Harmonic mean curvature flow and geometric inequalities

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This talk is based on the joint work with Ben Andrews (Australia National University) and Yingxiang Hu (YMSc, Tsinghua University).

OUTLINE

1. Isoperimetric inequality in Euclidean space

2. Alexandrov-Fenchel inequality in Euclidean space

3. Alexandrov-Fenchel inequality in hyperbolic space

4. Main results
ISOOPERIMETRIC INEQUALITY IN $\mathbb{R}^n$

One of the most well-known geometric inequalities for hypersurfaces in $\mathbb{R}^n$ is \textit{isoperimetric inequality}:

For any bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\Sigma = \partial \Omega$, we have

$$|\Sigma| \geq \omega_{n-1} \left( \frac{n |\Omega|}{\omega_n} - 1 \right)^{\frac{1}{n-1}},$$

where $\omega_{n-1}$ is the area of the unit sphere $S_{n-1} \subset \mathbb{R}^n$. Equality holds iff $\Omega$ is a geodesic ball.

Remark. Isoperimetric inequality imposes \textit{NO} convexity assumption on $\Sigma$.
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Isoperimetric inequality in $\mathbb{R}^n$

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Remark. Isoperimetric inequality imposes NO convexity assumption on $\Sigma$. 
Let $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ be the principal curvatures of a hypersurface $\Sigma \subset \mathbb{R}^n$. 

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**Alexandrov-Fenchel inequality**:

For any closed convex hypersurface $\Sigma \subset \mathbb{R}^n$, we have

$$
\int_{\Sigma} p_m(\kappa) \, d\mu \geq \omega_{m-l}^{n-1-l} \left( \int_{\Sigma} p_l(\kappa) \, d\mu \right)^{n-1-m},
$$

where $0 \leq l < m \leq n-1$. 

Equality holds iff $\Sigma$ is a geodesic sphere.
AF INEQUALITY IN $\mathbb{R}^n$ 

- Let $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ be the principal curvatures of a hypersurface $\Sigma \subset \mathbb{R}^n$. 
- The (normalized) $m$-th mean curvature $p_m$ of $\Sigma$ is 

$$p_0 = 1, \quad p_m(\kappa) = \frac{1}{C^m_{n-1}} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n-1} \kappa_{i_1} \cdots \kappa_{i_m}.$$
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For any closed convex hypersurface $\Sigma \subset \mathbb{R}^n$, we have

\[ \int_{\Sigma} p_m(\kappa) d\mu \geq \omega_{n-1}^{\frac{m}{n-1-l}} \left( \int_{\Sigma} p_l(\kappa) d\mu \right)^{\frac{n-1-m}{n-1-l}}, \quad 0 \leq l < m \leq n-1. \]

Equality holds iff $\Sigma$ is a geodesic sphere.
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Taking $l = 0$, for any closed convex hypersurface $\Sigma \subset \mathbb{R}^n$, we have

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- A hypersurface $\Sigma$ is called *starshaped*, if $\langle \partial_r, \nu \rangle > 0$ on $\Sigma$, where $\nu$ is the unit outward normal of $\Sigma$ and $\partial_r$ is the radial vector, respectively.
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- A hypersurface $\Sigma$ is called \textit{$m$-convex}, if $p_i(\kappa) > 0$ for $i = 1, \cdots, m$ everywhere on $\Sigma$. 
By using the smooth convergence of inverse curvature flows in $\mathbb{R}^n$ by C. Gerhardt (1990) and J. Urbas (1990), Pengfei Guan and Junfang Li proved that

**Theorem, P. Guan and J. Li, 2009, Adv. Math.**

If hypersurface $\Sigma \subset \mathbb{R}^n (n \geq 3)$ is *star-shaped* and *m-convex*, then

$$\int_{\Sigma} \rho_m(\kappa) d\mu \geq \omega_{n-1} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-1-m}{n-1}}, \quad 1 \leq m \leq n-1.$$  

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For $n = 2$, for a simple closed curve $\gamma$ in $\mathbb{H}^2$, we have

$$L^2 \geq 4\pi V + V^2,$$

where $L$ is the length of $\gamma$ and $V$ is the volume of the domain enclosed by $\gamma$. Moreover, Equality holds if and only if $\gamma$ is a circle.
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For $n \geq 3$, the isoperimetric inequality in $\mathbb{H}^n$ was proved by E. Schmidt (1940), but the explicit expression is rare.
AF INEQUALITY IN $\mathbb{H}^n$

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**Problem**

Establish an analogue of Alexandrov-Fenchel inequality in $\mathbb{H}^n$. 
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**PROBLEM**

Establish an analogue of Alexandrov-Fenchel inequality in $\mathbb{H}^n$.

Joint with Yong Wei and Changwei Xiong, we apply the inverse curvature flow to obtain


If hypersurface $\Sigma \subset \mathbb{H}^n (n \geq 3)$ is starshaped and 2-convex, then

$$\int_{\Sigma} p_2 \geq \omega_{n-1} \left[ \left( \frac{|\Sigma|}{\omega_{n-1}} \right) + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-3}{n-1}} \right].$$

Equality holds iff $\Sigma$ is a geodesic sphere.
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The proof of Li-Wei-Xiong’s inequality consists of four steps:

- The convergence result of inverse curvature flows \( \partial_t X = \frac{p_1}{p_2} \nu \) in \( \mathbb{H}^n \) by C. Gerhardt;
- The preservation of 2-convexity along this flow;
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- The monotonicity of

$$Q(t) = |\Sigma_t|^{-\frac{n-3}{n-1}} \int_{\Sigma} (p_2 - 1),$$

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convexity of $\Sigma_t$ plays an essential role;
- The Sobolev inequality by W. Beckner (1993), which is used to show

$$\lim_{t \to \infty} Q(t) \geq \omega^{\frac{2}{n-1}}_{n-1}.$$
Arising naturally from Li-Wei-Xiong’s inequality, the following Conjecture is still *open*:

\[ \text{AF INEQUALITY IN } \mathbb{H}^n \]

Equality holds iff \( \Sigma \) is a geodesic sphere.

Remark. This verifies the Conjecture for the case \( k = 2 \).
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Arising naturally from Li-Wei-Xiong’s inequality, the following Conjecture is still open:

**Conjecture**

Let $1 \leq k \leq n - 1$. Any *starshaped* and *k-convex* hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies

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**Remark.**

- This verifies the Conjecture for the case $k = 2$.
- With the Li-Wei-Xiong’s inequality and a result of Xu Cheng and Detang Zhou, the Conjecture for $k = 1$ holds for hypersurfaces with *nonnegative Ricci curvature* in $\mathbb{H}^n$. 
To state the recent developments on this Conjecture, we recall the various convexity for hypersurfaces in hyperbolic space.
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Different from a hypersurface in $\mathbb{R}^n$, there are *four* different kinds of convexity (in *strictly ascending order*) for a hypersurface $(\Sigma, g)$ in $\mathbb{H}^n$:
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Different from a hypersurface in $\mathbb{R}^n$, there are four different kinds of convexity (in **strictly ascending order** for a hypersurface $(\Sigma, g)$ in $\mathbb{H}^n$:

- **(strictly) convex** if $\kappa_i > 0$ for $i = 1, \cdots, n - 1$;
- **nonnegative Ricci curvature** if $\kappa_i \left( \sum_{j \neq i} \kappa_j \right) \geq n - 2$ for $i = 1, \cdots, n - 1$;
- **nonnegative sectional curvature** if $\kappa_i \kappa_j \geq 1$ for $1 \leq i < j \leq n - 1$;
- **horospherical convex (h-convex)** if $\kappa_i \geq 1$ for $i = 1, \cdots, n - 1$.

Remark. The strict convexity is equivalent to $(n - 1)$-convex and starshaped, and hence all these convexity conditions are stronger than $m$-convex and starshaped.
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**Remark.** The strict convexity is equivalent to $(n - 1)$-convex and starshaped, and hence all these convexity conditions are **stronger** than $m$-convex and starshaped.
In 2014, Yuxin Ge, Guofang Wang and Jie Wu investigated the $k$-th Gauss-Bonnet curvature $L_k$ on hypersurface $(\Sigma, g)$ in $\mathbb{H}^n$,

$$L_k := \frac{1}{2k} \delta_{i_1 j_1}^{i_2 j_2} \cdots \delta_{i_{2k-1} j_{2k-1}}^{i_{2k} j_{2k}} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}},$$

where $R_{ij}^{kl}$ is the Riemannian curvature tensor in the local coordinates w.r.t. the metric $g$, and the generalized Kronecker delta is defined by

$$\delta_{i_1 j_1}^{i_2 j_2} \cdots \delta_{i_r j_r}^{i_{r+1} j_{r+1}} = \det \begin{pmatrix} \delta_{i_1}^{i_1} & \delta_{i_1}^{i_2} & \cdots & \delta_{i_1}^{i_r} \\ \delta_{i_2}^{i_1} & \delta_{i_2}^{i_2} & \cdots & \delta_{i_2}^{i_r} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_r}^{i_1} & \delta_{i_r}^{i_2} & \cdots & \delta_{i_r}^{i_r} \end{pmatrix}. $$
AF INEQUALITY IN $\mathbb{H}^n$

By using the inverse curvature flows in $\mathbb{H}^n$, they proved an optimal Sobolev-type inequality for h-convex hypersurfaces in $\mathbb{H}^n$:

$$\int_{\Sigma} L_k \, d\mu \geq C_{2k}^n (2k)! \omega^{2k} n^{-1} |\Sigma|^{n-1} - 2^{k-n-1}.$$  

Equality holds iff $\Sigma$ is a geodesic sphere.

Remark. For any hypersurface $(\Sigma, g)$ in $\mathbb{H}^n$, the Gauss-Bonnet curvature $L_k$ can be expressed by

$$L_k = C_{2k}^n (2k)! \sum_{j=0}^k (-1)^j C_j^k 2^{k-j-2}.$$  

For $k=1$, the above inequality coincides with Li-Wei-Xiong's inequality.
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By using the inverse curvature flows in $\mathbb{H}^n$, they proved an optimal Sobolev-type inequality for h-convex hypersurfaces in $\mathbb{H}^n$:


Let $2 \leq 2k < n - 1$. Any **h-convex** hypersurface $(\Sigma, g) \subset \mathbb{H}^n$ satisfies

$$\int_{\Sigma} L_k d\mu \geq C_{n-1}^{2k}(2k)! \omega_{n-1}^{\frac{2k}{n-1}} |\Sigma|^{\frac{n-1-2k}{n-1}}.$$

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- The convergence result of inverse curvature flows $\partial_t X = \frac{p_{2k-1}}{p_{2k}} \nu$ in $\mathbb{H}^n$ by C. Gerhardt;

- The preservation of $h$-convexity along the inverse curvature flows;

- The monotonicity of $Q(t) = |\Sigma_t|^{\frac{n-1}{n-2}} - 2k^n - 1 - 2k n - 1 \int \Sigma_L^{k}$, where the $h$-convexity of $\Sigma$ plays an essential role;

- The generalized Sobolev inequality by P. Guan and G. Wang (2003), which is used to show $\lim_{t \to \infty} Q(t) \geq C_2^{2k} n - 1 (2k)! \omega_2^{n - 1} n - 1$. 

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The importance of this inequality is that it can be viewed as the bricks of other geometric inequalities. To observe this, we have

$$\int_{\Sigma} p_{2k} = \frac{1}{(2k)!} C_{2k}^{n-1} \sum_{i=0}^{k} C_k^i \int_{\Sigma} L_i.$$
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They proved AF inequality for curvature integrals in $\mathbb{H}^n$:


Let $2 \leq 2k \leq n - 1$. Any *h-convex* hypersurface $(\Sigma, g) \subset \mathbb{H}^n$ satisfies

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In a recent work joint with Yingxiang Hu, all Ge-Wang-Wu’s inequalities have been extended to hypersurfaces with nonnegative sectional curvature in hyperbolic space.
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**Theorem, Hu-L., 2019, Calc. Var.**

Let $2 \leq 2k \leq n - 1$. Any hypersurface $(\Sigma, g) \subset \mathbb{H}^n$ with *nonnegative sectional curvature* satisfies

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Equality holds iff $\Sigma$ is a geodesic sphere.

**Remark.** The Conjecture for $k = 2m$ ($2 \leq 2m \leq n - 1$) holds for hypersurfaces with nonnegative sectional curvature in $\mathbb{H}^n$. 

The proof consists of four ingredients:

1. The convergence result of inverse mean curvature flow $\partial_t X = \frac{1}{H} \nu$ in $\mathbb{H}^n$ by C. Gerhardt;
2. The preservation of nonnegative sectional curvature along the IMCF, which is inspired by the recent work of Andrews-Chen-Wei on volume preserving flows in hyperbolic space;
3. The monotonicity of $Q(t) = |\Sigma_t| - n - 1 - 2k_n - 1 \int_{\Sigma} L_k$, where the nonnegative sectional curvature of $\Sigma$ plays an essential role; this has already been observed by Ge-Wang-Wu;
4. The analysis of $\lim_{t \to \infty} Q(t)$ is the same as Ge-Wang-Wu.
AF INEQUALITY IN $\mathbb{H}^n$

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HMCF Haizhong Li THU
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In 2014, Guofang Wang and Chao Xia used the quermassintegral preserving curvature flows to prove AF inequality for curvature integrals in $\mathbb{H}^n$:
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Let $1 \leq k \leq n - 1$. If $\Omega \subset \mathbb{H}^n$ is a bounded domain with $\Sigma = \partial \Omega$ h-convex, then

$$\int_{\Sigma} p_k \geq \omega_{n-1} \left[ \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^\frac{2}{k} + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^\frac{2}{k} \frac{n-1-k}{n-1} \right]^\frac{k}{2},$$

Equality holds iff $\Sigma$ is a geodesic sphere.
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Equality holds iff $\Sigma$ is a geodesic sphere.

**Remark.** The Conjecture for $1 \leq k \leq n - 1$ holds for $h$-convex hypersurfaces in $\mathbb{H}^n$. 
Now we present our main results.

Let $0 < 2k < n - 1$. If $\Sigma$ is a strictly convex hypersurface in $H^n$, then

$$\int_{\Sigma} p^{n-1-2k} \omega^{n-1} \leq \int_{\Sigma} p^{n-1} \omega^{n-1} \left[1 - \left(\int_{\Sigma} p^{n-1} \omega^{n-1}\right)^{\frac{1}{n-1}}\right]^{-2(k-n)}.$$ 

Equality holds iff $\Sigma$ is a geodesic sphere.
Now we present our main results.

The first result is


Let $0 < 2k < n - 1$. If $\Sigma$ is a *strictly convex* hypersurface in $\mathbb{H}^n$, then

$$
\frac{\int_\Sigma p_{n-1-2k}}{\omega_{n-1}} \leq \frac{\int_\Sigma p_{n-1}}{\omega_{n-1}} \left[ 1 - \left( \frac{\int_\Sigma p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k.
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Equality holds iff $\Sigma$ is a geodesic sphere.
With the help of this AF inequality, we apply the inverse mean curvature flow to prove


Let $n - 1 > 2$. If $\Sigma$ is a *strictly convex* hypersurface in $\mathbb{H}^n$, then

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\int_{\Sigma} p_{n-1} \geq |\Sigma| \left[ 1 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{\frac{n-1}{2}}.
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With the help of this AF inequality, we apply the inverse mean curvature flow to prove


Let $n - 1 > 2$. If $\Sigma$ is a *strictly convex* hypersurface in $\mathbb{H}^n$, then

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\int_{\Sigma} \rho_{n-1} \geq |\Sigma| \left[ 1 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{\frac{n-1}{2}}.
$$

Equality holds iff $\Sigma$ is a geodesic sphere.

**Remark.** This verifies the Conjecture mentioned above for the case $k = n - 1$. 
HARMONIC MEAN CURVATURE FLOW

Here we give the proofs of Theorems A & B.
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The key difference from the previous work is that we choose the *contracting curvature flow* in $\mathbb{H}^n$. 
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The proof is the classical method for proving geometric inequalities by establishing the monotonicity properties along a suitable curvature flow, and the smooth convergence of this flow.

The key difference from the previous work is that we choose the *contracting curvature flow* in $\mathbb{H}^n$.

Given a smooth hypersurface $\Sigma_0$ in $\mathbb{H}^n$, parametrized by an embedding $X_0 : M^{n-1} \to \mathbb{H}^n$. The harmonic mean curvature flow (HMCF) is a family of embeddings $X : M^{n-1} \times [0, T) \to \mathbb{H}^n$ satisfying

$$
\frac{\partial}{\partial t} X(x, t) = - \frac{p_{n-1}}{p_{n-2}} (x, t) \nu(x, t),
$$

$$
X(\cdot, 0) = X_0(\cdot).
$$

\[\text{HMCF Haizhong Li THU}\]
Ben Andrews first proved the smooth convergence results for the flow of $h$-convex hypersurfaces in hyperbolic space, with speed given by functions with argument $\kappa_i - 1$, in particular the \textit{(shifted) harmonic mean curvature flow};
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Recently, Hao Yu proved the smooth convergence for a general class of contracting curvature flows in hyperbolic space.

A major ingredient in the proof of the smooth convergence of the HMCF is the pinching estimate. That is, if the initial hypersurface $\Sigma$ is strictly convex, then along the HMCF the evolving hypersurface $\Sigma_t$ satisfies

$$\kappa_{n-1}(x, t) \leq C\kappa_1(x, t), \quad \forall (x, t) \in M \times [0, T^*)$$

where $\kappa_1 \leq \cdots \leq \kappa_{n-1}$ and $C$ depends only on $\Sigma$. 

Recall that the inner radius $\rho_-$ and outer radius $\rho_+$ of a bounded domain $\Omega_t$ with boundary $\Sigma_t$ in $\mathbb{H}^n$ is defined by
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\rho_-(t) := \sup \{ \rho : B_\rho(p) \text{ is enclosed by } \Omega_t \text{ for some } p \in \mathbb{H}^n \},
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By the contracting property of the HMCF, together with the pinching estimate, we prove that the inner radius and outer radius is comparable as it shrinks to a point.
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$$

By the contracting property of the HMCF, together with the pinching estimate, we prove that the inner radius and outer radius is comparable as it shrinks to a point.


Let $\Sigma_t$ be a solution of the HMCF on a maximal time interval $[0, T^*)$. There exist positive constants $C$ and $\eta$, depending only on the initial hypersurface $\Sigma$, such that

$$
\rho_+(t) \leq C \rho_-(t), \quad \forall t \in [T^* - \eta, T^*).
$$
QUERMASSINTEGRALS IN $\mathbb{H}^n$

For a convex domain $\Omega \subset \mathbb{H}^n$, the quermassintegrals are defined by

$$W_r(\Omega) := \frac{(n - r)\omega_{r-1} \cdots \omega_0}{n\omega_{n-2} \cdots \omega_{n-r-1}} \int_{\mathcal{L}} \chi(L \cap \Omega) dL, \quad r = 1, \ldots, n-1,$$

where $\mathcal{L}_r$ is the space of $r$-dim totally geodesic subspaces $L$ in $\mathbb{H}^n$, and $dL$ is the natural measure on $\mathcal{L}_r$ which is invariant under the isometry group of $\mathbb{H}^n$. The function $\chi$ is defined to be 1 if $L \cap \Omega \neq \emptyset$ and to be 0 otherwise. Furthermore, we set $W_0(\Omega) = |\Omega|$ and $W_n(\Omega) = \omega_{n-1}/n$. 
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In $\mathbb{R}^n$, the quermassintegrals coincide with the curvature integrals up to a constant multiple. However, the quermassintegrals and the curvature integrals in $\mathbb{H}^n$ do not coincide. They are closely related by

$$\int_{\Sigma} p_k = n \left( W_{k+1}(\Omega) + \frac{k}{n-k+1} W_{k-1}(\Omega) \right), \quad k = 1, \cdots, n-1.$$
PROOF OF THEOREM A – LIMITING BEHAVIOR

Since $W_k$ is monotone under the set inclusion, i.e.,

$$W_k(\Omega_1) \leq W_k(\Omega_2), \quad \text{if } \Omega_1 \subset \Omega_2,$$

we prove the following asymptotic behavior along the HMCF.
**Proof of Theorem A – Limiting Behavior**

Since $W_k$ is monotone under the set inclusion, i.e.,

$$W_k(\Omega_1) \leq W_k(\Omega_2), \quad \text{if } \Omega_1 \subset \Omega_2,$$

we prove the following asymptotic behavior along the HMCF.


Let $\Sigma$ be a strictly convex hypersurface in $\mathbb{H}^n$. Let $\Sigma_t, \ t \in [0, T^*)$ be the solution of the HMCF with the initial hypersurface $\Sigma$. Then we have

$$\lim_{t \to T^*} \int_{\Sigma_t} p_j = \begin{cases} 
0, & 0 \leq j \leq n-2; \\
\omega_{n-1}, & j = n-1,
\end{cases}$$
To prove Theorem A, we only need to find suitable monotone quantities along the HMCF.
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To prove the AF inequality

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To prove the AF inequality

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we consider the functional

\[ P_k(t) := \left( \frac{\int_{\Sigma_t} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{n-1-2k}{n-1}} \left\{ \frac{\int_{\Sigma_t} p_{n-1} - 2k}{\omega_{n-1}} - \left( \frac{\int_{\Sigma_t} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma_t} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k \right\}. \]
PROOF OF THEOREM A – MONOTONICITY

Now we verify that $P_k(t)$ is monotone increasing along the HMCF.
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We first prove the case $k = 1$, i.e.,

$$\frac{\int_{\Sigma} p_{n-3}}{\omega_{n-1}} \leq \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right].$$
**Proof of Theorem A – Monotonicity**

Now we verify that $P_k(t)$ is monotone increasing along the HMCF.

We first prove the case $k = 1$, i.e.,

$$\frac{\int \Sigma p_{n-3}}{\omega_{n-1}} \leq \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \left[ 1 - \left( \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right].$$

Along the HMCF, we have

$$\frac{d}{dt} \int \Sigma p_{n-1} = -(n - 1) \int \Sigma p_{n-1},$$

and

$$\frac{d}{dt} \int \Sigma p_{n-3} = -2 \int \Sigma p_{n-1} - (n - 3) \int \Sigma \frac{p_{n-1}p_{n-4}}{p_{n-2}} \geq -2 \int \Sigma p_{n-1} - (n - 3) \int \Sigma p_{n-3}.$$
Then we have

\[
\frac{d}{dt} \left( \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \right) = -(n-1) \left( \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \right),
\]
**Proof of Theorem A – Monotonicity**

Then we have

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and

\[
\frac{d}{dt} \left[ \frac{\int \Sigma p_{n-3}}{\omega_{n-1}} - \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \left( 1 - \left( \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right) \right] \geq -(n-3) \left[ \frac{\int \Sigma p_{n-3}}{\omega_{n-1}} - \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \left( 1 - \left( \frac{\int \Sigma p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right) \right].
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**Proof of Theorem A – Monotonicity**

Then we have

\[
\frac{d}{dt} \left( \frac{\int \sum p_{n-1}}{\omega_{n-1}} \right) = -(n - 1) \left( \frac{\int \sum p_{n-1}}{\omega_{n-1}} \right),
\]

and

\[
\frac{d}{dt} \left[ \frac{\int \sum p_{n-3}}{\omega_{n-1}} - \frac{\int \sum p_{n-1}}{\omega_{n-1}} \left( 1 - \left( \frac{\int \sum p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right) \right]
\geq -(n - 3) \left[ \frac{\int \sum p_{n-3}}{\omega_{n-1}} - \frac{\int \sum p_{n-1}}{\omega_{n-1}} \left( 1 - \left( \frac{\int \sum p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right) \right].
\]

Therefore, we have

\[
\frac{d}{dt} P_1(t) \geq 0.
\]

By the limiting behavior of \(\int \sum p_j\), we have \(\lim_{t \to T^*} P_1(t) = 0\).
PROOF OF THEOREM A – MONOTONICITY

Thus we get $P_1(0) \leq \lim_{t \to T^*} P_1(t) = 0$, i.e.,

$$\left( \frac{\int p_{n-1}}{\omega_{n-1}} \right)^{-\frac{n-3}{n-1}} \left[ \frac{\int p_{n-3}}{\omega_{n-1}} - \frac{\int p_{n-1}}{\omega_{n-1}} \left( 1 - \left( \frac{\int p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right) \right] \leq 0,$$
Proof of Theorem A – Monotonicity

Thus we get $P_1(0) \leq \lim_{t \to T^*} P_1(t) = 0$, i.e.,

$$\left(\frac{\int \sum p_{n-1}}{\omega_{n-1}}\right)^{-\frac{n-3}{n-1}} \left[\frac{\int \sum p_{n-3}}{\omega_{n-1}} - \frac{\int \sum p_{n-1}}{\omega_{n-1}} \left(1 - \left(\frac{\int \sum p_{n-1}}{\omega_{n-1}}\right)^{-\frac{2}{n-1}}\right)\right] \leq 0,$$

which is equivalent to

$$\frac{\int \sum p_{n-3}}{\omega_{n-1}} \leq \frac{\int \sum p_{n-1}}{\omega_{n-1}} \left[1 - \left(\frac{\int \sum p_{n-1}}{\omega_{n-1}}\right)^{-\frac{2}{n-1}}\right].$$
Proof of Theorem A – Monotonicity

We prove the case $k \geq 2$ by induction.
PROOF OF THEOREM A – MONOTONICITY

We prove the case $k \geq 2$ by induction.

Assume that it holds for $k - 1$, i.e.,

$$\left( \frac{\int_{\Sigma} p_{n-1} - 2(k-1)}{\omega_{n-1}} \right) \leq \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^{k-1} ,$$

then we show that it also holds for $k$, i.e.,

$$\frac{\int_{\Sigma} p_{n-1} - 2k}{\omega_{n-1}} \leq \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k .$$
PROOF OF THEOREM A – MONOTONICITY

By the variational formula along the HMCF and Newton-MacLaurin inequality, we have

\[
\frac{d}{dt} \int_\Sigma p_{n-1-2k} = -2k \int_\Sigma p_{n-2k} \frac{p_{n-1}}{p_{n-2}} - (n - 1 - 2k) \int_\Sigma p_{n-2-2k} \frac{p_{n-1}}{p_{n-2}}
\]

\[
\geq -2k \int_\Sigma p_{n+1-2k} - (n - 1 - 2k) \int_\Sigma p_{n-1-2k}
\]

\[
= -2k \left( \frac{\int_\Sigma p_{n-1-2(k-1)}}{\omega_{n-1}} \right) - (n - 1 - 2k) \left( \frac{\int_\Sigma p_{n-1-2k}}{\omega_{n-1}} \right).
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By the variational formula along the HMCF and Newton-MacLaurin inequality, we have

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\geq -2k \int_{\Sigma} p_{n+1-2k} - (n - 1 - 2k) \int_{\Sigma} p_{n-1-2k} \\
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\]
PROOF OF THEOREM A – MONOTONICITY

For simplicity, we take

\[ x(t) = \frac{\int \Sigma p_{n-1}}{\omega_{n-1}}, \quad y(t) = \frac{\int \Sigma p_{n-1 - 2k}}{\omega_{n-1}}. \]
Proof of Theorem A – Monotonicity

For simplicity, we take

\[ x(t) = \frac{\int \sum p_{n-1}}{\omega_{n-1}}, \quad y(t) = \frac{\int \sum p_{n-1-2k}}{\omega_{n-1}}. \]

Combining with induction on \( k - 1 \), we have

\[ \frac{d}{dt} y = \frac{d}{dt} \left( \frac{\int \sum p_{n-1-2k}}{\omega_{n-1}} \right) \geq -2k \left( \frac{\int \sum p_{n-1-2(k-1)}}{\omega_{n-1}} \right) - (n - 1 - 2k) \left( \frac{\int \sum p_{n-1-2k}}{\omega_{n-1}} \right) \]

\[ \geq -2kx \left( 1 - x \frac{2}{n-1} \right)^{k-1} - (n - 1 - 2k)y. \]

and \( \frac{d}{dt} x = -(n - 1)x \).
PROOF OF THEOREM A – MONOTONICITY

A direct calculation gives

\[
\frac{d}{dt} \left[ y - x \left( 1 - x^{-\frac{2}{n-1}} \right)^k \right] \geq -(n - 1 - 2k) \left[ y - x \left( 1 - x^{-\frac{2}{n-1}} \right)^k \right],
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and hence \( \frac{d}{dt} P_k(t) \geq 0 \).
**Proof of Theorem A – Monotonicity**

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and hence \( \frac{d}{dt} P_k(t) \geq 0 \). By the limiting behavior of \( \int_{\Sigma_t} p_j \), we get

\[
\left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{n-1-2k}{n-1}} \left\{ \frac{\int_{\Sigma} p_{n-1-2k}}{\omega_{n-1}} - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k \right\}
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**Proof of Theorem A – Monotonicity**

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and hence \( \frac{d}{dt} P_k(t) \geq 0 \). By the limiting behavior of \( \int_{\Sigma} p_j \), we get

\[
\left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{n-1-2k}{n-1}} \left\{ \frac{\int_{\Sigma} p_{n-1-2k}}{\omega_{n-1}} - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k \right\}
\]

\[= P_k(0) \leq \lim_{t \to T^*} P_k(t) = 0, \]

which is equivalent to

\[
\frac{\int_{\Sigma} p_{n-1-2k}}{\omega_{n-1}} \leq \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^k.
\]
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Proof of Theorem B

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\frac{\partial}{\partial t} X(x, t) &= \frac{1}{H(x, t)} \nu(x, t), \\
X(\cdot, 0) &= X_0(\cdot),
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First, the strict convexity is preserved along the IMCF, so the following inequality in Theorem A holds on the evolving hypersurfaces:

\[
\frac{\int_{\Sigma} p_{n-3}}{\omega_{n-1}} \leq \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right].
\]

**Proof of Theorem B**

By the variational formula along the IMCF and Newton-MacLaurin inequality, we have
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\[
\frac{d}{dt} \left( \frac{|\Sigma_t|}{\omega_{n-1}} \right) = \frac{|\Sigma_t|}{\omega_{n-1}},
\]

Hence, we consider the monotone increasing (we omit the proof) functional

\[
Q(t) := \left( \frac{|\Sigma_t|}{\omega_{n-1}} \right) - 1 \left( \frac{|\Sigma_t|}{\omega_{n-1}} - \int \Sigma_{t \omega_{n-1}} \right) \left( 1 - \int \Sigma_{t \omega_{n-1}} - 2 \right) \left( \frac{|\Sigma_t|}{\omega_{n-1}} \right)^{\frac{n-1}{2}}.
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**Proof of Theorem B**

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\frac{d}{dt} \left( \frac{|\Sigma_t|}{\omega_{n-1}} \right) = \frac{|\Sigma_t|}{\omega_{n-1}},
\]

\[
\frac{d}{dt} \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) = \frac{1}{\omega_{n-1}} \int_{\Sigma} \frac{p_{n-2}}{p_1} \leq \frac{\int_{\Sigma} p_{n-3}}{\omega_{n-1}}
\]

\[
\leq \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right].
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By the variational formula along the IMCF and Newton-MacLaurin inequality, we have

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\frac{d}{dt} \left( \frac{\left| \Sigma_t \right|}{\omega_{n-1}} \right) = \frac{\left| \Sigma_t \right|}{\omega_{n-1}},
\]

\[
\frac{d}{dt} \left( \frac{\int_{\Sigma} \rho_{n-1}}{\omega_{n-1}} \right) = \frac{1}{\omega_{n-1}} \int_{\Sigma} \frac{\rho_{n-2}}{p_1} \leq \frac{\int_{\Sigma} \rho_{n-3}}{\omega_{n-1}}
\]

\[
\leq \left( \frac{\int_{\Sigma} \rho_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma} \rho_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right].
\]

Hence, we consider the monotone increasing (we omit the proof) functional

\[
Q(t) := \left( \frac{\left| \Sigma_t \right|}{\omega_{n-1}} \right)^{-1} \left[ \frac{\left| \Sigma_t \right|}{\omega_{n-1}} - \left( \frac{\int_{\Sigma_t} \rho_{n-1}}{\omega_{n-1}} \right) \left[ 1 - \left( \frac{\int_{\Sigma_t} \rho_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right] \right]^{\frac{n-1}{2}}
\]
Proof of Theorem B

Now we analyze the asymptotics of $Q(t)$ as $t \to \infty$. 

<table>
<thead>
<tr>
<th>Iso. ineq. in Euclidean space</th>
<th>AF ineq. in Euclidean space</th>
<th>AF ineq. in hyperbolic space</th>
<th>Main results</th>
</tr>
</thead>
</table>

HMCF  Haizhong Li  THU
**Proof of Theorem B**

Now we analyze the asymptotics of $Q(t)$ as $t \to \infty$. We have $|\Sigma_t| = |\Sigma| e^t$. The convergence result of Gerhardt gives

$$h_i^j = \left(1 + O(e^{-\frac{t}{n-1}})\right) \delta_i^j, \quad \text{on } \Sigma_t.$$
**Proof of Theorem B**

Now we analyze the asymptotics of $Q(t)$ as $t \to \infty$. We have $|\Sigma_t| = |\Sigma| e^t$. The convergence result of Gerhardt gives

$$h_i^j = \left(1 + O(e^{-\frac{t}{n-1}})\right) \delta_i^j, \quad \text{on } \Sigma_t.$$

As $p_{n-1}$ is homogeneous of degree $n - 1$, we get

$$p_{n-1}(h_i^j) = \left(1 + O(e^{-\frac{t}{n-1}})\right)^{n-1} = 1 + O(e^{-\frac{t}{n-1}}), \quad \text{on } \Sigma_t.$$

and

$$\frac{\int_{\Sigma_t} p_{n-1}}{\omega_{n-1}} = \frac{|\Sigma_t|}{\omega_{n-1}} \left(1 + O(e^{-\frac{t}{n-1}})\right) = O(e^t), \quad \text{on } \Sigma_t.$$
**Proof of Theorem B**

It follows that

\[ Q(t) = 1 - \left( \frac{\int \Sigma_t p_{n-1}}{|\Sigma_t|} \right) \left[ 1 - \left( \frac{\int \Sigma_t p_{n-1}}{\omega_{n-1}} \right)^{-\frac{2}{n-1}} \right]^\frac{n-1}{2} \]

\[ = 1 - \left( 1 + O(e^{-\frac{t}{n-1}}) \right) \left( 1 + O(e^{-\frac{2t}{n-1}}) \right)^\frac{n-1}{2} \]

\[ = 1 - \left( 1 + O(e^{-\frac{t}{n-1}}) \right) \left( 1 + O(e^{-\frac{2t}{n-1}}) \right) \]

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which gives \( \lim_{t \to \infty} Q(t) = 0. \)
PROOF OF THEOREM B

It follows that

\[ Q(t) = 1 - \left( \frac{\int_{\Sigma_t} p_{n-1}}{|\Sigma_t|} \right) \left[ 1 - \left( \frac{\int_{\Sigma_t} p_{n-1}}{\omega_{n-1}} \right)^{-2} \right]^{\frac{n-1}{2}} \]

\[ = 1 - \left( 1 + O(e^{-\frac{t}{n-1}}) \right) \left( 1 + O(e^{-\frac{2t}{n-1}}) \right)^{\frac{n-1}{2}} \]

\[ = 1 - \left( 1 + O(e^{-\frac{t}{n-1}}) \right) \left( 1 + O(e^{-\frac{2t}{n-1}}) \right) \]

\[ = O(e^{-\frac{t}{n-1}}), \]

which gives \( \lim_{t \to \infty} Q(t) = 0 \). Together with monotonicity of \( Q(t) \), we get

\[ Q(0) \leq Q(t) \leq \lim_{t \to \infty} Q(t) = 0, \]

which is equivalent to

\[ \int_{\Sigma} p_{n-1} \geq |\Sigma| \left[ 1 + \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{-2} \right]^{\frac{n-1}{2}}. \]


Thank you for your attention!