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Wellposedness and zero microrotation viscosity limit of the 2D micropolar fluid equations

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In this paper, we consider the 2D micropolar fluid equations in the whole space $\mathbb{R}^2$. We prove the global wellposedness of the system with rough initial data and show the vanishing microrotation viscosity limit in the case of zero kinematic viscosity or zero angular viscosity. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

The 2D incompressible micropolar fluid flow in the whole space is governed by the following equations (cf. [14])

$$\begin{cases} \partial_t u-(\nu+\kappa)\Delta u+u\cdot\nabla u+\nabla P=2\kappa\nabla\times\omega & in \quad D\times\mathbb{R}^+, \\ \partial_t\omega-\gamma\Delta\omega+u\cdot\nabla\omega+4\kappa\omega=2\kappa\nabla\times u & in \quad D\times\mathbb{R}^+, \\ \nabla\cdot u=0 & in \quad D\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

with the initial data

$$u|_{t=0}=u^0, \quad \omega|_{t=0}=\omega^0, \tag{1.2}$$

where $D=\mathbb{R}^2$, $u=(u_1,u_2)$ is the velocity field, P is the pressure, scalar ω denotes the microrotation field. Non-negative constants ν,κ,γ stand for the viscosity coefficients, ν is the Newtonian kinematic viscosity, κ is the microrotation viscosity and γ is called as the angular viscosity. $\nabla=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$, and

$$\nabla\times u=-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}, \quad \nabla\cdot u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}, \quad \nabla\times\omega=\left(\frac{\partial\omega}{\partial x_2},-\frac{\partial\omega}{\partial x_1}\right).$$

Here, the density of the fluid is assumed to be 1.

The 2D micropolar fluid motion is a special case of the corresponding 3D motion, that is,

$$(x,c_0)\in\mathbb{R}^3, \quad \big(u_1(x,c_0),u_2(x,c_0),0\big)\to u(x), \quad \big(0,0,\omega_3(x,c_0)\big)\to\omega(x).$$

The 3D micropolar fluid model, firstly introduced by Eringen in [9], is an essential generalization of the known Navier–Stokes/Euler model in the sense that the microstructure of the fluid is taken into account. It may better represent the fluids consisting of randomly oriented particles in a medium, for example, liquid crystals made up of dumbbell molecules.

There have been many works concerning the existence and uniqueness problems of the micropolar fluid model, for example, [4, 12, 13, 16, 17, 19, 21] and reference therein, especially in the 2D whole space case, the uniqueness of the global weak solutions and the global wellposedness of the smooth solution have been obtained in [13] when $\nu>0$ and $\gamma>0$. Dong and Zhang in [8] showed the global wellposedness of the smooth solution in the 2D whole space when $\gamma=0$ and $\nu>0$.

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When the microrotation viscosity κ equals to zero, then Equation (1.1) reduces to the following system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla P = 0 \\ \partial_t \omega + u\cdot\nabla\omega - \gamma\Delta\omega = 0 \\ \mathrm{div} u = 0. \end{cases} \tag{1.3}$$

Note that the equations of the velocity field in Equation (1.1) reduce to the incompressible Navier–Stokes/Euler system and are irrelevant with the microrotation field. Thus, the size of the microrotation viscosity may allow us to measure, in a certain sense, the deviation of flows of micropolar fluids from that of the Navier–Stokes/Euler model. In [16], Payne and Straughan proved the local convergence result when $\nu,\gamma>0$ and showed that the convergence rate is at least $O(\kappa)$ for the forward in time. Here, the local convergence is meant that for every $t\geq 0$, the solution $(u_\kappa(t),\omega_\kappa(t))$ of Equation (1.1) converges in L^2-norm to the solution $(u(t),\omega(t))$ of Equation (1.3), under the same initial data (1.2). In [14], in the case of 2D bounded domain D with homogeneous boundary condition and $\nu,\gamma>0$, Łukaszewicz moreover proved the global convergence result when ν was large enough, that is, the convergence in the above is uniform in t.

In this paper, we aim at proving the local convergence result of the 2D micropolar fluid systems (1.1)–(1.2) for the limiting cases (i.e., $\nu=0$ or $\gamma=0$). Compared with both positive cases, the situation is more delicate: when $\nu=0$, we have to consider the term like $\kappa\,\|\Delta u_\kappa\|_{L^1_tL^2}$ in the convergence part, and it seems hard to get the the appropriate a priori estimate simply from the classical L^2 energy inequality (e.g., Equation (3.2) below); when $\gamma=0$, because there is no smoothing effect in the equation of ω, we, here, only hope to get the global wellposedness of the strong solution with the rough initial regularity. Hence, we first consider the global wellposedness of system (1.1) with rough initial data and obtain some good a priori estimates, and then we show the zero microrotation viscosity limit of Equations (1.1)–(1.2). Precisely, our results are listed as follows:

Theorem 1.1

(1) Let $\kappa>0$, $\nu\geq 0$, $\gamma>0$, u^0 be a divergence-free vector field belonging to H^1 and $\omega^0\in L^2$. Then the 2D micropolar fluid systems (1.1)–(1.2) has a unique global solution (u,ω) such that for every $\sigma\in[1,2[$

$$u\in\mathcal{C}(\mathbb{R}^+,H^1)\cap\widetilde{L}^\sigma_{\mathrm{loc}}(\mathbb{R}^+,B^1_{\infty,1}),\quad \omega\in\mathcal{C}(\mathbb{R}^+,L^2)\cap\widetilde{L}^\sigma_{\mathrm{loc}}(\mathbb{R}^+,B^1_{2,1}).$$

(2) Let $\kappa>0$, $\nu\geq 0$, $\gamma=0$, u^0 be a divergence-free vector field belonging to H^1 and $\omega^0\in L^2\cap B^0_{\infty,1}$. Then the 2D micropolar fluid systems (1.1)–(1.2) has a unique global solution (u,ω) such that for every $\sigma\in[1,2[$

$$u\in\mathcal{C}(\mathbb{R}^+,H^1)\cap\widetilde{L}^\sigma_{\mathrm{loc}}(\mathbb{R}^+,B^1_{\infty,1}),\quad \omega\in\mathcal{C}(\mathbb{R}^+,L^2\cap B^0_{\infty,1}).$$

Theorem 1.2

Let $\nu\geq 0$, $\gamma\geq 0$, $(u^0,\omega^0)\in\Pi$ with

$$\Pi:=\begin{cases} H^1\times H^{1+\epsilon_0},\ \epsilon_0\in]0,1[, & \text{when}\quad \nu>0,\ \gamma=0, \\ B^2_{2,1}\times B^0_{2,1}, & \text{when}\quad \nu=0,\ \gamma>0, \\ B^2_{2,1}\times\left(B^1_{2,1}\cap B^1_{\infty,1}\right), & \text{when}\quad \nu=0,\ \gamma=0, \end{cases}$$

and (u_κ,ω_κ) and (u,ω) be the corresponding unique global solutions of Equations (1.1)–(1.2) and Equations (1.3)–(1.2), respectively. Then as $\kappa\to 0$, we have for every $T\geq 0$,

$$(u_\kappa,\omega_\kappa)\longrightarrow(u,\omega)\quad \text{in } L^\infty([0,T];L^2(\mathbb{R}^2)).$$

More precisely, we have

$$\|u_\kappa(t)-u(t)\|_{L^2}+\|\omega_\kappa(t)-\omega(t)\|_{L^2}\leq$$

$$\begin{cases} C\kappa t e^{\exp\{Ct\}}e^{f_1(\kappa t,z)}, & \text{when}\quad \nu>0,\ \gamma=0,\ \kappa>0, \\ C(\kappa t)^{\frac12}e^{\exp\{\exp\{Ct\}\}}e^{f_2(\kappa t,z)}, & \text{when}\quad \nu=0,\ \gamma>0,\ \kappa\in]0,\gamma[, \\ C(\kappa t)^{\frac12-}e^{\exp\{\exp\{Ct\}\}}e^{\exp\{\exp\{C\kappa t\}\}}, & \text{when}\quad \nu=0,\ \gamma=0,\ \kappa\in]0,1[, \end{cases}$$

where $t\geq 0$, $z:=-\frac{2\kappa}{\nu+\kappa-\gamma}$, and $f_1(\kappa t,z)\lesssim_z \kappa t+1$, $f_2(\kappa t,z)\lesssim_z\kappa t+1$ are defined in Equations (3.4) and (3.5), respectively, and C is the absolute constant that may depend on ν,γ but not depend on κ. Here, $\frac12-$ denotes a positive number strictly less than $\frac12$ and can be arbitrarily close to $\frac12$.

Remark 1.1

We note that when $\nu\neq\gamma$, we get $z\to 0$ as $\kappa\to 0$. Thus $f_1(\kappa t,z)$, $f_2(\kappa t,z)$ asymptotically behave not bad, indeed, for every i, $f_i(\kappa t,z)\to\kappa t$.

The proof of Theorem 1.1 mainly relies on the method of applying the hidden structures of the coupling system (1.1), which is a newly developed method in treating some kind of coupling systems dating from fluid mechanics, e.g. the generalized Boussinesq system (cf. [10, 11]), the compressible barotropic fluid equations (cf. [18]) and so on. The proof of Theorem 1.2 mainly bases on the a priori estimates obtained in Theorem 1.1.

Now we shall have a short discussion on how the method is used in proving Theorem 1.1. The vorticity $\Omega := \nabla \times u$ is an important physical quantity, and is closely related to the global continuation of the solution (e.g. BKM blowup criterion [2]). Thus firstly, by applying the *curl* operator to the equation of u and from $\nabla \times (\nabla \times \omega) = -\Delta\omega$, we see the coupling system of vorticity Ω and microrotation field ω

$$\begin{cases} \partial_t \Omega + u \cdot \nabla \Omega - (\nu + \kappa)\Delta\Omega = -2\kappa\Delta\omega \\ \partial_t \omega + u \cdot \nabla\omega - \gamma\Delta\omega + 4\kappa\omega = 2\kappa\Omega. \end{cases} \tag{1.4}$$

If one only views $\Delta\omega$ as a forcing term, then this high order term always plays a bad part and produces difficulty in some cases, and especially when $\gamma = 0$, because of its somewhat critical sense, the usual methods fail to obtain the global results. But fortunately, from the special structural property, we can construct a new interim quantity to avoid this bad term. Indeed, let $z \in \mathbb{R}$ be a number chosen later, and denote $\Gamma_z := \Omega + z\omega$, thus by a direct calculation we find

$$\partial_t \Omega + u \cdot \nabla\Omega - (\nu+\kappa)\Delta\Gamma_z = -(2\kappa + z(\nu+\kappa))\Delta\omega,$$
$$\partial_t(z\omega) + u \cdot \nabla(z\omega) = z\gamma\Delta\omega - 4z\kappa\omega + 2z\kappa\Omega,$$

and if $\gamma \neq \nu + \kappa$, we select z such that $z(\gamma - \nu - \kappa) - 2\kappa = 0$, that is, $z = -\frac{2\kappa}{\nu+\kappa-\gamma}$ (note that when $\gamma = 0$, it is just the quantity introduced in [8]), then

$$\partial_t \Gamma_z + u \cdot \nabla\Gamma_z - (\nu+\kappa)\Delta\Gamma_z = -4z\kappa\omega + 2z\kappa\Omega.$$

In what follows, we shall omit the subscript when there is no ambiguity. Clearly this Γ has very good structures, and by considering the coupling system of Γ and ω, we can get the necessary a priori estimates of Ω and finally reach the target.

The paper is organized as follows. Section 2 is devoted to present some preparatory knowledge on Besov spaces, and show some necessary estimates of smooth solutions of transport-diffusion equation and Stokes system. Then we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. Preliminaries

2.1. Notations

Throughout this paper the following notations will be used.

- ◇ The notion $X \lesssim Y$ means that there exists a positive harmless constant C_0 such that $X \le C_0 Y$. $X \sim Y$ means that both $X \lesssim Y$ and $Y \lesssim X$ are satisfied.
- ◇ $\mathcal{S}$ denotes the Schwartz class, $\mathcal{S}'$ the space of tempered distributions, and $\mathcal{S}'/\mathcal{P}$ the quotient space of tempered distributions, which are modulo polynomials.
- ◇ We use $\mathcal{F}f$ or $\widehat{f}$ to denote the Fourier transform of a tempered distribution f.
- ◇ For every $s \in \mathbb{R}$, $H^s(\mathbb{R}^n)$ (or $\dot{H}^s(\mathbb{R}^n)$) is the usual inhomogeneous (or homogeneous) Sobolev space in the L^2 framework.
- ◇ For any pair of operators A and B on some Banach space $\mathcal{X}$, the commutator $[A, B]$ is defined by $AB - BA$.

2.2. Littlewood–Paley decomposition and Besov spaces

To define the Besov space, we need the following dyadic partition of unity (cf. [3]). Choose two non-negative radial functions $\chi, \varphi \in C^\infty(\mathbb{R}^n)$ be supported respectively in the ball $\{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\}$ and the shell $\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n; \qquad \sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \neq 0.$$

For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the nonhomogeneous Littlewood–Paley operators

$$\Delta_{-1} f := \chi(D)f; \ \forall q \in \mathbb{N} \quad \Delta_q f := \varphi(2^{-q}D)f \ \text{ and } S_q f := \sum_{-1\le j\le q-1} \Delta_j f.$$

The homogeneous Littlewood–Paley operators are defined as follows:

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q f := \varphi(2^{-q}D)f, \quad \dot{S}_q f := \sum_{j\le q-1} \dot{\Delta}_j f.$$

Now we introduce the definition of Besov spaces . Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space $B^s_{p,r}$ is defined as the set of tempered distribution f such that

$$\|f\|_{B^s_{p,r}} := \|\{2^{qs}\|\Delta_q f\|_{L^p}\}_{q\ge -1}\|_{\ell^r} < \infty,$$

The homogeneous space $\dot{B}^s_{p,r}$ is the set of $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,r}} := \|\{2^{qs}\|\dot{\Delta}_q f\|_{L^p}\}_{q\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty.$$

We point out that for all $s \in \mathbb{R}$, $B^s_{2,2} = H^s$ and $\dot{B}^s_{2,2} = \dot{H}^s$.

Next, we introduce two kinds of coupled space–time Besov spaces. The first one $L^\varrho([0,T], B^s_{p,r})$, abbreviated by $L^\varrho_T B^s_{p,r}$, is the set of tempered distribution f such that

$$\|f\|_{L^\varrho_T B^s_{p,r}} := \left\| \|\{2^{qs}\left\|\Delta_q f\right\|_{L^p}\}_{q\geq -1}\|_{\ell^r}\right\|_{L^\varrho_T} < \infty.$$

The second one $\widetilde{L}^\varrho([0,T], B^s_{p,r})$, abbreviated by $\widetilde{L}^\varrho_T B^s_{p,r}$, is the set of tempered distribution f satisfying

$$\|f\|_{\widetilde{L}^\varrho_T B^s_{p,r}} := \left\|\{2^{qs}\|\Delta_q f\|_{L^\varrho_T L^p}\}_{q\geq -1}\right\|_{\ell^r} < \infty.$$

Because of the Minkowiski inequality, we immediately obtain

$$L^\varrho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\varrho_T B^s_{p,r}, \ \textit{if } r \geq \rho \quad \textit{and} \quad \widetilde{L}^\varrho_T B^s_{p,r} \hookrightarrow L^\varrho_T B^s_{p,r}, \ \textit{if } \varrho \geq r.$$

We can similarly extend to the homogeneous ones $L^\varrho_T \dot{B}^s_{p,r}$ and $\widetilde{L}^\varrho_T \dot{B}^s_{p,r}$.

Berstein's inequality is fundamental in the analysis involving Besov spaces (see e.g. [3])

Lemma 2.1

Let $f \in L^a(\mathbb{R}^n)$, $n \in \mathbb{N}$, $1 \leq a \leq b \leq \infty$. Then for every $(k,q) \in \mathbb{N}^2$ there exists a constant $C > 0$ such that

$$\sup_{|\alpha|=k} \left\|\partial^\alpha S_q f\right\|_{L^b} \leq C 2^{q(k+n(\frac{1}{a}-\frac{1}{b}))} \left\|S_q f\right\|_{L^a},$$

$$C^{-1} 2^{qk} \left\|\Delta_q f\right\|_{L^a} \leq \sup_{|\alpha|=k} \left\|\partial^\alpha \Delta_q f\right\|_{L^a} \leq C 2^{qk} \left\|\Delta_q f\right\|_{L^a}.$$

2.3. *On transport-diffusion equation and Stokes system*

In this subsection, we shall collect some useful estimates concerning the smooth solutions of the transport(-diffusion) equation and the Stokes system, which plays an important role in the existence and the uniqueness part. We consider the following transport(-diffusion) equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \kappa \Delta \omega + K\omega = f, \\ \mathrm{div} u = 0, \quad \omega|_{t=0} = \omega^0, \end{cases} \tag{2.1}$$

where $\kappa \geq 0$ and $K \in \mathbb{R}$.

First, we consider the basic L^p estimate.

Proposition 2.2

Let u be a smooth divergence-free vector field of $\mathbb{R}^n$ and ω be a smooth solution of Equation (2.1). Then for every $p \in [1,\infty]$ we have

$$\|\omega(t)\|_{L^p} \leq e^{\max\{-K,0\}t}\Big(\left\|\omega^0\right\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p}\,\mathrm{d}\tau\Big).$$

Proof

Denote $\widetilde{\omega} := e^{Kt}\omega$ and $\widetilde{f} := e^{Kt} f$, then Equation (2.1) reduces to

$$\begin{cases} \partial_t \widetilde{\omega} + u \cdot \nabla \widetilde{\omega} - \kappa \Delta \widetilde{\omega} = \widetilde{f}, \\ \mathrm{div} u = 0, \quad \widetilde{\omega}|_{t=0} = \omega^0. \end{cases}$$

This is just the standard transport(-diffusion) equation, thus from the classical estimate in [5]

$$\|\widetilde{\omega}(t)\|_{L^p} \leq \left\|\omega^0\right\|_{L^p} + \int_0^t \|\widetilde{f}(\tau)\|_{L^p} \mathrm{d}\tau,$$

we further get

$$\begin{aligned} \|\omega(t)\|_{L^p} &\leq e^{-Kt}\Big(\left\|\omega^0\right\|_{L^p} + \int_0^t e^{K\tau}\,\|f(\tau)\|_{L^p}\,\mathrm{d}\tau\Big) \\ &\leq \begin{cases} \left\|\omega^0\right\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p}\,\mathrm{d}\tau & \text{if} \quad K \geq 0 \\ e^{-Kt}\big(\left\|\omega^0\right\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p}\,\mathrm{d}\tau\big) & \text{if} \quad K < 0. \end{cases} \end{aligned}$$

□

The following smoothing effect is useful in the main proof.

Proposition 2.3
Let u be a smooth divergence-free vector field of $\mathbb{R}^n$ with vorticity $\Omega = \nabla \times u$, and ω be a smooth solution of the transport-diffusion equation (2.1) with $\kappa > 0$. Then for every $\varrho \in [1,\infty]$ and $p \in [2,\infty[$ we have

$$\sup_{q\in\mathbb{N}} 2^{2q/\varrho} \left\| \Delta_q \omega \right\|_{L^\varrho_t L^p} \lesssim \kappa^{-\frac{1}{\varrho}} e^{\max\{-K,0\}t} \Big(\|\omega^0\|_{L^p} + \|\omega\|_{L^\infty_t L^\infty} \|\Omega\|_{L^1_t L^p} + \|f\|_{L^1_t L^p} \Big).$$

Proof
Let $q \in \mathbb{N}$ and denote $\Delta_q \omega := \omega_q$ and $\Delta_q f := f_q$. Then applying Δ_q to the equation, we obtain

$$\partial_t \omega_q + u \cdot \nabla \omega_q - \kappa \Delta \omega_q + K \omega_q = -[\Delta_q, u \cdot \nabla]\omega + f_q. \tag{2.2}$$

From the definition of the dyadic operator, ω_q is real-valued. Multiplying the upper equation by $|\omega_q|^{p-2}\omega_q$ and integrating over the whole space, we find

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \left\|\omega_q(t)\right\|^p_{L^p} + \kappa(p-1)\int_{\mathbb{R}^2} |\nabla \omega_q(t,x)|^2 |\omega_q(t,x)|^{p-2}\mathrm{d}x + K\left\|\omega_q(t)\right\|^p_{L^p} \\
&= -\int_{\mathbb{R}^2} [\Delta_q, u\cdot\nabla]\omega \cdot |\omega_q|^{p-2}\omega_q(t,x)\mathrm{d}x + \int_{\mathbb{R}^2} f_q \cdot |\omega_q|^{p-2}\omega_q(t,x)\mathrm{d}x \\
&\leq \Big(\left\|[\Delta_q, u\cdot\nabla]\omega(t)\right\|_{L^p} + \left\|f_q(t)\right\|_{L^p} \Big) \left\|\omega_q(t)\right\|^{p-1}_{L^p}.
\end{aligned}$$

By virtue of the generalized Bernstein inequality (cf. [7]), there exists an absolute constant c independent of q such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\omega_q(t)\right\|_{L^p} + c\kappa 2^{2q}\left\|\omega_q(t)\right\|_{L^p} + K\left\|\omega_q(t)\right\|_{L^p} \leq \left\|[\Delta_q, u\cdot\nabla]\omega(t)\right\|_{L^p} + \left\|f_q(t)\right\|_{L^p}.$$

It directly leads to

$$\begin{aligned}
\left\|\omega_q(t)\right\|_{L^p} &\leq e^{-c\kappa 2^{2q}t - Kt}\left\|\omega^0_q\right\|_{L^p} + \int_0^t e^{-c\kappa 2^{2q}(t-\tau) - K(t-\tau)} \Big(\left\|[\Delta_q, u\cdot\nabla]\omega(\tau)\right\|_{L^p} + \left\|f_q(\tau)\right\|_{L^p}\Big)\mathrm{d}\tau \\
&\leq e^{\max\{-K,0\}t}\Big(e^{-c\kappa 2^{2q}t}\left\|\omega^0_q\right\|_{L^p} + \int_0^t e^{-c\kappa 2^{2q}(t-\tau)}\Big(\left\|[\Delta_q, u\cdot\nabla]\omega\right\|_{L^p} + \left\|f_q\right\|_{L^p}\Big)\mathrm{d}\tau\Big).
\end{aligned}$$

Taking the L^ϱ norm over $[0,t]$ and from the Young's inequality, we find

$$\|\omega_q\|_{L^\varrho t L^p} \lesssim e^{\max\{-K,0\}t}\kappa^{-1/\varrho}2^{-2q/\varrho}\Big(\|\omega^0_q\|_{L^p} + \int_0^t \big(\|[\Delta_q, u\cdot\nabla]\omega(\tau)\|_{L^p} + \|f_q(\tau)\|_{L^p}\big)\mathrm{d}\tau\Big). \tag{2.3}$$

From a simple paraproduct computation, we get (cf. [10])

$$\left\|[\Delta_q, u\cdot\nabla]\omega\right\|_{L^p} \lesssim \|\Omega\|_{L^p}\,\|\omega\|_{L^\infty}, \tag{2.4}$$

thus combining with the Bernstein inequality, we have for every $q \in \mathbb{N}$,

$$2^{2q/\varrho}\left\|\omega_q\right\|_{L^\varrho t L^p} \lesssim \kappa^{-\frac{1}{\varrho}} e^{\max\{-K,0\}t}\Big(\|\omega^0\|_{L^p} + \|\Omega\|_{L^1_t L^p}\,\|\omega\|_{L^\infty_t L^\infty} + \|f_q\|_{L^1_t L^p}\Big).$$

□

We also have the regularization effect as follows.

Proposition 2.4
Let $(s,r) \in]-1,1[\times[1,\infty] \cup \{(1,1),(-1,\infty)\}$, $p \in [1,\infty]$, u be a smooth divergence-free vector field and ω be a smooth solution of the Equation (2.1). Then there exists $C > 0$ such that for every $t \in \mathbb{R}^+$,

$$\|\omega\|_{\widetilde{L}^\infty_t B^s_{p,r}} \leq Ce^{CU(t) + \max\{-K,0\}t}\Big(\left\|\omega^0\right\|_{B^s_{p,r}} + \|f\|_{\widetilde{L}^1_t B^s_{p,r}}\Big),$$

where

$$U(t) := \begin{cases} \|\nabla u\|_{L^1_t L^\infty} & \text{if} \quad s \in]-1,1[, \\ \|u\|_{L^1_t B^1_{\infty,1}} & \text{if} \quad s = 1,\ r = 1, \\ \|u\|_{L^1_t B^1_{\infty,1}} & \text{if} \quad s = -1,\ r = \infty. \end{cases}$$

Proof

Applying Proposition 2.2 to Equation (2.2), we have for every $q \geq -1$

$$\|\omega_q\|_{L_t^\infty L^p} \lesssim e^{\max\{-K,0\}t}\Big(\left\|\omega_q^0\right\|_{L^p} + \int_0^t \left\|[\Delta_q, u\cdot\nabla]\omega(\tau)\right\|_{L^p} \mathrm{d}\tau + \|f_q\|_{L_t^1 L^p} \Big).$$

The remaining part is classical (cf. [6] and [1]), and thus we omit it. □

When $s = 0$, we also have a logarithmic improvement of the upper estimate.

Proposition 2.5

Let $(p, r) \in [1, \infty]^2$, u be a smooth divergence-free vector field of $\mathbb{R}^n$ and ω be a smooth solution of (2.1). Then for every $t \geq 0$

$$\|\omega\|_{\widetilde{L}_t^\infty B_{p,r}^0} \lesssim e^{\max\{-K,0\}t}\Big(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau\Big)\Big(\left\|\omega^0\right\|_{B_{p,r}^0} + \|f\|_{L_t^1 B_{p,r}^0} \Big).$$

Proof

The proof is from the classical process (cf. [10]) combining with Proposition 2.2 and Proposition 2.4, and we omit it. □

Next we shall consider the regularization effect of the following Stokes system

$$\begin{cases} \partial_t u + v\cdot\nabla u - \nu\Delta u + \nabla P = F \\ \mathrm{div} v = 0, \quad u|_{t=0} = u^0, \end{cases} \tag{2.5}$$

where v is a smooth divergence-free vector field of $\mathbb{R}^n$ and F is a smooth forcing term.

Proposition 2.6

Let $s \in]-1, 1[$, $\rho \in [1, \infty]$, and u be a smooth solution of the Stokes system (2.5). Then for every $t \geq 0$, there exists an absolute constant $C > 0$ such that

$$\|u\|_{L_t^\infty B_{2,\infty}^s} \leq Ce^{C\|\nabla v\|_{L_t^1 L^\infty}} \Big(\left\|u^0\right\|_{B_{2,\infty}^s} + \big((1/\nu)^{1-\frac{1}{\rho}} + t^{1-\frac{1}{\rho}}\big) \|F\|_{\widetilde{L}^{\rho}{}_t B_{2,\infty}^{s-2+\frac{2}{\rho}}} \Big).$$

Remark 2.1

The proof can be carried out in a similar way as obtaining Proposition 4.2 in [11], and we omit it. We note that if $F := F_1 + F_2$, we can choose different ρ_1 and ρ_2 to match F_1, F_2, respectively.

3. Proof of Theorem 1.1

The outline of the proof is as follows: first, we give some appropriate a priori estimates, then we prove the uniqueness in a weaker framework, and at last, we show the existence.

3.1. A priori estimates

Proposition 3.1

Let $\nu \geq 0$, $\gamma \geq 0$ and (u, ω) be a solution of the 2D micropolar fluid equations in Equation (1.1) with $(u^0, \omega^0) \in L^2 \times L^2$. Then for every $t \geq 0$

$$\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \mathrm{d}\tau + 2\gamma \int_0^t \|\nabla\omega(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq \left\|u^0\right\|_{L^2}^2 + \left\|\omega^0\right\|_{L^2}^2. \tag{3.1}$$

Besides, when $\nu = 0$, we also get

$$\kappa \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C_0 e^{C_0\kappa t}. \tag{3.2}$$

Proof

Multiplying the first equation of Equation (1.1) by u, the second by ω, and integrating in the spatial variable, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\Big) + (\nu+\kappa)\|\nabla u(t)\|_{L^2}^2 + \gamma\|\nabla\omega(t)\|_{L^2}^2 + 4\kappa\|\omega(t)\|_{L^2}^2 \\ = 2\kappa \int_{\mathbb{R}^2} (\nabla\times\omega)\cdot u(t,x)\mathrm{d}x + 2\kappa \int_{\mathbb{R}^2} (\nabla\times u)\omega(t,x)\mathrm{d}x \\ = 4\kappa \int_{\mathbb{R}^2} (\nabla\times u)\omega(t,x)\mathrm{d}x. \end{aligned}$$

From the Young inequality, we find

$$4\kappa \int (\nabla \times u)\omega \le \kappa \|\nabla \times u\|_{L^2}^2 + 4\kappa \|\omega\|_{L^2}^2 \le \kappa \|\nabla u\|_{L^2}^2 + 4\kappa \|\omega\|_{L^2}^2 .$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + \nu \|\nabla u(t)\|_{L^2}^2 + \gamma \|\nabla \omega(t)\|_{L^2}^2 \le 0.$$

Then integrating in time leads to Equation (3.1). When $\nu = 0$, Equation (3.2) is naturally from

$$4\kappa \int (\nabla \times u)\omega \le \frac{\kappa}{2} \|\nabla u\|_{L^2}^2 + 8\kappa \|\omega\|_{L^2}^2 .$$

□

Proposition 3.2
Let $\nu \ge 0$, $\gamma \ge 0$, (u^0, ω^0) be a solution of Equation (1.1) with $u^0 \in H^1$ and $\omega^0 \in L^2$. Then for every $t \ge 0$

$$\|\nabla \times u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \le C_0 e^{2f_1(\kappa t, z)}. \tag{3.3}$$

where C_0 is a absolute constant depending only on the data, $z := -\frac{2\kappa}{\nu+\kappa-\gamma}$,

$$f_1(\kappa t, z) := \begin{cases} \min\left\{\frac{\kappa}{4}t, f_2(\kappa t, z)\right\} & \text{if} \quad \gamma \ge \frac{\kappa^2}{\nu+\kappa}, \gamma \ne \nu+\kappa \\ f_2(\kappa t, z) & \text{if} \quad \gamma < \frac{\kappa^2}{\nu+\kappa}, \\ \frac{\kappa}{4}t & \text{if} \quad \gamma = \nu+\kappa, \end{cases} \tag{3.4}$$

and

$$f_2(\kappa t, z) := g_1(z)\kappa t + \max\{0, \log|z|\}, \tag{3.5}$$

and

$$\begin{aligned} g_1(z) &:= \max\{|z^2+2z-1| + 2z, |z^2+2z-1| - 2z - 4\} \\ &= \begin{cases} z^2+4z-1 & \text{if} \quad z \in [\sqrt{2}-1, \infty[, \\ 1-z^2 & \text{if} \quad z \in [-1, 0[\cup]0, \sqrt{2}-1[, \\ -4z-z^2-3 & \text{if} \quad z \in [-1-\sqrt{2}, -1[, \\ z^2-5 & \text{if} \quad z \in]-\infty, -1-\sqrt{2}[. \end{cases} \end{aligned} \tag{3.6}$$

Besides, when $\nu + \kappa < \gamma$, we also have

$$\|\nabla \times u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\nabla \times u(\tau)\|_{\dot{H}^1}^2 \mathrm{d}\tau + \gamma \int_0^t \|\nabla \omega(\tau)\|_{L^2}^2 \mathrm{d}\tau \le C_0 e^{2f_2(\kappa t, z)}.$$

Proof
We shall first take the new idea mentioned in the Section 1 to consider this problem. Indeed, we have the following coupling system of $\Gamma := \Omega + z\omega = \nabla \times u + z\omega$ and ω

$$\begin{cases} \partial_t \Gamma + u \cdot \nabla \Gamma - (\nu+\kappa)\Delta\Gamma - 2\kappa z \Gamma = -2\kappa(z^2+2z)\omega, \\ \partial_t \omega + u \cdot \nabla \omega - \gamma \Delta \omega + 2\kappa(z+2)\omega = 2\kappa \Gamma, \end{cases} \tag{3.7}$$

By using the usual L^2 method, we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Gamma(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + (\nu+\kappa)\|\nabla\Gamma(t)\|_{L^2}^2 + \gamma \|\nabla\omega(t)\|_{L^2}^2 \\ &\quad = 2\kappa z \|\Gamma(t)\|_{L^2}^2 - 2\kappa(z^2+2z-1)\int_{\mathbb{R}^2} \Gamma \cdot \omega(t,x)\mathrm{d}x - 2\kappa(z+2)\|\omega(t)\|_{L^2}^2 \\ &\quad \le \kappa(|z^2+2z-1| + 2z)\|\Gamma(t)\|_{L^2}^2 + \kappa(|z^2+2z-1| - 2z - 4)\|\omega(t)\|_{L^2}^2 \end{aligned}$$

Hence, we infer

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Gamma(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + (\nu+\kappa)\|\nabla\Gamma(t)\|_{L^2}^2 + \gamma\|\nabla\omega(t)\|_{L^2}^2 \\ &\le \kappa g_1(z)(\|\Gamma(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2), \end{aligned}$$

where $g_1(z)$ is defined in Equation (3.6). Thus

$$\begin{aligned}&\|\Gamma(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+(\nu+\kappa)\int_0^t\|\nabla\Gamma(\tau)\|_{L^2}^2\mathrm{d}\tau+\gamma\int_0^t\|\nabla\omega(\tau)\|_{L^2}^2\mathrm{d}\tau\\&\leq(\|\Omega^0\|_{L^2}^2+\|\omega^0\|_{L^2}^2)\max\{z^2,1\}e^{2g_1(z)\kappa t},\end{aligned}$$

and from Equation (3.1)

$$\begin{aligned}&\|\Omega(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\min\{\nu+\kappa,\gamma\}\int_0^t\|\nabla\Omega(\tau)\|_{L^2}^2\mathrm{d}\tau+\gamma\int_0^t\|\nabla\omega(\tau)\|_{L^2}^2\mathrm{d}\tau\\&\leq\|\Gamma(t)\|_{L^2}^2+(1+z^2)\|\omega(t)\|_{L^2}^2+(\nu+\kappa)\int_0^t\|\nabla\Gamma(\tau)\|_{L^2}^2\mathrm{d}\tau+\gamma(1+z^2)\int_0^t\|\nabla\omega(\tau)\|_{L^2}^2\mathrm{d}\tau\\&\leq C_0\max\{z^2,1\}e^{2g_1(z)\kappa t}.\end{aligned}$$

On the other hand, we can consider the coupling system (1.4) of vorticity Ω and microrotation field ω. Similarly from the L^2 method, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2)+(\nu+\kappa)\|\nabla\Omega(t)\|_{L^2}^2+\gamma\|\nabla\omega(t)\|_{L^2}^2+4\kappa\|\omega(t)\|_{L^2}^2\\&=2\kappa\int_{\mathbb{R}^2}\nabla\Omega\cdot\nabla\omega(t,x)\mathrm{d}x+2\kappa\int_{\mathbb{R}^2}\Omega\cdot\omega(t,x)\mathrm{d}x.\end{aligned}$$

Because of the Young inequality, we get

$$2\kappa\left|\int\nabla\Omega\cdot\nabla\omega\mathrm{d}x\right|\leq(\nu+\kappa)\|\nabla\Omega\|_{L^2}^2+\frac{\kappa^2}{\nu+\kappa}\|\nabla\omega\|_{L^2}^2,$$

and

$$2\kappa\left|\int\Omega\cdot\omega\mathrm{d}x\right|\leq 4\kappa\|\omega\|_{L^2}^2+\frac{\kappa}{4}\|\Omega\|_{L^2}^2.$$

Thus if $\gamma\geq\frac{\kappa^2}{\nu+\kappa}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2)\leq\frac{\kappa}{2}(\|\Omega(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2).$$

Integrating in time yields a part of Equation (3.3).

□

Proposition 3.3
Let $\nu\geq 0$, $\gamma\geq 0$, $\nu+\kappa\neq\gamma$ and (u^0,ω^0) be a solution of Equation (1.1) with $(\nabla\times u^0,\omega^0)\in L^\infty\times L^\infty$. Then for every $t\geq 0$

$$\|\nabla\times u(t)\|_{L^\infty}+\|\omega(t)\|_{L^\infty}\leq\left(\left\|\nabla\times u^0\right\|_{L^\infty}+\left\|\omega^0\right\|_{L^\infty}\right)e^{2f_3(\kappa t,z)},$$

where $z=-\frac{2\kappa}{\nu+\kappa-\gamma}$,

$$f_3(\kappa t,z):=g_3(z)\kappa t+\log(1+|z|),\tag{3.8}$$

and

$$\begin{aligned}g_3(z)&:=\max\{|z^2+2z|,1\}+\max\{z,-z-2,0\}\\&=\begin{cases}z^2+3z & \text{if}\quad z\in[\sqrt{2}-1,\infty[,\\ z+1 & \text{if}\quad z\in]0,\sqrt{2}-1[,\\ 1 & \text{if}\quad z\in[-2,0[,\\ -z-1 & \text{if}\quad z\in[-\sqrt{2}-1,-2[,\\ z^2+z-2 & \text{if}\quad z\in]-\infty,-\sqrt{2}-1[.\end{cases}\end{aligned}\tag{3.9}$$

Proof
It seems very difficult to get this type estimates simply from the coupling system (1.4), but we can consider the interim system (3.7) to tackle this problem. Denote

$$\delta_1:=\max\{z,0\}\quad\text{and}\quad\delta_2:=\max\{-z-2,0\}.$$

For the equation of Γ, by applying Proposition 2.2, we have

$$\|\Gamma(t)\|_{L^\infty} \le e^{2\kappa\delta_1 t}\Big(\left\|\Gamma^0\right\|_{L^\infty} + 2\kappa|z^2+2z|\int_0^t \|\omega(\tau)\|_{L^\infty}\,\mathrm{d}\tau\Big).$$

Similarly, for the equation of ω,

$$\|\omega(t)\|_{L^\infty} \le e^{2\kappa\delta_2 t}\Big(\left\|\omega^0\right\|_{L^\infty} + 2\kappa\int_0^t \|\Gamma(\tau)\|_{L^\infty}\,\mathrm{d}\tau\Big).$$

Combining the upper two estimates, we find

$$\begin{aligned}\|\Gamma(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty} \le & e^{2\kappa\max\{\delta_1,\delta_2\}t}\Big(\left\|\Gamma^0\right\|_{L^\infty} + \left\|\omega^0\right\|_{L^\infty} \\ & + 2\kappa\max\{|z^2+2z|,1\}\int_0^t (\|\Gamma(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^\infty})\mathrm{d}\tau\Big).\end{aligned}$$

The Gronwall inequality ensures that

$$\|\Gamma(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty} \le (\left\|\Gamma^0\right\|_{L^\infty} + \left\|\omega^0\right\|_{L^\infty})e^{2g_3(z)\kappa t},$$

where $g_3(z)$ $(z\in\mathbb{R}\setminus\{0\})$ defined in Equation (3.9). Furthermore, from the relationship $\Omega = \Gamma - z\omega$, we obtain

$$\|\Omega(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty} \le \left(\left\|\Omega^0\right\|_{L^\infty} + \left\|\omega^0\right\|_{L^\infty}\right)(1+|z|)^2 e^{2g_3(z)\kappa t}.$$

□

Proposition 3.4

(1) Let $\nu \ge 0$, $\gamma > 0$ and (u,ω) be a solution of Equation (1.1) with $(u^0,\omega^0)\in H^1\times L^2$. Then for every $t\ge 0$ and $\sigma\in[1,2[$, there exists a constant $C>0$ depending only on κ,ν,γ and σ such that

$$\|\nabla\times u\|_{\widetilde{L}_t^\sigma B_{2,1}^1} + \|\omega\|_{\widetilde{L}_t^\sigma B_{2,1}^1} + \|u\|_{\widetilde{L}_t^\sigma B_{\infty,1}^1} \le Ce^{Ct}, \tag{3.10}$$

and

$$\|u\|_{\widetilde{L}_t^\infty H^1} + \|\omega\|_{\widetilde{L}_t^\infty L^2} \le Ce^{e^{Ct}}. \tag{3.11}$$

(2) Let $\nu\ge 0$, $\gamma = 0$ and (u,ω) be a solution of Equation (1.1) with $(u^0,\omega^0)\in H^1\times(L^2\cap B^0_{\infty,1})$. Then for every $t\ge 0$ and $\sigma\in[1,2[$, there exists a constant $C>0$ depending only on κ,ν,σ such that

$$\|u\|_{\widetilde{L}_t^\sigma B_{\infty,1}^1} + \|\omega\|_{\widetilde{L}_t^\infty B_{\infty,1}^0} \le C\exp\{e^{Ct}\}, \tag{3.12}$$

$$\|\omega\|_{\widetilde{L}_t^\infty L^2} + \|u\|_{\widetilde{L}_t^\infty H^1} \le Ce^{\exp\{e^{Ct}\}}. \tag{3.13}$$

Proof

(1) We first consider the following coupling system to get the desired estimates

$$\begin{cases}\partial_t\Omega + u\cdot\nabla\Omega - (\nu+\kappa)\Delta\Omega = -2\kappa\Delta\omega \\ \partial_t\omega + u\cdot\nabla\omega - \gamma\Delta\omega + 4\kappa\omega = 2\kappa\Omega.\end{cases} \tag{3.14}$$

Denote $\Omega_q := \Delta_q\Omega$, $\omega_q := \Delta_q\omega$ with $q\in\mathbb{N}$, then for the equation of ω, similarly as obtaining Equation (2.3), we have

$$\|\omega_q\|_{L_t^\sigma L^2} \lesssim 2^{-q\frac{2}{\sigma}}\gamma^{-\frac{1}{\sigma}}\left(\|\omega_q^0\|_{L^2} + \int_0^t \|[\Delta_q, u\cdot\nabla]\omega(\tau)\|_{L^2}\mathrm{d}\tau + 2\kappa\|\omega_q\|_{L_t^1 L^2}\right), \tag{3.15}$$

From Equation (2.4) and the Besov embeddimg we find

$$\int_0^t \|[\Delta_q, u\cdot\nabla]\omega(\tau)\|_{L^2}\mathrm{d}\tau \le \|\Omega\|_{L_t^\infty L^2}\|\omega\|_{L_t^1 L^\infty} \lesssim \|\Omega\|_{L_t^\infty L^2}\|\omega\|_{L_t^1 B_{2,1}^1}.$$

Hence, taking account of Equation (3.3), we infer

$$\|\omega_q\|_{L_t^\sigma L^2} \lesssim 2^{-q\frac{2}{\sigma}}\gamma^{-\frac{1}{\sigma}}\left(1 + e^{f_1(\kappa t,z)}\|\omega\|_{L_t^1 B_{2,1}^1} + e^{f_1(\kappa t,z)}\kappa t\right).$$

Then set $\sigma \in [1,2[$ such that $1-\frac{2}{\sigma}<0$, and $Q_1 \in \mathbb{N}$ be a number chosen later, we see

$$\begin{aligned}\|\omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} &= \sum_{-1\le q<Q_1} 2^q\|\omega_q\|_{L_t^\sigma L^2} + \sum_{q\ge Q_1} 2^q\|\omega_q\|_{L_t^\sigma L^2}\\ &\lesssim 2^{Q_1} t^{\frac{1}{\sigma}}\|\omega\|_{L_t^\infty L^2} + \sum_{q\ge Q_1} 2^{q(1-\frac{2}{\sigma})}\gamma^{-\frac{1}{\sigma}}\left(1+e^{f_1(\kappa t,z)}\|\omega\|_{L_t^1 B^1_{2,1}} + e^{f_1(\kappa t,z)}\kappa t\right)\\ &\le C_0 2^{Q_1} t^{\frac{1}{\sigma}} + C_0 2^{Q_1(1-\frac{2}{\sigma})}\gamma^{-\frac{1}{\sigma}}\left(1+e^{f_1(\kappa t,z)} t^{1-\frac{1}{\sigma}}\|\omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} + e^{f_1(\kappa t,z)}\kappa t\right).\end{aligned}$$

We can select $Q_1 \in \mathbb{N}$ such that

$$2^{Q_1(1-\frac{2}{\sigma})} C_0 \gamma^{-1/\sigma} e^{f_1(\kappa t,z)}(1+t^{1-\frac{1}{\sigma}}) \approx \frac{1}{2},$$

thus for every $\sigma \in [1,2[$,

$$\begin{aligned}\|\omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} &\lesssim \gamma^{-\frac{1}{2-\sigma}} e^{\frac{\sigma}{2-\sigma}f_1(\kappa t,z)}(1+t^{1-\frac{1}{\sigma}})^{\frac{\sigma}{2-\sigma}} t^{\frac{1}{\sigma}} + 1 + \kappa t\\ &\lesssim_\sigma (1+\gamma^{-\frac{1}{2-\sigma}}) e^{\frac{\sigma}{2-\sigma}f_1(\kappa t,z)}(1+t^{\frac{1}{2-\sigma}}+\kappa t).\end{aligned} \tag{3.16}$$

Now for the equation of Ω, similarly we have

$$\|\Omega_q\|_{L_t^\sigma L^2} \lesssim 2^{-q\frac{2}{\sigma}}(\nu+\kappa)^{-\frac{1}{\sigma}}\left(\left\|\Omega_q^0\right\|_{L^2} + \|\Omega\|_{L_t^\infty L^2}\|\Omega\|_{L_t^1 B^1_{2,1}} + 2\kappa 2^{2q}\|\omega_q\|_{L_t^1 L^2}\right).$$

Note that from Equations (3.15) and (3.16), we find

$$2^{2q}\|\omega\|_{L_t^1 L^2} \lesssim \gamma^{-1}(1+\gamma^{-1})e^{2f_1(\kappa t,z)}(1+t+\kappa t).$$

Also taking account of Equation (3.3), we infer

$$\|\Omega_q\|_{L_t^\sigma L^2} \lesssim 2^{-q\frac{2}{\sigma}}\frac{1+\kappa\gamma^{-2}}{(\nu+\kappa)^{1/\sigma}} e^{2f_1(\kappa t,z)}\left(1+\|\Omega\|_{L_t^1 B^1_{2,1}} + t + \kappa t\right). \tag{3.17}$$

Similarly let $Q_2 \in \mathbb{N}$ be a number chosen later, we have

$$\begin{aligned}\|\Omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} &= \sum_{q<Q_2} 2^q\|\Omega_q\|_{L_t^\sigma L^2} + \sum_{q\ge Q_2} 2^q\|\Omega_q\|_{L_t^\sigma L^2}\\ &\le C_0 2^{Q_2} t^{\frac{1}{\sigma}} e^{f_1(\kappa t,z)} + C_0 2^{Q_2(1-\frac{2}{\sigma})}\frac{1+\kappa\gamma^{-2}}{(\nu+\kappa)^{1/\sigma}} e^{2f_1(\kappa t,z)}\left(1+t^{1-\frac{1}{\sigma}}\|\Omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} + t + \kappa t\right).\end{aligned}$$

Hence, we select $Q_2 \in \mathbb{N}$ such that

$$2^{Q_2(1-\frac{2}{\sigma})} C_0 \frac{1+\kappa\gamma^{-2}}{(\nu+\kappa)^{1/\sigma}} e^{2f_1(\kappa t,z)}(1+t^{1-\frac{1}{\sigma}}) \approx \frac{1}{2},$$

then for every $\sigma \in [1,2[$

$$\begin{aligned}\|\Omega\|_{\widetilde{L}_t^\sigma B^1_{2,1}} &\lesssim \left(\frac{1+\kappa\gamma^{-2}}{(\nu+\kappa)^{1/\sigma}}\right)^{\frac{\sigma}{2-\sigma}} e^{\frac{2+\sigma}{2-\sigma}f_1}(1+t^{1-\frac{1}{\sigma}})^{\frac{\sigma}{2-\sigma}} t^{\frac{1}{\sigma}} + 1 + t + \kappa t\\ &\lesssim e^{Ct}.\end{aligned} \tag{3.18}$$

In particular, when $\sigma = 1$, we get

$$\|\Omega\|_{L_t^1 B^1_{2,1}} \lesssim \left(1+\frac{1+\kappa\gamma^{-2}}{\nu+\kappa}\right) e^{3f_1(\kappa t,z)}(1+t+\kappa t). \tag{3.19}$$

On the other hand, we can consider the new coupling system (3.7) of Γ and ω to get a similar estimate as Equation (3.19), but because of that, it will not improve the bound essentially, we here omit it.

Thus by a high–low frequency decomposition, we obtain

$$\begin{aligned}\|\nabla u\|_{L_t^1 L^\infty} \le \|u\|_{L_t^1 B^1_{\infty,1}} &= \|\Delta_{-1} u\|_{L_t^1 L^\infty} + \sum_{q\in\mathbb{N}} 2^q\left\|\Delta_q u\right\|_{L_t^1 B^1_{\infty,1}}\\ &\lesssim t\|u\|_{L_t^\infty L^2} + \|\Omega\|_{L_t^1 B^1_{2,1}}\\ &\lesssim e^{Ct}.\end{aligned}$$

Then from the equation of Ω, we get

$$\begin{aligned}\|\Omega\|_{\widetilde{L}_t^\infty L^2} &\lesssim e^{C\|\nabla u\|_{L_t^1 L^\infty}}(\|\Omega^0\|_{L^2}+\|\Delta\omega\|_{\widetilde{L}_t^\infty \dot{B}_{2,2}^{-2}})\\ &\lesssim e^{C\|\nabla u\|_{L_t^1 L^\infty}}(\|\Omega^0\|_{L^2}+\|\omega\|_{\widetilde{L}_t^\infty L^2}),\end{aligned}$$

where we have used the regularization effect of the transport-diffusion equation (cf. Theorem 1.1 in [7]). From Proposition 2.4, we also find

$$\|\omega\|_{\widetilde{L}_t^\infty L^2} \lesssim e^{C\|\nabla u\|_{L_t^1 L^\infty}+2\delta_2\kappa t}\Big(\left\|\omega^0\right\|_{L^2}+\int_0^t \|\Gamma(\tau)\|_{L^2}\,d\tau\Big) \lesssim e^{e^{Ct}}.$$

This further leads to

$$\|\Omega\|_{\widetilde{L}_t^\infty L^2} \lesssim_{\kappa,\nu,\gamma} e^{e^{Ct}}.$$

Therefore,

$$\|u\|_{\widetilde{L}_t^\infty H^1} \lesssim \|\Delta_{-1}u\|_{L_t^\infty L^2}+\|\Omega\|_{\widetilde{L}_t^\infty L^2} \lesssim e^{e^{Ct}}.$$

(2) When $\gamma=0$, we have to consider the new coupling system

$$\begin{aligned}&\partial_t\Gamma+u\cdot\nabla\Gamma-(\nu+\kappa)\Delta\Gamma-2\kappa z\Gamma=-2\kappa(z^2+2z)\omega,\\ &\partial_t\omega+u\cdot\nabla\omega+2\kappa(z+2)\omega=2\kappa\Gamma,\end{aligned}$$

where $\Gamma=\Omega+z\omega$ and $z=-\frac{2\kappa}{\nu+\kappa}\in[-2,0[$. Note that in this case $f_1(\kappa t,z)\lesssim\kappa t+1$ (especially when $z=-1, f_1=0$), and $f_3(\kappa t,z)\sim\kappa t+1$. Denote $\Gamma_q:=\Delta_q\Gamma$ for every $q\in\mathbb{N}$, in a similar way as obtaining Equation (3.17), we have

$$\begin{aligned}\|\Gamma_q\|_{L_t^\sigma L^2} &\lesssim 2^{-q\frac{2}{\sigma}}(\nu+\kappa)^{-\frac{1}{\sigma}}\Big(1+|z|+e^{f_1(\kappa t,z)}\|\Gamma\|_{L_t^1 B_{2,1}^1}+\kappa|z^2+2z|\|\omega_q\|_{L_t^1 L^2}\Big)\\ &\lesssim 2^{-q\frac{2}{\sigma}}(\nu+\kappa)^{-\frac{1}{\sigma}}\Big(1+\kappa t+e^{C\kappa t}\|\Gamma\|_{L_t^1 B_{2,1}^1}\Big).\end{aligned}$$

Furthermore, similarly as obtaining Equation (3.18), we get for every $\sigma\in[1,2[$

$$\|\Gamma\|_{\widetilde{L}_t^\sigma B_{2,1}^1} \le C_0\Big(1+\frac{1}{(\nu+\kappa)^{1/(2-\sigma)}}\Big)e^{\frac{C_0}{2-\sigma}\kappa t}, \tag{3.20}$$

where C_0 is an absolute constant depending on κ,ν,σ. By virtue of the following fact

$$\begin{aligned}\|\nabla u\|_{L_t^1 L^\infty} &\lesssim t\,\|\Delta_{-1}u\|_{L_t^\infty L^2}+\|\Omega\|_{L_t^1 B_{\infty,1}^0}\\ &\lesssim t+\|\Gamma\|_{L_t^1 B_{\infty,1}^0}+\|\omega\|_{L_t^1 B_{\infty,1}^0},\end{aligned} \tag{3.21}$$

and Proposition 2.5, we have

$$\|\omega(t)\|_{B_{\infty,1}^0} \le \|\omega\|_{\widetilde{L}_t^\infty B_{\infty,1}^0} \lesssim_\kappa \Big(e^{Ct}+\int_0^t \|\omega(\tau)\|_{B_{\infty,1}^0}\,d\tau\Big)\Big(\left\|\omega^0\right\|_{B_{\infty,1}^0}+e^{Ct}\Big).$$

Thus the Gronwall inequality yields

$$\|\omega(t)\|_{B_{\infty,1}^0} \lesssim_\kappa \exp\{e^{Ct}\}.$$

By a direct decomposition, we further get

$$\|u\|_{\widetilde{L}_t^\sigma B_{\infty,1}^1} \lesssim \|\Delta_{-1}u\|_{L_t^\sigma L^2}+\|\Gamma\|_{\widetilde{L}_t^\sigma B_{\infty,1}^0}+\|\omega\|_{\widetilde{L}_t^\sigma B_{\infty,1}^0} \lesssim_\kappa \exp\{e^{Ct}\}.$$

Then we can use Proposition 2.4 to infer

$$\|\omega\|_{\widetilde{L}_t^\infty L^2} \lesssim_\kappa e^{C\|\nabla u\|_{L_t^1 L^\infty}}\Big(\left\|\omega^0\right\|_{L^2}+\int_0^t \|\Gamma(\tau)\|_{L^2}\,d\tau\Big) \lesssim_\kappa e^{\exp\{e^{Ct}\}},$$

and

$$\|\Gamma\|_{\widetilde{L}_t^\infty L^2} \lesssim e^{C\|\nabla u\|_{L_t^1 L^\infty}}\Big(\left\|\Gamma^0\right\|_{L^2}+\int_0^t \|\omega(\tau)\|_{L^2}\,d\tau\Big) \lesssim_\kappa e^{\exp\{e^{Ct}\}}.$$

This estimate, combined with Equation (3.20), leads to

$$\|u\|_{\widetilde{L}_t^\infty H^1} \lesssim \|\Delta_{-1}u\|_{L_t^\infty L^2}+\|\Gamma\|_{\widetilde{L}_t^\infty L^2}+\|\omega\|_{\widetilde{L}_t^\infty L^2} \lesssim_\kappa e^{\exp\{e^{Ct}\}}.$$

□

3.2. Uniqueness

Let $\nu \geq 0$, $\gamma \geq 0$, we prove the uniqueness of the 2D micropolar fluid equations (1.1) in the following working space

$$\mathcal{Z}_T := L_T^\infty H^1 \cap L_T^1 B^1_{\infty,1} \times L_T^\infty L^2 \cap L_T^1 L^\infty.$$

Assume that $(u^i, \omega^i) \in \mathcal{Z}_T$ are two solutions of Equation (1.1) with initial data $(u^{i,0}, \omega^{i,0})$, $i = 1, 2$. Denote $u := u^1 - u^2$, $\omega := \omega^1 - \omega^2$ and $P := P^1 - P^2$. Then the difference system writes

$$\begin{aligned} \partial_t u + u^1 \cdot \nabla u - (\nu + \kappa)\Delta u + \nabla P &= 2\kappa \nabla \times \omega - u \cdot \nabla u^2 \\ \partial_t \omega + u^1 \cdot \nabla \omega - \gamma \Delta \omega + 4\kappa \omega &= 2\kappa \nabla \times u - u \cdot \nabla \omega^2 \\ (u, \omega)|_{t=0} &= (u^0, \omega^0). \end{aligned}$$

First, by virtue of Proposition 2.6 (with its remark), we choose $\rho_1 = 1$ for the term $-u \cdot \nabla u^2$ and ρ_2 for term $2\kappa \nabla \times \omega$ to get

$$\|u\|_{L_t^\infty B^0_{2,\infty}} \lesssim e^{C\|\nabla u^1\|_{L_t^1 L^\infty}} \left(\left\|u^0\right\|_{B^0_{2,\infty}} + \int_0^t \left\|u \cdot \nabla u^2(\tau)\right\|_{B^0_{2,\infty}} \mathrm{d}\tau + (1 + \kappa t) \|\nabla \times \omega\|_{L_t^\infty B^{-2}_{2,\infty}} \right) \tag{3.22}$$

For the integral term of the right-hand side, from a direct computation, we find

$$\left\|u \cdot \nabla u^2\right\|_{B^0_{2,\infty}} \leq \left\|u \cdot \nabla u^2\right\|_{L^2} \leq \|u\|_{L^2} \left\|\nabla u^2\right\|_{L^\infty}.$$

By a high–low frequency decomposition, we obtain the following logarithmic interpolation inequality (cf. [11])

$$\|u\|_{L^2} \lesssim \|u\|_{B^0_{2,\infty}} \log\left(e + \frac{1}{\|u\|_{B^0_{2,\infty}}}\right) \log(e + \|u\|_{H^1}). \tag{3.23}$$

Hence

$$\left\|u \cdot \nabla u^2\right\|_{B^0_{2,\infty}} \lesssim \left\|\nabla u^2\right\|_{L^\infty} \log\left(e + \|u\|_{H^1}\right) \mu\left(\|u\|_{B^0_{2,\infty}}\right), \tag{3.24}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a function defined by $\mu(x) := x \log(e + \frac{1}{x})$. Then, for the last term of the right-hand side of Equation (3.22), from the endpoint case of Proposition 2.4, we infer

$$\begin{aligned} \|\nabla \times \omega\|_{L_t^\infty B^{-2}_{2,\infty}} &\lesssim \|\omega\|_{L_t^\infty B^{-1}_{2,\infty}} \\ &\lesssim e^{C\|u^1\|_{L_t^1 B^1_{\infty,1}}} \left(\left\|\omega^0\right\|_{B^{-1}_{2,\infty}} + \kappa \int_0^t \|\nabla \times u(\tau)\|_{B^{-1}_{2,\infty}} \mathrm{d}\tau + \int_0^t \left\|u \cdot \nabla \omega^2(\tau)\right\|_{B^{-1}_{2,\infty}} \mathrm{d}\tau \right). \end{aligned}$$

From a simple computation and Equation (3.23), we obtain

$$\begin{aligned} \left\|u \cdot \nabla \omega^2\right\|_{B^{-1}_{2,\infty}} \lesssim \left\|u\, \omega^2\right\|_{B^0_{2,\infty}} &\lesssim \|u\|_{L^2} \left\|\omega^2\right\|_{L^\infty} \\ &\lesssim \|\omega^2\|_{L^\infty} \log(e + \|u\|_{H^1})\, \mu\left(\|u\|_{B^0_{2,\infty}}\right). \end{aligned}$$

Thus from $x \leq \mu(x)$,

$$\begin{aligned} \|\nabla \times \omega\|_{L_t^\infty B^{-1}_{2,\infty}} &\lesssim \|\omega^0\|_{B^{-1}_{2,\infty}} + \kappa \int_0^t \|u(\tau)\|_{B^0_{2,\infty}} \mathrm{d}\tau + \int_0^t \|\omega^2(\tau)\|_{L^\infty} \mu\left(\|u(\tau)\|_{B^0_{2,\infty}}\right) \mathrm{d}\tau \\ &\lesssim \|\omega^0\|_{B^{-1}_{2,\infty}} + \int_0^t (1 + \|\omega^2(\tau)\|_{L^\infty}) \mu\left(\|u(\tau)\|_{L_t^\infty B^0_{2,\infty}}\right) \mathrm{d}\tau. \end{aligned} \tag{3.25}$$

Denote $Z(t) := \|u\|_{L_t^\infty B^0_{2,\infty}} + \|\omega\|_{L_t^\infty B^{-1}_{2,\infty}}$. Inserting Equations (3.24) and (3.25) into Equation (3.22), we get

$$Z(t) \leq f(t)\left(Z(0) + \int_0^t g(\tau) \mu(Z(\tau)) \mathrm{d}\tau\right),$$

where $g(t) := 1 + \|u^2(t)\|_{B^1_{\infty,1}} + \|\omega^2(t)\|_{L^\infty}$ and $f(t)$ is an explicit function, which is continuously and increasingly depended on time t and $\left\|(u^i, \omega^i)\right\|_{\mathcal{Z}_t}$. From

$$\mathcal{M}(x) := \int_x^{1/e} \frac{1}{\mu(r)} \mathrm{d}r = \int_x^{1/e} \frac{1}{r \log(e + 1/r)} \mathrm{d}r = \int_e^{1/x} \frac{1}{r \log(e + r)} \mathrm{d}r, \quad x \in]0, 1/e[,$$

we know that

$$\mathcal{M}(x) \le \int_e^{1/x} \frac{1}{r\log r}\mathrm{d}r = \log(\log(1/x)), \quad \mathcal{M}(x) \ge \int_e^{1/x} \frac{1}{r(1+\log r)}\mathrm{d}r \ge \log(\log(e/x)) - 1,$$

then the classical Osgood Lemma (cf. Theorem 5.2.1 of [3]) can be applied and it ensures the uniqueness result. Moreover, the Lemma shows the following quantified estimate

$$-\mathcal{M}(Z(T)) + \mathcal{M}(f(T)Z(0)) \le f(T)\int_0^T g(t)\mathrm{d}t,$$

thus

$$-\log\Big(\log\Big(\frac{1}{Z(T)}\Big)\Big) + \log\Big(\log\Big(\frac{e}{f(T)Z(0)}\Big)\Big) - 1 \le f(T)\int_0^T g(t)\mathrm{d}t.$$

By a direct computation, we get

$$Z(0) \le a(T) \Longrightarrow Z(T) \le b(T)\big(Z(0)\big)^{\gamma(T)}, \tag{3.26}$$

where a, b, γ are explicit functions depending continuously on time T and $\|(u^i, \omega^i)\|_{\mathcal{Z}_T}$ (noting that a is from the condition that $f(T)Z(0) < 1/e$ and $Z(T) < 1/e$).

3.3. Existence

We smooth the data to get the following approximate system

$$\begin{cases} \partial_t u_n + u_n\cdot\nabla u_n - (\nu+\kappa)\Delta u_n + \nabla P_n = 2\kappa\nabla\times\omega_n \\ \partial_t\omega_n + u_n\cdot\nabla\omega_n - \gamma\Delta\omega_n + 4\kappa\omega_n = 2\kappa\nabla\times u_n \\ \mathrm{div} u_n = 0, \quad (u_n,\omega_n)|_{t=0} = (S_n u^0, S_n\omega^0), \end{cases} \tag{3.27}$$

Because $S_n u^0, S_n\omega^0 \in H^s$ for every $s \in \mathbb{R}$, from the classical theory of quasi-linear hyperbolic systems (cf. [8, 15]), we can get the local wellposedness of the approximate system (3.27). We also have a natural continuation criterion as follows: the solution can go beyond the time T if the quantity $\|\nabla u_n\|_{L^1_T L^\infty}$ is finite. Then for the both cases, the a priori estimates (3.10) and (3.12) with $\sigma = 1$ guarantee the system (3.27) is globally defined. Moreover, we have the following uniform estimates that when $\gamma > 0$, for every $\sigma \in [1, 2[$,

$$\|u_n\|_{\widetilde{L}^\infty_T H^1 \cap \widetilde{L}^\sigma_T B^1_{\infty,1}} + \|\omega_n\|_{\widetilde{L}^\infty_T L^2 \cap \widetilde{L}^\sigma_T B^1_{2,1}} \lesssim_{\kappa,\nu,\gamma,\sigma} e^{e^{CT}},$$

and when $\gamma = 0$, for every $\sigma \in [1, 2[$,

$$\|u_n\|_{\widetilde{L}^\infty_T H^1 \cap \widetilde{L}^\sigma_T B^1_{\infty,1}} + \|\omega_n\|_{\widetilde{L}^\infty_T (L^2 \cap B^0_{\infty,1})} \lesssim_{\kappa,\sigma} e^{\exp\{e^{CT}\}}.$$

Thus there exists (u, ω) satisfying the above estimates such that (u_n, ω_n) weakly converges to (u, ω) up to the extraction of a subsequence. Furthermore, from Equation (3.26), if

$$\mathrm{d}_{n,m} := \left\|S_n u^0 - S_m u^0\right\|_{L^\infty_T B^0_{2,\infty}} + \left\|S_n\omega^0 - S_m\omega^0\right\|_{L^\infty_T B^{-1}_{2,\infty}} \le a(T),$$

then we get

$$\|u_n - u_m\|_{L^\infty_T B^0_{2,\infty}} + \|\omega_n - \omega_m\|_{L^\infty_T B^{-1}_{2,\infty}} \le b(T)(d_{n,m})^{\gamma(T)},$$

where a, b, γ are the explicit functions introduced in the upper subsection. This means that u_n is a Cauchy sequence and it converges strongly to u in $L^\infty_T B^0_{2,\infty}$. By interpolation, we further obtain the strong convergence of u_n to u in $L^2([0,T]\times\mathbb{R}^2)$. Thus $u_n \otimes u_n$ strongly converges to $u \otimes u$ in $L^1([0,T]\times\mathbb{R}^2)$. Meanwhile, because of the weak convergence of ω_n to ω in $L^2([0,T]\times\mathbb{R}^2)$, we have that $u_n\omega_n$ converges weakly to $u\omega$. It then suffices to pass to the limit in Equation (3.27) and we finally get that (u, ω) is a solution of the original system (1.1).

For the continuity-in-time issue, because we have Equations (3.11), (3.12), and (3.13), the proof is standard and we omit it (cf. [1]).

4. Proof of Theorem 1.2

Let $\kappa > 0$, $(u_\kappa, \omega_\kappa)$ be a solution of the following 2D micropolar fluid equations

$$\begin{cases} \partial_t u_\kappa + u_\kappa.\nabla u_\kappa - (\nu+\kappa)\Delta u_\kappa + \nabla P_\kappa = 2\kappa\nabla\times\omega_\kappa \\ \partial_t\omega_\kappa + u_\kappa.\nabla_{\omega_\kappa} - \gamma\Delta\omega_\kappa + 4\kappa\omega_\kappa = 2\kappa\nabla\times u_\kappa \\ \mathrm{div} u_\kappa = 0, \qquad (u_\kappa,\omega_\kappa)|_{t=0} = (u^0,\omega^0), \end{cases} \tag{4.1}$$

and (u,ω) be a solution of the limiting system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla P = 0 \\ \partial_t\omega + u\cdot\nabla\omega - \gamma\Delta\omega = 0 \\ \mathrm{div} u = 0, \quad (u,\omega)|_{t=0} = (u^0,\omega^0). \end{cases} \tag{4.2}$$

Denote $U_\kappa := u_\kappa - u$, $W_\kappa := \omega_\kappa - \omega$ and $\widetilde{P}_\kappa := P_\kappa - P$, then the difference system can be written as follows

$$\begin{cases} \partial_t U_\kappa + u_\kappa.\nabla U_\kappa - \nu\Delta U_\kappa + \nabla\widetilde{P}_\kappa = \kappa\Delta u_\kappa + 2\kappa\nabla\times\omega_\kappa - U_\kappa.\nabla u \\ \partial_t W_\kappa + u_\kappa.\nabla W_\kappa - \gamma\Delta W_\kappa = 2\kappa\nabla\times u_\kappa - 4\kappa\omega_\kappa - U_\kappa.\nabla\omega \\ \mathrm{div} U_\kappa = 0, \qquad (U_\kappa, W_\kappa)|_{t=0} = (0,0). \end{cases} \tag{4.3}$$

In the sequel, we shall consider the L^2 estimates of U_κ and W_κ according to different cases of ν,γ.

4.1. Case $\nu > 0, \gamma = 0, \kappa \in]0,\infty[$

because $(u^0,\omega^0) \in H^1 \times H^{1+\epsilon_0}$ with $\epsilon_0 \in]0,1[$, then from Theorem 1.1, the system (4.1) is uniquely and globally defined, and the solution satisfies

$$\|\nabla\times u_\kappa\|_{L_t^\infty L^2} \le C_0 e^{f_1(\kappa t,z)}, \quad \|\omega_\kappa\|_{L_t^\infty L^2} \le C_0,$$

where $f_1(\kappa t,z) \lesssim_z \kappa t + 1$ is defined by (3.4). For the same initial data (u^0,ω^0), clearly the system (4.2) is also uniquely and globally defined, and in particular, from the equation of vorticity $\Omega := \nabla\times u$

$$\partial_t\Omega + u\cdot\nabla\Omega - \nu\Delta\Omega = 0,$$

we have

$$\|u\|_{L_t^\infty H^1} + \|\omega\|_{L_t^\infty L^2} \le C_0;$$

and in a similar way as obtaining Equation (3.18),

$$\begin{aligned} \nu\|u\|_{L_t^1 B_{\infty,1}^{1+\epsilon_0}} &\lesssim \nu\|u\|_{L_t^1 L^2} + \nu\sum_{q\in\mathbb{N}} 2^{q(1+\epsilon_0)}\|\Delta_q\Omega\|_{L_t^1 L^2} \\ &\lesssim \nu t + \nu\sum_{q\le Q} 2^{q(1+\epsilon_0)}\|\Omega^0\|_{L^2} t + \sum_{q\ge Q} 2^{q(\epsilon_0-1)}\left(\|\Omega^0\|_{L^2} + \|\Omega\|_{L_t^1 L^\infty}\|\Omega\|_{L_t^\infty L^2}\right) \\ &\lesssim \nu t 2^{Q(1+\epsilon_0)} + 2^{-Q(1-\epsilon_0)}(1 + \|u\|_{L_t^1 B_{\infty,1}^{1+\epsilon_0}}), \end{aligned}$$

for a suitably chosen number $Q \in \mathbb{N}$, we get

$$\|u\|_{L_t^1 B_{\infty,1}^{1+\epsilon_0}} \lesssim 1 + \nu^{\frac{1+\epsilon_0}{1-\epsilon_0}} t;$$

and from the classical regularity effect of the transport equation (cf. Proposition 2.1 in [6]), we obtain

$$\|\omega\|_{L_t^\infty H^{1+\epsilon_0}} \le \|\omega^0\|_{H^{1+\epsilon_0}} e^{C\|\nabla u\|_{L_t^1 B_{\infty,1}^{\epsilon_0}}} \lesssim e^{C_\nu t}.$$

Then from the L^2 energy method, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|U_\kappa(t)\|_{L^2}^2 + \|W_\kappa(t)\|_{L^2}^2) + \nu\|\nabla U_\kappa(t)\|_{L^2}^2 \\ &\quad = -\kappa\int \nabla u_\kappa\cdot\nabla U_\kappa + 2\kappa\int\omega_\kappa(\nabla\times U_\kappa) - \int(U_\kappa.\nabla u)\cdot U_\kappa \\ &\qquad + 2\kappa\int(\nabla\times u_\kappa)W_\kappa - 4\kappa\int\omega_\kappa W_\kappa - \int(U_\kappa.\nabla\omega)W_\kappa. \end{aligned}$$

Because of $\nabla\omega \in L^\infty_t H^{\epsilon_0}$ and the Sobolev embedding $H^{\epsilon_0} \hookrightarrow L^{\frac{2}{1-\epsilon_0}}$, we have $\nabla\omega \in L^\infty_t L^{\frac{2}{1-\epsilon_0}}$ and $\|\nabla\omega\|_{L^\infty_t L^{\frac{2}{1-\epsilon_0}}} \lesssim e^{C_\nu t}$, thus

$$\begin{aligned}\left|\int (U_\kappa.\nabla\omega)W_\kappa\right| &\le \|U_\kappa\|_{L^{\frac{2}{\epsilon_0}}} \|\nabla\omega\|_{L^{\frac{2}{1-\epsilon_0}}} \|W_\kappa\|_{L^2} \\ &\lesssim \|U_\kappa\|_{L^2}^{\epsilon_0}\|\nabla U_\kappa\|_{L^2}^{1-\epsilon_0}\|\nabla\omega\|_{L^{\frac{2}{1-\epsilon_0}}}\|W_\kappa\|_{L^2} \\ &\le \|W_\kappa\|_{L^2}^2 + \frac{\nu}{4}\|\nabla U_\kappa\|_{L^2}^2 + C_0\frac{\epsilon_0}{1-\epsilon_0}\nu^{-\frac{1-\epsilon_0}{\epsilon_0}}\|U_\kappa\|_{L^2}^2\|\nabla\omega\|_{L^{\frac{2}{1-\epsilon_0}}}^{\frac{2}{\epsilon_0}}.\end{aligned}$$

Using the integration by parts, Gagliardo–Nirenberg's inequality, Sobolev's inequality ($H^1 \hookrightarrow L^4$) and Young's inequality, we have

$$\begin{aligned}&\left|\int (U_\kappa.\nabla u)\cdot U_\kappa\right| = \left|\int u\cdot(U_\kappa\cdot\nabla U_\kappa)\right| \\ &\le \|U_\kappa\|_{L^4}\|u\|_{L^4}\|\nabla U_\kappa\|_{L^2} \lesssim \|U_\kappa\|_{L^2}^{1/2}\|\nabla U_\kappa\|_{L^2}^{3/2} \le \frac{C_0}{\nu^3}\|U_\kappa\|_{L^2}^2 + \frac{\nu}{4}\|\nabla U_\kappa\|_{L^2}^2.\end{aligned}$$

Also we can directly get

$$\left|\kappa\int \nabla u_\kappa\cdot\nabla U_\kappa\right| + \left|2\kappa\int \omega_\kappa(\nabla\times U_\kappa)\right| \le \frac{\kappa^2}{\nu}\|\nabla u_\kappa\|_{L^2}^2 + \frac{4\kappa^2}{\nu}\|\omega_\kappa\|_{L^2}^2 + \frac{\nu}{2}\|\nabla U_\kappa\|_{L^2}^2,$$

and

$$\left|2\kappa\int(\nabla\times u_\kappa)W_\kappa\right| + \left|4\kappa\int \omega_\kappa W_\kappa\right| \le 2\kappa^2\|\nabla\times u_\kappa\|_{L^2}^2 + 8\kappa^2\|\omega_\kappa\|_{L^2}^2 + \|W_\kappa\|_{L^2}^2.$$

Thus gathering the upper estimates, we find

$$\begin{aligned}&\frac{d}{dt}(\|U_\kappa(t)\|_{L^2}^2 + \|W_\kappa(t)\|_{L^2}^2) \\ &\lesssim \kappa^2\Big(1+\frac{1}{\nu}\Big)\Big(\|\omega_\kappa(t)\|_{L^2}^2 + \|\nabla\times u_\kappa\|_{L^2}^2\Big) + \Big(1+\frac{1}{\nu^3}+e^{C_{\nu,\epsilon_0}t}\Big)\Big(\|U_\kappa(t)\|_{L^2}^2 + \|W_\kappa(t)\|_{L^2}^2\Big).\end{aligned}$$

Therefore the Gronwall inequality yields

$$\|U_\kappa(t)\|_{L^2}^2 + \|W_\kappa(t)\|_{L^2}^2 \le C_0\Big(1+\frac{1}{\nu}\Big)e^{C_0t/\nu^3}e^{\exp\{C_{\nu,\epsilon_0}t\}}e^{f_1(\kappa t,z)}(\kappa t)^2.$$

4.2. Case $\nu = 0, \gamma > 0, \kappa \in]0,\gamma[$

Because $(u^0,\omega^0) \in B^2_{2,1}\times B^0_{2,1}$, then from Theorem 1.1 and the result in [20], the system (4.1) and the limiting system (4.2) are both globally and uniquely defined. Moreover, the corresponding solutions satisfy (noting that in this case, $z = \frac{2\kappa}{\gamma-\kappa} > 0$)

$$\|\Omega_\kappa\|_{L^\infty_t L^2} + \|\omega_\kappa\|_{L^\infty_t L^2} + \kappa^{1/2}\|\Omega_\kappa\|_{L^2_t\dot H^1} + \gamma^{1/2}\|\omega_\kappa\|_{L^2_t\dot H^1} \lesssim e^{f_2(\kappa t,z)}, \tag{4.4}$$

and

$$\|\nabla u\|_{L^\infty_t L^\infty} \lesssim \|u\|_{L^\infty_t L^2} + \|\nabla\times u\|_{L^\infty_t B^0_{\infty,1}} \lesssim e^{C_0t},$$

$$\begin{aligned}\|\nabla\omega\|_{L^1_tL^\infty} &\lesssim \|\omega\|_{L^1_tL^2} + \|(\mathrm{Id}-\Delta_{-1})\omega\|_{L^1_tB^2_{2,1}} \\ &\lesssim t + \gamma^{-1}e^{C\|\nabla u\|_{L^1_tL^\infty}}\|\omega^0\|_{B^0_{2,1}} \lesssim (1+\gamma^{-1})e^{\exp C_0t},\end{aligned}$$

where $\Omega_\kappa := \nabla\times u_\kappa$ and $f_2(\kappa t,z) \sim_z \kappa t + 1$ is defined by Equation (3.5). By using the L^2 method, we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}(\|U_\kappa(t)\|_{L^2} + \|W_\kappa(t)\|_{L^2}) \le &\kappa\|\Delta u_\kappa\|_{L^2} + 2\kappa\|\nabla\times\omega_\kappa\|_{L^2} + \|U_\kappa\|_{L^2}\|\nabla u\|_{L^\infty} \\ &+ 2\kappa\|\Omega_\kappa\|_{L^2} + 4\kappa\|\omega_\kappa\|_{L^2} + \|U_\kappa\|_{L^2}\|\nabla\omega\|_{L^\infty}.\end{aligned}$$

Thus the Gronwall inequality and Equation (4.4) ensures

$$\begin{aligned}\|U_\kappa(t)\|_{L^2} + \|W_\kappa(t)\|_{L^2} &\lesssim e^{C_\gamma\exp\{\exp\{C_0t\}\}}\kappa\Big(\|\Omega_\kappa\|_{L^1_t\dot H^1} + \|\omega_\kappa\|_{L^1_t\dot H^1} + \|\Omega_\kappa\|_{L^1_tL^2} + \|\omega_\kappa\|_{L^1_tL^2}\Big) \\ &\lesssim e^{C_\gamma\exp\{\exp^{\{C_0t\}}\}}e^{f_2(\kappa t,z)}(\kappa t + (\kappa t)^{1/2}).\end{aligned} \tag{4.5}$$

4.3. Case $\nu=\gamma=0, \kappa\in]0,1[$

Let $(u^0,\omega^0)\in B^2_{2,1}\times(B^1_{2,1}\cap B^1_{\infty,1})$, then the associated solutions (u_κ,ω_κ) and (u,ω) are globally defined. Moreover, we have the following explicit estimates.

Lemma 4.1
For the solution (u_κ,ω_κ) of the 2D micropolar fluid equation (4.1), we have that for every $p\in]2,\infty[$

$$\|\omega_\kappa\|_{L^\infty_t B^1_{2,1}}\lesssim(\kappa t)^{-\frac{p}{2(p-1)}}e^{\exp\{\exp\{C_0\kappa t\}\}}e^{\exp\{C_0 t\}}. \tag{4.6}$$

Whereas for the solution (u,ω) of the limiting equation (4.2), we have

$$\|u\|_{L^\infty_t B^1_{\infty,1}}\lesssim e^{C_0 t},\quad \|\omega\|_{L^\infty_t B^1_{\infty,1}}\lesssim e^{\exp\{C_0 t\}}. \tag{4.7}$$

Clearly, based on this Lemma and in a similar way as obtaining Equation (4.5), we get that for every $p\in]2,\infty[$

$$\begin{aligned}\|U_\kappa(t)\|_{L^2}+\|W_\kappa(t)\|_{L^2}&\lesssim e^{\|\nabla(u,\omega)\|_{L^1_tL^\infty}}\kappa\Big(\|\Omega_\kappa\|_{L^1_t\dot H^1}+\|\omega_\kappa\|_{L^1_t\dot H^1}+\|\Omega_\kappa\|_{L^1_tL^2}+\|\omega_\kappa\|_{L^1_tL^2}\Big)\\&\lesssim\big((\kappa t)^{\frac{p-2}{2(p-1)}}+\kappa t\big)e^{\exp\{\exp\{C_0t\}\}}e^{\exp\{\exp\{C_0\kappa t\}\}}.\end{aligned}$$

Now, it suffices to prove this Lemma.

Proof of Lemma 4.1
We first prove Equation (4.6). From the equation of $\Gamma_\kappa:=\Omega_\kappa-2\omega_\kappa$ (noting that $z=-2$)

$$\partial_t\Gamma_\kappa+u_\kappa.\nabla\Gamma_\kappa-\kappa\Delta\Gamma_\kappa+4\kappa\Gamma_\kappa=0,$$

from Proposition 2.5, Proposition 3.2 and Proposition 3.3, we find that for every $q\in\mathbb{N}$

$$\begin{aligned}\|\Delta_q\Gamma_\kappa\|_{L^1_tL^p}&\lesssim\kappa^{-1}2^{-2q}\Big(\|\Delta_q\Gamma^0\|_{L^p}+\|\Omega_\kappa\|_{L^\infty_tL^p}\|\Gamma_\kappa\|_{L^1_tL^\infty}\Big)\\&\lesssim\kappa^{-1}2^{-2q}\Big(1+e^{C_0\kappa t}t\Big).\end{aligned}$$

By a high–low frequency decomposition, and for every $\tilde Q\in\mathbb{N}$ and $p\in]1,\infty[$, we have

$$\begin{aligned}\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}&\le\sum_{-1\le q\le\tilde Q}\|\Delta_q\Gamma_\kappa\|_{L^1_tL^\infty}+\sum_{q>\tilde Q}2^{2q/p}\|\Delta_q\Gamma_\kappa\|_{L^1_tL^p}\\&\lesssim(1+\tilde Q)t\|\Gamma_\kappa\|_{L^\infty_tL^\infty}+\sum_{q>\tilde Q}2^{-q(2-\frac{2}{p})}\kappa^{-1}e^{C_0\kappa t}(1+t)\\&\lesssim(1+\tilde Q)e^{C_0\kappa t}t+2^{-\tilde Q(2-\frac{2}{p})}\kappa^{-1}e^{C_0\kappa t}(1+t).\end{aligned}$$

We choose $\tilde Q$ such that $2^{\tilde Q(2-\frac{2}{p})}\sim\kappa^{-1}$, thus

$$\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}\lesssim(1+\log(\kappa^{-\frac{p}{2(p-1)}}))e^{C_0\kappa t}(1+t). \tag{4.8}$$

Now, from the equation of ω_κ

$$\partial_t\omega_\kappa+u_\kappa.\nabla\omega_\kappa=-4\kappa\Gamma_\kappa,$$

and using Proposition 2.5 and Equation (3.21), we get

$$\begin{aligned}\|\omega_\kappa(t)\|_{B^0_{\infty,1}}&\lesssim\Big(\|\omega^0\|_{B^0_{\infty,1}}+\kappa\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}\Big)\Big(1+\|\nabla u_\kappa\|_{L^1_tL^\infty}\Big)\\&\lesssim\Big(1+\kappa\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}\Big)\Big(1+t+\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}+\int_0^t\|\omega_\kappa(\tau)\|_{B^0_{\infty,1}}\mathrm{d}\tau\Big).\end{aligned}$$

The Gronwall inequality yields

$$\|\omega_\kappa(t)\|_{B^0_{\infty,1}}\lesssim(1+\log(\kappa^{-\frac{p}{2(p-1)}}))e^{\exp\{C_0\kappa t\}}e^{C_0t}.$$

This combining with Equation (4.8) further leads to

$$\begin{aligned}\|u_\kappa\|_{L^1_tB^1_{\infty,1}}&\lesssim\|\Delta_{-1}u_\kappa\|_{L^1_tL^2}+\|\Gamma_\kappa\|_{L^1_tB^0_{\infty,1}}+\|\omega_\kappa\|_{L^1_tB^0_{\infty,1}}\\&\lesssim(1+\log(\kappa^{-\frac{p}{2(p-1)}}))e^{\exp\{C_0\kappa t\}}e^{C_0t}.\end{aligned}$$

Hence, from Proposition 2.4 and the following estimation

$$\begin{aligned}\kappa\|\Gamma_\kappa\|_{L^1_tB^1_{2,1}} &\lesssim \kappa\|\Delta_{-1}\Gamma_\kappa\|_{L^1_tL^2}+\kappa\sum_{q\in\mathbb{N}}2^q\|\Delta_q\Gamma_\kappa\|_{L^1_tL^2}\\ &\lesssim \kappa t+\sum_{q\in\mathbb{N}}2^{-q}(1+t)e^{C_0\kappa t}\lesssim e^{C_0\kappa t}(1+t),\end{aligned}$$

we obtain

$$\begin{aligned}\|\omega_\kappa\|_{L^\infty_tB^1_{2,1}} &\lesssim e^{C_0\|u_\kappa\|_{L^1_tB^1_{\infty,1}}}\left(\|\omega^0_\kappa\|_{B^1_{2,1}}+\kappa\|\Gamma_\kappa\|_{L^1_tB^1_{2,1}}\right)\\ &\lesssim \kappa^{-\frac{p}{2(p-1)}}e^{\exp\{\exp\{C_0\kappa t\}\}}e^{\exp\{C_0t\}}.\end{aligned}$$

We then treat Equation (4.7). From the equation of vorticity $\partial_t\Omega+u\cdot\nabla\Omega=0$, by applying Proposition 2.5 and (3.21), we get

$$\begin{aligned}\|\Omega(t)\|_{B^0_{\infty,1}}\le\|\Omega\|_{\widetilde{L}^\infty_tB^0_{\infty,1}} &\lesssim \|\Omega^0\|_{B^0_{\infty,1}}\left(1+\|\nabla u\|_{L^1_tL^\infty}\right)\\ &\lesssim 1+t+\int_0^t\|\Omega(\tau)\|_{B^0_{\infty,1}}\,\mathrm{d}\tau.\end{aligned}$$

The Gronwall inequality leads to

$$\|\Omega(t)\|_{B^0_{\infty,1}}\lesssim e^{C_0t}.$$

This further implies that

$$\|u\|_{L^\infty_tB^1_{\infty,1}}\lesssim\|\Delta_{-1}u\|_{L^\infty_tL^2}+\|\Omega\|_{L^\infty_tB^0_{\infty,1}}\lesssim e^{C_0t}.$$

Thus from the equation $\partial_t\omega+u\cdot\nabla\omega=0$ and Proposition 2.4, we have

$$\|\omega\|_{L^\infty_tB^1_{\infty,1}}\lesssim\|\omega^0\|_{B^1_{\infty,1}}e^{C_0\|u\|_{L^1_tB^1_{\infty,1}}}\lesssim e^{\exp\{C_0t\}}.$$

□

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