Homogeneous and inhomogeneous isoparametric hypersurfaces in symmetric spaces of noncompact type

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Main new results

Joint work with J. Carlos Díaz-Ramos and Alberto Rodríguez-Vázquez
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- **Classification of cohomogeneity one actions** on $\mathbb{H}H^n$
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- Classification of cohomogeneity one actions on $\mathbb{H}H^n$
  $\implies$ Classification of cohomogeneity one actions on symmetric spaces of rank one
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- **Classification of cohomogeneity one actions** on $\mathbb{H}H^n$
  $\implies$ Classification of cohomogeneity one actions on symmetric spaces of rank one

- Uncountably many **inhomogeneous isoparametric** families of hypersurfaces with **constant principal curvatures**
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1 Homogeneous and isoparametric hypersurfaces
2 Symmetric spaces of noncompact type and rank one
   1 Cohomogeneity one actions
   2 Isoparametric hypersurfaces
3 The quaternionic hyperbolic space
Cohomogeneity one actions

\( \tilde{M} \) complete Riemannian manifold

**Definition**

A **cohomogeneity one action** on \( \tilde{M} \) is a proper isometric action on \( \tilde{M} \) with codimension one orbits.
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**Properties**

- All the orbits, except at most two, are hypersurfaces.
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\[
\begin{align*}
\text{SO}(2) \, \circ \, \mathbb{R}^2 & \quad A \cdot v = Av \\
\mathbb{R} \, \circ \, \mathbb{R}^2 & \quad t \cdot v = v + tw \\
\text{SO}(2) \times \mathbb{R} \, \circ \, \mathbb{R}^3 & \quad (A, t) \cdot v = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ t \end{pmatrix} \\
\text{SO}(2) \, \circ \, S^2 & \quad A \cdot v = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} v
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Homogeneous hypersurfaces

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**Definition**

Two isometric actions of groups \(G_1, G_2\) on \(\tilde{M}\) are **orbit equivalent** if there exists \(\varphi \in \text{Isom}(\tilde{M})\) that maps each \(G_1\)-orbit to a \(G_2\)-orbit.

**Problem**

Classify cohomogeneity one actions on \(\tilde{M}\) up to orbit equivalence.

**Definition**

A submanifold is a **homogeneous submanifold** if it is an orbit of an isometric action. Homogeneous hypersurfaces are precisely the codimension one orbits of cohomogeneity one actions.

**Equivalent problem**

Classify homogeneous hypersurfaces in a given Riemannian manifold \(\tilde{M}\).
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Classify homogeneous hypersurfaces in a given Riemannian manifold \( \tilde{M} \).
Classification of cohomogeneity one actions

Cohomogeneity one actions have been classified, up to orbit equivalence, in:

- Euclidean spaces $\mathbb{R}^n$ [Somigliana (1918), Segre (1938)]
- Real hyperbolic spaces $\mathbb{R}H^n$ [Cartan (1939)]
- Round spheres $\mathbb{S}^n$ [Hsiang, Lawson (1971), Takagi, Takahashi (1972)]

Irreducible symmetric spaces of compact type [Kollross (2002)]

Homogeneous 3-manifolds with 4-dimensional isometry group (\(E(\kappa,\tau)\)-spaces) [DV, Manzano (2018)]
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**Question**

What happens in symmetric spaces of noncompact type?
\( \bar{M} \) Riemannian manifold

**Definition [Levi-Civita (1937)]**

A hypersurface \( M \) in \( \bar{M} \) is **isoparametric** if \( M \) and its nearby equidistant hypersurfaces have constant mean curvature.

Equivalently, if \( M \) is a regular level set of a function \( f: \mathcal{U}^{\text{open}} \subset \bar{M} \to \mathbb{R} \) such that \( |\nabla f| = a \circ f \) and \( \Delta f = b \circ f \), for smooth functions \( a, b \).
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Isoparametric hypersurfaces

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Let $M$ be a hypersurface in a real space form $\tilde{M} \in \{\mathbb{R}^n, \mathbb{R}H^n, \mathbb{S}^n\}$. Then:

- $M$ is isoparametric $\iff M$ has constant principal curvatures
- If $\tilde{M} \in \{\mathbb{R}^n, \mathbb{R}H^n\}$, $M$ is isoparametric $\iff M$ is homogeneous
Isoparametric hypersurfaces in space forms

Theorem [Cartan (1939), Segre (1938)]

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Classification in the Euclidean space $\mathbb{R}^n$ [Segre (1938)]

- Parallel hyperplanes $\mathbb{R}^{n-1}$
- Concentric spheres $\mathbb{S}^{n-1}$
- Generalized cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$
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Classification in the real hyperbolic space $\mathbb{R}H^n$ [Cartan (1939)]

- Tot. geod. $\mathbb{R}H^{n-1}$ and equidistant hypersurfaces
- Tubes around a tot. geod. $\mathbb{R}H^k$
- Geodesic spheres
- Horospheres
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**Classification in spheres $\mathbb{S}^n$**

- **There are inhomogeneous examples** [Ferus, Karcher, Münzner (1981)]
- **All isoparametric hypersurfaces are homogeneous or of FKM-type** [Cartan; Münzner; Takagi; Ozeki, Takeuchi; Tang; Fang; Stolz; Cecil, Chi, Jensen; Immervoll; Abresch; Dorfmeister, Neher; Miyaoka; Chi]
Isoparametric hypersurfaces in nonconstant curvature

Isoparametric hypersurfaces in nonconstant curvature

- Classification in $\mathbb{C}P^n, n \neq 15$ [DV (2016)] and in $\mathbb{H}P^n, n \neq 7$ [DV, Gorodski (2018)]
  - There are countably many inhomogeneous examples, all of them with nonconstant principal curvatures
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**Question**

What happens in symmetric spaces of noncompact type?
Symmetric spaces of noncompact type

1. Homogeneous and isoparametric hypersurfaces
2. Symmetric spaces of noncompact type and rank one
   1. Cohomogeneity one actions
   2. Isoparametric hypersurfaces
3. The quaternionic hyperbolic space
Definition [Cartan (1926)]

A **symmetric space** is a Riemannian manifold \( \tilde{M} \) whose geodesic symmetry \( \sigma_p : \exp_p(v) \mapsto \exp_p(-v) \), \( v \in T_p\tilde{M} \), around each \( p \in \tilde{M} \) is a global isometry of \( \tilde{M} \).
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![Diagram of a symmetric space with a geodesic symmetry around point $p$.]
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![Symmetric space diagram](image.png)
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Symmetric spaces are complete and homogeneous
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$\tilde{M} \cong G/K$, where $G = \text{Isom}^0(\tilde{M})$ and $K = \{g \in G : g(o) = o\}$ are Lie groups, and $o \in \tilde{M}$ is a base point
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- **noncompact type**
- **Euclidean type**

**duality**
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- **compact type**
  - $\tilde{M}$ compact,
  - $\sec(\tilde{M}) \geq 0$,
  - $g$ compact semisimple

- **noncompact type**
  - $\tilde{M}$ noncompact,
  - $\sec(\tilde{M}) \leq 0$,
  - $g$ noncompact semisimple

- **Euclidean type**
  - $\tilde{M} = \mathbb{R}^n/\Gamma$ flat
Symmetric spaces of noncompact type

\[ \bar{M} \cong G/K \] symmetric space of noncompact type
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\[ \tilde{M} \cong G/K \text{ symmetric space of noncompact type} \implies \tilde{M} \cong \mathbb{B}^n \]
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**Iwasawa decomposition**

\[ g = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \]

\( \mathfrak{n} \) is nilpotent
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\( \text{diffeo.} \)
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\( \mathfrak{a} \oplus \mathfrak{n} \) Lie subalgebra of \( g \sim AN \) Lie subgroup of \( G \)
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Symmetric spaces of noncompact type

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**The solvable model of a symmetric space of noncompact type**

\( \tilde{M} \) is isometric to \( AN \) endowed with a left-invariant metric.
\[ \tilde{M} \cong G/K \cong AN \text{ symmetric space of noncompact type, rank } \tilde{M} = 1 \]
Symmetric spaces of noncompact type and rank one

\[ \tilde{M} \cong G/K \cong AN \] symmetric space of noncompact type, rank \( \tilde{M} = 1 \)

\[ \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}, \quad \mathfrak{a} \cong \mathbb{R}, \quad \mathfrak{z} = Z(n) \]
Symmetric spaces of noncompact type and rank one

\[ \tilde{M} \cong G/K \cong AN \] symmetric space of noncompact type, rank \( \tilde{M} = 1 \)

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\begin{align*}
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\end{align*}
\]

### Symmetric spaces of noncompact type and rank 1

<table>
<thead>
<tr>
<th>( \tilde{M} )</th>
<th>( \mathbb{R}H^n )</th>
<th>( \mathbb{C}H^n )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\text{SO}^0(1,n)}{\text{SO}(n)} )</td>
<td>( \frac{\text{SU}(1,n)}{\text{S(U(1) \times U(n))}} )</td>
<td>( \frac{\text{Sp}(1,n)}{\text{Sp(1) \times Sp(n)}} )</td>
<td>( \frac{\text{F}_{4-20}}{\text{Spin}(9)} )</td>
<td></td>
</tr>
<tr>
<td>( \mathfrak{v} )</td>
<td>( \mathbb{R}^{n-1} )</td>
<td>( \mathbb{C}^{n-1} )</td>
<td>( \mathbb{H}^{n-1} )</td>
<td>( \mathbb{O} )</td>
</tr>
<tr>
<td>( \dim \mathfrak{z} )</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>
Symmetric spaces of noncompact type

1. Homogeneous and isoparametric hypersurfaces
2. Symmetric spaces of noncompact type and rank one
   1. Cohomogeneity one actions
   2. Isoparametric hypersurfaces
3. The quaternionic hyperbolic space
Cohomogeneity one actions on hyperbolic spaces

$\mathbb{F}H^n$ symmetric space of noncompact type and rank one, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$

Cohomogeneity one actions with a totally geodesic singular orbit
[Berndt, Brück (2001)]

Tubes around totally geodesic submanifolds $P$ in $\mathbb{F}H^n$ are homogeneous if and only if

- in $\mathbb{R}H^n$: $P = \{\text{point}\}, \mathbb{R}H^1, \ldots, \mathbb{R}H^{n-1}$
- in $\mathbb{C}H^n$: $P = \{\text{point}\}, \mathbb{C}H^1, \ldots, \mathbb{C}H^{n-1}, \mathbb{R}H^n$
- in $\mathbb{H}H^n$: $P = \{\text{point}\}, \mathbb{H}H^1, \ldots, \mathbb{H}H^{n-1}, \mathbb{C}H^n$
- in $\mathbb{O}H^2$: $P = \{\text{point}\}, \mathbb{O}H^1, \mathbb{H}H^2$
Cohomogeneity one actions on hyperbolic spaces

$FH^n \cong G/K \cong AN$ symmetric space of noncompact type and rank one

$a \oplus n = a \oplus v \oplus \mathfrak{z}$, $\quad a \cong \mathbb{R}$, $\quad \mathfrak{z} = Z(n)$, $\quad K_0 = N_K(a)$

Symmetric spaces of noncompact type and rank 1

<table>
<thead>
<tr>
<th>$FH^n$</th>
<th>$RH^n$</th>
<th>$CH^n$</th>
<th>$HH^n$</th>
<th>$OH^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>$\mathbb{R}^{n-1}$</td>
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Cohomogeneity one actions on hyperbolic spaces

\[ \mathbb{F}H^n \simeq G/K \simeq AN \text{ symmetric space of noncompact type and rank one} \]

\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = Z(n), \quad K_0 = N_K(a) \]

Symmetric spaces of noncompact type and rank 1

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</table>

Cohomogeneity one actions without singular orbits [Berndt, Brück (2001), Berndt, Tamaru (2003)]

Orbit equivalent to the action of:

- $N \sim$ horosphere foliation
- The connected subgroup of $G$ with Lie algebra $a \oplus \mathfrak{w} \oplus z$, where $\mathfrak{w}$ is a (real) hyperplane in $v$
F$H^n \cong G/K \cong \text{AN}$ symmetric space of noncompact type and rank one

$\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$, \hspace{1em} \mathfrak{a} \cong \mathbb{R}, \hspace{1em} \mathfrak{z} = Z(\mathfrak{n}), \hspace{1em} K_0 = N_K(\mathfrak{a})$

Symmetric spaces of noncompact type and rank 1

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Cohomogeneity one actions with a non-totally singular orbit [Berndt, Brück (2001)]

$\mathfrak{w} \subsetneq \mathfrak{v}$ (real) subspace $\implies s_{\mathfrak{tv}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ is a Lie algebra

$S_{\mathfrak{tv}}$ connected subgroup of $AN$ with Lie algebra $s_{\mathfrak{tv}}$
Cohomogeneity one actions on hyperbolic spaces

\[ \mathbb{F}H^n \cong G/K \cong AN \text{ symmetric space of noncompact type and rank one} \]
\[ a \oplus n = a \oplus v \oplus \mathfrak{z}, \quad a \cong \mathbb{R}, \quad \mathfrak{z} = \mathbb{Z}(n), \quad K_0 = N_K(a) \]

### Symmetric spaces of noncompact type and rank 1

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\( \mathfrak{w} \subsetneq v \) (real) subspace \( \implies \mathfrak{s}_{\mathfrak{tv}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z} \) is a Lie algebra

\( S_{\mathfrak{tv}} \) connected subgroup of \( AN \) with Lie algebra \( \mathfrak{s}_{\mathfrak{tv}} \)

The tubes around \( S_{\mathfrak{tv}} \) are homogeneous if and only if \( N_{K_0}(\mathfrak{w}) \) acts transitively on the unit sphere of \( \mathfrak{w}^\perp \) (the orthogonal complement of \( \mathfrak{w} \) in \( v \))
Cohomogeneity one actions on hyperbolic spaces

\[ \mathbb{F}H^n \cong G/K \cong AN \] symmetric space of noncompact type and rank one
\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = Z(n), \quad K_0 = N_K(a) \]

**Symmetric spaces of noncompact type and rank 1**

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$\mathfrak{w} \subsetneq v$ (real) subspace \( \implies \mathfrak{s}_\mathfrak{w} = a \oplus \mathfrak{w} \oplus z \) is a Lie algebra

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The tubes around $S_\mathfrak{w}$ are homogeneous if and only if $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$ (the orthogonal complement of $\mathfrak{w}$ in $v$)
Cohomogeneity one actions on hyperbolic spaces

$\mathbb{FH}^n \cong AN$ symmetric space of noncompact type and rank one

$a \oplus n = a \oplus v \oplus \mathfrak{z}, \quad a \cong \mathbb{R}, \quad \mathfrak{z} = Z(n), \quad K_0 = N_K(a)$

Theorem [Berndt, Tamaru (2007)]

For a cohomogeneity one action on $\mathbb{FH}^n$, one of the following holds:

1. There is a totally geodesic singular orbit.
2. Its orbit foliation is regular.
3. There is a non-totally geodesic singular orbit $S_w$, where $w \subset v$ is such that $N_K(\mathfrak{w})$ acts transitively on the unit sphere of $w \perp \mathfrak{w}$. 

Cohomogeneity one actions on hyperbolic spaces

\[ \mathbb{F}H^n \cong A\mathbb{N} \] symmetric space of noncompact type and rank one
\[ \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}, \quad \mathfrak{a} \cong \mathbb{R}, \quad \mathfrak{z} = Z(\mathfrak{n}), \quad K_0 = N_K(\mathfrak{a}) \]

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For a cohomogeneity one action on \( \mathbb{F}H^n \), one of the following holds:
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Cohomogeneity one actions on hyperbolic spaces

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Cohomogeneity one actions on hyperbolic spaces

\( \mathbb{F}H^n \) isom. \( AN \) symmetric space of noncompact type and rank one
\( a \oplus n = a \oplus v \oplus \mathbb{R} \), \( a \sim \mathbb{R}, \mathbb{R} = Z(n), K_0 = N_K(a) \)

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Cohomogeneity one actions on hyperbolic spaces

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The study of the last case was carried out for \( \mathbb{R}H^n, \mathbb{C}H^n, \mathbb{H}H^2 \) and \( \mathbb{O}H^2 \).
Cohomogeneity one actions on hyperbolic spaces

\[ \mathbb{F}H^n \ \text{isom.} \cong AN \] symmetric space of noncompact type and rank one
\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = Z(n), \quad K_0 = N_K(a) \]

**Theorem [Berndt, Tamaru (2007)]**

For a cohomogeneity one action on \( \mathbb{F}H^n \), one of the following holds:

- There is a totally geodesic singular orbit. ✓
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The study of the last case was carried out for \( \mathbb{R}H^n, \mathbb{C}H^n, \mathbb{H}H^2 \) and \( \mathbb{O}H^2 \)

**Problem**

Analyze the last case for \( \mathbb{H}H^n, \ n \geq 3 \), to conclude the classification.
Symmetric spaces of noncompact type

1. Homogeneous and isoparametric hypersurfaces
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New isoparametric hypersurfaces

$\mathbb{F}H^n \cong AN$ symmetric space of noncompact type and rank one

$\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$, \quad \mathfrak{a} \cong \mathbb{R}, \quad \mathfrak{z} = Z(n)$
New isoparametric hypersurfaces

$F^H_n \isom AN$ symmetric space of noncompact type and rank one

$a \oplus n = a \oplus v \oplus z$, $a \cong \mathbb{R}$, $z = Z(n)$

New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]

$v \subset v$ real subspace $\implies s_v = a \oplus w \oplus z$ is a Lie algebra
New isoparametric hypersurfaces

\[ F^H \cong AN \] symmetric space of noncompact type and rank one

\[ \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}, \quad \mathfrak{a} \cong \mathbb{R}, \quad \mathfrak{z} = Z(\mathfrak{n}) \]

New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]

\( \mathfrak{w} \subsetneq \mathfrak{v} \) real subspace \( \Rightarrow s_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z} \) is a Lie algebra

\( S_\mathfrak{w} \) connected subgroup of \( AN \) with Lie algebra \( s_\mathfrak{w} \)
New isoparametric hypersurfaces

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\( S_\mathfrak{w} \) connected subgroup of \( AN \) with Lie algebra \( \mathfrak{s}_\mathfrak{w} \)

- \( S_\mathfrak{w} \) is a homogeneous minimal submanifold
New isoparametric hypersurfaces

$FH^n \simeq AN$ symmetric space of noncompact type and rank one

$a \oplus n = a \oplus v \oplus \mathfrak{z}$, $a \simeq \mathbb{R}$, $\mathfrak{z} = Z(n)$

New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]

$\mathfrak{w} \subsetneq v$ real subspace $\implies \mathfrak{s}_\mathfrak{w} = a \oplus \mathfrak{w} \oplus \mathfrak{z}$ is a Lie algebra

$S_{\mathfrak{w}}$ connected subgroup of $AN$ with Lie algebra $\mathfrak{s}_{\mathfrak{w}}$

- $S_{\mathfrak{w}}$ is a homogeneous minimal submanifold
- The tubes around $S_{\mathfrak{w}}$ are isoparametric
New isoparametric hypersurfaces

\[ F H^n \cong AN \] symmetric space of noncompact type and rank one

\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = Z(n) \]

New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]

\( w \subsetneq v \) real subspace \( \implies s_{vw} = a \oplus w \oplus z \) is a Lie algebra

\( S_{vw} \) connected subgroup of \( AN \) with Lie algebra \( s_{vw} \)

- \( S_{vw} \) is a homogeneous minimal submanifold
- The tubes around \( S_{vw} \) are isoparametric

- In \( RH^n \) such hypersurfaces are homogeneous
New isoparametric hypersurfaces

\[ \mathbb{F}H^n \overset{\text{isom.}}{\cong} AN \] symmetric space of noncompact type and rank one

\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = \mathbb{Z}(n) \]

New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]

\( w \subsetneq v \) real subspace \( \implies s_w = a \oplus w \oplus z \) is a Lie algebra

- \( S_w \) connected subgroup of \( AN \) with Lie algebra \( s_w \)
  - \( S_w \) is a homogeneous minimal submanifold
  - The tubes around \( S_w \) are isoparametric

- In \( \mathbb{R}H^n \) such hypersurfaces are homogeneous
- In \( \mathbb{C}H^n \) and \( \mathbb{H}H^n, n \geq 3 \), there are inhomogeneous isoparametric families of hypersurfaces with nonconstant principal curvatures
New isoparametric hypersurfaces

\[ \mathbb{F}H^n \cong AN \text{ symmetric space of noncompact type and rank one} \]

\[ a \oplus n = a \oplus v \oplus z, \quad a \cong \mathbb{R}, \quad z = Z(n) \]

**New isoparametric hypersurfaces [Díaz-Ramos, DV (2013)]**

\[ \mathfrak{w} \subsetneq \mathfrak{v} \text{ real subspace} \implies \mathfrak{s}_\mathfrak{w} = a \oplus \mathfrak{w} \oplus z \text{ is a Lie algebra} \]

- \( S_\mathfrak{w} \) connected subgroup of \( AN \) with Lie algebra \( \mathfrak{s}_\mathfrak{w} \)
  - \( S_\mathfrak{w} \) is a homogeneous minimal submanifold
  - The tubes around \( S_\mathfrak{w} \) are isoparametric

- In \( \mathbb{R}H^n \) such hypersurfaces are homogeneous
- In \( \mathbb{C}H^n \) and \( \mathbb{H}H^n \), \( n \geq 3 \), there are **inhomogeneous** isoparametric families of hypersurfaces with nonconstant principal curvatures
- In \( \mathbb{O}H^2 \) there is one **inhomogeneous isoparametric family** of hypersurfaces with **constant principal curvatures** (when \( \dim \mathfrak{w} = 3 \))
Theorem [Díaz-Ramos, DV, Sanmartín-López (2017)]

A connected hypersurface $M$ in the complex hyperbolic space $\mathbb{C}H^n$ is isoparametric if and only if it is an open subset of:

- A tube around a totally geodesic complex hyperbolic space $\mathbb{C}H^k$
- A tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$
- A horosphere
- A tube around a homogeneous minimal submanifold $S_m$
Classification in the complex hyperbolic space

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Classical examples [Montiel (1985)]: all are homogeneous
A connected hypersurface $M$ in the complex hyperbolic space $\mathbb{CH}^n$ is isoparametric if and only if it is an open subset of:

- A tube around a totally geodesic complex hyperbolic space $\mathbb{CH}^k$
- A tube around a totally geodesic real hyperbolic space $\mathbb{RH}^n$
- A horosphere
- A tube around a homogeneous minimal submanifold $S_w$

Classical examples [Montiel (1985)]: all are homogeneous

New examples: there are both (uncountably many) homogeneous [Berndt, Brück (2001)] and inhomogeneous [Díaz-Ramos, DV (2012)] examples, depending on $\mathfrak{w} \subset \mathfrak{v}$
Homogeneous and isoparametric hypersurfaces
Symmetric spaces of noncompact type and rank one
  1. Cohomogeneity one actions
  2. Isoparametric hypersurfaces
The quaternionic hyperbolic space
Problem

Classify cohomogeneity one actions on $\mathbb{H}H^{n+1}, n \geq 2$. 

Equivalent problem [Berndt, Tamaru (2007)]

Classify real subspaces $w \subset v \sim = \mathbb{H}^n$ such that $N_K \circ (w)$ acts transitively on the unit sphere of $w \perp$.

$K \sim = \text{Sp}(n)\text{Sp}(1)$ acts on $v \sim = \mathbb{H}^n$ via $(A, q) \cdot v = Avq^{\frac{1}{2}}$.
The quaternionic hyperbolic space

Problem

Classify cohomogeneity one actions on $\mathbb{H}H^{n+1}$, $n \geq 2$.

Equivalent problem [Berndt, Tamaru (2007)]

Classify real subspaces $\mathfrak{v} \subset \mathfrak{v} \cong \mathbb{H}^n$ such that $N_{K_0}(\mathfrak{v})$ acts transitively on the unit sphere of $\mathfrak{v}^\perp$. 

Definition

A real subspace $V$ of $\mathbb{H}^n$ is protohomogeneous if there is a (connected) subgroup of $\textrm{Sp}(n)\textrm{Sp}(1)$ that acts transitively on the unit sphere of $V$. 

Problem
Classify cohomogeneity one actions on $\mathbb{H}H^{n+1}$, $n \geq 2$.

Equivalent problem [Berndt, Tamaru (2007)]
Classify real subspaces $\mathfrak{w} \subset \mathfrak{v} \cong \mathbb{H}^n$ such that $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$.

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The quaternionic hyperbolic space

Problem

Classify cohomogeneity one actions on $\mathbb{H}H^{n+1}$, $n \geq 2$.

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Definition

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The quaternionic hyperbolic space

**Problem**

Classify cohomogeneity one actions on $\mathbb{H}H^{n+1}$, $n \geq 2$.

**Equivalent problem [Berndt, Tamaru (2007)]**

Classify real subspaces $\mathfrak{w} \subset \mathfrak{v} \cong \mathbb{H}^n$ such that $N_{K_0}(\mathfrak{w})$ acts transitively on the unit sphere of $\mathfrak{w}^\perp$.

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**Definition**

A real subspace $V$ of $\mathbb{H}^n$ is **protohomogeneous** if there is a (connected) subgroup of $\text{Sp}(n)\text{Sp}(1)$ that acts transitively on the unit sphere of $V$.

**Equivalent problem**

Classify protohomogeneous subspaces of $\mathbb{H}^n$. 

Getting intuition in the complex setting

Analogous definition in $\mathbb{C}^n$

A real subspace $V$ of $\mathbb{C}^n$ is **protohomogeneous** if there is a (connected) subgroup of $U(n)$ that acts transitively on the unit sphere of $V$.

Analogous problem in $\mathbb{C}^n$

Classify protohomogeneous subspaces of $\mathbb{C}^n$.
Getting intuition in the complex setting

Analogous definition in $\mathbb{C}^n$

A real subspace $V$ of $\mathbb{C}^n$ is **protohomogeneous** if there is a (connected) subgroup of $U(n)$ that acts transitively on the unit sphere of $V$.

Analogous problem in $\mathbb{C}^n$

Classify protohomogeneous subspaces of $\mathbb{C}^n$.

$\{e_1, \ldots, e_n\}$ $\mathbb{C}$-orthonormal basis of $\mathbb{C}^n$, $J$ complex structure of $\mathbb{C}^n$.
Getting intuition in the complex setting

**Analogous definition in \( \mathbb{C}^n \)**

A real subspace \( V \) of \( \mathbb{C}^n \) is **protohomogeneous** if there is a (connected) subgroup of \( U(n) \) that acts transitively on the unit sphere of \( V \).

**Analogous problem in \( \mathbb{C}^n \)**

Classify protohomogeneous subspaces of \( \mathbb{C}^n \)

\( \{e_1, \ldots, e_n\} \) \( \mathbb{C} \)-orthonormal basis of \( \mathbb{C}^n \), \( J \) complex structure of \( \mathbb{C}^n \)

- Totally real subspaces \( V = \text{span}_\mathbb{R} \{e_1, \ldots, e_k\} \) are protohomogeneous
  \( \leadsto \text{SO}(k) \subset U(n) \) acts transitively on \( S^{k-1} \)
### Analogous definition in $\mathbb{C}^n$

A real subspace $V$ of $\mathbb{C}^n$ is **protohomogeneous** if there is a (connected) subgroup of $U(n)$ that acts transitively on the unit sphere of $V$.

### Analogous problem in $\mathbb{C}^n$

Classify protohomogeneous subspaces of $\mathbb{C}^n$

- $\{e_1, \ldots, e_n\}$ $\mathbb{C}$-orthonormal basis of $\mathbb{C}^n$, $J$ complex structure of $\mathbb{C}^n$

  - **Totally real subspaces** $V = \text{span}_\mathbb{R}\{e_1, \ldots, e_k\}$ are protohomogeneous
    - $\sim SO(k) \subset U(n)$ acts transitively on $S^{k-1}$
  - **Complex subspaces** $V = \text{span}_\mathbb{C}\{e_1, \ldots, e_k\}$ are protohomogeneous
    - $\sim U(k) \subset U(n)$ acts transitively on $S^{2k-1}$
Getting intuition in the complex setting

**Analogous definition in \( \mathbb{C}^n \)**

A real subspace \( V \) of \( \mathbb{C}^n \) is **protohomogeneous** if there is a (connected) subgroup of \( U(n) \) that acts transitively on the unit sphere of \( V \).

**Analogous problem in \( \mathbb{C}^n \)**

Classify protohomogeneous subspaces of \( \mathbb{C}^n \)

\[ \{ e_1, \ldots, e_n \} \text{ \( \mathbb{C} \)-orthonormal basis of} \mathbb{C}^n, \quad J \text{ complex structure of} \mathbb{C}^n \]

- Totally real subspaces \( V = \text{span}_\mathbb{R}\{ e_1, \ldots, e_k \} \) are protohomogeneous
  \( \leadsto \) \( \text{SO}(k) \subset U(n) \) acts transitively on \( S^{k-1} \)

- Complex subspaces \( V = \text{span}_\mathbb{C}\{ e_1, \ldots, e_k \} \) are protohomogeneous
  \( \leadsto \) \( U(k) \subset U(n) \) acts transitively on \( S^{2k-1} \)

- \( V = \text{span}_\mathbb{R}\{ e_1, Je_1, e_2 \} \) is not protohomogeneous
  \( \leadsto \) \( N_{U(n)}(V) = U(1) \times U(n-2) \) does not act transitively on \( S^2 \)
Getting intuition in the complex setting

\( V \) real subspace of \( \mathbb{C}^n \), \( \pi : \mathbb{C}^n \to V \) orthogonal projection, \( v \in V \setminus \{0\} \)

**Definition**

The **Kähler angle** of \( v \) with respect to \( V \) is the angle \( \varphi \in [0, \pi/2] \) between \( Jv \) and \( V \). Equivalently, \( \langle \pi Jv, \pi Jv \rangle = \cos^2 \varphi \langle v, v \rangle \).

\( V \) has **constant Kähler angle** \( \varphi \) if the Kähler angle of any \( v \in V \setminus \{0\} \) with respect to \( V \) is \( \varphi \).
Getting intuition in the complex setting

$V$ real subspace of $\mathbb{C}^n$, $\pi : \mathbb{C}^n \rightarrow V$ orthogonal projection, $v \in V \setminus \{0\}$

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The **Kähler angle** of $v$ with respect to $V$ is the angle $\varphi \in [0, \pi/2]$ between $Jv$ and $V$. Equivalently, $\langle \pi Jv, \pi Jv \rangle = \cos^2 \varphi \langle v, v \rangle$.

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- Totally real subspaces have constant Kähler angle $\pi/2$
- Complex subspaces have constant Kähler angle $0$
- $V = \text{span}_\mathbb{R}\{e_1, Je_1, e_2\}$ does *not* have constant Kähler angle
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**Proposition [Berndt, Brück (2001)]**

\( V \subset \mathbb{C}^n \) is protohomogeneous if and only if it has constant Kähler angle.
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**Proposition [Berndt, Brück (2001)]**

\( V \subset \mathbb{C}^n \) is protohomogeneous if and only if it has constant Kähler angle. Moreover, \( V \) has constant Kähler angle \( \varphi \in [0, \pi/2) \) if and only if \( V = \text{span}\{e_1, \cos \varphi Je_1 + \sin \varphi Je_2, \ldots, e_k, \cos \varphi Je_{2k-1} + \sin \varphi Je_{2k}\} \).
Problem

Classify protohomogeneous subspaces of $\mathbb{H}^n$. 
Problem

Classify protohomogeneous subspaces of $\mathbb{H}^n$.

$\mathcal{J} \subset \text{End}_\mathbb{R}(\mathbb{H}^n)$ quaternionic structure of $\mathbb{H}^n$

$\{J_1, J_2, J_3\}$ canonical basis of $\mathcal{J}$: $J_i^2 = -\text{Id}$ and $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$
Back to the quaternionic setting

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**Definition**
Consider the symmetric bilinear form

$$L_v: \mathcal{J} \times \mathcal{J} \to \mathbb{R}, \quad L_v(J, J') := \langle \pi Jv, \pi J'v \rangle.$$ 

The **quaternionic Kähler angle** of $v$ with respect to $V$ is the triple $(\varphi_1, \varphi_2, \varphi_3)$, with $\varphi_1 \leq \varphi_2 \leq \varphi_3$, such that the eigenvalues of $L_v$ are $\cos^2 \varphi_i \langle v, v \rangle$, $i = 1, 2, 3$. 
Problem
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$\mathcal{J} \subset \text{End}_\mathbb{R}(\mathbb{H}^n)$ quaternionic structure of $\mathbb{H}^n$

$\{J_1, J_2, J_3\}$ canonical basis of $\mathcal{J}$: $J_i^2 = -\text{Id}$ and $J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i$

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There is a canonical basis $\{J_1, J_2, J_3\}$ of $\mathcal{J}$ made of eigenvectors of $L_v$
Back to the quaternionic setting

Proposition [Berndt, Brück (2001)]

\( V \subset \mathbb{H}^n \) protohomogeneous \( \Rightarrow V \) has constant quaternionic Kähler angle.

Partial subspaces with constant quaternionic Kähler angle:

- \((0, 0, 0), (0, 0, \pi/2), (0, \pi/2, \pi/2), (\pi/2, \pi/2, \pi/2), (\phi, \pi/2, \pi/2), (0, \phi, \phi), \ldots\)

But not every triple arises as the constant quaternionic Kähler angle of a subspace \( V \), e.g. \((0, 0, \phi), \phi \in (0, \pi/2)\)

Problem: Classify real subspaces of \( \mathbb{H}^n \) with constant quaternionic Kähler angle.

Question: Does constant quaternionic Kähler angle imply protohomogeneous?

Theorem [Díaz-Ramos, DV (2013)]

The tubes around \( S_w \) have constant principal curvatures if and only if \( w \perp \subset v \) has constant quaternionic Kähler angle.
Back to the quaternionic setting

**Proposition [Berndt, Brück (2001)]**

\[ V \subset \mathbb{H}^n \text{ protohomogeneous} \Rightarrow V \text{ has constant quaternionic Kähler angle.} \]

There are subspaces \( V \) with constant quaternionic Kähler angle \((0, 0, 0), (0, 0, \pi/2), (0, \pi/2, \pi/2), (\pi/2, \pi/2, \pi/2), (\varphi, \pi/2, \pi/2), (0, \varphi, \varphi)\ldots\)
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Protohomogeneous subspaces in $\mathbb{H}^n$

$V$ protohomogeneous real subspace of $\mathbb{H}^n$, $\dim V = k$

Applying the generalized hairy ball theorem [Adams (1963)]

If $k \geq 5$ is odd, then $\Phi(V) = (\pi/2, \pi/2, \pi/2)$.

If $k \equiv 2 \pmod{4}$, then $\Phi(V) = (\varphi, \pi/2, \pi/2)$, for some $\varphi \in [0, \pi/2]$.

If $k = 3$, then $\Phi(V) = (\varphi, \varphi, \pi/2)$, for some $\varphi \in [0, \pi/2]$.

Remaining cases

Classify subspaces $V$ with $k = 3$ and $\Phi(V) = (\varphi, \varphi, \pi/2)$. 

✓ Case $k \equiv 0 \pmod{4}$. 

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$\mathbb{S}^{k-1}$ unit sphere of $\mathcal{V}$, $\pi : \mathbb{H}^n \to \mathcal{V}$ orthogonal projection onto $\mathcal{V}$

$\nu \in \mathbb{S}^{k-1} \Rightarrow \langle J\nu, \nu \rangle = 0 \Rightarrow \langle \pi J\nu, \nu \rangle = 0$ for any $J \in \mathfrak{J}$

$\Delta_\nu := \{ \pi J\nu : J \in \mathfrak{J} \}$ smooth distribution on $\mathbb{S}^{k-1}$, $\text{rank} \Delta \in \{0, 1, 2, 3\}$

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Remaining cases

- Classify subspaces $V$ with $k = 3$ and $\Phi(V) = (\varphi, \varphi, \pi/2)$.
- Case $k \equiv 0 \pmod{4}$. 

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- Case $k \equiv 0 \pmod{4}$. ?
Problem

Classify protohomogeneous real subspaces $V \subset \mathbb{H}^n$ with $\dim V = k = 4r$. 

There exists a canonical basis $\{J_1, J_2, J_3\}$ of $J$ (i.e. $J_i J_i = -J_{i+2}$) such that the Kähler angle of any $v \in S_{k-1}$ with respect to $V$ and the complex structure $J_i$ is $\phi_i$. 

$P_i = \frac{1}{\cos \phi_i} J_i : V \to V$.

Then $P_i P_j + P_j P_i = -2 \delta_{ij} \text{Id}$. 

$\{P_1, P_2, P_3\}$ induces a structure of $\text{Cl}(3)$-module on $V$. 

$V = (\bigoplus V^+) \oplus (\bigoplus V^-)$, where $V^+$ and $V^-$ are the two inequivalent irreducible $\text{Cl}(3)$-modules, $\dim V^{\pm} = 4$. 

Each factor has constant quaternionic Kähler angle $(\phi_1, \phi_2, \phi_3)$. 

Protohomogeneous subspaces in $\mathbb{H}^n$
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**Problem**

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Assume $k \geq 5$. 
Protohomogeneous subspaces in $\mathbb{H}^n$

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Assume $k \geq 5$. For simplicity, assume $\varphi_3 \neq \pi/2$. 
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1. There exists a canonical basis $\{J_1, J_2, J_3\}$ of $\mathbb{J}$
Protohomogeneous subspaces in $\mathbb{H}^n$

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**Problem**

Classify protohomogeneous real subspaces $V \subset \mathbb{H}^n$ with $\dim V = k = 4r$

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2. Define \( P_i = \frac{1}{\cos \varphi_i} \pi J_i : V \to V \).
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**Note:** The page contains a mix of text and tables, suggesting a structured approach to solving the problem. The text describes the classification of protohomogeneous subspaces and details the necessary conditions and properties. The tables are likely used to organize the steps or results in a structured manner.
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4. $V = \left( \bigoplus V_+ \right) \oplus \left( \bigoplus V_- \right)$, where $V_+$ and $V_-$ are the two inequivalent irreducible $Cl(3)$-modules, $\dim V_\pm = 4$. 
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5. Each factor has constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$. 
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There are two types of subspaces $V$ of dimension 4:

- $V_+$, which exists if and only if $\cos \varphi_1 + \cos \varphi_2 + \cos \varphi_3 \leq 1$.
- $V_-$, which exists if and only if $\cos \varphi_1 + \cos \varphi_2 - \cos \varphi_3 \leq 1$. 
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7. $\nexists T \in Sp(n)Sp(1)$ such that $TV_+ = V_-$. 
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7. $\not\exists T \in \text{Sp}(n)\text{Sp}(1)$ such that $TV_+ = V_-$.

8. If $V$, with $\dim V = 4r$, then either $V = \bigoplus V_+$ or $V = \bigoplus V_-$. 
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From this, one can obtain the classification of protohomogeneous subspaces of $\mathbb{H}^n$, and hence of cohomogeneity one actions on $\mathbb{H}H^{n+1}$. 


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From this, one can obtain the classification of protohomogeneous subspaces of $\mathbb{H}^n$, and hence of cohomogeneity one actions on $\mathbb{H}H^{n+1}$.

**Question**

What if we mix both types of 4-dimensional subspaces, $V_+$ and $V_-$?
New isoparametric hypersurfaces

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**Theorem [Díaz-Ramos, DV, Rodríguez-Vázquez (2019)]**

If \( r_+, r_- \geq 1 \), then \( V \) is a non-protohomogeneous subspace of \( \mathbb{H}^n \) with constant quaternionic Kähler angle.
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\( \mathbb{H}H^{n+1} \overset{\text{isom.}}{\cong} AN, \quad a \oplus n = a \oplus v \oplus j, \quad v \cong \mathbb{H}^n \)
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\[ \mathbb{H}^{n+1} \cong \mathbb{A} \mathbb{N}, \quad a \oplus n = a \oplus v \oplus z, \quad v \cong \mathbb{H}^n \]

\( w := \) orthogonal complement of \( V \) in \( v \)

\( s_w = a \oplus w \oplus z \leadsto S_w \) connected subgroup of \( \mathbb{A} \mathbb{N} \)
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**Theorem** [Díaz-Ramos, DV, Rodríguez-Vázquez (2019)]

\( S_w \) and the tubes around it define an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in \( \mathbb{H}H^{n+1} \).