ such that $\epsilon^{1/(q-p)}u_\epsilon \leq \tilde{\eta}_{d\lambda}$ for $\epsilon$ small and all large positive solution $u_\epsilon$ of $(I_\epsilon)$ with $\epsilon^{1/(q-p)}\|u_\epsilon\|_\infty < 1$. It is easy to see that this reduces to show that there is a $K > 0$ such that $\epsilon^{1/(q-p)}u_\epsilon \leq K\epsilon^{-(p-1)/(2(q-p))}t$ if $u_\epsilon$ is a positive large solution of $(I_\epsilon)$, $x$ is near $\partial \Omega$ and $\epsilon$ is small. Obviously, it suffices to prove the result for $t \leq \tilde{t}_1\epsilon^{(p-1)/(2(q-p))}$. Now, for any $x_0 \in \partial \Omega$, let $X = \lambda^{1/2}(x - x_0)$ and $\tilde{u}_\epsilon(X) = \epsilon^{1/(q-p)}u_\epsilon(x)$; then

$$
-\Delta \tilde{u}_\epsilon = \xi(\tilde{u}_\epsilon) \quad \text{in } \tilde{\Omega}, \quad \tilde{u}_\epsilon = 0 \quad \text{on } \partial \tilde{\Omega},
$$

where $\tilde{\Omega} = \{X: \epsilon^{(p-1)/(2(q-p))}X + x_0 \in \Omega\}$. By a blow up argument as in [12], we have that the stretching only flattens the boundary as $\epsilon \to 0$. Since $0 \in \partial \tilde{\Omega}$ and $\|\tilde{u}_\epsilon\|_\infty \leq 1$, we apply the regularity result of $-\Delta$ to obtain that $\nabla \tilde{u}_\epsilon$ is bounded on bounded subsets of $\tilde{\Omega}$ which contain neighbourhoods of 0 on $\partial \tilde{\Omega}$. Hence, in the original variables, $\|\nabla \epsilon^{1/(q-p)}u_\epsilon\|_\infty \leq K_1\epsilon^{-(p-1)/(2(q-p))}$ on the subsets of $\Omega$ which contain neighbourhoods of $x_0$ on $\partial \Omega$, where $K_1 > 0$ is suitable large. Since $\partial \Omega$ is compact, we obtain our estimate for $u_\epsilon$ near $\partial \Omega$. This completes the proof. \(\square\)

4. Uniqueness of large positive solutions of $(I_\epsilon)$

In this section we will use contradiction arguments to prove the uniqueness of large positive solutions of $(I_\epsilon)$ for $\epsilon$ sufficiently small.

**Theorem 4.1.** Problem $(I_\epsilon)$ has exact one large positive solution $\tilde{\eta}_\epsilon$ for $\epsilon$ sufficiently small.

**Proof.** We use contradiction arguments here. Suppose there are sequences $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ and $\{u_{\epsilon_n}\} \equiv \{u_n\}$, $\{u_{\epsilon_n}^*\} \equiv \{u_n^*\}$ which are large solutions
of \((I_{ε_n})\) with \(u_n^* \neq u_n\) in \(Ω\) and \(\max u_n < ε_n^{-1/(q−p)}\), \(\max u_n^* < ε_n^{-1/(q−p)}\). Then Theorem 3.7 implies that for any compact set \(K \subset Ω\),

\[ε_n u_n^{−p}(x) \to 1 \quad \text{for} \ x \in K, \ \text{as} \ n \to ∞,\]

\[ε_n (u_n^*)^{−p}(x) \to 1 \quad \text{for} \ x \in K, \ \text{as} \ n \to ∞.\]

Moreover, by the monotone method as in [14], there exists a maximal positive solution of \((I_{ε})\) in \((0, ε_n^{−1/(q−p)})\). Without loss of generality, we assume that \(u_n\) is the maximal solution for each \(n\), thus, \(u_n^* < u_n\) in \(Ω\). Let \(w_n = (u_n − u_n^*)/∥u_n − u_n^*∥\). Then \(∥w_n∥ = 1\) and \(w_n\) satisfies the problem

\[-Δw_n = \left[pξ_n^{−p} − ε_n qξ_n^{−q} \right]w_n \quad \text{in} \ Ω, \quad w_n = 0 \ \text{on} \ ∂Ω \quad \text{(4.1)}\]

where \(ξ_n ∈ [u_n^*, u_n]\). Multiplying the both sides of (4.1) by \(ε_n^{−(p−1)/(q−p)}\), we have

\[-Δw_n = ε_n^{−(p−1)/(q−p)} \left[p(ε_n^{1/(q−p)}ξ_n^{−p})^{−1} − q(ε_n^{1/(q−p)}ξ_n)^{−q} \right]w_n. \quad \text{(4.2)}\]

Now we show that if \(η_n \in Ω\) is such that \(w_n(η_n) = 1\), then

\[\text{dist}(η_n, ∂Ω) \to 0 \quad \text{as} \ n \to ∞. \quad \text{(4.3)}\]

In the contrary case, there exists a compact set \(K \subset Ω\) such that \(η_n \in K\) for all large \(n\) (choose a subsequence if necessary). Since \(ε_n u_n^{−p}(x) \to 1\) and \(ε_n (u_n^*)^{−p}(x) \to 1\) for \(x \in K\) as \(n \to ∞\), we have

\[p(ε_n^{1/(q−p)}ξ_n(x))^{−p} − q(ε_n^{1/(q−p)}ξ_n(x))^{−q} < 0 \]

for \(x \in K\) and all \(n\) sufficiently large (since \(p < q\)). Thus, \(−Δw_n < 0\) in the neighbourhood of \(η_n\). This contradicts the fact that \(w_n\) attains maximum at \(η_n \in K\).

Now we use the blow up argument as in [15] and [12] to deduce contradictions when (4.3) holds.

Let \(\tilde{η}_n\) be the point of \(∂Ω\) closest to \(η_n\). Choose a subsequence such that \(\tilde{η}_n \to \tilde{η} \in ∂Ω\). Choose coordinates such that \(T_{\tilde{η}}(∂Ω) = \{x ∈ R^N: x_1 = 0\}\) and \(v(\tilde{η}) = e_1 = (1, 0, \ldots, 0)\). By choosing subsequences if necessary, there are two cases to be considered:

(i) \(\lim_{n \to ∞} ε_n^{−(p−1)/(2(q−p))}\text{dist}(η_n, \tilde{η}_n) = ∞.\)

(ii) \(\lim_{n \to ∞} ε_n^{−(p−1)/(2(q−p))}\text{dist}(η_n, \tilde{η}_n) = Z, \ 0 ≤ Z < ∞.\)

For the first case, we have from Theorem 3.7 that

\[ε_n^{1/(q−p)} u_n(η_n) \to 1, \quad ε_n^{1/(q−p)} u_n^*(η_n) \to 1 \ \text{as} \ n \to ∞.\]

If we make a change of variables

\[X_n = ε_n^{−(p−1)/(2(q−p))}(x − η_n), \quad \tilde{w}_n(X_n) = w_n(x),\]

we have that \(\tilde{w}_n\) satisfies the problem

\[-Δ\tilde{w}_n = \tilde{f}_n(X_n)\tilde{w}_n \quad \text{in} \ Ω_n, \ \tilde{w}_n = 0 \ \text{on} \ ∂Ω_n, \quad \text{(4.4)}\]
where
\[ \tilde{f}_n(X^n) = \left[ p \left( \epsilon_n^{1/(q-p)} \xi_n(x) \right)^{p-1} - q \left( \epsilon_n^{1/(q-p)} \xi_n(x) \right)^{q-1} \right], \]
\[ \xi_n \in [u_n^*, u_n], \tilde{\Omega}_n = \{ X^n : x \in \Omega \}. \] We know that
\[ \epsilon_n^{1/(q-p)} \xi_n(\eta_n) \to 1 \text{ as } n \to \infty, \]
thus, the right-hand side of (4.4) is negative near \( X^n = 0 \) if \( n \) is sufficiently large.
This is a contradiction since \( \tilde{w}_n \) attains its maximum at \( X^n = 0 \) for \( n \) sufficiently large and \( \text{dist}(0, \partial \tilde{\Omega}_n) \to \infty \) as \( n \to \infty \).

For the second case, making the changes of variables
\[ X_n = \epsilon_n^{-(p-1)/(2(q-p))} (x - \tilde{\eta}_n), \quad \tilde{w}_n(X^n) = w_n(x), \]
we have that \( \tilde{w}_n \) satisfies the problem
\[ -\Delta \tilde{w}_n = \tilde{f}_n(X^n) \tilde{w}_n \quad \text{in} \quad \tilde{\Omega}_n, \quad \tilde{w}_n = 0 \quad \text{on} \ \partial \tilde{\Omega}_n, \quad (4.5) \]
where \( \tilde{\Omega}_n = \{ X^n : x \in \Omega \}. \) Note that in the new coordinates, \( \tilde{w}(Z_n) = 1 \), where \( Z_n = \epsilon_n^{-(p-1)/(2(q-p))} (\eta_n - \tilde{\eta}_n) \) is at distance at most \( Z \) from 0. Now we use a very similar blow up argument to that in the proofs of Theorem 1.1 of [15] and Theorem 2 of [12] to deduce that \( \tilde{w}_n \) converges in \( C^1_{\text{loc}}(T_1) \) to a non-trivial non-negative bounded solution \( \tilde{w} \) of the problem
\[ -\Delta \tilde{w} = \left[ pz_0^{p-1}(x_1) - qz_0^{q-1}(x_1) \right] \tilde{w} \quad \text{in} \quad T_1, \quad \tilde{w} = 0 \quad \text{on} \ \partial T_1. \quad (4.6) \]
where \( T_1 = \{ x \in \mathbb{R}^N : x_1 \geq 0 \}, z_0(x_1) \) satisfies the problem
\[ -z''_0 = z''_0 - z_0^{q} \quad \text{in} \ (0, \infty), \quad z_0(0) = 0, \quad z_0(\infty) = 1. \quad (4.7) \]
Here \( \tilde{w} \) is non-trivial because \( \tilde{w}_n(Z_n) = 1 \) and \( \text{dist}(0, Z_n) \leq Z \). The fact that
\[ \tilde{f}_n(X^n) \to \left[ pz_0^{p-1}(x_1) - qz_0^{q-1}(x_1) \right] \quad \text{as} \ n \to \infty \]
can be obtained by arguments similar to those in the proof of Theorem 2 of [12].
Now we show that \( \tilde{w} \) can not exist by the three steps similar to that in the proof of Proposition 2 of [12].

Step 1. We first find a solution \( \hat{q} \) of
\[ -\hat{q}'' = \left[ pz_0^{p-1} - qz_0^{q-1} \right] \hat{q} \]
which is positive on \( [0, \infty) \) and is not bounded as \( x_1 \to \infty \).

Step 2. If (4.6) has a non-trivial bounded non-negative solution \( \tilde{w} \) and \( x_1 > 0 \), then \( \tilde{w} \) can be chosen so that \( T(x_1) = \sup_{y \in \mathbb{R}^{N-1}} \tilde{w}(x_1, y) \) is continuous in \( (0, \infty) \).

Step 3. We show that \( \tilde{w} \) can not exist by using \( \tilde{w}, \hat{q} \) and the second step.

The proof of these steps is just a variant of the proof of Proposition 2 of [12].
We omit the details here. This completes the proof of uniqueness of large positive solutions of \( (I_\epsilon) \) for \( \epsilon \) sufficiently small and hence the proof of Theorem 4.1. \( \square \)
5. Boundary derivative estimate of the large solution

In this section we will find the boundary behaviour of the derivative of the large positive solution \( \bar{u}_\epsilon \) as \( \epsilon \to 0 \).

**Theorem 5.1.** Let \( \bar{u}_\epsilon \) be the large positive solution of (I\( \epsilon \)). Then, if, for any \( x \in \partial \Omega \), \( \nu(x) \) stands for the outward unit normal vector at \( x \),

\[
\lim_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial \bar{u}_\epsilon}{\partial \nu}(x) = -\left( \frac{2(q - p)}{(p + 1)(q + 1)} \right)^{1/2}.
\]  

(5.1)

To prove this theorem, we first present the following lemmas.

**Lemma 5.2.** Let \( a, b \in \mathbb{R} \) be such that \( a < b \). Then for \( \epsilon \) sufficiently small, there exists a positive solution \( u_\epsilon \in C^1_0(a, b) \) for the problem

\[
- u'' = u^p - \epsilon u^q \quad \text{in } (a, b), \quad u(a) = u(b) = 0
\]  

(5.2)

such that \( \epsilon u_\epsilon^q - p ((a + b)/2) \to 1 \) and

\[
\lim_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} u_\epsilon(a) = - \lim_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} u_\epsilon'(b) = \left( \frac{2(q - p)}{(p + 1)(q + 1)} \right)^{1/2}.
\]  

(5.3)

**Proof.** It is easily known that if \( u_\epsilon \) is a positive solution of (5.2), \( u_\epsilon \) is symmetric about \( x = (a + b)/2 \). We first consider the initial value problem

\[
- v'' = v^p - cv^q \quad \text{in } (0, \infty), \quad v(0) = 1, \quad v'(0) = 0.
\]  

(5.4)

It follows from theory of ordinary differential equations that (5.4) has a solution \( v_\epsilon \) such that \( \lim_{c \uparrow 1} v_\epsilon(x) = 1 \) for all \( x \in (0, \infty) \). Define

\[
X(c) = \sup \{ x > 0: v(\cdot, c) > 0 \text{ on } (0, x) \}.
\]

By arguments similar to those in the proof of Lemma 2.3 of [1], we know that \( X(c) < \infty \) in a left-neighbourhood of \( c = 1 \) and

\[
\lim_{c \uparrow 1} X(c) = \infty.
\]

Now, setting

\[
\gamma_\epsilon = \left( \frac{c}{\epsilon} \right)^{1/(q-p)}, \quad (b - a)/2 = \gamma_\epsilon^{-(p-1)/2} X(c)
\]

and

\[
y = \frac{(a + b)}{2} + \gamma_\epsilon^{-(p-1)/2} x, \quad u_\epsilon(y) = \gamma_\epsilon v_\epsilon(x),
\]

\[
\lim_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial \bar{u}_\epsilon}{\partial \nu}(x) = -\left( \frac{2(q - p)}{(p + 1)(q + 1)} \right)^{1/2}.
\]  

(5.1)
we have that \( \gamma_\epsilon \to \infty \) as \( c \to 1 \) and \( \epsilon \to 0 \), \( u_\epsilon(y) \) satisfies the equation in (5.2) on \( ((a+b)/2, b) \) and
\[
\begin{align*}
u_\epsilon((a+b)/2) &= \gamma_\epsilon, \\
u_\epsilon(b) &= 0.
\end{align*}
\]
Define
\[
u_\epsilon(y) = u_\epsilon(a + b - y) \quad \text{for} \quad y \in (a, (a+b)/2).
\]
We easily know that \( u_\epsilon \) is a positive solution of (5.2) with \( \epsilon \|u_\epsilon\|^{q-p}_\infty \to 1 \) as \( \epsilon \to 0 \).

Since the energy
\[
E(u_\epsilon, u_\epsilon') = \frac{1}{2} |u_\epsilon'|^2 + \frac{u_\epsilon^{p+1}}{p+1} - \frac{\epsilon u_\epsilon^{q+1}}{q+1}
\]
is constant along the trajectories of the first-order system associated to (5.2), so
\[
\frac{1}{2} |u_\epsilon'(a)|^2 = \frac{1}{2} |u_\epsilon'(b)|^2 = \frac{u_\epsilon^{p+1}((a+b)/2)}{p+1} - \frac{\epsilon u_\epsilon^{q+1}((a+b)/2)}{q+1}.
\]
(5.5)
Since \( \epsilon u_\epsilon^{q-p}((a+b)/2) \to 1 \) as \( \epsilon \to 0 \), we can easily obtain (5.3). This completes the proof. \( \square \)

The following result provides us with some lower estimates of the boundary layer of the large positive solution \( \theta_\epsilon \) of \( (I_\epsilon) \) when \( \Omega \) is an \( N \)-ball. Notice that when \( \Omega \) is a ball \( B_R(0) \), we know from [1] that \( (I_\epsilon) \) has a large positive solution \( \theta_\epsilon \), which is radially symmetric and decreasing about the radius \( r \). We claim that \( \theta_\epsilon \) is the unique positive solution of \( (I_\epsilon) \) such that
\[
\lim_{\epsilon \to 0} \epsilon \theta_\epsilon^{q-p}(0) \to 1.
\]
(5.6)
Indeed, since \( \theta_\epsilon \) satisfies the equation
\[
-\Delta (\epsilon^{1/(q-p)} \theta_\epsilon) = \epsilon^{-(p-1)/(q-p)} \left[ (\epsilon^{1/(q-p)} \theta_\epsilon)^p - (\epsilon^{1/(q-p)} \theta_\epsilon)^q \right],
\]
by making the changes of variables
\[
y = \epsilon^{-(p-1)/(2(q-p))} r, \quad v_\epsilon(y) = \epsilon^{1/(q-p)} \theta_\epsilon(r),
\]
we can easily know that \( v_\epsilon \) satisfies \( \lim_{\epsilon \to 0} v_\epsilon(0) = 1 \) and \( v_\epsilon(y) \to 1 \) for any \( y \in \mathbb{R}^N \). This implies that
\[
\epsilon \theta_\epsilon^{q-p}(x) \to 1
\]
for \( x \) in any compact set of \( B_R(0) \). Our claim can be obtained from Theorem 4.1.

**Lemma 5.3.** Assume that \( \Omega = B_R(x_0) \) with \( x_0 \in \mathbb{R}^N \) and \( R > 0 \) is an arbitrary number. Then, for any \( z \in \partial B_R(x_0) \),
\[
\liminf_{\epsilon \to 0} -\epsilon^{(p+1)/(2(q-p))} \left[ \frac{\partial \theta_\epsilon}{\partial v}(z) \right] \geq \left( \frac{2(q-p)}{(p+1)(q+1)} \right)^{1/2},
\]
\[
\]
where $v = v(z)$ is the outward unit normal vector to $\partial B_R(x_0)$ at $z$ and $\theta_\epsilon$ is the unique large positive solution of $(I_\epsilon)$ on $B_R(x_0)$.

**Proof.** Without loss of generality we can assume $x_0 = 0$. By [1], the large positive solution of $(I_\epsilon)$ in $\Omega = B_R$ is radially symmetric and so it is given by the positive solution, say $\theta_\epsilon$, of

$$
-\theta'' - \frac{N - 1}{r} \theta' = \theta^p - \epsilon \theta^q, \quad 0 < r < R,
$$

$$
\theta'(0) = 0, \quad \theta(R) = 0.
$$

(5.7)

The change of independent variables

$$
\begin{align*}
\omega_\epsilon(\rho) &= \theta_\epsilon(r), \\
\rho &= h(r) = \frac{1}{N - 2} \left[ R^{2 - N} - r^{2 - N} \right]
\end{align*}
$$

transforms (5.7) into

$$
\begin{align*}
\omega''(\rho) + g(\rho) \left[ \omega^p - \epsilon \omega^q \right] &= 0, \quad -\infty < \rho < 0, \\
\omega'(-\infty) &= \omega(0) = 0
\end{align*}
$$

(5.8)

where $g(\rho) = [h^{-1}(\rho)]^{2(N-1)}$. Thus, $\omega_\epsilon$ is the unique large positive solution of (5.8).

On the other hand, given $d > 0$, by arguments similar to those in the proof of Lemma 5.2, the auxiliary problem

$$
-\omega'' = g(-d) \left[ \omega^p - \epsilon \omega^q \right], \quad -d < \rho < 0, \quad \omega(-d) = \omega(0) = 0
$$

(5.9)

has a positive solution $\omega_\epsilon(\rho)$ if $\epsilon$ is sufficiently small. Moreover,

$$
\epsilon \|\omega_\epsilon\|_{q-p} \to 1 \quad \text{as} \quad \epsilon \to 0.
$$

Since $g(\rho) \geq g(-d)$ for $-d \leq \rho < 0$, then $\omega_\epsilon$ is a subsolution to the problem

$$
-\omega'' = g(\rho) \left[ \omega^p - \epsilon \omega^q \right], \quad -d < \rho < 0, \quad \omega(-d) = \omega(0) = 0.
$$

(5.10)

It is clear that $\epsilon^{-1/(q-p)}$ is a supersolution of (5.10). Then the monotone method as in [14] implies that there exists a maximal positive solution $w_1^\epsilon$ of (5.10). Extending $w_1^\epsilon$ by 0 for $\rho < -d$, we know that $w_1^\epsilon$ is a solution of (5.8). On the other hand, since $w_\epsilon$ is the unique large positive solution of (5.8), we have

$$
\omega_\epsilon \geq w_1^\epsilon \quad \text{in} \quad (-\infty, 0).
$$

Therefore,

$$
\omega_\epsilon \geq w_1^\epsilon \geq \omega_\epsilon \quad \text{in} \quad (-d, 0).
$$

(5.11)

The strong maximum principle then implies $(\omega_\epsilon - \omega_\epsilon)'(0) < 0$. Thus,

$$
-\omega'_{\epsilon}(0) < -\omega'_{\epsilon}(0).
$$

(5.12)
By arguments similar to those in the proof of Lemma 5.2, we have that
\[ \lim_{\epsilon \to 0} -\epsilon^{(p+1)/(2(q-p))}w'_\epsilon(0) = \left(\frac{2(q-p)}{(p+1)(q+1)}\right)^{1/2} \]. \hspace{1cm} (5.13)

From (5.12) and (5.13) we find
\[ \lim_{\epsilon \to 0} \inf -\epsilon^{(p+1)/(2(q-p))}w'_\epsilon(0) \geq \left(\frac{2(q-p)}{(p+1)(q+1)}\right)^{1/2} \], \hspace{1cm} (5.14)

and since the left-hand side of (5.14) does not depend on \(d\), passing to the limit as \(d \to 0^+\) we obtain
\[ \lim_{\epsilon \to 0} \inf -\epsilon^{(p+1)/(2(q-p))}w'_\epsilon(0) \geq R^{N-1}\left(\frac{2(q-p)}{(p+1)(q+1)}\right)^{1/2} \]. \hspace{1cm} (5.15)

On the other hand,
\[ \frac{\partial \theta_\epsilon}{\partial \nu}(z) = R^{1-N}w'_\epsilon(0). \hspace{1cm} (5.16) \]

Therefore, from (5.15) and (5.16) we get
\[ \lim_{\epsilon \to 0} \inf -\epsilon^{(p+1)/(2(q-p))}\frac{\partial \theta_\epsilon}{\partial \nu}(z) \geq \left(\frac{2(q-p)}{(p+1)(q+1)}\right)^{1/2} \]. \hspace{1cm} (5.17)

The proof is completed. \(\Box\)

We are next confining ourselves in obtaining a lower estimate of the limit (5.1) in the case of the annulus \(A(a,R) = \{x \in \mathbb{R}^N: 0 < a < |x| < R\}\) and at points located in the part \(|x| = a\) of its boundary \(\partial A\). This will allow us, when dealing with the case of a smooth domain \(\Omega \subset \mathbb{R}^N\), to produce a lower estimate of the inferior limit \(\lim_{\epsilon \to 0} \inf (\partial u_\epsilon/\partial \nu)(x_0)\) at any \(x_0 \in \partial \Omega\). The main idea is similar to that in [16]. To proceed in the general situation, a suitable annulus \(A(a,R;y_0) := A(a,R) + y_0, \Omega \subset A(a,R; y_0)\), will be chosen to be tangent to \(\partial \Omega\) at \(x_0\).

As we construct the large positive solution in the proof of Theorem 3.1, we can find a large positive radial solution to \((I_\epsilon)\) in \(\Omega = A(a,R)\) for \(\epsilon\) sufficiently small. To see this, we only need to choose \(w(\epsilon, y)(x)\) in the proof of Lemma 3.3 to be a radial function. In fact, for any \(r_0 \in (a, R)\) and
\[ 0 < \epsilon < \epsilon_2 := \min\left\{\epsilon_0, g^{-1}(\beta_\epsilon^{-2}\left(\min(|a - r_0|, |R - r_0|)\right)^2)\right\}, \]
we define
\[ w(\epsilon, r_0)(r) := v_\epsilon\left((g(\epsilon))^{-1/2}(r - r_0)\right), \hspace{1cm} r \in (a, R). \]

It can be easily verified that \(w(\epsilon, r_0)(r)\) is a subsolution of \((I_\epsilon)\) in \(\Omega = A(a,R)\). Therefore, the unique large positive solution of \((I_\epsilon)\) in \(A(a,R)\) is a radially symmetric solution.

Let us now state our next result.
Lemma 5.4. Let \( u = u_\epsilon(r) \) be the unique large positive (radial) solution of \((I_\epsilon)\) in \( A(a, R) \) for \( \epsilon \) sufficiently small. Then
\[
\lim_{\epsilon \to 0} \sup \epsilon^{(p+1)/(2(q-p))} u_\epsilon'(a) \leq \left( \frac{2(q-p)}{(p+1)(q+1)} \right)^{1/2}.
\] (5.18)

Proof. We consider the problem
\[
-(r^{N-1}u')' = r^{N-1}(u^p - \epsilon u^q), \quad a < r < R,
\]
\[
u(a) = u(R) = 0.
\] (5.19)

Following the idea used in Lemma 5.3, the variable \( r \) can be removed from the left-hand side of (5.19) by introducing a new variable \( \rho = \rho(r) \) by means of the expression
\[
\rho = g(r) = \frac{1}{2-N}\left[ r^{2-N} - a^{2-N} \right].
\]
Observe that now \( d\rho/dr = r^{1-N} \) and that \( 0 < \rho < T \) as \( a < r < R \), where \( T = g(R) \). By setting \( v(\rho) = u(g^{-1}(\rho)) \) the problem (5.19) is re-written as
\[
-v'' = \left( g^{-1}(\rho) \right)^{2(N-1)} \left[ v^p - \epsilon v^q \right], \quad 0 < \rho < T,
\]
\[
v(0) = v(T) = 0,
\] (5.20)
where \( ' = \frac{d}{d\rho} \).

If \( v_\epsilon \) designates the unique large positive solution to (5.20), then \( v_\epsilon \) is a supersolution of the problem
\[
-v'' = a^{2(N-1)} \left[ v^p - \epsilon v^q \right], \quad 0 < \rho < T, \quad v(0) = v(T) = 0.
\] (5.21)

We can conclude then
\[
0 < v_\epsilon^{-}(\rho) \leq v_\epsilon(\rho), \quad 0 < \rho < T,
\] (5.22)
where \( v_\epsilon^{-} \) is a large positive solution of (5.22). (The existence of \( v_\epsilon^{-} \) can be known from Lemma 5.2.) Choose a positive number \( 0 < b < T/2 \) and consider the auxiliary boundary value problem in the interval \( 0 < \rho < b \),
\[
-w'' = \left( g^{-1}(b) \right)^{2(N-1)} \left[ w^p - \epsilon w^q \right], \quad 0 < \rho < b,
\]
\[
w(0) = 0, \quad w(b) = v_\epsilon(b),
\] (5.23)
where \( v_\epsilon \) is the large positive solution of (5.20). (We know that \( \epsilon v_\epsilon^{-p}(b) \to 1 \) as \( \epsilon \to 0 \).) Then, by arguments similar to those in the proof of Lemma 5.2, we can obtain a maximal positive solution \( w_\epsilon \) of (5.23) such that \( w_\epsilon(b) = v_\epsilon(b) \) and \( w_\epsilon(\rho) < w_\epsilon(b) \) for \( 0 < \rho < b \).

On the other hand, the large positive solution of (5.20) is a subsolution to (5.23) while \( \epsilon^{-1/(q-p)} \) is a supersolution of (5.23). Thus,
\[
0 < v_\epsilon \leq w_\epsilon, \quad 0 < \rho < b
\] (5.24)
for \( \epsilon \) sufficiently small which implies
\[
v'_\epsilon(0) \leq w'_\epsilon(0) = \left( (g^{-1}(b))^{2(N-1)} \frac{2(q-p)}{(p+1)(q+1)} v^{p+1}_\epsilon(b) \right)^{1/2}.
\] (5.25)

and the fact that \( \epsilon v^{q-p}_\epsilon(b) \to 1 \) imply
\[
\limsup_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} v'_\epsilon(0) \leq \left( (g^{-1}(b))^{N-1} \frac{2(q-p)}{aN-1} \right)^{1/2}.
\]

Therefore,
\[
\limsup_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} u'_\epsilon(a) \leq \left( (g^{-1}(b))^{N-1} \frac{2(q-p)}{aN-1} \right)^{1/2}.
\] (5.26)

Sending \( b \) in (5.26) to zero, we obtain (5.18). \( \square \)

**Proof of Theorem 5.1.** Fix an arbitrary \( x_0 \in \partial \Omega \) with associated outward unit normal \( \nu = \nu(x_0) \). Since \( \partial \Omega \) is a smooth surface it is possible to find points \( y_i \in \Omega, \ y_e \in \mathbb{R}^N \setminus \overline{\Omega} \) and positive numbers \( R, a, R_e \) such that the ball \( B_i := B_R(y_i) \subset \Omega \), the annulus \( A_e := A(a, R_e, y_e) = \{ a < |x - y_e| < R_e \} \) contains \( \Omega \), i.e., \( \Omega \subset A_e \), while finally both \( B_i \) and \( A_e \) are tangent to \( \partial \Omega \) at \( x_0 \).

If, following the previous notation, we introduce the function \( \theta^{ Bi}_\epsilon \) it is found that \( \theta^{ Bi}_\epsilon(x) \leq \overline{u}_\epsilon(x) \) for \( x \in B_i \). Then \( (\partial \overline{u}_\epsilon / \partial \nu)(x_0) \leq (\partial \theta^{ Bi}_\epsilon / \partial \nu)(x_0) \). Lemma 5.3 then implies that
\[
\limsup_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial \overline{u}_\epsilon}{\partial \nu}(x_0) \leq \limsup_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial \theta^{ Bi}_\epsilon}{\partial \nu}(x_0)
\]
\[
\leq -\left( \frac{2(q-p)}{(p+1)(q+1)} \right)^{1/2}.
\] (5.27)

In a symmetric way, consider the problem \( (I_\epsilon) \) taking the annulus \( A_e \) as the domain \( \Omega \) and let \( u_{\epsilon,A_e} \) be its unique large positive solution for \( \epsilon \) sufficiently small. Then, since \( u_{\epsilon,A_e} \) is a supersolution to \( (I_\epsilon) \), it follows as before that \( \overline{u}_\epsilon \leq u_{\epsilon,A_e} \) for every \( x \in \Omega \). So, \( (\partial u_{\epsilon,A_e} / \partial \nu)(x_0) \leq (\partial \overline{u}_\epsilon / \partial \nu)(x_0) \). On the other hand, if \( r = |x - y_e| \), then \( u_{\epsilon,A_e} = u_{\epsilon,A_e}(r) \), while \( (\partial u_{\epsilon,A_e} / \partial \nu)(x_0) = -u'_\epsilon(a) \) \( (\ = d/dr) \). Thus, Lemma 5.4 permits to conclude
\[
-\left( \frac{2(q-p)}{(p+1)(q+1)} \right)^{1/2} \leq \liminf_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial u_{\epsilon,A_e}}{\partial \nu}(x_0)
\]
\[
\leq \liminf_{\epsilon \to 0} \epsilon^{(p+1)/(2(q-p))} \frac{\partial \overline{u}_\epsilon}{\partial \nu}(x_0).
\] (5.28)

Finally, the combination of (5.27) and (5.28) gives the desired identity (5.1) at \( x_0 \in \partial \Omega \). On the other hand, the compactness of \( \partial \Omega \) allows us to perform a choice of \( a, R_i, R_e \) not depending upon the position of \( x_0 \) in \( \partial \Omega \). Therefore, the limit (5.1) is indeed uniform. This completes the proof of Theorem 5.1. \( \square \)
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