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On the locally self-similar solution of the surface quasi-geostrophic equation with decaying or non-decaying profiles

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Abstract

Motivated by the numerical simulation and the study on several 1D models, we consider the locally self-similar singular solutions for the surface quasi-geostrophic equation with decaying or non-decaying blowup profiles. Based on a suitable local L^p-inequality in terms of the profile and the bootstrapping method, we show some exclusion results and derive the asymptotic behavior of the possible blowup profiles.

MSC: 76B03; 35Q31; 35Q35; 35Q86

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1. Introduction

In this paper we address the Cauchy problem of the surface quasi-geostrophic (abbr. SQG) equation

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$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ u = (u_1, u_2) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\ \theta|_{t=0} = \theta_0, \end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^2\times\mathbb{R}^+$, $\mathcal{R}_i = \partial_i|D|^{-1}$ $(i=1,2)$ is the usual Riesz transform, $\theta:\mathbb{R}^2\to\mathbb{R}$ is a scalar field understood as temperature or density field, and $u:\mathbb{R}^2\to\mathbb{R}^2$ is the velocity field of $\mathbb{R}^2$. The SQG equation arises from the geostrophic study of the highly rotating flow (cf. [17]) and is viewed as a 2D simple model sharing much formal analogy with the 3D Euler equations (cf. [8]). It is known for some time that the SQG equation associated with smooth initial data generates a local smooth solution, e.g., for $\theta_0\in H^s(\mathbb{R}^2)$, $s>2$, there exists a unique solution $\theta\in C([0,T[;H^s(\mathbb{R}^2))$ with some $T>0$ and u is expressed as

$$u(x,t) = \mathrm{p.v.}\int_{\mathbb{R}^2} K^\perp(x-z)\theta(z,t)\mathrm{d}z, \tag{1.2}$$

with $K^\perp(z) := \frac{1}{2\pi}\frac{z^\perp}{|z|^3}$, $\forall z\neq 0$; moreover, if additionally $\theta_0\in L^p(\mathbb{R}^2)$ with $p\in[1,\infty]$, the solution also satisfies $\theta\in L^\infty([0,T[;L^p(\mathbb{R}^2))$ with $\|\theta\|_{L^\infty([0,T[;L^p(\mathbb{R}^2))}\le\|\theta_0\|_{L^p(\mathbb{R}^2)}$ (e.g. [14, Proposition 6.2]). In recent years there have been intense mathematical works on the SQG equation and its dissipative cases (one can see [2] for a long list of references), but so far the fundamental problem: whether the local smooth solutions remain regular forever or develop blowup singularity at finite time, remains completely open.

Some numerical simulation was also made to understand the issue of finite-time blowup. It was suggested by [15,18,13], and very recently by [19,20] that the finite-time singularity formulation for the SQG equation is possible to happen, via a self-similar cascade of filament instabilities of geometrically decreasing spatial and temporal scales. Other past numerical studies [8,9,16] mainly focused on the flow of a closing saddle geometry, and through a much higher resolution than [9,16], Scott in [19] pointed out that the self-similar type filament instability mentioned above was also potentially important in this scenario.

Motivated by the numerical work, we here mainly focus on the self-similar singular solution for the SQG equation (1.1), i.e., the solution of the form

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right), \quad \forall(x,t)\in D\times\,]0,T[, \tag{1.3}$$

where $\alpha>-1$, $x_0\in D$, $D\subset\mathbb{R}^2$ is the blowup region, $T>0$ is the finite blowup time, $\Theta(\cdot)$ is the stationary profiles of θ, and the solution θ is regular enough on $(\mathbb{R}^2\setminus D)\times\,]0,T[$. If $D=\mathbb{R}^2$, the self-similar solution (1.3) is referred to as the globally self-similar solution; while if $D\subsetneqq\mathbb{R}^2$, (1.3) is called the locally self-similar solution. For the globally self-similar solution (1.3), the velocity field u is also globally self-similar satisfying that

$$u(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}}U\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right), \quad \forall(x,t)\in\mathbb{R}^2\times\,]0,T[, \tag{1.4}$$

with $U(\cdot)$ the velocity profile. In terms of (Θ,U), we formally get

$$\begin{cases} \frac{\alpha}{1+\alpha}\Theta + \frac{1}{1+\alpha} y \cdot \nabla\Theta + U \cdot \nabla\Theta = 0, \\ U = \mathcal{R}^\perp \Theta = (-\mathcal{R}_2\Theta, \mathcal{R}_1\Theta). \end{cases} \tag{1.5}$$

The self-similar singularity is an important and popular type of finite-time singularities that have been abundantly studied for Navier–Stokes/Euler equations, reaction–diffusion equations, dispersive equations and so on, and one can see the recent review paper [12] for more introductions. For the globally self-similar solution (i.e. $D = \mathbb{R}^2$ in (1.3)) of the SQG equation, Chae in [5,6] based on the transport formula of θ represented by the back to label map, proved that $\Theta \equiv 0$ for all $\alpha > -1$ under the assumption $\Theta \in L^{p_1} \cap L^{p_2}(\mathbb{R}^2)$ with $p_1, p_2 \in]0, \infty]$ and $p_1 < p_2$, which excludes the profiles with decaying asymptotics. One can see Cannone and Xue [3] for a similar result which relied on the local L^p-inequality of profiles and the bootstrapping method. However, there is not much work on the self-similar solution of SQG equation with profiles having non-decaying asymptotics, and the more physical locally self-similar solution was also not much addressed. Noting that for the Burgers equation $\partial_t\theta + \theta\partial_x\theta = 0$, $x \in \mathbb{R}$, and the CCF equation (cf. [10])

$$\partial_t\theta + H(\theta)\,\partial_x\theta = 0, \quad x \in \mathbb{R},\ H \text{ the usual Hilbert transform,} \tag{1.6}$$

which can be viewed as the 1D models of the SQG equation, recent studies [12,11] reveal that the finite-time singularities of both systems are of locally self-similar type and the associated blowup profiles have some growing asymptotics near infinity, thus the corresponding scenario for the SQG equation is also worthwhile to investigate.

In this paper we consider the locally self-similar solution of the SQG equation, and show some excluding results and derive the spatially asymptotic behavior of possible profiles. Our main result is as follows.

Theorem 1.1. *Assume that $D \supset B_\rho(x_0)$, $\rho > 0$, and $\theta \in C([0, T[; H^s(\mathbb{R}^2)) \cap L^\infty([0, T[; L^1(\mathbb{R}^2))$, $s > 2$ is the locally self-similar singular solution for the SQG equation which satisfies (1.3) on $D \times]0, T[$ for $\alpha > -1$ with profile $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Let $p \in]1, \infty[$ be fixed, and additionally suppose that for some $r \geq p + 1$, $0 \leq \gamma < r - p$,*

$$\int_{|y| \leq L} |\Theta(y)|^r \,\mathrm{d}y \lesssim L^\gamma, \quad \forall L \gg 1. \tag{1.7}$$

Then either $\Theta \equiv 0$, or the index α admitting nontrivial profiles belongs to $\frac{2-\gamma}{r} \leq \alpha \leq \frac{2}{p}$, and the profile corresponding to each $\alpha \in [\frac{2-\gamma}{r}, \frac{2}{p}[$ satisfies

$$\int_{|y| \leq L} |\Theta(y)|^p \,\mathrm{d}y \sim L^{2-p\alpha}, \quad \forall L \gg 1. \tag{1.8}$$

As a consequence of Theorem 1.1, we have the following result about the profile with spatially decaying or non-decaying asymptotics.

Theorem 1.2. *Assume that $D \supset B_\rho(x_0)$, $\rho > 0$, and $\theta \in C([0, T[; H^s(\mathbb{R}^2)) \cap L^\infty([0, T[; L^1(\mathbb{R}^2))$, $s > 2$ is the locally self-similar singular solution for the SQG equation which satis-*

fies (1.3) on $D \times]0, T[$ for $\alpha > -1$ with profile $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$. The following statements hold true.

(1) If there is some $\mu > 0$ so that $|\Theta(y)| \lesssim \frac{1}{|y|^\mu}$ for all $|y| \gg 1$, then we necessarily have $\Theta \equiv 0$ on $\mathbb{R}^2$.

(2) If there is some number $\sigma \in [0, 1[$ so that

$$1 \lesssim |\Theta(y)| \lesssim |y|^\sigma, \quad \forall |y| \gg 1, \tag{1.9}$$

then the index α admitting nontrivial profiles belongs to $-\sigma \le \alpha \le 0$, and each profile corresponding to such an α satisfies that for every $p \in]1, \infty[$,

$$\int_{|y|\le L} |\Theta(y)|^p \,\mathrm{d}y \sim L^{2-p\alpha}, \quad \forall L \gg 1. \tag{1.10}$$

The method of proving Theorem 1.1, similarly as [3], is from the local L^p-inequality of the profile Θ and the bootstrapping method. We first derive a suitable basic local L^p-inequality (2.12) from the local L^p-equality (2.1) of the original quantity θ, and then we start with (2.12) by appropriately choosing l_1, l_2, and we iteratively improve the upper bound on $\int_{|y|\le L} |\Theta(y)|^p \mathrm{d}y$ by a careful analysis according to the scope of α and (1.7), so that we eventually obtain the desired result. In the process, the term $U^{(1)}(y)$ appearing in the right-hand-side of (2.12) needs to be well estimated again and again, and we tackle this point in Lemma 2.2. For Theorem 1.2, we suitably select the values of p, r, γ so that the assumption (1.7) is satisfied, and thus we can apply Theorem 1.1 to obtain Theorem 1.2.

The use of local inequalities of self-similar profiles in showing the exclusion results has already appeared in the contexts of self-similar solutions of Euler equations [7,1,21], and also the previous work [3], but here the local L^p-inequality (2.12) is derived in a slightly different manner. The reason is that if $D \ne \mathbb{R}^2$, the formula (1.4) in general no longer holds in the domain $D \times]0, T[$ and we also do not have the explicit expression formula of U in terms of Θ (see also Remark 1.5 below), while both counterparts were obtained and employed in the local inequalities of [7,1,21,3]. In order to overcome this difficulty, we observe that in the local equality (2.1) we only need to treat the velocity u defined by (1.2) in a small region $B_{\frac{\rho}{4}}(x_0) \times]0, T[$, and we can divide the whole integral region of (1.2) into $B_\rho(x_0)$ and $\mathbb{R}^2 \setminus B_\rho(x_0)$ to estimate the contribution respectively: for the inner part we can use the self-similar scenario (1.3) to get a good bound; while for the outer part, we can simply apply the conservation of $\|\theta(t)\|_{L^1}$ and the support property to control it. It turns out that the local L^p-inequality (2.12) derived in this way is sufficient for our purpose here.

A few remarks are listed as follows.

Remark 1.3. From (1.10), we can expect that the typical possible asymptotics of the nontrivial profiles at the case $-1 < \alpha \le 0$ is $|\Theta(y)| \sim |y|^{-\alpha}$, $\forall |y| \gg 1$, which thanks to (1.3) further implies $|\theta(x,t)| \sim |x - x_0|^{-\alpha}$ for $x \in D \setminus \{x_0\}$ and t sufficiently close to T. Such a scenario clearly is consistent with the blowup criterion $\int_0^T \|\nabla\theta(t)\|_{L^\infty} \mathrm{d}t = \infty$ for $T > 0$ the first blowup time. We also see that the uniform-in-t bound of $\|\theta(t)\|_{C^\beta(\mathbb{R}^2)}$, $\beta \in]0, 1[$ (which of course has not been a priori proved so far) is sufficient to exclude the locally self-similar scenario (1.3) with typical asymptotics of profiles at the case $0 \le \alpha < \beta$, but still can not deal with the case $\beta \le \alpha < 1$.

Remark 1.4. According to the numerical simulation [19] and the work [11] on the CCF equation (1.6), the blowup scenario more likely to happen is of the peaked self-similar structure:

$$\theta(x,t) = A_0 - \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right), \quad \forall (x,t) \in D \times]0,T[, \tag{1.11}$$

where $A_0 := \|\theta_0\|_{L^\infty}$ and $x_0, T, \alpha, \Theta, D$ are as above. By virtue of Remark 2.1 below, we indeed can repeat the proof of Theorem 1.1 to achieve the same conclusion for the profile Θ in (1.11).

Remark 1.5. Under the assumption of Theorem 1.2-(2) with $D = \mathbb{R}^2$, and additionally assuming that $\Theta \in C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\nabla\Theta(y)| \lesssim |y|^{\sigma_1 - 1}$, $\forall |y| \gg 1$ with some $\sigma_1 \in]0,1[$, one can use the method of [21, Lemma 2.1] or [1, Lemma 2.1] to show that the formula (1.4) holds, and the profile U on $D = \mathbb{R}^2$ is given by

$$U(y) = \mathrm{p.v.}\int\limits_{|z|\le M} K^\perp(y-z)\Theta(z)\mathrm{d}z + \int\limits_{|z|\ge M} (K^\perp(y-z) - K^\perp(z))\Theta(z)\mathrm{d}z + (C_1, C_2) \tag{1.12}$$

where $M > 0$, C_1, C_2 are some pure numbers, and $K^\perp(z) = \frac{1}{2\pi}\frac{(-z_2,z_1)}{|z|^3}$, $\forall z \neq 0$. But it is not clear whether or not we can get a similar result for the locally self-similar scenario (1.3) with $D = B_\rho(x_0) \subsetneq \mathbb{R}^2$.

The plan of this paper is as follows: we show the local energy inequality of profile Θ and also Lemma 2.2 concerning the estimate of $U^{(1)}$ in the section 2, and then we prove Theorems 1.1 and 1.2 in the sections 3 and 4 respectively.

Throughout this paper, C stands for a constant which may be different from line to line, $X \lesssim Y$ means that there is a harmless constant C such that $X \le CY$, and $X \sim Y$ means that $X \lesssim Y$ and $Y \lesssim X$ simultaneously. We use $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| \le r\}$ to denote the ball of $\mathbb{R}^2$.

2. Local L^p-inequality in terms of the profile

We start with the local L^p-equality of the original qualities (θ, u)

$$\begin{aligned} &\int\limits_{\mathbb{R}^2} |\theta|^p \chi(x,t_2)\mathrm{d}x - \int\limits_{\mathbb{R}^2} |\theta|^p \chi(x,t_1)\mathrm{d}x \\ &= \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^2} |\theta|^p \partial_t \chi(x,t)\mathrm{d}x\mathrm{d}t + \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^2} |\theta|^p u\cdot\nabla\chi(x,t)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.1}$$

where $\chi \in C_c^\infty(\mathbb{R}^2 \times]0,T[)$ and $0 < t_1 < t_2 < T$. Since $\theta \in C([0,T[; H^s(\mathbb{R}^2))$, $s > 2$ is smooth enough, we can derive (2.1) simply from the approximation (if needed) and the integration by parts, and one can see [4] for a similar result concerning the weak solution θ under less regular conditions.

Without loss of generality we assume $x_0 = 0$. Let $\phi \in C_c^\infty(\mathbb{R}^2)$ be a test function such that $\operatorname{supp}\phi \subset B_1(0)$, $\phi \equiv 1$ on $B_{\frac{1}{2}}(0)$ and $0 \le \phi \le 1$. Set $\phi_R(\cdot) = \phi(\frac{\cdot}{R})$ for $R > 0$, and $\chi(x,t) \equiv \phi_{\frac{\rho}{4}}(x)$, then (2.1) reduces to

$$\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \phi_{\frac{\rho}{4}}(x)\,\mathrm{d}x - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \phi_{\frac{\rho}{4}}(x)\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta|^p u(x,t)\cdot\nabla\phi_{\frac{\rho}{4}}(x)\,\mathrm{d}x\mathrm{d}t. \tag{2.2}$$

Since $B_\rho(0) \subset D$ and θ is expressed as (1.3) on $D \times]0,T[$, we see that for all $0 < t_1 < t_2 < T$,

$$\begin{aligned}
\int_{\mathbb{R}^2} |\theta(x,t_i)|^p \phi_{\frac{\rho}{4}}(x)\mathrm{d}x &= \frac{1}{(T-t_i)^{\frac{\alpha p}{1+\alpha}}} \int_{\mathbb{R}^2} \left|\Theta\left(\frac{x}{(T-t_i)^{1/(1+\alpha)}}\right)\right|^p \phi_{\frac{\rho}{4}}(x)\,\mathrm{d}x \\
&= \frac{1}{(T-t_i)^{\frac{\alpha p-2}{1+\alpha}}} \int_{\mathbb{R}^2} |\Theta(y)|^p\, \phi_{\frac{\rho}{4}}\left(y(T-t_i)^{\frac{1}{1+\alpha}}\right)\mathrm{d}y \\
&= l_i^{\alpha p-2}\int_{\mathbb{R}^2} |\Theta(y)|^p\, \phi_{\frac{\rho}{4}}(y l_i^{-1})\,\mathrm{d}y,
\end{aligned} \tag{2.3}$$

with $l_i = (T-t_i)^{-\frac{1}{1+\alpha}}$, $i = 1, 2$. Taking advantage of (1.2), we decompose u as

$$\begin{aligned}
u(x,t) &= \mathrm{p.v.}\int_{\mathbb{R}^2} K^\perp(x-z)\theta(z,t)\phi_\rho(z)\mathrm{d}z + \int_{\mathbb{R}^2} K^\perp(x-z)\theta(z,t)\left(1-\phi_\rho(z)\right)\mathrm{d}z \\
&:= u^{(1)}(x,t) + u^{(2)}(x,t),
\end{aligned} \tag{2.4}$$

where $u^{(1)}$ according to (1.3) satisfies

$$\begin{aligned}
u^{(1)}(x,t) &= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}}\mathrm{p.v.}\int_{\mathbb{R}^2} K^\perp(x-z)\Theta\left(\frac{z}{(T-t)^{\frac{1}{1+\alpha}}}\right)\phi_\rho(z)\,\mathrm{d}z \\
&= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}}\,\mathrm{p.v.}\int_{\mathbb{R}^2} K^\perp\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}} - \tilde{z}\right)\Theta(\tilde{z})\,\phi_\rho\left(\tilde{z}(T-t)^{\frac{1}{1+\alpha}}\right)\mathrm{d}\tilde{z} \\
&= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} U^{(1)}\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}, t\right),
\end{aligned} \tag{2.5}$$

with

$$U^{(1)}(y,t) := \mathrm{p.v.}\int_{\mathbb{R}^2} K^\perp(y-\tilde{z})\,\Theta(\tilde{z})\,\phi_\rho\left(\tilde{z}(T-t)^{\frac{1}{1+\alpha}}\right)\mathrm{d}\tilde{z}. \tag{2.6}$$

Inserting (1.3) and (2.4)–(2.5) into the right-hand-side term of (2.2), we find

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta|^p u(x,t)\cdot\nabla\phi_{\frac{\rho}{4}}(x)\,\mathrm{d}x\mathrm{d}t\right| \\
&\leq \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{(p+1)\alpha}{1+\alpha}}}\int_{\mathbb{R}^2}\left|\Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right)\right|^p \left|U^{(1)}\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}},t\right)\right| |\nabla\phi_{\frac{\rho}{4}}(x)|\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{p\alpha}{1+\alpha}}}\int_{\mathbb{R}^2}\left|\Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right)\right|^p |u^{(2)}(x,t)|\,|\nabla\phi_{\frac{\rho}{4}}(x)|\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{(p+1)\alpha-2}{1+\alpha}}}\int_{\mathbb{R}^2} |\Theta(y)|^p\,|U^{(1)}(y,t)|\,|\nabla\phi_{\frac{\rho}{4}}(y(T-t)^{\frac{1}{1+\alpha}})|\mathrm{d}y\mathrm{d}t \\
&\quad + \frac{C}{\rho^2}\int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{p\alpha-2}{1+\alpha}}}\int_{\mathbb{R}^2} |\Theta(y)|^p\,|\nabla\phi_{\frac{\rho}{4}}(y(T-t)^{\frac{1}{1+\alpha}})|\mathrm{d}y\mathrm{d}t,
\end{aligned}
\tag{2.7}
$$

where in the last line we used the estimate that for every $x\in B_{\frac{\rho}{4}}(0)$,

$$
\begin{aligned}
|u^{(2)}(x,t)| &\leq \int_{\mathbb{R}^2}\frac{1}{|x-z|^2}|\theta(z,t)|\,|1-\phi_\rho(z)|\mathrm{d}z \\
&\leq C\int_{|z|\geq\frac{\rho}{2}}\frac{1}{|z|^2}|\theta(z,t)|\mathrm{d}z \leq \frac{C}{\rho^2}\|\theta(\cdot,t)\|_{L^1} \leq \frac{C}{\rho^2}\|\theta_0\|_{L^1}.
\end{aligned}
\tag{2.8}
$$

In (2.7), by using the support property of $\nabla\phi_\rho$ and integrating on the t-variable, and denoting

$$
B(t) := \left\{t : \frac{\rho}{8}\frac{1}{|y|} \leq (T-t)^{\frac{1}{1+\alpha}} \leq \frac{\rho}{4}\frac{1}{|y|}\right\},
\tag{2.9}
$$

we further deduce that for every $0<t_1<t_2<T$,

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta|^p u(x,t)\cdot\nabla\phi_{\frac{\rho}{4}}(x)\,\mathrm{d}x\mathrm{d}t\right| \\
&\leq C_\rho\int_{t_1}^{t_2}\int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\Theta(y)|^p|U^{(1)}(y,t)|}{|y|^{2-(p+1)\alpha}}1_{B(t)}\,\mathrm{d}y\mathrm{d}t + C_\rho\int_{t_1}^{t_2}\int_{\frac{\rho}{8}l_1\leq|y|\leq\frac{\rho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{2-p\alpha}}1_{B(t)}\,\mathrm{d}y\mathrm{d}t
\end{aligned}
$$

$$\leq C_\rho \int\limits_{\frac{\rho}{8}l_1 \leq |y| \leq \frac{\rho}{4}l_2} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} \, \mathrm{d}y + C_\rho \int\limits_{\frac{\rho}{8}l_1 \leq |y| \leq \frac{\rho}{4}l_2} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}} \, \mathrm{d}y, \tag{2.10}$$

with $1_{B(t)}$ denoting the standard indicator function on $B(t)$, and $U^{(1)}(y)$ defined by

$$\begin{aligned} U^{(1)}(y) &:= \int\limits_{t_1}^{t_2} |U^{(1)}(y,t)| \, 1_{B(t)} \, \mathrm{d}t \\ &= \int\limits_{t_1}^{t_2} \left| \text{p.v.} \int\limits_{\mathbb{R}^2} K^\perp(y-\tilde{z}) \, \Theta(\tilde{z}) \, \phi_\rho\left(\tilde{z}(T-t)^{\frac{1}{1+\alpha}}\right) \mathrm{d}\tilde{z} \right| 1_{B(t)} \, \mathrm{d}t. \end{aligned} \tag{2.11}$$

Gathering (2.2), (2.3) and (2.10) leads to

$$\begin{aligned} &\left| l_2^{\alpha p-2} \int\limits_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\rho}{4}}(y l_2^{-1}) \, \mathrm{d}y - l_1^{\alpha p-2} \int\limits_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\rho}{4}}(y l_1^{-1}) \mathrm{d}y \right| \\ &\leq C_\rho \int\limits_{\frac{\rho}{8}l_1 \leq |y| \leq \frac{\rho}{4}l_2} \frac{|\Theta(y)|^p \, U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} \mathrm{d}y + C_\rho \int\limits_{\frac{\rho}{8}l_1 \leq |y| \leq \frac{\rho}{4}l_2} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}} \, \mathrm{d}y, \end{aligned} \tag{2.12}$$

where $l_i = (T-t_i)^{-\frac{1}{1+\alpha}}$, $i=1,2$, and $U^{(1)}(y)$ is given by (2.11).

Remark 2.1. Denoting $\tilde{\theta} := A_0 - \theta$, we see that $\partial_t \tilde{\theta} + u \cdot \nabla \tilde{\theta} = 0$ with $u = \mathcal{R}^\perp \theta = \mathcal{R}^\perp(A_0 + \tilde{\theta})$. By multiplying both sides of the equation with $|\tilde{\theta}|^{p-2}\tilde{\theta}\phi_{\rho/4}$ and integrating on $\mathbb{R}^2 \times [t_1, t_2]$, we can get an L^p-equality of $\tilde{\theta}$ similar to (2.2). We then decompose $u = \mathcal{R}^\perp \theta = \mathcal{R}^\perp(\tilde{\theta} + A_0)$ as

$$\begin{aligned} u(x,t) &= \text{p.v.} \int\limits_{\mathbb{R}^2} K^\perp(x-z)\tilde{\theta}(z,t)\phi_\rho(z)\mathrm{d}z + \int\limits_{\mathbb{R}^2} K^\perp(x-z)\theta(z,t)\left(1-\phi_\rho(z)\right)\mathrm{d}z \\ &\quad + A_0 \, \text{p.v.} \int\limits_{\mathbb{R}^2} K^\perp(x-z)\phi_\rho(z)\mathrm{d}z \\ &:= \tilde{u}^{(1)}(x,t) + \tilde{u}^{(2)}(x,t) + \tilde{u}^{(3)}(x,t); \end{aligned}$$

the terms $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$ can be estimated exactly as (2.5) and (2.8) respectively, whereas thanks to the zero-average property of $K^\perp$ on the circle, the term $\tilde{u}^{(3)}$ has the following upper bound

$$|\tilde{u}^{(3)}(x,t)| = A_0 \left| \text{p.v.} \int\limits_{\mathbb{R}^2} K^\perp(x-z)\left(\phi_\rho(z) - \phi_\rho(x)\right)\mathrm{d}z \right|$$

$$\leq \frac{A_0}{2\pi} \int\limits_{|x-z|\leq\rho} \frac{1}{|x-z|^2} |x-z| \|\nabla\phi_\rho\|_{L^\infty} \mathrm{d}z \leq C A_0.$$

Thus similarly as above, we can obtain an inequality analogous to (2.12) with a slightly different C_ρ.

Before ending this section, we include an auxiliary lemma concerning the term $U^{(1)}(y)$, which is of great use in the sequel.

Lemma 2.2. *Assume that $U^{(1)}(y)$ is defined by (2.11), and $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$ satisfies that for some $r \geq p+1$ and $b \geq 0$,*

$$\int\limits_{|y|\lesssim L} |\Theta(y)|^r \,\mathrm{d}y \lesssim L^b, \quad \forall L \gg 1. \tag{2.13}$$

Then we have

$$\int\limits_{L\leq|y|\leq 2L} |U^{(1)}(y)|^r \,\mathrm{d}y \lesssim L^{b-r(1+\alpha)}, \quad \forall L \gg 1. \tag{2.14}$$

Proof of Lemma 2.2. Noting that $B(t) = \left\{t \in [t_1, t_2] : T - \frac{(\rho/4)^{1+\alpha}}{|y|^{1+\alpha}} \leq t \leq T - \frac{(\rho/8)^{1+\alpha}}{|y|^{1+\alpha}}\right\}$, we use the Minkowski inequality and Calderón–Zygmund theorem to get

$$\begin{aligned}
&\left(\int\limits_{L\leq|y|\leq 2L} |U^{(1)}(y)|^r \mathrm{d}y \right)^{1/r} \\
&\leq \left(\int\limits_{L\leq|y|\leq 2L} \left(\int\limits_{T-\frac{(\rho/4)^{1+\alpha}}{|y|^{1+\alpha}}}^{T-\frac{(\rho/8)^{1+\alpha}}{|y|^{1+\alpha}}} \left| \mathrm{p.v.} \int\limits_{\mathbb{R}^2} K^{\perp}(y-\tilde{z})\, \Theta(\tilde{z})\, \phi_\rho\left(\tilde{z}(T-t)^{\frac{1}{1+\alpha}}\right) \mathrm{d}\tilde{z} \right| \mathrm{d}t \right)^r \mathrm{d}y \right)^{1/r} \\
&\leq \int\limits_{T-\frac{(\rho/4)^{1+\alpha}}{L^{1+\alpha}}}^{T-\frac{(\rho/8)^{1+\alpha}}{(2L)^{1+\alpha}}} \left(\int\limits_{L\leq|y|\leq 2L} \left| \mathrm{p.v.} \int\limits_{\mathbb{R}^2} K^{\perp}(y-\tilde{z})\, \Theta(\tilde{z})\, \phi_\rho\left(\tilde{z}(T-t)^{\frac{1}{1+\alpha}}\right) \mathrm{d}\tilde{z} \right|^r \mathrm{d}y \right)^{1/r} \mathrm{d}t \\
&\leq C \int\limits_{T-\frac{(\rho/4)^{1+\alpha}}{L^{1+\alpha}}}^{T-\frac{(\rho/16)^{1+\alpha}}{L^{1+\alpha}}} \left(\int\limits_{\mathbb{R}^2} \left| \Theta(z)\, \phi_\rho\left(z(T-t)^{\frac{1}{1+\alpha}}\right) \right|^r \mathrm{d}z \right)^{1/r} \mathrm{d}t
\end{aligned}$$

$$\le C \int_{T-\frac{(\rho/4)^{1+\alpha}}{L^{1+\alpha}}}^{T-\frac{(\rho/16)^{1+\alpha}}{L^{1+\alpha}}} \left(\int_{|z|\le\rho(T-t)^{-\frac{1}{1+\alpha}}} |\Theta(z)|^r \, dz \right)^{1/r} dt$$

$$\le \frac{C\rho^{1+\alpha}}{L^{1+\alpha}} \left(\int_{|z|\le 16L} |\Theta(z)|^r \, dz \right)^{1/r} \le C\rho^{1+\alpha} L^{\frac{b}{r}-(1+\alpha)}. \qquad \square$$

3. Proof of Theorem 1.1

We divide the proof into three steps according to the value of α.
Step 1: first we show that

$$\Theta \equiv 0 \text{ on } \mathbb{R}^2, \quad \text{for all } -1 < \alpha < \frac{2-\gamma}{r}. \tag{3.1}$$

By virtue of (1.7), (2.3) and Hölder's inequality, we get that as $l_2 \to \infty$,

$$\begin{aligned} l_2^{\alpha p-2} \int_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\rho}{4}}(y l_2^{-1}) \, dy &\le l_2^{\alpha p-2} \int_{|y|\le\frac{\rho}{4}l_2} |\Theta(y)|^p \, dy \\ &\le C l_2^{\alpha p-2} \left(\int_{|y|\le\frac{\rho}{4}l_2} |\Theta(y)|^r \, dy \right)^{p/r} l_2^{2(1-\frac{p}{r})} \\ &\le C l_2^{(\alpha+\frac{\gamma}{r}-\frac{2}{r})p} \longrightarrow 0. \end{aligned}$$

Thus by choosing $l_2 \to \infty$ and $\frac{\rho}{8} l_1 = L \gg 1$ in (2.12), and using the support property of $\phi_{\frac{\rho}{4}}$, we obtain

$$\frac{1}{L^{2-\alpha p}} \int_{|y|\le L} |\Theta(y)|^p \, dy \le C \int_{|y|\ge L} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} dy + C \int_{|y|\ge L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}} dy, \tag{3.2}$$

where we have suppressed the dependence on the constant ρ. By applying the dyadic decomposition, Hölder's inequality and Lemma 2.2 (noting that (2.13) holds with $b = \gamma$), we infer that

$$\begin{aligned} &\int_{|y|\ge L} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} dy \\ &\le C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-(p+1)\alpha}} \int_{2^k L\le|y|\le 2^{k+1}L} |\Theta(y)|^p \, U^{(1)}(y) dy \end{aligned}$$

$$\le C\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-(p+1)\alpha}}\left(\int\limits_{|y|\sim 2^kL}|\Theta(y)|^r\,\mathrm{d}y\right)^{p/r}\left(\int\limits_{|y|\sim 2^kL}|U^{(1)}(y)|^r\,\mathrm{d}y\right)^{1/r}(2^kL)^{2(1-\frac{p+1}{r})}$$

$$\le C\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-(p+1)\alpha}}(2^kL)^{\frac{p\gamma}{r}+\frac{\gamma}{r}-\alpha+1-\frac{2(p+1)}{r}}\le CL^{\frac{(p+1)(\gamma-2)}{r}-1+p\alpha}, \tag{3.3}$$

where we used the fact that $\frac{(p+1)(\gamma-2)}{r}-1+p\alpha<0$, equivalently, $\alpha<\frac{r-(\gamma-2)}{pr}+\frac{2-\gamma}{r}$, which is ensured for all $-1<\alpha<\frac{2-\gamma}{r}$. For the second term of the right-hand-side of (3.3), we similarly get

$$\int\limits_{|y|\ge L}\frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}}\,\mathrm{d}y\le\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{3-(p-1)\alpha}}\int\limits_{2^kL\le|y|\le 2^{k+1}L}|\Theta(y)|^p\,\mathrm{d}y$$

$$\le C\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{3-(p-1)\alpha}}\left(\int\limits_{|y|\sim 2^kL}|\Theta(y)|^r\,\mathrm{d}y\right)^{p/r}(2^kL)^{2(1-\frac{p}{r})}$$

$$\le C\sum_{k=0}^{\infty}(2^kL)^{-1+(p-1)\alpha+\frac{p\gamma}{r}-\frac{2p}{r}}\le CL^{-1+(p-1)\alpha+\frac{(\gamma-2)p}{r}}, \tag{3.4}$$

where the last inequality is guaranteed by the fact that $-1+(p-1)\alpha+\frac{(\gamma-2)p}{r}=(p-1)(\alpha+\frac{\gamma-2}{r})-\frac{r-(\gamma-2)}{r}<0$ for all $-1<\alpha<\frac{2-\gamma}{r}$. Gathering (3.2)–(3.4), and noticing that $-\alpha+\frac{(\gamma-2)p}{r}<\frac{(p+1)(\gamma-2)}{r}$ for all $\alpha\in\,]-1,\frac{2-\gamma}{r}[$, we have that for all $L\gg 1$,

$$\int\limits_{|y|\le L}|\Theta(y)|^p\,\mathrm{d}y\le CL^{a_0},\quad \text{with } a_0:=\frac{(p+1)(\gamma-2)}{r}+1. \tag{3.5}$$

If $a_0<0$, we conclude that $\Theta(y)\equiv 0$ for all $y\in\mathbb{R}^2$; we also note that if $r=p+1$, the condition on γ becomes $0\le\gamma<1$, which leads to $a_0<0$, thus in the following the scope of r needed to consider is $r>p+1$. Otherwise, for $a_0\ge 0$, by using (1.7) and the interpolation inequality, we see that

$$\begin{aligned}\int\limits_{|y|\le L}|\Theta(y)|^{p+1}\,\mathrm{d}y&\le C\left(\int\limits_{|y|\le L}|\Theta(y)|^p\,\mathrm{d}y\right)^{\delta}\left(\int\limits_{|y|\le L}|\Theta(y)|^r\,\mathrm{d}y\right)^{1-\delta}\\&\le CL^{a_0\delta+\gamma(1-\delta)},\quad \text{with } \delta:=\frac{r-p-1}{r-p}\in\,]0,1[.\end{aligned} \tag{3.6}$$

By arguing as (3.3), we apply the Hölder inequality and Lemma 2.2 with $r=p+1$, $b=a_0\delta+\gamma(1-\delta)$ to find

$$
\begin{aligned}
&\int\limits_{|y|\geq L} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} \mathrm{d}y \\
&\leq \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-(p+1)\alpha}} \left(\int\limits_{|y|\sim 2^k L} |\Theta|^{p+1} \mathrm{d}y \right)^{\frac{p}{p+1}} \left(\int\limits_{|y|\sim 2^k L} |U^{(1)}|^{p+1} \mathrm{d}y \right)^{\frac{1}{p+1}} \\
&\leq C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-(p+1)\alpha}} (2^k L)^{a_0\delta+\gamma(1-\delta)-(1+\alpha)} \\
&\leq C \sum_{k=0}^{\infty} (2^k L)^{a_0-(2-p\alpha)-(a_0-\gamma)(1-\delta)-1}. \qquad (3.7)
\end{aligned}
$$

Noticing that $a_0 - (2 - p\alpha) < 0$ from (3.3), and thanks to $\gamma < r - p$,

$$
\begin{aligned}
a_1 := (a_0 - \gamma)(1-\delta) + 1 &= \left(\frac{(\gamma-2)(p+1)}{r} + 1 - \gamma \right) \frac{1}{r-p} + 1 \\
&= \left(-\frac{\gamma(r-p-1)}{r} - \frac{2(p+1)}{r} + 1 + (r-p) \right) \frac{1}{r-p} \qquad (3.8) \\
&= \frac{(r-p-1)(r+2-\gamma)}{r(r-p)} > \frac{(p+2)(r-p-1)}{r(r-p)},
\end{aligned}
$$

we see that $a_0 - (2 - p\alpha) - a_1 < 0$, and thus

$$
\int\limits_{|y|\geq L} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} \mathrm{d}y \leq C L^{a_0-(2-p\alpha)-a_1}. \qquad (3.9)
$$

By virtue of the dyadic decomposition and (3.5), we obtain

$$
\begin{aligned}
\int\limits_{|y|\geq L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}} \mathrm{d}y &\leq \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3-(p-1)\alpha}} \int\limits_{|y|\sim 2^k L} |\Theta(y)|^p \mathrm{d}y \\
&\leq C \sum_{k=0}^{\infty} (2^k L)^{a_0-(2-p\alpha)-(1+\alpha)} \leq C L^{a_0-(2-p\alpha)-(1+\alpha)},
\end{aligned} \qquad (3.10)
$$

where the last equality is implied by the fact that $a_0 < 2 - p\alpha$ and $1 + \alpha > 0$. Inserting (3.9) and (3.10) into (3.2) leads to

$$
\int\limits_{|y|\leq L} |\Theta(y)|^p \mathrm{d}y \leq C L^{a_0-b_0}, \quad \text{with } b_0 := \min\{a_1, 1+\alpha\} > 0. \qquad (3.11)
$$

If $a_0 - b_0 < 0$, the proof is finished. Otherwise, for $a_0 - b_0 \geq 0$, similarly as obtaining (3.6), (3.7), (3.9) and (3.10), we get

$$\int_{|y|\le L} |\Theta(y)|^{p+1}\,\mathrm{d}y \le CL^{(a_0-b_0)\delta+\gamma(1-\delta)},$$

and

$$\begin{aligned}\int_{|y|\ge L} \frac{|\Theta(y)|^p U^{(1)}(y)}{|y|^{2-(p+1)\alpha}}\,\mathrm{d}y &\le C\sum_{k=0}^{\infty}(2^k L)^{a_0-(2-p\alpha)-(a_0-\gamma)(1-\delta)-1-b_0\delta} \\ &\le CL^{a_0-(2-p\alpha)-a_1-b_0\delta},\end{aligned} \tag{3.12}$$

and

$$\int_{|y|\ge L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}}\,\mathrm{d}y \le CL^{a_0-(2-p\alpha)-b_0-(1+\alpha)}. \tag{3.13}$$

If $b_0 = a_1$, i.e., $a_1 \le 1+\alpha$, then from (3.2), (3.12) and (3.13), we deduce

$$\int_{|y|\le L} |\Theta(y)|^p\,\mathrm{d}y \le CL^{a_0-a_1(1+\delta)}, \tag{3.14}$$

which implies $\Theta \equiv 0$ on $\mathbb{R}^2$ if $a_0 - a_1(1+\delta) < 0$, otherwise for all $a_1(1+\delta) \ge a_0$, we can repeat this process for many times to obtain that for $n \in \mathbb{N}$,

$$\int_{|y|\le L} |\Theta(y)|^p\,\mathrm{d}y \le CL^{a_0-a_1(1+\delta+\delta^2+\cdots+\delta^n)}; \tag{3.15}$$

thus noting that $1+\delta+\cdots+\delta^n = \frac{1-\delta^{n+1}}{1-\delta} \to \frac{1}{1-\delta} = r-p$ as $n \to \infty$, and from (3.5), (3.8),

$$\begin{aligned} a_0 - a_1(r-p) &= \frac{(p+1)(\gamma-2)}{r} + 1 - \frac{(r-(p+1))(r-(\gamma-2))}{r} \\ &= -(r-p-\gamma) < 0,\end{aligned} \tag{3.16}$$

we infer that there exists some number $n \in \mathbb{N}$ (depending only on p, γ, r) so that the power index $a_0 - a_1(1+\delta+\cdots+\delta^n) < 0$, which thanks to (3.15) implies $\Theta \equiv 0$ on $\mathbb{R}^2$. If $b_0 = 1+\alpha < a_1$, then we have

$$\int_{|y|\le L} |\Theta(y)|^p\,\mathrm{d}y \le \begin{cases} CL^{a_0-2(1+\alpha)}, & \text{if } a_1 \ge (1+\alpha)(2-\delta), \\ CL^{a_0-a_1-(1+\alpha)\delta}, & \text{if } a_1 < (1+\alpha)(2-\delta), \end{cases} \tag{3.17}$$

thus if the power index of L on the right-hand-side is negative, the proof is over; otherwise, denoting by $m_0 \in \mathbb{N}^+$ the smallest integer such that

$$m_0(1+\alpha)(1-\delta) > a_1 - (1+\alpha), \tag{3.18}$$

which implies that $m_0(1+\alpha)+(1+\alpha)\delta \le a_1+m_0(1+\alpha)\delta<(m_0+1)(1+\alpha)$, by reiterating the above process for (m_0-1)-times, then (3.12) and (3.13) respectively reduce to

$$\int_{|y|\ge L} \frac{|\Theta(y)|^p\, U^{(1)}(y)}{|y|^{2-(p+1)\alpha}}\,\mathrm{d}y \le C L^{a_0-(2-p\alpha)-a_1-m_0(1+\alpha)\delta}, \tag{3.19}$$

and

$$\int_{|y|\ge L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}}\,\mathrm{d}y \le C L^{a_0-(2-p\alpha)-(m_0+1)(1+\alpha)}, \tag{3.20}$$

which combined with (3.2) yield

$$\int_{|y|\le L} |\Theta(y)|^p\,\mathrm{d}y \le C L^{a_0-b_1}, \tag{3.21}$$

with $b_1 := a_1+m_0(1+\alpha)\delta$; then it suffices to treat the case $a_0-b_1\ge 0$, and similarly as above, denoting by $m_1\in\mathbb{N}$ the smallest integer such that

$$m_1(1+\alpha)(1-\delta) > a_1-b_1(1-\delta)-(1+\alpha), \tag{3.22}$$

which guarantees that $b_1+m_1(1+\alpha)+(1+\alpha)\delta \le a_1+(b_1+m_1(1+\alpha))\delta < b_1+(m_1+1)(1+\alpha)$, and through repeating the above process for m_1-times (at the worst situation) once again, we find

$$\int_{|y|\ge L} \frac{|\Theta(y)|^p\, U^{(1)}(y)}{|y|^{2-(p+1)\alpha}}\,\mathrm{d}y \le C L^{a_0-(2-p\alpha)-a_1-(b_1+m_1(1+\alpha))\delta}, \tag{3.23}$$

and

$$\int_{|y|\ge L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}}\,\mathrm{d}y \le C L^{a_0-(2-p\alpha)-b_1-(m_1+1)(1+\alpha)}, \tag{3.24}$$

and thus

$$\int_{|y|\le L} |\Theta(y)|^p\,\mathrm{d}y \le C L^{a_0-b_2}, \quad \forall L\gg 1, \tag{3.25}$$

with $b_2 = a_1+(b_1+m_1(1+\alpha))\delta$ satisfying $b_2\ge a_1+a_1\delta$; then the remaining case is $a_0-b_2\ge 0$, and by iteratively repeating the above process, and denoting by $m_n\in\mathbb{N}$ $(n=1,2,\cdots)$ the smallest integer such that

$$m_n(1+\alpha)(1-\delta) > a_1-b_n(1-\delta)-(1+\alpha), \tag{3.26}$$

which ensures that $b_n + m_n(1+\alpha) + (1+\alpha)\delta \le a_1 + (b_n + m_n(1+\alpha))\delta < b_n + (m_n+1)(1+\alpha)$, we deduce that

$$\int_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \le C L^{a_0 - b_{n+1}}, \quad \forall L \gg 1, \tag{3.27}$$

where $b_{n+1} := a_1 + (b_n + m_n(1+\alpha))\delta$ satisfying

$$b_{n+1} \ge a_1 + a_1\delta + \cdots + a_1\delta^n, \tag{3.28}$$

and the iteration can be stopped provided that the power of L on the r.h.s. of (3.27) is negative; according to (3.28) and (3.16), we infer that there exists some number $n = \tilde{n} \in \mathbb{N}$ (depending only on p, γ, r) so that $a_1 + a_1\delta + \cdots + a_1\delta^{\tilde{n}} > a_0$ and the power index $a_0 - b_{\tilde{n}+1}$ on the r.h.s. of (3.27) satisfies $a_0 - b_{\tilde{n}+1} < 0$, which clearly implies $\Theta \equiv 0$ on $\mathbb{R}^2$.

Step 2: we show that for all $\alpha \ge \frac{2-\gamma}{r}$, the profile Θ satisfies

$$\int_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \le C L^{2-p\alpha}, \quad \forall L \gg 1. \tag{3.29}$$

Indeed, this can be simply deduced from the L^p-inequality of the original quantity θ which reads $\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p} \le \|\theta_0\|_{H^s \cap L^1}$ for all $t \in [0, T[$: by using the blowup scenario (1.3), we infer that

$$\begin{aligned} &\frac{1}{(T-t)^{\frac{p\alpha}{1+\alpha}}} \int_{|x-x_0|\le\rho} \left| \Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}} \right) \right|^p \mathrm{d}x \\ &= \frac{1}{(T-t)^{\frac{p\alpha-2}{1+\alpha}}} \int_{|y|\le \rho (T-t)^{-\frac{1}{1+\alpha}}} |\Theta(y)|^p \, \mathrm{d}y \le C, \end{aligned} \tag{3.30}$$

thus (3.29) is implied by denoting $L = \rho(T-t)^{-\frac{1}{1+\alpha}}$, $t \in [0, T[$. We remark that (3.29) can also be proved from the local energy inequality (2.12), where the assumptions like (1.7) truly take part in the proof, but we here omit the details (one can see [21] for a similar treating).

As a consequence of (3.29), we deduce that $\Theta \equiv 0$ on $\mathbb{R}^2$ for all $\alpha > \frac{2}{p}$, which combined with (3.1) yields that the scope of α admitting nontrivial blowup profiles is $\left\{ \frac{2-\gamma}{r} \le \alpha \le \frac{2}{p} \right\}$ for $p \in]1, \infty[$ and for some $r \in [p+1, \infty[$, $\gamma \in [0, r-p[$.

Step 3: we prove the desired estimate (1.8) for every $\frac{2-\gamma}{r} \le \alpha < \frac{2}{p}$.

According to (3.29), under the assumption that $\Theta \not\equiv 0$, it suffices to show that for all $L \gg 1$,

$$\frac{1}{L^{2-p\alpha}} \int_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \gtrsim 1. \tag{3.31}$$

We prove (3.31) by contradiction: suppose that it is not true, then there exists a sequence of numbers $L_i \gg 1$ $(i = 1, 2, \cdots)$ such that

$$\frac{1}{L_i^{2-p\alpha}} \int\limits_{|y|\le L_i} |\Theta(y)|^p \, \mathrm{d}y \to 0, \quad \text{as } L_i \to \infty. \tag{3.32}$$

By letting $l_2 = L_i \to \infty$ and $\frac{\rho}{8} l_1 = L \gg 1$ in (2.12), we get

$$\int\limits_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \le C L^{2-p\alpha} \int\limits_{|y|\ge L} \frac{|\Theta(y)|^p \, U^{(1)}(y)}{|y|^{2-(p+1)\alpha}} \mathrm{d}y + C L^{2-p\alpha} \int\limits_{|y|\ge L} \frac{|\Theta(y)|^p}{|y|^{3-(p-1)\alpha}} \, \mathrm{d}y. \tag{3.33}$$

Similarly as obtaining (3.6), (3.7), (3.9) and (3.10), and by using (3.29) and Lemma 2.2 with $r = p+1$, $b = (2-p\alpha)\delta + \gamma(1-\delta)$, we deduce that

$$\begin{aligned} \int\limits_{|y|\le L} |\Theta(y)|^{p+1} \, \mathrm{d}y &\le C \Bigg(\int\limits_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \Bigg)^{\delta} \Bigg(\int\limits_{|y|\le L} |\Theta(y)|^r \, \mathrm{d}y \Bigg)^{1-\delta} \\ &\le C L^{(2-p\alpha)\delta + \gamma(1-\delta)}, \quad \text{with } \delta = \frac{r-p-1}{r-p}, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{L^{2-p\alpha}} \int\limits_{|y|\le L} |\Theta|^p \mathrm{d}y \\ &\le C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-(p+1)\alpha}} \int\limits_{|y|\sim 2^k L} |\Theta|^p \, U^{(1)} \mathrm{d}y + C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3-(p-1)\alpha}} \int\limits_{|y|\sim 2^k L} |\Theta|^p \mathrm{d}y \\ &\le C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-(p+1)\alpha}} (2^k L)^{(2-p\alpha)\delta + \gamma(1-\delta) - (1+\alpha)} + C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3-(p-1)\alpha}} (2^k L)^{2-p\alpha} \\ &\le C \sum_{k=0}^{\infty} (2^k L)^{(\gamma-(2-p\alpha))(1-\delta)-1} + C \sum_{k=0}^{\infty} (2^k L)^{-(1+\alpha)} \\ &\le C L^{(\gamma-(2-p\alpha))(1-\delta)-1} + C L^{-(1+\alpha)}, \end{aligned}$$

where in the last inequality we used the fact that $(\gamma - (2-p\alpha))(1-\delta) - 1 = ((\gamma + p - r) - (2-p\alpha))(1-\delta) < 0$ for all $\alpha \le \frac{2}{p}$ and $\gamma < r - p$. Noting that for all $\frac{2-\gamma}{r} \le \alpha \le \frac{2}{p}$,

$$(\gamma - (2-p\alpha))(1-\delta) - 1 + (1+\alpha) = (\gamma - 2 + p\alpha) \frac{1}{r-p} + \alpha = \frac{\gamma - 2 + r\alpha}{r-p} \ge 0,$$

we obtain

$$\int\limits_{|y|\le L} |\Theta(y)|^p \, \mathrm{d}y \le C L^{(2-p\alpha)-\tilde{a}_0}, \quad \text{with } \tilde{a}_0 := -(\gamma - (2-p\alpha))(1-\delta) + 1 > 0. \tag{3.34}$$

If $2 - p\alpha - \tilde{a}_0 < 0$, then the proof is over; we also note that if $r = p + 1$, i.e. $\delta = 0$, then $2 - p\alpha - \tilde{a}_0 = \gamma - 1 < 0$, thus in the following the scope of δ we need to treat is $\delta \in]0, 1[$. Otherwise, for the case $2 - p\alpha - \tilde{a}_0 \geq 0$, by applying this improved estimate and Lemma 2.2, we repeat the above process to derive that

$$\int\limits_{|y|\leq L} |\Theta(y)|^{p+1}\, \mathrm{d}y \leq C L^{(2-p\alpha-\tilde{a}_0)\delta+\gamma(1-\delta)},$$

and

$$\begin{aligned}\frac{1}{L^{2-p\alpha}} \int\limits_{|y|\leq L} |\Theta(y)|^{p}\, \mathrm{d}y &\leq C \sum_{k=0}^{\infty} \frac{(2^k L)^{(2-p\alpha-\tilde{a}_0)\delta+\gamma(1-\delta)-(1+\alpha)}}{(2^k L)^{2-(p+1)\alpha}} + C \sum_{k=0}^{\infty} \frac{(2^k L)^{2-p\alpha-\tilde{a}_0}}{(2^k L)^{3-(p-1)\alpha}} \\ &\leq C \sum_{k=0}^{\infty} (2^k L)^{(\gamma-(2-p\alpha))(1-\delta)-\tilde{a}_0\delta-1} + C \sum_{k=0}^{\infty} (2^k L)^{-(1+\alpha)-\tilde{a}_0} \\ &\leq C L^{-\tilde{a}_0-\tilde{a}_0\delta} + C L^{-\tilde{a}_0-(1+\alpha)} \leq C L^{-\tilde{a}_0-\tilde{a}_0\delta},\end{aligned}$$

which implies that

$$\int\limits_{|y|\leq L} |\Theta(y)|^{p}\, \mathrm{d}y \leq C L^{2-p\alpha-\tilde{a}_0-\tilde{a}_0\delta}. \tag{3.35}$$

It suffices to consider the case $2 - p\alpha - \tilde{a}_0(1+\delta) \geq 0$, and by iteratively repeating the above process we find that for every $n \in \mathbb{N}$,

$$\int\limits_{|y|\leq L} |\Theta(y)|^{p}\, \mathrm{d}y \leq C L^{2-p\alpha-\tilde{a}_0(1+\delta+\cdots+\delta^n)}. \tag{3.36}$$

Since $1 + \delta + \cdots + \delta^n \to \frac{1}{1-\delta} = r - p$, and

$$\tilde{a}_0(r-p) = 2 - p\alpha + r - p - \gamma > 2 - p\alpha,$$

there exists a sufficiently large number $n \in \mathbb{N}$ so that $2 - p\alpha - \tilde{a}_0(1 + \delta + \cdots + \delta^n) < 0$, which obviously implies $\Theta \equiv 0$ on $\mathbb{R}^2$ and contradicts with the nontrivial assumption of Θ. Hence we prove (1.8) under the assumption that $\Theta \not\equiv 0$, and thus conclude Theorem 1.1.

4. Proof of Theorem 1.2

For the first part, since $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$ and

$$|\Theta(y)| \lesssim \frac{1}{|y|^{\mu}}, \quad \forall |y| \geq L_0, \tag{4.1}$$

with $L_0 > 0$ some pure number, we get that for every $r > \max\{\frac{2}{\mu}, 2\}$,

$$\int_{|y|\le L} |\Theta(y)|^r \, dy \lesssim \int_{|y|\le L_0} |\Theta(y)|^r \, dy + \int_{|y|\ge L_0} \frac{1}{|y|^{\mu r}} \, dy \le C, \quad \forall L \gg 1,$$

which corresponds to (1.7) with $\gamma = 0$. Let $p_1 := \max\{\frac{2}{\mu}, 2\}$, and set $p = p_1$, $r = p_1 + 1$, $\gamma = 0$ in Theorem 1.1, which clearly contains the case (4.1), then the scope of α admitting nontrivial profiles is $\frac{2}{p_1+1} \le \alpha \le \frac{2}{p_1}$. On the other hand, we set $p = p_1 + 2$, $r = p_1 + 3$, $\gamma = 0$ in Theorem 1.1, which also includes the considered case (4.1), then the range of α admitting nontrivial profiles is $\frac{2}{p_1+3} \le \alpha \le \frac{2}{p_1+2}$. Since $[\frac{2}{p_1+1}, \frac{2}{p_1}] \cap [\frac{2}{p_1+3}, \frac{2}{p_1+2}] = \emptyset$, we conclude that $\Theta \equiv 0$ on $\mathbb{R}^2$.

We next consider the second part. Let $L_0 > 0$ be a constant such that (1.9) holds for all $|y| \ge L_0$, then under the assumptions (1.9) and $\Theta \in C^1_{\text{loc}}(\mathbb{R}^2)$, we have that for every $p \in]1, \infty[$ and some $r \ge p + 1$,

$$\int_{|y|\le L} |\Theta(y)|^r \, dy \lesssim \int_{|y|\le L_0} |\Theta(y)|^r \, dy + \int_{L_0\le |y|\le L} |y|^{\sigma r} \, dy \le C L^{\sigma r+2}, \quad \forall L \gg 1.$$

We set $\gamma = \sigma r + 2$, and we need that $\sigma r + 2 < r - p$, and it suffices to choose $r = \frac{p+2}{1-\sigma} + 1$. Thus for every $p \in]1, \infty[$, and such r and γ, we can apply Theorem 1.1 to see that the scope of α admitting nontrivial profiles is $-\sigma = \frac{2-\gamma}{r} \le \alpha \le \frac{2}{p}$. On the other hand, due to the lower bound at the assumption (1.9), we see that $\int_{|y|\le L} |\Theta(y)|^r \, dy \ge C L^2$, $\forall L \gg 1$, which means that the range $0 < \alpha \le \frac{2}{p}$ is not admissible. Hence, the scope of α admitting nontrivial profiles is $-\sigma \le \alpha \le 0$, and the profile corresponding to each $\alpha \in [-\sigma, 0]$ satisfies (1.10) for every $p \in]1, \infty[$.

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