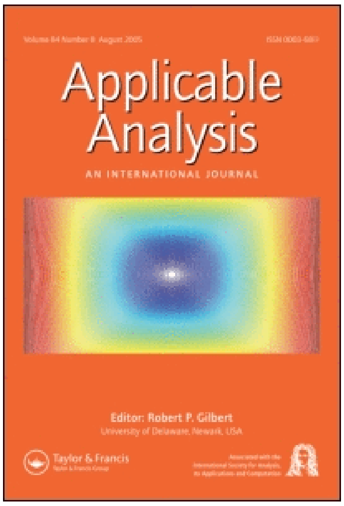

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On the well-posedness of incompressible flow in porous media with supercritical diffusion

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In this article we study the heat transfer equation with a supercritical diffusion term of an incompressible fluid in porous media governed by Darcy's law. We obtain the global well-posedness for small initial data belonging to critical Besov spaces and the local well-posedness for arbitrary initial data. We further show the pointwise blowup criterion.

Keywords: porous media; well-posedness; Besov space; blowup criterion

AMS Subject Classifications: 76S05; 76D03

1. Introduction

We use Darcy's law to describe the flow velocity, which reads

$$v = -k(\nabla p + g\gamma\theta)$$

where $v \in \mathbb{R}^N$ is the liquid discharge, p is the scalar pressure, θ is the liquid temperature, k is the matrix position-independent medium permeabilities in the different directions, respectively, divided by the viscosity, g is the acceleration due to gravity and $\gamma \in \mathbb{R}^N$ is the last canonical vector e_N. For brevity, we only consider $k = g = 1$.

In this article, we study the system of heat transfer with a fractional diffusion in an incompressible N (2 or 3), dimensional flow [1]

$$(DPM_\alpha)\begin{cases} \partial_t\theta + v\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \\ v = -(\nabla p + \gamma\theta), \qquad \operatorname{div} v = 0, \\ \theta(0,x) = \theta^0(x), \end{cases} \tag{1}$$

where $\nu > 0$ is the dissipative coefficient and the differential operator $|D|^\alpha$ is given by $|D|^\alpha := (-\Delta)^{\frac{\alpha}{2}}$. Considering the scaling transform $\theta(t,x) \to \theta_\lambda(t,x) := \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x)$ for $\lambda > 0$, the system will be divided into three cases: The case $\alpha = 1$ is called the critical case, the case $\alpha > 1$ is subcritical and the case $\alpha < 1$ is supercritical.

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Next by rewriting Darcy's law we obtain the expression of velocity v only in terms of temperature θ [2,3]. In the 2D case, thanks to the incompressibility, taking the *curl* operator first and the $\nabla^{\perp} := (-\partial_{x_2}, \partial_{x_1})$ operator second on both sides of Darcy's law, we have

$$-\Delta v = \nabla^{\perp}(\partial_{x_1}\theta) = (-\partial_{x_1}\partial_{x_2}\theta, \partial_{x_1}^2\theta),$$

thus the velocity v can be recovered as

$$v(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\Big(-\frac{\partial^2\theta}{\partial y_2 \partial y_1}(t,y), \frac{\partial^2\theta}{\partial y_1^2}(t,y)\Big)\mathrm{d}y \quad x \in \mathbb{R}^2.$$

Through integration by parts we finally get

$$v(t,x) = -\frac{1}{2}(0, \theta(t,x)) + \frac{1}{2\pi} PV \int_{\mathbb{R}^2} H(x-y)\theta(t,y)\mathrm{d}y \quad x \in \mathbb{R}^2, \tag{2}$$

where the kernel $H(\cdot)$ is defined by

$$H(x) = \left(\frac{2x_1x_2}{|x|^4}, \frac{x_2^2 - x_1^2}{|x|^4}\right).$$

Similarly, in 3D case, applying the *curl* operator twice to Darcy's law, we get

$$-\Delta v = (-\partial_1\partial_3\theta, -\partial_2\partial_3\theta, \partial_1^2\theta + \partial_2^2\theta),$$

where $\partial_i := \frac{\partial}{\partial x_i}$, thus

$$v(t,x) = -\frac{2}{3}(0,0,\theta(t,x)) + \frac{1}{4\pi} PV \int_{\mathbb{R}^3} K(x-y)\theta(t,y)\mathrm{d}y \quad x \in \mathbb{R}^3, \tag{3}$$

where

$$K(x) = \left(\frac{3x_1x_3}{|x|^5}, \frac{3x_2x_3}{|x|^5}, \frac{2x_3^2 - x_1^2 - x_2^2}{|x|^5}\right).$$

We observe that, in general, each coefficient of $v(\cdot, t)$ (with t as parameter) is only the linear combination of the Calderón–Zygmund singular integral (with the definition see the sequel) of θ and θ itself. We write the general version as

$$v := \mathcal{T}(\theta) = \mathcal{C}(\theta) + \mathcal{S}(\theta) \tag{4}$$

where $\mathcal{T} = (\mathcal{T}_k)$, $\mathcal{C} = (\mathcal{C}_k)$, $\mathcal{S} = (\mathcal{S}_k)$, $1 \le k \le N$ are all operators mapping scalar functions to vector-valued functions and $\mathcal{C}_k$ equals a constant multiplication operator whereas $\mathcal{S}_k$ means a Calderón–Zygmund singular integral operator. Especially the corresponding specific forms in 2D or 3D are shown as (2) or (3).

We observe that the system (DPM$_\alpha$) is not more than a dissipative transport-diffusion equation with non-local divergence-free velocity field (the specific relationship between velocity and temperature as (4) shows). It shares many similarities with another flow model – 2D dissipative quasi-geostrophic (QG) equation, which has been intensively studied by many authors [4–9]. From a mathematical view, the system (DPM$_\alpha$) is somewhat a generalization of (QG) equation. Very recently, the system (DPM$_\alpha$) was introduced and investigated by Córdoba and his group. In [2], the authors obtained some results on strong solutions, weak solutions and attractors for the dissipative system (DPM_α). For finite energy they obtained global existence and uniqueness in the subcritical and critical cases. In the supercritical case, they obtained local results in H^s, $s>(N-\alpha)/2+1$ and extended to be global under a small condition $\|\theta^0\|_{H^s}\leq c\nu$, for $s>N/2+1$, where c is a small fixed constant. In [3], they treated the non-dissipative ($\nu=0$) 2D case and obtained the local existence and uniqueness in Hölder space C^δ for $0<\delta<1$ by the particle-trajectory method and gave some blowup criteria of smooth solutions.

In this article we focus on the supercritical case ($0<\alpha<1$). Basically using the method in [7], we give a detailed and slightly different iterative process to derive the local and global results. Further, we show the pointwise blowup criterion. We state our main results as follows.

THEOREM 1.1 *Let* $0<\alpha<1$, $1\leq p\leq\infty$ *and* $s\geq s_c^p:=\frac{N}{p}+1-\alpha$. *We* define

$$\mathcal{Y}_{s,p}:=\begin{cases} B^s_{p,1}, & p<\infty \\ B^s_{\infty,1}\cap \dot{B}^0_{\infty,1}, & p=\infty,\end{cases}$$

then for $\theta^0(x)\in\mathcal{Y}_{s,p}$, *there exists a positive time* T *such that the system* (DPM_α) *has a unique solution* θ *in* $C([0,T);\mathcal{Y}_{s,p})\cap L^1([0,T);\dot{B}^{s+\alpha}_{p,1})$.

Additionally, if there exists an absolute small constant $\zeta>0$ *such that*

$$\|\theta^0\|_{\dot{B}^{1-\alpha}_{\infty,1}}\leq\zeta\nu,$$

then one can take $T=\infty$.

THEOREM 1.2 *Let* T^* *be a maximal local existence time of* θ *in* $\tilde{L}^\infty_T\mathcal{Y}_{s,p}\cap L^1_T\dot{B}^{s+\alpha}_{p,1}$, $s\geq 1+\frac{N}{p}-\alpha$. *There exists an absolute constant* $\epsilon_0>0$ *such that if* $T^*<\infty$

- $s>s_c^p=1+\frac{N}{p}-\alpha$, *equally*, $s\geq 1+\frac{N}{p}-\frac{\alpha}{r}$, $\exists r>1$, *then*

$$\liminf_{t\to T^*}(T^*-t)^{\frac{1}{r}}\|\nabla\theta(t)\|_{L^\infty}^{1-\frac{\alpha}{r}}\geq\epsilon_0, \tag{5}$$

especially, if $s\geq 1+\frac{N}{p}$, *then*

$$\liminf_{t\to T^*}(T^*-t)\|\nabla\theta(t)\|_{L^\infty}\geq\epsilon_0,$$

- $s=s_c^p=1+\frac{N}{p}-\alpha$, *then*

$$\liminf_{t\to T^*}(T^*-t)^{\alpha}\|\nabla\theta(t)\|_{L^\infty}^{2-\alpha}\geq\epsilon_0. \tag{6}$$

Remark 1.1 Compared to the results in [2], our results in Theorem 1.1 are more generalized and elegant. We obtain the existence in the generalized Besov space ($1\leq p\leq\infty$) with all possible regularity index ($s\geq s_c^p$), and for all these cases only taking a small assumption on the smallest scaling invariant norm ($\|\cdot\|_{\dot{B}^{1-\alpha}_{\infty,1}}$) is enough to extend globally.

Remark 1.2 In fact, from the proof of the Theorem 1.2, we shall see that the claim also holds if $\|\nabla\theta(t)\|_{L^\infty}$ is substituted by a smaller norm $\|\theta(t)\|_{\dot{B}^1_{\infty,\infty}}$. But because it does not have internal advantage (the regularity indexes of both norms are equal), we shall still use the clearer and simpler form.

Remark 1.3 For the critical case ($\alpha = 1$) in the critical space $\tilde{L}^\infty \dot{B}^{\frac{N}{p}}_{p,1} \cap L^1 \dot{B}^{1+\frac{N}{p}}_{p,1}$, we can similarly prove the local existence and the pointwise blowup criterion in terms of (6). Moreover, using the method in [4], we can show $\|\nabla\theta(t)\|_{L^\infty}$ is uniformly bounded and from blowup criterion $T^* = \infty$. But it seems very hard to obtain the uniform boundedness in the supercritical case.

Remark 1.4 The method we use can apply to case $\alpha = 0$. But since it is relatively ordinary and needs an additional statement in Theorem 1.2, we omit it here.

Notation: Throughout this article, C stands for a constant which may be different from line to line. The notation $X \lesssim Y$ means $X \leq CY$.

2. Preliminaries

In this preparatory section, we give the definition of the Besov spaces based on the Littlewood–Paley decomposition and introduce the Calderón–Zygmund singular integral and finally we review some important results that will be used in the following.

We start with the dyadic unity partition. Choose two non-negative radial functions $\chi, \varphi \in C^\infty(\mathbb{R}^N)$ be supported respectively in the ball $\{\xi \in \mathbb{R}^N : |\xi| \leq \frac{4}{3}\}$ and the shell $\{\xi \in \mathbb{R}^N : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^N,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

For every tempered distribution u we define the non-homogeneous Littlewood–Paley operators

$$\Delta_{-1}u := \chi(D)u;\ \forall j \in \mathbb{Z}^+ \cup \{0\},\ \Delta_j u := \varphi(2^{-j}D)u \quad S_j u := \sum_{-1\leq k\leq j-1} \Delta_k u,$$

the homogeneous Littlewood–Paley operators can be defined as follows

$$\forall j \in \mathbb{Z},\ \dot{\Delta}_j u := \varphi(2^{-j}D)u \quad \dot{S}_j u := \sum_{k\leq j-1} \dot{\Delta}_k u,$$

where S_j or $\dot{S}_j$ is the corresponding low-frequency cutoff operator.

Now we introduce the definition of Besov spaces through the above dyadic decomposition. Let $(p, q) \in [1, \infty]^2$, $s \in \mathbb{R}$, the non-homogeneous Besov space $B^s_{p,q}$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p,q}} := \left(2^{js}\|\Delta_j u\|_{L^p}\right)_{\ell^q(\{j\geq -1\}\cap\mathbb{Z})} < \infty,$$

where the ℓ^q norm denotes the concrete version of L^q norm. For the homogeneous case, we first denote by $\mathcal{S}'/\mathcal{P}$ the space of tempered distribution modulo polynomials. Then the homogeneous space $\dot{B}^s_{p,q}$ is the set of distribution $u \in \mathcal{S}'/\mathcal{P}$ such that

$$\|u\|_{\dot{B}^s_{p,q}} := \left(2^{js}\left\|\dot{\Delta}_j u\right\|_{L^p}\right)_{\ell^q(\mathbb{Z})} < \infty.$$

We point out that if $s > 0$ then $B^s_{p,q} = \dot{B}^s_{p,q} \cap L^p$, and $\|f\|_{B^s_{p,q}} \approx \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$.

Next we define the two kinds of coupled space-time Besov spaces. The first one $L^r([0, T], B^s_{p,q})$, abbreviated by $L^r_T B^s_{p,q}$, is defined in the usual sense. The second one, Chemin–Lerner's space-time space $\tilde{L}^r([0, T], B^s_{p,q})$, abbreviated by $\tilde{L}^r_T B^s_{p,q}$, is the set of tempered distribution u satisfying

$$\|u\|_{\tilde{L}^r_T B^s_{p,q}} := \left(2^{js}\left\|\Delta_j u\right\|_{L^r_T L^p}\right)_{\ell^q} < \infty.$$

Due to Minkowiski's inequality, we immediately obtain the following embeddings:

$$L^r_T B^s_{p,q} \hookrightarrow \tilde{L}^r_T B^s_{p,q}, \quad \text{if } q \geq r,$$
$$\tilde{L}^r_T B^s_{p,q} \hookrightarrow L^r_T B^s_{p,q}, \quad \text{if } r \geq q.$$

The homogeneous ones $L^r([0, T], \dot{B}^s_{p,q})$ and $\tilde{L}^r([0, T], \dot{B}^s_{p,q})$ can similarly extend.

We next introduce the classical Berstein's inequality [10].

LEMMA 2.1 *Let $\mathcal{B}$ be a ball, $\mathcal{R}$ be a ring, $0 \leq a \leq b \leq \infty$. Then $\forall k \in \mathbb{Z}^+ \cup \{0\}$, $\forall \lambda > 0$ there exists a constant $C > 0$ such that*

$$\sup_{|\alpha|=k}\left\|\partial^\alpha f\right\|_{L^b} \leq C\lambda^{k+N(\frac{1}{a}-\frac{1}{b})}\left\|f\right\|_{L^a} \quad \textit{if } \operatorname{supp}\mathcal{F}f \subset \lambda\mathcal{B},$$

$$C^{-1}\lambda^k\left\|f\right\|_{L^a} \leq \sup_{|\alpha|=k}\left\|\partial^\alpha f\right\|_{L^a} \leq C\lambda^k\left\|f\right\|_{L^a} \quad \textit{if } \operatorname{supp}\mathcal{F}f \subset \lambda\mathcal{R}.$$

Similar inequalities hold for the fractional derivative $|D|^\beta$.

The next lemma also concerns the Fourier supported functions (see e.g. [7]).

LEMMA 2.2 *Let $\mathcal{R}$ be a ring and $(\alpha, t, \lambda) \in (0, \infty)^3$. Then there exists two positive constants C, c such that*

$$\left\|e^{-t|D|^\alpha} f\right\|_{L^p} \leq Ce^{-ct\lambda^\alpha}\left\|f\right\|_{L^p} \quad \textit{if } \operatorname{supp}\mathcal{F}f \subset \lambda\mathcal{R}.$$

The classical Calderón–Zygmund singular integrals are operators of the form

$$T_{cz} f(x) := PV\int_{\mathbb{R}^N} \frac{\Omega(y')}{|y|^N} f(x-y)\mathrm{d}y = \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^N} f(x-y)\mathrm{d}y,$$

where Ω is defined on the unit sphere of $\mathbb{R}^N$, $\mathbb{S}^{N-1}$, is integrable with zero average and where $y' := \frac{y}{|y|} \in \mathbb{S}^{N-1}$. Clearly, the definition is meaningful for Schwartz functions. Moreover if $\Omega \in C^1(\mathbb{S}^{N-1})$, T_{cz} is L^p bounded, $1 < p < \infty$.

The general version (4) *of the relationship between v and θ is in fact ensured by the following result* (see *e.g.* [11]).

LEMMA 2.3 *Let $m \in C^\infty(\mathbb{R}^N \setminus \{0\})$ be a homogeneous function of degree* 0, *and T_m be the corresponding multiplier operator defined by $(T_m f)^\wedge = m\hat{f}$, then* *there exist $a \in \mathbb{C}$ and $\Omega \in C^\infty(\mathbb{S}^{N-1})$ with zero average such that for any Schwartz function f,*

$$T_m f = af + PV\frac{\Omega(x')}{|x|^N} * f.$$

Remark 2.1 Since $-\Delta v = (-\partial_1\partial_N\theta, \ldots, -\partial_{N-1}\partial_N\theta, \partial_1^2\theta + \cdots + \partial_{N-1}^2\theta)$, the Fourier multiplier of the operator $\mathcal{T}$ is rather clear. In fact, each component of its multiplier is the linear combination of the term like $\frac{\xi_i\xi_j}{|\xi|^2}$, $i, j \in \{1, 2, \ldots, N\}$, which of course belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$ and is homogeneous of degree 0.

Next for the transport-diffusion equation

$$(TD_\alpha)\begin{cases} \partial_t\theta + v\cdot\nabla\theta + \nu|D|^\alpha\theta = f \\ \theta(0, x) = \theta^0(x) \end{cases}$$

where θ is the unknown scalar function, we have the following regularization effect estimates.

PROPOSITION 2.4 *Let $s \in (-1, 1)$, $\alpha \in [0, 1), (p, r) \in [1, \infty]^2$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^s_{p,1})$ and v be a divergence-free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{Lip}(\mathbb{R}^N))$. We consider a smooth solution θ of the equation* (TD_α), *then there exists a constant C depending only on s and α such that for each $t \in \mathbb{R}^+$*

$$\nu^{\frac{1}{r}}\|\theta\|_{\tilde{L}^r_t\dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \leq Ce^{CV(t)}\left(\|\theta^0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1_t\dot{B}^s_{p,1}}\right), \tag{7}$$

where $V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau$.

Besides, if $v = \mathcal{C}(\theta) + \mathcal{S}(\theta)$ as (4) *shows, then for all $s \geq 1$ we also have*

$$\nu^{\frac{1}{r}}\|\theta\|_{\tilde{L}^r_t\dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \leq Ce^{C\tilde{V}(t)}\left(\|\theta^0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1_t\dot{B}^s_{p,1}}\right), \tag{8}$$

where $\widetilde{V}(t) := \int_0^t\left(\|\nabla v(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty}\right)\mathrm{d}\tau$.

Remark 2.2 The proof relies on the para-differential calculus combined with the Lagrangian coordinate method and two key commutator estimates. We here omit the proof, and for details see [4,7]. We just point out that the most fundamental and important result in the proof is a short-time estimate, that is, if $V(t) \leq C_0$, where C_0 is a chosen absolute constant, then

$$\begin{aligned}\nu^{\frac{1}{r}}\|\theta\|_{\tilde{L}^r_t\dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \lesssim &\sum_{j\in\mathbb{Z}}(1 - 2^{-ct2^{j\alpha}})^{\frac{1}{r}}2^{js}\left\|\theta^0_j\right\|_{L^p} + \|f\|_{L^1_t\dot{B}^s_{p,1}} \\ &+ \sum_{j\in\mathbb{Z}} 2^{js}\int_0^t \left\|[\dot{\Delta}_j, v\cdot\nabla]\theta(\tau)\right\|_{L^p}\mathrm{d}\tau.\end{aligned} \tag{9}$$

We also notice that the limitation in s only comes from the estimate of the commutator term $\sum_{j\in\mathbb{Z}} 2^{js}\left\|[\dot{\Delta}_j, v\cdot\nabla]\theta\right\|_{L^p}$, and especially the upper bound of s from the estimate of the term $\sum_{j\in\mathbb{Z}}\sum_{|k-j|\leq 4} 2^{js}\left\|\dot{\Delta}_j\left(\dot{\Delta}_k v\cdot\nabla\dot{S}_{k-1}\theta\right)\right\|_{L^p}$. Thus for our specific relationship between v and θ, one can breakthrough the ordinary limitation of s with a necessary modification.

The important maximal principle for (TD_α) equation is shown in [12].

Proposition 2.5 *Let v be* a smooth *divergence-free vector field and f be a smooth function. Assume that θ is the smooth solution of* (TD_α) *equation with* $\nu \geq 0$, $0 \leq \alpha \leq 2$, *then for* $p \in [1, \infty]$ *we have*

$$\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} \mathrm{d}\tau.$$

At last we recall a commutator estimate [13].

Lemma 2.6 *Let* $\rho_1 < 1$, $\rho_2 < 1$, $(p, r) \in [1, \infty]^2$, $\frac{1}{r} + \frac{1}{r'} = 1$ *and v be a divergence-free vector field in* $\mathbb{R}^N$. *In addition, assume that*

$$\rho_1 + \rho_2 + N \min(1, 1/p) > 0 \quad \textit{and} \quad \rho_1 + N/p > 0.$$

Then we have

$$\sum_{j \in \mathbb{Z}} 2^{j(\rho_1 + \rho_2 + \frac{N}{p} - 1)} \|[\dot{\Delta}_j, v \cdot \nabla]\theta\|_{L_t^1 L^p} \lesssim \|v\|_{\tilde{L}_t^r \dot{B}_{p,1}^{\frac{N}{p}+\rho_1}} \|\theta\|_{\tilde{L}_t^{r'} \dot{B}_{p,1}^{\frac{N}{p}+\rho_2}}.$$

3. Proof of Theorem 1.1

3.1. *Local existence*

The proof is based on the iterative method and Proposition 2.4. First, we obtain the global linear approximate solutions to the approximate system of (1) in the work spaces, then we show the uniform bounds of the solution sequence for some positive time T independent of the parameter n and further that the sequence is of Cauchy under an appropriate topology, which are enough to pass to the limit in the approximate system to get the local result.

Step 1: Global linear approximate solutions.

The approximate linear scheme is as follows: set $\theta_0(t, x) := e^{-\nu t|D|^\alpha}\theta^0(x)$, $v_0 := \mathcal{C}(\theta_0) + \mathcal{S}(\theta_0)$, and θ_{n+1} is the solution of the system

$$\begin{cases} \partial_t \theta_{n+1} + v_n \cdot \nabla \theta_{n+1} + \nu |D|^\alpha \theta_{n+1} = 0, \\ v_n := \mathcal{C}(\theta_n) + \mathcal{S}(\theta_n), \\ \theta_{n+1}(0, x) = \theta^0(x) \in \mathcal{Y}_{s,p}. \end{cases}$$

Since for $r \in [1, \infty]$, from Lemma 2.2

$$\begin{aligned} \|\theta_0\|_{\tilde{L}^r_{\mathrm{loc}}\left(\mathbb{R}^+, \dot{B}_{p,1}^{s+\frac{\alpha}{r}}\right)} &= \sum_{j \in \mathbb{Z}} 2^{j(s+\frac{\alpha}{r})} \left\| e^{-\nu\tau|D|^\alpha} \dot{\Delta}_j \theta^0(x) \right\|_{L^r([0,t], L^p)} \\ &\lesssim \sum_{j \in \mathbb{Z}} 2^{j(s+\frac{\alpha}{r})} \left\| e^{-c\nu\tau 2^{j\alpha}} \right\|_{L_t^r} \|\dot{\Delta}_j \theta^0(x)\|_{L^p} \lesssim \|\theta^0\|_{\dot{B}_{p,1}^s}, \end{aligned}$$

where $s \in \mathbb{R}$, $t \in \mathbb{R}^+$. Especially, for our special use $s \geq s_c^p$, the upper estimate is satisfied and if we let $r = 1$, $s = 1 - \alpha$, $p = \infty$ and $r = \infty$, $s = 0$, $p = \infty$ in, we have

$$\|\theta_0\|_{L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^1_{\infty,1})} \lesssim \|\theta^0\|_{\dot{B}^{1-\alpha}_{\infty,1}},$$

$$\|\theta_0\|_{\tilde{L}^\infty_{\mathbb{R}^+} \dot{B}^0_{\infty,1}} \lesssim \|\theta^0\|_{\dot{B}^0_{\infty,1}}.$$

Further, for $s \geq 1 + \frac{N}{p} - \alpha > 0$, then

$$\begin{aligned}\|\theta_0\|_{\tilde{L}^\infty_{\mathbb{R}^+} B^s_{p,1}} &= 2^{-s}\|\Delta_{-1}\theta_0\|_{L^\infty(\mathbb{R}^+, L^p)} + \sum_{j\geq 0} 2^{js}\|\Delta_j\theta_0\|_{L^r(\mathbb{R}^+, L^p)} \\ &\lesssim \|e^{-\nu t|D|^\alpha}\theta^0\|_{L^\infty(\mathbb{R}^+, L^p)} + \|\theta_0\|_{\tilde{L}^\infty(\mathbb{R}^+, \dot{B}^s_{p,1})} \\ &\lesssim \|\theta^0\|_{L^p} + \|\theta^0\|_{\dot{B}^s_{p,1}} \approx \|\theta^0\|_{B^s_{p,1}}.\end{aligned}$$

Thus

$$\begin{aligned}&\|\theta_0\|_{L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^{s+\alpha}_{p,1})} + \|\theta_0\|_{\tilde{L}^\infty_{\mathbb{R}^+}\mathcal{Y}_{s,p}} \\ &\quad = \|\theta_0\|_{L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^{s+\alpha}_{p,1})} + \begin{cases} \|\theta_0\|_{\tilde{L}^\infty_{\mathbb{R}^+} B^s_{p,1}} \\ \|\theta_0\|_{\tilde{L}^\infty_{\mathbb{R}^+}(B^s_{\infty,1}\cap \dot{B}^0_{\infty,1})} \end{cases} \lesssim \begin{cases} \|\theta^0\|_{B^s_{p,1}}, \\ \|\theta^0\|_{B^s_{\infty,1}\cap\dot{B}^0_{\infty,1}}, \end{cases}\end{aligned}$$

hence $\theta_0 \in \tilde{L}^\infty(\mathbb{R}^+; \mathcal{Y}_{s,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{s+\alpha}_{p,1})$.

Then suppose for some $n \in \mathbb{Z}^+ \cup \{0\}$ we have $\theta_n \in \tilde{L}^\infty(\mathbb{R}^+; \mathcal{Y}_{s,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{s+\alpha}_{p,1})$, we shall deduce that θ_{n+1} also belongs to this space.

For all $s \geq s^p_c = 1 + \frac{N}{p} - \alpha$, we directly have $L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^{s+\alpha}_{\infty,1}) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^+, \dot{B}^1_{\infty,1})$, where $\hookrightarrow$ denotes continuous embedding, thus $\theta_n \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^1_{\infty,1})$. From Remark 2.1 the Fourier multiplier of the operator $\mathcal{T} : \theta_n \to v_n$ has singularity only at the origin point, thus it maps $\dot{B}^1_{\infty,1}$ into $\dot{B}^1_{\infty,1}$, we further have $v_n \in L^1(\mathbb{R}^+; \dot{B}^1_{\infty,1})$. Then using Proposition 2.4 we obtain for any $t \in \mathbb{R}^+$

$$\begin{aligned}\|\theta_{n+1}\|_{\tilde{L}^r_t \dot{B}^{s+\frac{\alpha}{r}}_{p,1}} &\leq C e^{C\tilde{V}(t)}\|\theta^0\|_{\dot{B}^s_{p,1}} \\ &\lesssim e^{C(\|v_n\|_{L^1_t\dot{B}^1_{\infty,1}} + \|\theta_n\|_{L^1_t\dot{B}^1_{\infty,1}})}\|\theta^0\|_{\dot{B}^s_{p,1}} \\ &\lesssim \|\theta^0\|_{\dot{B}^s_{p,1}},\end{aligned} \tag{10}$$

where the estimate holds for all $s > -1$, which contains our choice $s \geq s^p_c$, and especially,

$$\|\theta_{n+1}\|_{\tilde{L}^\infty_{\mathbb{R}^+}\dot{B}^0_{\infty,1}} \lesssim \|\theta^0\|_{\dot{B}^0_{\infty,1}}.$$

Due to the fact that θ_{n+1} satisfies a standard transport-diffusion equation, from Lemma 2.5. $\|\theta_{n+1}(t)\|_{L^p} \leq \|\theta^0(x)\|_{L^p}$, $\forall p \in [1, \infty]$, and a high–low frequency decomposition leads for all $s \geq s^p_c$

$$\begin{aligned}\|\theta_{n+1}\|_{\tilde{L}^\infty_{\mathbb{R}^+} B^s_{p,1}} &\lesssim \|\Delta_{-1}\theta_{n+1}\|_{L^\infty(\mathbb{R}^+, L^p)} + \|\theta_{n+1}\|_{\tilde{L}^\infty(\mathbb{R}^+, \dot{B}^s_{p,1})} \\ &\lesssim \|\theta^0\|_{L^p} + \|\theta^0\|_{\dot{B}^s_{p,1}} \lesssim \|\theta^0\|_{B^s_{p,1}},\end{aligned} \tag{11}$$

thus $\theta_{n+1} \in \tilde{L}^\infty(\mathbb{R}^+; \mathcal{Y}_{s,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{s+\alpha}_{p,1})$.

Hence, the standard mathematical induction method concludes $\forall n \in \mathbb{Z}^+ \cup \{0\}$

$$\theta_n \in \tilde{L}^\infty(\mathbb{R}^+; \mathcal{Y}_{s,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{s+\alpha}_{p,1}).$$

Step 2: Uniform bounds.

To begin with, we set $V_n(t) := \|\nabla v_n\|_{L^1_t L^\infty}$ and $\Theta_n(t) := \|\theta_n\|_{L^1_t \dot{B}^1_{\infty,1}}$. Then we intend to obtain the uniform bounds of $(v_n, \theta_n)_{n\in\mathbb{N}}$ (with respect to n) for some positive time T independent of n. For this purpose, we divide into two parts concerning the regularity index s.

- First case: Subcritical index $s > s_c^p = 1 + \frac{N}{p} - \alpha$.

In this case we notice that there exists $r > 1$ such that $s \geq 1 + \frac{N}{p} - \frac{\alpha}{r}$. Then from the fact $V_n(t) \lesssim \Theta_n(t)$, the Hölder's inequality and estimate (7) ($s = 1 - \frac{\alpha}{r} < 1, p = \infty$) we have

$$\Theta_n(t) \leq t^{\frac{1}{r'}} \|\theta_n\|_{L^r_t \dot{B}^1_{\infty,1}} \leq C t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} e^{C\Theta_{n-1}(t)} \tag{12}$$

where r' is the dual number of r.
Then we deduce that there exists some $\widetilde{C}(= 2C'\eta)$, $\eta > 0$ such that for all $n \in \mathbb{Z}^+ \cup \{0\}$

$$t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \leq \eta \Rightarrow \Theta_n(t) \leq \widetilde{C}. \tag{13}$$

We also use the ordinary mathematical induction to prove. First for all t satisfying $t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \leq \eta$, where η is a small constant chosen later, we have

$$\begin{aligned} \Theta_0(t) &= \left\|e^{-\nu\tau|D|^\alpha}\theta^0\right\|_{L^1_t \dot{B}^1_{\infty,1}} \\ &\leq t^{\frac{1}{r'}} \left\|e^{-\nu\tau|D|^\alpha}\theta^0\right\|_{L^r_t \dot{B}^1_{\infty,1}} \leq C_1 t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \leq C_1\eta. \end{aligned}$$

Furthermore from (12), we iterate forward as follows

$$\Theta_1(t) \leq C_2 t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} e^{C_3\Theta_0(t)} \leq C_2\eta e^{C_3C_1\eta} \leq 2C'\eta,$$

$$\Theta_2(t) \leq C_2\eta e^{C_3\Theta_1(t)} \leq C_2\eta e^{C_3 2C'\eta} \leq 2C'\eta,$$

where $C' := \max\{C_1, C_2\}$ and where both the last inequalities in the upper two estimates hold as long as η is small enough such that $\min\{e^{C_3C_1\eta}, e^{2C_3C'\eta}\} \leq 2$. We also note that C_1, C_2, C_3 are absolute constants independent of n. Thus for this η and $\widetilde{C} = 2C'\eta$, a standard induction argument will conclude the statement.

Moreover, since we have bounded uniformly the quantity $\Theta_n(t)$, as estimating (10) and (11), we have

$$\|\theta_n\|_{\tilde{L}^r_T \dot{B}^{s+\frac{\alpha}{r}}_{p,1}} + \|\theta_n\|_{\tilde{L}^\infty_T \mathcal{Y}_{s,p}} \leq C\|\theta^0\|_{\mathcal{Y}_{s,p}}, \tag{14}$$

where C is independent of the parameter n and where

$$T := \sup\left\{t > 0 : t^{\frac{1}{r'}} \|\theta^0\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \leq \eta\right\}.$$

- Second case: Critical index $s = s_c^p = 1 + \frac{N}{p} - \alpha$.

The critical case is somewhat more subtle in proof. The key is to obtain the uniform boundedness of $V_n(t)$ ($\Theta_n(t)$) for some positive time T. We use the induction method to complete the proof.

From the Remark 2.2, if for some $n \in \mathbb{Z}^+ \cup \{0\}$ the condition $V_n(t) \le C_0$ is satisfied, then applying the estimate (9) ($s = 1-\alpha, r = 1, p = \infty$) we have

$$\Theta_{n+1}(t) \lesssim \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}}) 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} + \sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)} \left\|[\dot\Delta_j, v_n\cdot\nabla]\theta_{n+1}\right\|_{L^1_t L^\infty},$$

where $\theta_j^0 := \dot\Delta_j\theta^0$. From Lemma 2.6 (taking $\rho_1 = \rho_2 = 1-\frac{\alpha}{2} < 1, p = \infty, r = 2$), the second term of the right-hand side can be estimated as

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)} \left\|[\dot\Delta_j, v_n\cdot\nabla]\theta_{n+1}\right\|_{L^1_t L^\infty} &\lesssim \|v_n\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \\ &\lesssim \|\theta_n\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}}. \end{aligned} \tag{15}$$

Hence we get

$$\Theta_{n+1}(t) \lesssim \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}}) 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} + \|\theta_n\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}}.$$

Also by virtue of (9) ($s = 1-\alpha, r = 2, p = \infty$), we obtain

$$\begin{aligned}&\left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \\ &\lesssim \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}})^{\frac12} 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} + \sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)} \left\|[\dot\Delta_j, v_n\cdot\nabla]\theta_{n+1}\right\|_{L^1_t L^\infty} \\ &\lesssim \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}})^{\frac12} 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} + \|\theta_n\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}}.\end{aligned}$$

Combining these two estimates, we get

$$\begin{aligned}\Theta_{n+1}(t) + \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \le &C_4 \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}})^{\frac12} 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} \\ &+ C_4 \|\theta_n\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \left\|\theta_{n+1}\right\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}}.\end{aligned} \tag{16}$$

For $n = 0$, since $\|\theta_0\|_{\tilde L^2_{\mathbb{R}^+} \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} + \Theta_0(\infty) \lesssim \left\|\theta^0\right\|_{\dot B^{1-\alpha}_{\infty,1}} \lesssim \mathcal{Y}_{s,p}$, we can choose a sufficiently small t such that

$$C_4 \|\theta_0\|_{\tilde L^2_t \dot B^{1-\frac{\alpha}{2}}_{\infty,1}} \le \frac12 \quad \text{and} \quad V_0(t) \le C_5\Theta_0(t) \le C_0. \tag{17}$$

Since from the Lebesgue dominated convergence theorem,

$$\lim_{t\to 0+} \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}})^{\frac12} 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} = 0, \tag{18}$$

thus the smallness of t can also be expressed in the following sense: Let $\eta > 0$ be an sufficiently small absolute constant chosen later, denoted by

$$T := \sup\left\{ t > 0 : C_4 \sum_{j\in\mathbb{Z}} (1-2^{-ct2^{j\alpha}})^{\frac12} 2^{j(1-\alpha)} \left\|\theta_j^0\right\|_{L^\infty} \le \eta \right\}, \tag{19}$$

then for all $t \leq T$, the estimates (17) are satisfied. Thus from (16), we obtain

$$\Theta_1(t) + \|\theta_1\|_{\tilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq 2\eta.$$

We next iterate forward, and since $V_1(t) \leq C_5\Theta_1(t) \leq 2C_5\eta$ and $C_4\|\theta_1\|_{\tilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq 2C_4\eta$, as long as $\eta > 0$ is sufficiently small (precisely s.t. $2C_5\eta \leq C_0$, $2C_4\eta \leq \frac{1}{2}$ and (17) holds), we have

$$C_4\|\theta_1\|_{\tilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq \frac{1}{2} \quad \text{and} \quad V_1(t) \leq C_0.$$

Thus we can use (16) again and get

$$\Theta_2(t) + \|\theta_2\|_{\tilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq 2\eta.$$

We also note that C_4 and C_5 are absolute constants independent of n. For this suitable η (also n independent), by induction we have $\forall n \in \mathbb{Z}^+ \cup \{0\}$ and $\forall t \leq T$

$$\Theta_n(t) + \|\theta_n\|_{\tilde{L}^2_t \dot{B}^{1-\frac{\alpha}{2}}_{\infty,1}} \leq 2\eta. \tag{20}$$

In a similar way as estimating (10) and (11), we have

$$\|\theta_n\|_{\tilde{L}^r_T \dot{B}^{s^p_c+\frac{\alpha}{r}}_{p,1}} + \|\theta_n\|_{\tilde{L}^\infty_T \mathcal{Y}_{s^p_c,p}} \leq C\|\theta^0\|_{\mathcal{Y}_{s^p_c,p}}. \tag{21}$$

Until now, for both cases we have obtained the uniform estimates of θ_n in the work spaces. Next, for all $s \geq s^p_c$ we treat the corresponding problem for the velocity v_n. Since the operator $\mathcal{T} : \theta_n \to v_n$ maps the homogeneous $\dot{B}^s_{p,1}$ into itself, naturally we have $\forall s \geq s^p_c$ and $\forall p \in [1, \infty]$

$$\|v_n\|_{\tilde{L}^r_T \dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \lesssim \|\theta_n\|_{\tilde{L}^r_T \dot{B}^{s+\frac{\alpha}{r}}_{p,1}} \lesssim \|\theta^0\|_{\mathcal{Y}_{s,p}}.$$

Furthermore, in a similar way as estimating (11) and due to the L^p bounded property of $\mathcal{T}$ we obtain for $p \in (1, \infty)$

$$\begin{aligned}
\|v_n\|_{\tilde{L}^\infty_T B^s_{p,1}} &\leq \|\Delta_{-1}v_n\|_{L^\infty_T L^p} + \|v_n\|_{\tilde{L}^\infty_T \dot{B}^s_{p,1}} \\
&\lesssim \|\theta_n\|_{L^\infty_T L^p} + \|\theta_n\|_{\tilde{L}^\infty_T \dot{B}^s_{p,1}} \\
&\lesssim \|\theta^0\|_{L^p} + \|\theta^0\|_{\dot{B}^s_{p,1}} \lesssim \|\theta^0\|_{B^s_{p,1}},
\end{aligned}$$

and similarly for $p = \infty$,

$$\begin{aligned}
\|v_n\|_{\tilde{L}^\infty_T B^s_{\infty,1}} &\lesssim \|v_n\|_{L^\infty_T L^\infty} + \|v_n\|_{\tilde{L}^\infty_T \dot{B}^s_{\infty,1}} \\
&\lesssim \|\theta_n\|_{\tilde{L}^\infty_T \dot{B}^0_{\infty,1}} + \|\theta_n\|_{\tilde{L}^\infty_T \dot{B}^s_{\infty,1}} \\
&\lesssim \|\theta^0\|_{B^s_{\infty,1} \cap \dot{B}^0_{\infty,1}},
\end{aligned}$$

and for $p=1$, we cannot bound the quantity $\|v_n\|_{\tilde{L}^\infty_T B^s_{1,1}}$ uniformly, but alteratively from the Calderón–Zygmund theorem and the embedding $B^s_{1,1} \hookrightarrow L^{p_1}$ for all $p_1 > 1$ we have

$$\begin{aligned}\|v_n\|_{\tilde{L}^\infty_T \dot{B}^s_{1,1}} + \|v_n\|_{L^\infty_T L^{p_1}} &\leq \|\theta_n\|_{\tilde{L}^\infty_T \dot{B}^s_{1,1}} + \|\theta_n\|_{L^\infty_T L^{p_1}} \\ &\lesssim \|\theta_n\|_{\tilde{L}^\infty_T B^s_{1,1}} \lesssim \left\|\theta^0\right\|_{B^s_{1,1}}.\end{aligned}$$

Step 3: *Strong convergence*

We shall prove that $(\theta_n, v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\tilde{L}^\infty_t \dot{B}^0_{\infty,1}$. Notice that for all $s \geq s^p_c$, $\mathcal{Y}_{s,p} \hookrightarrow \dot{B}^0_{\infty,1}$ ($p=\infty$, obvious; $p<\infty, \mathcal{Y}_{s,p} \hookrightarrow B^{\frac{n}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{n}{p}}_{p,1} \hookrightarrow \dot{B}^0_{\infty,1}$).

For $n, m \in \mathbb{Z}^+ \cup \{0\}$, $n > m$, let $\theta^{n,m} := \theta_{n+1} - \theta_{m+1}$ and $v^{n,m} := v_n - v_m$, then the difference function satisfies

$$\begin{cases} \partial_t \theta^{n,m} + v_n \cdot \nabla\theta^{n,m} + \nu|D|^\alpha \theta^{n,m} = -v^{n,m}\cdot\nabla\theta_{m+1}, \\ \theta^{n,m}(0,x) = 0. \end{cases}$$

From the estimate (7), we deduce

$$\|\theta^{n,m}\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}} \lesssim e^{C\Theta_n(t)} \int_0^t \left\| v^{n,m}\cdot\nabla\theta_{m+1}(\tau)\right\|_{\dot{B}^0_{\infty,1}} \mathrm{d}\tau. \tag{22}$$

Thanks to Bony's decomposition, we have (see e.g. [6,7])

$$\left\| v^{n,m}\cdot\nabla\theta_{m+1}\right\|_{\dot{B}^0_{\infty,1}} \lesssim \|v^{n,m}\|_{\dot{B}^0_{\infty,1}} \left\|\theta_{m+1}\right\|_{\dot{B}^1_{\infty,1}}, \tag{23}$$

and since the operator $\mathcal{T} : \theta^{n-1,m-1} \to v^{n,m}$ continuously maps $\dot{B}^1_{\infty,1}$ into $\dot{B}^1_{\infty,1}$, then

$$\left\| v^{n,m}\cdot\nabla\theta_{m+1}(\tau)\right\|_{\dot{B}^0_{\infty,1}} \lesssim \left\|\theta^{n-1,m-1}\right\|_{\dot{B}^0_{\infty,1}} \left\|\theta_{m+1}\right\|_{\dot{B}^1_{\infty,1}}.$$

Thus we obtain

$$\|\theta^{n,m}\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}} \lesssim e^{C\Theta_n(t)} \left\|\theta^{n-1,m-1}\right\|_{L^\infty_t \dot{B}^0_{\infty,1}} \Theta_{m+1}(t).$$

Taking advantage of (13) and (20) (for all $n, \Theta_n(t) \leq \mathrm{const}\cdot\eta$), one can further choose η sufficiently small such that

$$\|\theta^{n,m}\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}} \leq \delta \left\|\theta^{n-1,m-1}\right\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}},$$

with $\delta < 1$. By iteration we have

$$\|\theta^{n,m}\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}} \leq \delta^{m+1} \|\theta_{n-m} - \theta_0\|_{\tilde{L}^\infty_t \dot{B}^0_{\infty,1}} \leq C\delta^{m+1} \left\|\theta^0\right\|_{\dot{B}^0_{\infty,1}}.$$

Thus $(\theta_n)_{n\in\mathbb{Z}^+\cup\{0\}}$ is of Cauchy in $\tilde{L}^\infty_t \dot{B}^0_{\infty,1}$, hence there exist $\theta \in \tilde{L}^\infty_t \dot{B}^0_{\infty,1}$ and $v := \mathcal{C}(\theta) + \mathcal{S}(\theta) \in \tilde{L}^\infty_t \dot{B}^0_{\infty,1}$ such that (θ_n, v_n) strongly converges to (θ, v) in $\tilde{L}^\infty_t \dot{B}^0_{\infty,1}$. Using Fatou's lemma, from the uniform estimates in *Step 2* we get

$$\theta \in \tilde{L}^\infty_t \mathcal{Y}_{s,p} \cap L^1_t \dot{B}^{s+\alpha}_{p,1},$$

$$v \in L^1_t \dot{B}^{s+\alpha}_{p,1} \cap \begin{cases} \tilde{L}^\infty_t \mathcal{Y}_{s,p}, & p \in (1,\infty] \\ \tilde{L}^\infty_t \dot{B}^s_{1,1} \cap \tilde{L}^\infty_t L^{p_1}, & p=1,\ \forall p_1 > 1. \end{cases}$$

This information allow us to pass to limit in the equation.

The continuity in time issue, that is $\theta \in C_t^0 \mathcal{Y}_{s,p}$, is a standard process and one can refer to [2] for a detailed proof.

Step 4: Uniqueness

We prove the uniqueness issue in the space $Y_T := \tilde{L}_T^\infty \dot{B}^0_{\infty,1} \cap L_T^1 \dot{B}^1_{\infty,1}$, where T is arbitary possible finite constant. Also notice that for all $s \geq s_c^p$ the space $\tilde{L}_T^\infty \mathcal{Y}_{s,p} \cap L_T^1 \dot{B}^{s+\alpha}_{p,1}$ is continuously embedded in Y_T.

Let $(\theta, v), (\theta', v') \in Y_T$ be two solutions of the system (DPM$_\alpha$) with the same initial data, and denote $(\delta\theta, \delta v) := (\theta - \theta', v - v')$, then the difference equation becomes

$$\begin{cases} \partial_t \delta\theta + v \cdot \nabla \delta\theta + \nu |D|^\alpha \delta\theta = -\delta v \cdot \nabla \theta', \\ \delta\theta(0, x) = 0. \end{cases}$$

Also using Proposition 2.4 and the estimate as (23), we have

$$\|\delta\theta\|_{\tilde{L}_T^\infty \dot{B}^0_{\infty,1}} \lesssim e^{C\|\theta\|_{L_T^1 \dot{B}^1_{\infty,1}}} \int_0^T \|\delta\theta\|_{\tilde{L}_t^\infty \dot{B}^0_{\infty,1}} \left\|\theta'(t)\right\|_{\dot{B}^1_{\infty,1}} \mathrm{d}t.$$

Then Gronwall's inequality ensures $\|\delta\theta\|_{\tilde{L}_T^\infty \dot{B}^0_{\infty,1}} = 0$, thus $(\theta, v) = (\theta', v')$.

3.2. *Global existence*

Now we intend to prove that for appropriate sufficiently small initial data the system (DPM$_\alpha$) generates a global solution.

We notice that in order to obtain global result for small data we use the iterative process only once. From the natural blowup criterion in the iterative procedure, it suffices to bound *a priori* the quantity $V(t) := \|\nabla v\|_{L_t^1 L^\infty}$ for all $t \in \mathbb{R}^+$.

We apply Proposition 2.4 ($s = 1 - \alpha, r = 1, p = \infty$), then

$$V(t) \leq C\|\theta\|_{L_t^1 \dot{B}^1_{\infty,1}} \leq C\nu^{-1} \left\|\theta^0\right\|_{\dot{B}^{1-\alpha}_{\infty,1}} e^{CV(t)}.$$

Since $V(0) = 0$ and $V(t)$ is continuous in time, using the standard continuity method (in a spirit as obtaining (12)), we shall conclude that there exist two absolute constants $\bar{C}$, $\zeta > 0$ such that

$$\left\|\theta^0\right\|_{\dot{B}^{1-\alpha}_{\infty,1}} \leq \nu\zeta \Rightarrow V(t) \leq \overline{C} \left\|\theta^0\right\|_{\dot{B}^{1-\alpha}_{\infty,1}}, \forall t \in \mathbb{R}^+.$$

We also note that $\overline{C} = \text{const} \cdot \nu^{-1}$ is an absolute constant independent of t. Thus this means the global result.

4. Proof of Theorem 1.2

As a byproduct of the local existence part, we prove the blowup criteria.

For the $s > s_c^p = 1 + \frac{N}{p} - \alpha$ case, due to the local existence theory and especially (13), if $T^* < \infty$, then necessarily we have

$$\liminf_{t \to T^*} (T^* - t)^{\frac{1}{r}} \left\|\theta(t)\right\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \geq \epsilon_0, \tag{24}$$

where ϵ_0 is an absolute positive constant, otherwise the solution can continue past T^*. Next, by a direct decomposition and using maximal principle Proposition 2.5 and Bernstein's inequality, we have for $t < T^*$

$$\begin{aligned}\|\theta(t)\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} &= \sum_{j\le M} 2^{j(1-\frac{\alpha}{r})}\|\dot{\Delta}_j\theta(t)\|_{L^\infty} + \sum_{j>M} 2^{j(1-\frac{\alpha}{r})}\|\dot{\Delta}_j\theta(t)\|_{L^\infty}\\ &\lesssim \sum_{j\le M} 2^{j(1-\frac{\alpha}{r})}\|\theta(t)\|_{L^\infty} + \sum_{j>M} 2^{-j\frac{\alpha}{r}}\|\dot{\Delta}_j\nabla\theta(t)\|_{L^\infty}\\ &\lesssim 2^{M(1-\frac{\alpha}{r})}\|\theta^0\|_{L^\infty} + 2^{-M\frac{\alpha}{r}}\|\nabla\theta(t)\|_{L^\infty}\end{aligned}$$

choosing appropriate M such that $2^{M(1-\frac{\alpha}{r})}\|\theta^0\|_{L^\infty} \approx 2^{-M\frac{\alpha}{r}}\|\nabla\theta(t)\|_{L^\infty}$, then

$$\|\theta(t)\|_{\dot{B}^{1-\frac{\alpha}{r}}_{\infty,1}} \lesssim \|\nabla\theta(t)\|^{1-\frac{\alpha}{r}}_{L^\infty},$$

thus we can rewrite the blowup criterion (24) to get the desired result

$$\liminf_{t\to T^*}(T^* - t)^{\frac{1}{r}}\|\nabla\theta(t)\|^{1-\frac{\alpha}{r}}_{L^\infty} \ge \epsilon_0'.$$

For the critical case $s = s_c^p$, similarly, if $T^* < \infty$, then from (19) we necessarily have

$$\liminf_{t\to T^*}\sum_{j\in\mathbb{Z}}(1 - e^{-c(T^*-t)2^{j\alpha}})^{\frac{1}{2}}2^{j(1-\alpha)}\|\dot{\Delta}_j\theta(t)\|_{L^\infty} \ge \epsilon_0.$$

By Lebesgue dominated convergence theorem, it leads to a direct blowup criterion $\|\theta\|_{\tilde{L}^\infty_{T^*}\dot{B}^{1-\alpha}_{\infty,1}} = \infty$. Further, using mean value theorem, maximal principle and Bernstein's inequality, we get

$$\begin{aligned}\epsilon_0 &\le \liminf_{t\to T^*}\Bigg\{\sum_{j\le M}(1 - e^{-c(T^*-t)2^{j\alpha}})^{\frac{1}{2}}2^{j(1-\alpha)}\|\dot{\Delta}_j\theta(t)\|_{L^\infty}\\ &\quad + \sum_{j>M}(1 - e^{-c(T^*-t)2^{j\alpha}})^{\frac{1}{2}}2^{j(1-\alpha)}\|\dot{\Delta}_j\theta(t)\|_{L^\infty}\Bigg\}\\ &\lesssim \liminf_{t\to T^*}\Bigg\{\sum_{j\le M}(T^* - t)^{\frac{1}{2}}2^{j(1-\frac{\alpha}{2})}\|\theta^0\|_{L^\infty} + \sum_{j>M}2^{-j\alpha}\|\dot{\Delta}_j\nabla\theta(t)\|_{L^\infty}\Bigg\}\\ &\lesssim \liminf_{t\to T^*}\Bigg\{(T^* - t)^{\frac{1}{2}}2^{M(1-\frac{\alpha}{2})}\|\theta^0\|_{L^\infty} + 2^{-M\alpha}\|\nabla\theta(t)\|_{L^\infty}\Bigg\}.\end{aligned}$$

Also choosing suitable M such that both last terms nearly equals, we obtain the desired result

$$\liminf_{t\to T^*}(T^* - t)^{\alpha}\|\nabla\theta(t)\|^{2-\alpha}_{L^\infty} \ge \epsilon_0'.$$

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