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On the discretely self-similar solutions for the 3D Navier–Stokes equations

Liutang Xue

School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

E-mail: xuelt@bnu.edu.cn

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Abstract
We consider the discretely self-similar blowup solutions of the three-dimensional Navier–Stokes equations, and under suitable assumptions we show some estimates on the asymptotic behavior of the possible nontrivial velocity profiles.

Keywords: Navier–Stokes equations, self-similar type solutions, local energy inequality
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1. Introduction

In this paper we consider the Cauchy problem of the three-dimensional (3D) incompressible Navier–Stokes equations

$$(3\mathrm{D}-\mathrm{NSE})\begin{cases}\partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla p = 0,\\ \operatorname{div} v = 0,\\ v|_{t=0}(x) = v_0(x),\end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$ and $\nu>0$ is the viscosity coefficient which is always normalized to 1 in the sequel. $v=(v_1,v_2,v_3)$ is the velocity vector field of $\mathbb{R}^3$ and the scalar function p denotes the pressure field. (3D-NSE) describes the motion of incompressible viscous flows and is a fundamental model in fluid mechanics. There are numerous works on the theoretical studies of (3D-NSE), but so far the global regularity problem of smooth solutions remains an outstanding open problem, that is, we do not know whether the smooth solutions will always exist or they break down at finite time.

We here are mainly concerned with the possible self-similar type singular solutions of (3D-NSE), which is an important potential type of finite-time blowup solutions for these equations (see [9, 13]). Such singular solutions are mostly related to the scaling property of (3D-NSE), that is, the system (1.1) is invariant under the following scaling transformation

$$\begin{aligned} v(x,t) &\mapsto v_\lambda(x,t) := \lambda v(\lambda x, \lambda^2 t), \quad \lambda > 0, \\ p(x,t) &\mapsto p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t). \end{aligned} \tag{1.2}$$

Singular solutions (v, p) of (3D-NSE) are called (backward) self-similar with respect to a spacetime point (x_0, T) on the domain $D := \mathbb{R}^3 \times]0, T[$ if for all $(x,t) \in D$,

$$v(x,t) = \frac{1}{\sqrt{T-t}} V\left(\frac{x-x_0}{\sqrt{T-t}}\right), \quad p(x,t) = \frac{1}{T-t} P\left(\frac{x-x_0}{\sqrt{T-t}}\right), \tag{1.3}$$

where $(V, P) = (V, P)(y)$ correspondingly are stationary functions solving that

$$\begin{cases} \frac{1}{2}V + \frac{1}{2}y\cdot\nabla V + V\cdot\nabla V - \Delta V + \nabla P = 0, \quad y \in \mathbb{R}^3, \\ \operatorname{div} V = 0. \end{cases} \tag{1.4}$$

Combined with the time translation transformation $\mathcal{T}$ which is defined by $\mathcal{T}f(t) = f(T-t)$, we see that solutions with the form (1.3) satisfy $\forall (x,t) \in D$, $\forall \lambda > 0$,

$$\mathcal{T}v(x,t) = \mathcal{T}v_\lambda(x,t), \quad \mathcal{T}p(x,t) = \mathcal{T}p_\lambda(x,t), \tag{1.5}$$

which accounts for the terminology 'self-similar solutions'. We also consider a more general case that (1.5) holds only for a single $\lambda > 1$, that is, $\forall (x,t) \in D$,

$$v(x, T-t) = \lambda v(\lambda x, T-\lambda^2 t), \quad p(x,t) = \lambda^2 p(\lambda x, T-\lambda^2 t), \quad \text{for some } \lambda > 1, \tag{1.6}$$

which is called 'discretely self-similar solutions' with respect to a spacetime point (x_0, T) on the space-time domain D. In terms of the similarity variables

$$y := \frac{x-x_0}{\sqrt{T-t}}, \quad s := \log\frac{T}{T-t}, \tag{1.7}$$

we see that such singular solutions satisfy

$$v(x,t) = \frac{1}{\sqrt{T-t}} V(y,s), \quad p(x,t) = \frac{1}{T-t} P(y,s), \tag{1.8}$$

where $(V, P)(y, s)$ are the time periodic functions defined on $\mathbb{R}^3 \times \mathbb{R}^+$ with the period $S_0 := 2\log\lambda > 0$, and they solve the following equations

$$\begin{cases} \partial_s V + \frac{1}{2}V + \frac{1}{2}y\cdot\nabla V + V\cdot\nabla V - \Delta V + \nabla P = 0, \\ \operatorname{div} V = 0, \\ V|_{s=0}(y) = \sqrt{T}\, v_0(\sqrt{T}y + x_0). \end{cases} \tag{1.9}$$

Under some suitable assumption on V (e.g. the $L^3_s L^p_y$-condition in theorem 1.1 below), the pressure profile up to a function depending only on s can be expressed as

$$P(y,s) = -\frac{1}{3}|V(y,s)|^2 + \text{p.v.}\int_{\mathbb{R}^3} K_{ij}(y-z) V_i(z,s) V_j(z,s)\,\mathrm{d}z, \tag{1.10}$$

where $K_{ij}(y) = \frac{1}{4\pi}\frac{3y_iy_j - |y|^2\delta_{ij}}{|y|^5}$ $(i, j = 1, 2, 3)$ are the standard Calderón–Zygmund kernels.

Self-similar solutions (1.3) of (3D-NSE) were firstly proposed as a possible candidate of finite-time blowup solutions by Leray in his famous article [9]. It was about 60 years later the problem was finally solved, by Nečas–Ružička–Šverák [12] which excluded such a scenario under the condition $V \in L^3(\mathbb{R}^3)$, and later by Tsai [15] under the more general condition $V \in L^q(\mathbb{R}^3)$, $q > 3$. Both papers rely on a key Liouville-type lemma of the quantity $\Pi(y) := \frac{1}{2}|V(y)|^2 + P(y) + \frac{1}{2}y \cdot V(y)$. However, as pointed out by Plechac–Šverák [13] (also by Tsai [16]), the possible occurrence of discretely self-similar solution (1.8) remains to be an important blowup ansatz. So far there is not much work on this topic in the literature, and the main known results, to the best of the author's knowledge, are the direct consequence of the regularity criteria developed by Escauriaza–Seregin–Šverák [7] and its generalizations [8, 17]. Indeed, in the blowup scenario (1.8) and under the condition $V \in L^\infty_s(\mathbb{R}^+; L^3(\mathbb{R}^3))$, we can deduce $v \in L^\infty_t([0, T[; L^3(\mathbb{R}^3))$, which implies the solutions are regular beyond the time T from [7], thus only the trivial velocity profile $V \equiv 0$ is admitted; we also notice that the Lebesgue space $L^3(\mathbb{R}^3)$ at above can be replaced by the Besov space $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$ $(3 < p, q < \infty)$ according to [8] and by the Lorentz space $L^{3,\ell}(\mathbb{R}^3)$ $(3 < \ell < \infty)$ according to [17].

We also mention an another type of self-similar solutions called the forward self-similar solutions, which are defined by (1.3) with t $(t > 0)$ in place of $T - t$, and the associated existence problem is similarly faced. In a recent work [10], Jia and Šverák make a significant progress in proving the global existence of large forward self-similar solutions for (3D-NSE) in the framework of local-Leray weak solutions, and they also conjecture the non-uniqueness of such solutions. One can see Tsai [16] for a similar result concerning the global existence of large discretely forward self-similar solutions of (3D-NSE).

Recently, there are also much works studying the backward self-similar type solutions of the 3D Euler system, which is defined by (1.1) with $\nu = 0$ and is an another fundamental system in fluid mechanics, and one can refer to [1–6, 14, 18] and references therein for various nonexistence results and some interesting properties of the possible velocity profiles. However, unlike (3D-NSE), even for the self-similar solutions of the 3D Euler equations, the existence problem under natural assumptions (e.g. the velocity profiles belonging to $L^p(\mathbb{R}^3)$, $p > 2$) remains widely open.

In this note, partially motivated by the recent work on the self-similar type solutions of the Euler equations, we adapt the method introduced in [5], and later developed by [1, 6, 18], to consider the discretely self-similar solutions of (3D-NSE) to obtain some estimates on the asymptotic behavior of the possible velocity profiles. Our main result reads as follows.

Theorem 1.1. *Assume that $v \in C([0, T[; H^s(\mathbb{R}^3))$, $s > \frac{5}{2}$ satisfying (1.8) is the discretely self-similar solutions for the (3D-NSE). Suppose that for some $p \in [3, \infty[$, the velocity profile $V \in C^1_s C^1_{y,\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^+) \cap L^3([0, S_0]; L^p(\mathbb{R}^3))$ is a time periodic vector field with period $S_0 = 2\log\lambda > 0$, and the pressure profile P is defined from V by (1.10) up to a function depending only on s. Then we have*

$$\sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y, s)|^2\, \mathrm{d}y + \int_0^{S_0}\int_{|y|\leqslant L} |\nabla V(y, s)|^2\, \mathrm{d}y\mathrm{d}s \lesssim L, \quad \forall L \gg 1. \tag{1.11}$$

If additionally $V \not\equiv 0$ (i.e. V is a nontrivial velocity profile) satisfies that

$$\lim_{L\to\infty} \frac{\sum_{k=0}^{\infty} \frac{1}{\lambda^k} \int_0^{S_0} \int_{|y|\leqslant \lambda^{k+2}L} |\nabla V(y,s)|^2 \,\mathrm{d}y\mathrm{d}s}{\sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y,s)|^2 \,\mathrm{d}y} \leqslant \frac{1}{4\lambda}, \tag{1.12}$$

we also have

$$\sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y,s)|^2 \,\mathrm{d}y \sim L, \quad \forall L \gg 1. \tag{1.13}$$

The proof of (1.11) is an easy consequence of the energy inequality (3.1) of the velocity field v. The proof of (1.13) mainly relies on the local energy inequality (2.23) of the profiles (V, P), which in turn is derived from the local energy equality (2.1) of the original solutions (v, p). Then by obtaining the dominant term of the left-hand-side of (2.23) under the assumption (1.12), and by estimating the associated right-hand-side terms, we can conclude (1.13). We also notice that (1.11) can similarly be proved from the local energy inequality (2.20), where the assumption $V \in L^3_s L^p_y$ truly takes part in the proof, and we place this interesting and more complicated proof in the appendix A.

Theorem 1.1 can be applicable in identifying the nontrivial velocity profiles with typical asymptotics. Indeed, since we have the $L^3_s L^p_y$ $(p \in [3, \infty[)$-assumption of V and may additionally assume that $V \notin L^\infty_s(\mathbb{R}^+; L^3(\mathbb{R}^3))$, the candidates with typical asymptotics will be like

$$|V(y,s)| \sim |y|^{-\gamma} + o(|y|^{-\gamma}), \quad \text{for some } \gamma \in]0,1], \quad \forall s \in [0,S_0], \forall |y| \gg 1. \tag{1.14}$$

By scaling, we can expect that

$$|\nabla V(y,s)| \sim |y|^{-\gamma-1} + o(|y|^{-\gamma-1}), \quad \text{for some } \gamma \in]0,1], \quad \forall s \in [0,S_0], \forall |y| \gg 1. \tag{1.15}$$

Using these estimates we see that (1.12) holds true, and thanks to theorem 1.1, we moreover necessarily have (1.13), which is only compatible with the $\gamma = 1$ case, i.e.

$$|V(y,s)| \sim |y|^{-1} + o(|y|^{-1}), \quad \forall s \in [0,S_0], \forall |y| \gg 1. \tag{1.16}$$

A few remarks are as follows.

Remark 1.2. The assumption on the velocity field v in theorem 1.1 in fact can be weaken as the local Leray weak solution v (constructed in [11, chapters 32 and 33]) satisfying the local energy equality (2.1) and the discretely self-similar blowup ansatz (1.8). Note that the local Leray solution may not satisfy the energy inequality (3.1), but we can still conclude (1.11) thanks to the appendix A.

Remark 1.3. If the velocity profile $V \in C^1_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^+) \cap L^3([0,S_0]; L^p(\mathbb{R}^3))$, $p \in [3, \infty[$ additionally satisfies that for all $L \gg 1$,

$$\sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y,s)|^2 \,\mathrm{d}y \leqslant \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{1}{\lambda^k} \int_0^{S_0} \int_{|y|\leqslant \lambda^{k-2}L} |\nabla V(y,s)|^2 \,\mathrm{d}y\mathrm{d}s, \tag{1.17}$$

then by arguing as obtaining (1.13) in theorem 1.1, we have $V \equiv 0$ for all $(y,s) \in \mathbb{R}^3 \times [0,S_0]$.

Remark 1.4. Under the blowup ansatz (1.8), we infer that

$$\|v(t)\|^2_{L^2_x(B_r(x_0))} = \frac{1}{L} \int_{|y|\leqslant rL} |V(y,s)|^2 \,\mathrm{d}y,$$

where $L=(T-t)^{-1/2}$, $r>0$, $s=\log\frac{T}{T-t}$ and $V(y,s)$ is a time periodic vector field with period S_0, thus for the nontrivial velocity profile, (1.13) guarantees that

$$\limsup_{t\to T}\|v(t)\|_{L^2_x(B_r(x_0))}\sim 1,$$

that is, the discretely self-similar blowup contains some positive amount of energy as time tending to the blowup time.

Remark 1.5. The numerator of the left-hand-side term of (1.12) is virtually from the contribution of the local dissipation term in (2.1), while the denominator is from the estimation of the local energy term in (2.1), thus the assumption (1.12) means that in the blowup scenario (1.8) and as time approaching to the blowup time, the contribution from the local dissipation term is much weaker than that from the local energy term. At least for the nontrivial velocity profiles with typical asymptotics, this assumption seems reasonable.

Remark 1.6. Note that the term $\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y$ in (1.12) and (1.13) can also simultaneously be replaced by the term $\int_0^{S_0}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\mathrm{d}s$, by starting from (2.22) instead of (2.23) in the proof of (1.13).

The outline of the paper is as follows: in section 2 we show the key local energy inequalities of profiles (V,P), and based on these inequalities we prove theorem 1.1 in the section 3, and at last we present an another proof of (1.11) from the local energy inequality (2.20) in the appendix A.

Throughout this paper, C denotes a harmless constant which may be of different value from line to line. The formula $X\lesssim Y$ denotes that there is a constant $C>0$ such that $X\leqslant CY$, and $X\sim Y$ means that $X\lesssim Y$ and $Y\lesssim X$. For a real number a, denote by $[a]$ its integer part. For $x_0\in\mathbb{R}^3$, $r>0$, denote by $B_r(x_0)$ the open ball of $\mathbb{R}^3$ centered at x_0 with radius r.

2. Local energy inequality

We start with the following local energy equality of the original Navier–Stokes equations (1.1):

$$\begin{aligned}&\int_{\mathbb{R}^3}|v(x,t_2)|^2\chi(x,t_2)\,\mathrm{d}x-\int_{\mathbb{R}^3}|v(x,t_1)|^2\chi(x,t_1)\,\mathrm{d}x+2\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|\nabla v|^2\chi(x,t)\,\mathrm{d}x\mathrm{d}t\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|v|^2\partial_t\chi(x,t)\,\mathrm{d}x\mathrm{d}t+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(|v|^2v+2pv)\cdot\nabla\chi(x,t)\,\mathrm{d}x\mathrm{d}t+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|v|^2\Delta\chi(x,t)\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.1}$$

where $\chi\in C_c^\infty(\mathbb{R}^3\times[0,T[)$ and $0\leqslant t_1<t_2<T$.

Without loss of generality, we set $x_0=0$ for brevity. Let $\phi\in C_c^\infty(\mathbb{R}^3)$ be a test function such that $0\leqslant\phi\leqslant 1$, $\operatorname{supp}\phi\subset B_1(0)$ and $\phi\equiv 1$ on $B_{1/\lambda}(0)$ (note that this λ is just the number in (1.6)). By setting $\chi(x,t)\equiv\phi(x)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}|v(x,t_2)|^2\phi(x)\,\mathrm{d}x-\int_{\mathbb{R}^3}|v(x,t_1)|^2\phi(x)\,\mathrm{d}x+2\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|\nabla v(x,t)|^2\phi(x)\,\mathrm{d}x\mathrm{d}t\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(|v|^2v+2pv)(x,t)\cdot\nabla\phi(x)\,\mathrm{d}x\mathrm{d}t+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|v(x,t)|^2\Delta\phi(x)\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.2}$$

Using the blowup ansatz (1.8) and setting $s_i:=\log\frac{T}{T-t_i}$, $i=1,2$, we get

$$\int_{\mathbb{R}^3} |v(x,t_i)|^2 \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{T-t_i} \left| V\left(\frac{x}{\sqrt{T-t_i}}, s_i\right)\right|^2 \phi(x)\,\mathrm{d}x$$
$$= \sqrt{T}\mathrm{e}^{-\frac{s_i}{2}} \int_{\mathbb{R}^3} |V(y,s_i)|^2 \phi\left(y\sqrt{T}\mathrm{e}^{-\frac{s_i}{2}}\right)\mathrm{d}y. \tag{2.3}$$

Similarly, by the change of variables, we also see that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\nabla v|^2 \phi(x)\,\mathrm{d}x\mathrm{d}t = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \frac{1}{(T-t)^2} \left|(\nabla V)\left(\frac{x}{\sqrt{T-t}}, \log\frac{T}{T-t}\right)\right|^2 \phi(x)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{s_1}^{s_2}\int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V(y,s)|^2 \phi\left(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}}\right)\mathrm{d}y\mathrm{d}s. \tag{2.4}$$

Thus local energy equality (2.2) under the blowup ansatz (1.8) reduces to

$$\begin{aligned}
&\sqrt{T}\mathrm{e}^{-\frac{s_2}{2}} \int_{\mathbb{R}^3} |V(y,s_2)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s_2}{2}})\,\mathrm{d}y - \sqrt{T}\mathrm{e}^{-\frac{s_1}{2}} \int_{\mathbb{R}^3} |V(y,s_1)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s_1}{2}})\,\mathrm{d}y \\
&+ 2\sqrt{T}\int_{s_1}^{s_2}\int_{\mathbb{R}^3} \mathrm{e}^{-\frac{s}{2}} |\nabla V(y,s)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s \\
= {} & \int_{t_1}^{t_2}\int_{\mathbb{R}^3} (|V|^2 V + 2PV)\left(y, \log\frac{T}{T-t}\right) \cdot \nabla\phi(y\sqrt{T-t})\,\mathrm{d}y\mathrm{d}t \\
&+ \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \sqrt{T-t}\left|V\left(y, \log\frac{T}{T-t}\right)\right|^2 \Delta\phi(y\sqrt{T-t})\,\mathrm{d}y\mathrm{d}t \\
= {} & T\int_{s_1}^{s_2}\int_{\mathbb{R}^3} \mathrm{e}^{-s}(|V|^2 V + 2PV)(y,s) \cdot \nabla\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s \\
&+ T^{\frac{3}{2}}\int_{s_1}^{s_2}\int_{\mathbb{R}^3} \mathrm{e}^{-\frac{3s}{2}} |V(y,s)|^2 \Delta\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s,
\end{aligned} \tag{2.5}$$

where $0 \leqslant s_1 < s_2 < \infty$. With no loss of generality we assume that $s_2 - s_1 \gg S_0$ with $S_0 = 2\log\lambda$ the time period of the profiles $V(y, s)$, then for any real number $\tau_i \in [0, S_0]$, by replacing s_i in (2.5) with $s_i + \tau_i$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\sqrt{T}\mathrm{e}^{-\frac{s_2+\tau_2}{2}} \int_{\mathbb{R}^3} |V(y,s_2+\tau_2)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s_2+\tau_2}{2}})\,\mathrm{d}y \\
&\quad - \sqrt{T}\mathrm{e}^{-\frac{s_1+\tau_1}{2}} \int_{\mathbb{R}^3} |V(y,s_1+\tau_1)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s_1+\tau_1}{2}})\,\mathrm{d}y \\
&+ 2\int_{s_1+\tau_1}^{s_2+\tau_2}\int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V(y,s)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s \\
= {} & T\int_{s_1+\tau_1}^{s_2+\tau_2}\int_{\mathbb{R}^3} \mathrm{e}^{-s}(|V|^2 V + 2PV)(y,s) \cdot \nabla\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s \\
&+ T^{\frac{3}{2}}\int_{s_1+\tau_1}^{s_2+\tau_2}\int_{\mathbb{R}^3} \mathrm{e}^{-\frac{3s}{2}} |V(y,s)|^2\, \Delta\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\,\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{2.6}$$

For $i = 1, 2$, denote by

$$I_i := \int_0^{S_0} \int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s_i+\tau_i}{2}} |V(y, s_i + \tau_i)|^2 \phi\left(y\sqrt{T}\mathrm{e}^{-\frac{s_i+\tau_i}{2}}\right) \mathrm{d}y\mathrm{d}\tau_i,$$

$$J_i := \sup_{\tau_i\in[0,S_0]} \sqrt{T}\mathrm{e}^{-\frac{s_i+\tau_i}{2}} \int_{\mathbb{R}^3} |V(y, s_i + \tau_i)|^2 \phi\left(y\sqrt{T}\mathrm{e}^{-\frac{s_i+\tau_i}{2}}\right) \mathrm{d}y. \tag{2.7}$$

By virtue of the time periodicity property of V and the support property of ϕ, and by setting

$$l_i := T^{-\frac{1}{2}}\mathrm{e}^{s_i/2},\ i = 1, 2, \tag{2.8}$$

we infer that (recalling $\lambda = \mathrm{e}^{\frac{S_0}{2}}$)

$$\frac{1}{\lambda l_i} \int_0^{S_0} \int_{|y|\leqslant \frac{l_i}{\lambda}} |V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s \leqslant I_i \leqslant \frac{1}{l_i} \int_0^{S_0} \int_{|y|\leqslant \lambda l_i} |V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s,$$

$$\frac{1}{\lambda l_i} \sup_{s\in[0,S_0]} \int_{|y|\leqslant \frac{l_i}{\lambda}} |V(y, s)|^2 \,\mathrm{d}y \leqslant J_i \leqslant \frac{1}{l_i} \sup_{s\in[0,S_0]} \int_{|y|\leqslant \lambda l_i} |V(y, s)|^2 \,\mathrm{d}y. \tag{2.9}$$

Denote by

$$K(\tau_1, \tau_2) := \int_{s_1+\tau_1}^{s_2+\tau_2} \int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V(y, s)|^2 \phi\left(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}}\right) \mathrm{d}y\mathrm{d}s, \quad \forall\, \tau_1, \tau_2 \in [0, S_0], \tag{2.10}$$

then we roughly have the following estimate

$$\int_{s_1+S_0}^{s_2} \int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\, \mathrm{d}y\mathrm{d}s \leqslant K(\tau_1, \tau_2) \leqslant \int_{s_1}^{s_2+S_0} \int_{\mathbb{R}^3} \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\, \mathrm{d}y\mathrm{d}s. \tag{2.11}$$

By letting $B_k := \{s : l_1\lambda^k \leqslant T^{-\frac{1}{2}}\mathrm{e}^{\frac{s}{2}} \leqslant l_1\lambda^{k+1}\}$, $k \in \mathbb{N}$, and

$$k_1 := \left[\log_\lambda \frac{l_2}{\lambda l_1}\right], \qquad k_2 := \left[\log_\lambda \frac{\lambda l_2}{l_1}\right], \tag{2.12}$$

and thanks to the dyadic decomposition, the time periodic property of V, (2.8) and $S_0 = 2 \log \lambda$, we further get

$$\begin{aligned} K(\tau_1, \tau_2) &\geqslant \sum_{k=1}^{k_1} \int_{s_1+S_0}^{s_2} \int_{\mathbb{R}^3} 1_{B_k}(s)\, \sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V(y, s)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\, \mathrm{d}y\mathrm{d}s \\ &\geqslant \sum_{k=1}^{k_1} \frac{1}{l_1\lambda^{k+1}} \int_{s_1+2k\log\lambda}^{s_1+2(k+1)\log\lambda} \int_{|y|\leqslant l_1\lambda^{k-1}} |\nabla V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s \\ &\geqslant \sum_{k=1}^{k_1} \frac{1}{l_1\lambda^{k+1}} \int_0^{S_0} \int_{|y|\leqslant l_1\lambda^{k-1}} |\nabla V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} K(\tau_1, \tau_2) &\leqslant \sum_{k=0}^{k_2} \int_{s_1}^{s_2+S_0} \int_{\mathbb{R}^3} 1_{B_k}(s)\sqrt{T}\mathrm{e}^{-\frac{s}{2}} |\nabla V(y, s)|^2 \phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})\, \mathrm{d}y\mathrm{d}s \\ &\leqslant \sum_{k=0}^{k_2} \frac{1}{l_1\lambda^k} \int_{s_1+2k\log\lambda}^{s_1+2(k+1)\log\lambda} \int_{|y|\leqslant l_1\lambda^{k+1}} |\nabla V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s \\ &\leqslant \sum_{k=0}^{k_2} \frac{1}{l_1\lambda^k} \int_0^{S_0} \int_{|y|\leqslant l_1\lambda^{k+1}} |\nabla V(y, s)|^2 \,\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{2.14}$$

where $1_{B_k}(s)$ is the standard indicator function. In (2.6), by taking the supremum over $\tau_2 \in [0, S_0]$ and integrating on the variable $\tau_1 \in [0, S_0]$, we obtain

$$\left| S_0 J_2 - I_1 + 2\int_0^{S_0} \sup_{\tau_2\in[0,S_0]} K(\tau_1,\tau_2)\,\mathrm{d}\tau_1 \right| \leqslant K_1 + K_2, \tag{2.15}$$

where K_1 and K_2 are given by

$$K_1 := \sup_{\tau_2\in[0,S_0]} \int_0^{S_0}\int_{s_1+\tau_1}^{s_2+\tau_2}\int_{\mathbb{R}^3} T\mathrm{e}^{-s}(|V|^3 + 2|P||V|)(y,s)\,|\nabla\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})|\,\mathrm{d}y\mathrm{d}s\mathrm{d}\tau_1, \tag{2.16}$$

and

$$K_2 := \sup_{\tau_2\in[0,S_0]} \int_0^{S_0}\int_{s_1+\tau_1}^{s_2+\tau_2}\int_{\mathbb{R}^3} T^{\frac{3}{2}}\mathrm{e}^{-\frac{3s}{2}}|V(y,s)|^2|\Delta\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})|\,\mathrm{d}y\mathrm{d}s\mathrm{d}\tau_1. \tag{2.17}$$

By arguing as obtaining (2.14), we find

$$\begin{aligned} K_1 &\leqslant S_0 \int_{s_1}^{s_2+S_0}\int_{\mathbb{R}^3} T\mathrm{e}^{-s}(|V|^3 + 2|P||V|)(y,s)\,|\nabla\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})|\,\mathrm{d}y\mathrm{d}s \\ &\leqslant S_0 \sum_{k=0}^{k_2}\int_{s_1}^{s_2+S_0}\int_{\mathbb{R}^3} 1_{B_k}(s)\,T\mathrm{e}^{-s}(|V|^3 + 2|P||V|)(y,s)\,|\nabla\phi(y\sqrt{T}\mathrm{e}^{-\frac{s}{2}})|\,\mathrm{d}y\mathrm{d}s \\ &\leqslant \frac{C\lambda S_0}{\lambda-1}\sum_{k=0}^{k_2}\frac{1}{(l_1\lambda^k)^2}\int_{s_1+2k\log\lambda}^{s_1+2(k+1)\log\lambda}\int_{l_1\lambda^{k-1}\leqslant|y|\leqslant l_1\lambda^{k+1}} (|V|^3 + |P||V|)\,\mathrm{d}y\mathrm{d}s \\ &\leqslant \frac{C\lambda S_0}{\lambda-1}\sum_{k=0}^{k_2}\frac{1}{(l_1\lambda^k)^2}\int_0^{S_0}\int_{l_1\lambda^{k-1}\leqslant|y|\leqslant l_1\lambda^{k+1}} (|V|^3 + |P||V|)\,\mathrm{d}y\mathrm{d}s =: K_3, \end{aligned} \tag{2.18}$$

and

$$K_2 \leqslant \frac{C\lambda^2 S_0}{(\lambda-1)^2}\sum_{k=0}^{k_2}\frac{1}{(l_1\lambda^k)^3}\int_0^{S_0}\int_{l_1\lambda^{k-1}\leqslant|y|\leqslant l_1\lambda^{k+1}} |V(y,s)|^2\,\mathrm{d}y\mathrm{d}s =: K_4, \tag{2.19}$$

where in the third line of (2.18) we used $|\nabla\phi| \leqslant \frac{C\lambda}{\lambda-1}$ and in (2.19) $|\Delta\phi| \leqslant \frac{C\lambda^2}{(\lambda-1)^2}$. Thus we get

$$\left| S_0 J_2 - I_1 + \int_0^{S_0} \sup_{\tau_2\in[0,S_0]} 2K(\tau_1,\tau_2)\,\mathrm{d}\tau_1 \right| \leqslant K_3 + K_4. \tag{2.20}$$

Similarly, by using the different treating in $\tau_1, \tau_2 \in [0, S_0]$, we also obtain

$$\left| I_2 - I_1 + \frac{1}{S_0}\int_0^{S_0}\int_0^{S_0} 2K(\tau_1,\tau_2)\,\mathrm{d}\tau_1\mathrm{d}\tau_2 \right| \leqslant K_3 + K_4, \tag{2.21}$$

and

$$\left| I_2 - S_0\,J_1 + \sup_{\tau_1\in[0,S_0]}\int_0^{S_0} 2K(\tau_1,\tau_2)\,\mathrm{d}\tau_2 \right| \leqslant K_3 + K_4, \tag{2.22}$$

and

$$\left| J_2 - J_1 + \sup_{\tau_1,\tau_2\in[0,S_0]} 2K(\tau_1,\tau_2) \right| \leqslant \frac{K_3}{S_0} + \frac{K_4}{S_0}, \tag{2.23}$$

where I_i-J_i, $K(\tau_1, \tau_2)$, K_3, K_4 are defined by (2.7), (2.10), (2.18) and (2.19) respectively.

3. Proof of theorem 1.1

First we prove (1.11), and we see that it is a direct consequence of the classical energy inequality

$$\sup_{t\in[0,T[}\int_{\mathbb{R}^3}|v(x,t)|^2\mathrm{d}x+2\int_0^T\int_{\mathbb{R}^3}|\nabla v(x,t)|^2\mathrm{d}x\mathrm{d}t\leqslant\|v_0\|_{L^2}^2\,. \tag{3.1}$$

Indeed, concerning the local ball $B_1(x_0)$ instead of $\mathbb{R}^3$, and inserting the blowup ansatz (1.8) into (3.1), we similarly as (2.3) and (2.4) obtain

$$\sup_{t\in[0,T[}\int_{B_1(x_0)}|v(x,t)|^2\mathrm{d}x=\sup_{s\in[0,\infty[}T^{\frac12}\mathrm{e}^{-\frac{s}{2}}\int_{|y|\leqslant T^{-\frac12}\mathrm{e}^{\frac{s}{2}}}|V(y,s)|^2\mathrm{d}y\lesssim 1,$$

and

$$\int_0^T\int_{B_1(x_0)}|\nabla v(x,t)|^2\mathrm{d}x=\int_0^\infty\int_{|y|\leqslant T^{-\frac12}\mathrm{e}^{\frac{s}{2}}}T^{\frac12}\mathrm{e}^{-\frac{s}{2}}|\nabla V(y,s)|^2\mathrm{d}y\mathrm{d}s\lesssim 1,$$

then by considering $s\in[s_1,s_1+S_0]$ with $s_1\in\mathbb{N}$, and denoting $L=T^{-\frac12}\mathrm{e}^{\frac{s_1}{2}}$, we infer that

$$\frac{1}{\lambda L}\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\mathrm{d}y\leqslant\sup_{s\in[s_1,s_1+S_0]}T^{\frac12}\mathrm{e}^{-\frac{s}{2}}\int_{|y|\leqslant T^{-\frac12}\mathrm{e}^{\frac{s}{2}}}|V(y,s)|^2\mathrm{d}y\lesssim 1,$$

and

$$\frac{1}{\lambda L}\int_0^{S_0}\int_{|y|\leqslant L}|\nabla V(y,s)|^2\mathrm{d}y\mathrm{d}s\leqslant\int_{s_1}^{s_1+S_0}\int_{|y|\leqslant T^{-\frac12}\mathrm{e}^{\frac{s}{2}}}T^{\frac12}\mathrm{e}^{-\frac{s}{2}}|\nabla V(y,s)|^2\mathrm{d}y\mathrm{d}s\lesssim 1,$$

which imply the desired estimate (1.11). We remark that (1.11) can also be derived from the local energy inequality (2.20), and the proof may have its own interest (see remark 1.2), thus we place it in the appendix.

Next we prove (1.13) under the assumption (1.12), and for this purpose, it suffices to show that

$$\frac{1}{L}\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\mathrm{d}s\gtrsim 1,\quad\forall L\gg 1. \tag{3.2}$$

The argument is by contradiction: suppose that (3.2) is not true for all $L\gg 1$, then there exists a sequence of numbers $\{L_n\}_{n\in\mathbb{N}}$ tending to ∞ such that

$$\frac{1}{L_n}\sup_{s\in[0,S_0]}\int_{|y|\leqslant L_n}|V(y,s)|^2\,\mathrm{d}y\to 0,\quad\text{as }\;L_n\to\infty. \tag{3.3}$$

We then begin with the local energy inequality (2.23): by setting $l_1=\lambda L$ and $l_2=L_n\to\infty$, and according to (3.3), (2.9), (2.14), we infer that $J_2\to 0$, and

$$\begin{aligned} J_1&\geqslant\frac{1}{\lambda^2 L}\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y,\\ \sup_{\tau_1,\tau_2\in[0,S_0]}K(\tau_1,\tau_2)&\leqslant\frac{1}{\lambda L}\sum_{k=0}^{\infty}\frac{1}{\lambda^k}\int_0^{S_0}\int_{|y|\leqslant\lambda^{k+2}L}|\nabla V(y,s)|^2\,\mathrm{d}y\mathrm{d}s,\end{aligned} \tag{3.4}$$

and thus

$$\frac{1}{3\lambda^2 L}\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\leqslant \frac{C\lambda}{\lambda-1}\sum_{k=0}^{\infty}\frac{1}{(L\lambda^{k+1})^2}\int_0^{S_0}\int_{\lambda^k L\leqslant|y|\leqslant\lambda^{k+2}L}(|V|^3+|V||P|)\,\mathrm{d}y\mathrm{d}s$$
$$+\frac{C\lambda^2}{(\lambda-1)^2}\sum_{k=0}^{\infty}\frac{1}{(L\lambda^{k+1})^3}\int_0^{S_0}\int_{\lambda^k L\leqslant|y|\leqslant\lambda^{k+2}L}|V(y,s)|^2\,\mathrm{d}y\mathrm{d}s, \tag{3.5}$$

where in obtaining the left-hand-side term of (3.5) from (3.4) we have used the assumption (1.12) and taken L sufficiently large so that $\sup_{\tau_1,\tau_2\in[0,S_0]}2K(\tau_1,\tau_2)\leqslant\frac{2}{3}J_1$. In the following we will suppress the dependence on the constants λ. Since we already have (1.11), this estimate combined with the interpolation inequality guarantees that

$$\begin{aligned}\int_0^{S_0}\int_{|y|\leqslant L}|V(y,s)|^3\,\mathrm{d}y\mathrm{d}s &\lesssim \int_0^{S_0}\left(\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\right)^{\beta_p}\left(\int_{|y|\leqslant L}|V(y,s)|^p\,\mathrm{d}y\right)^{\frac{1}{p-2}}\mathrm{d}s\\ &\lesssim\left(\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\right)^{\beta_p}\left(\int_0^{S_0}\left(\int_{|y|\leqslant L}|V(y,s)|^p\,\mathrm{d}y\right)^{\frac{3}{p}}\mathrm{d}s\right)^{\frac{p}{3(p-2)}}\\ &\lesssim L^{\beta_p},\qquad\text{with}\quad \beta_p:=\frac{p-3}{p-2}.\end{aligned}$$

For the treating of the pressure profile, we turn to the following lemma, whose proof is placed in the end of this section.

Lemma 3.1. *Suppose that $V\in C^1_sC^1_{y,\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}^+)$ is a locally periodic-in-s vector field with period $S_0>0$, which additionally satisfies that for every $L\gg 1$, $2<p<\infty$ and $2\leqslant r\leqslant\infty$,*

$$\left\|\left(\int_{|y|\leqslant L}|V(y,s)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right\|_{L^r([0,S_0])}\lesssim L^{\frac{a}{p}},\quad\text{with}\quad 0\leqslant a<3.$$

Let $P(y, s)$ be a scalar-valued function defined from V by

$$P(y,s):=c_0|V(y,s)|^2+\mathrm{p.v.}\int_{\mathbb{R}^3}K_{ij}(y-z)V_i(z,s)V_j(z,s)\,\mathrm{d}z \tag{3.6}$$

with $c_0\in\mathbb{R}$ and $K_{ij}(z)$ $(i, j=1, 2, 3)$ some Calderón–Zygmund kernels, then we have

$$\left\|\left(\int_{|y|\leqslant L}|P(y,s)|^{\frac{p}{2}}\,\mathrm{d}y\right)^{\frac{2}{p}}\right\|_{L^{\frac{r}{2}}([0,S_0])}\lesssim L^{\frac{2a}{p}}. \tag{3.7}$$

Applying lemma 3.1 leads to

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{(L\lambda^{k+1})^2}\int_0^{S_0}\int_{|y|\sim\lambda^k L}(|V|^3+|V||P|)\,\mathrm{d}y\mathrm{d}s\\ &\lesssim\sum_{k=0}^{\infty}\frac{1}{(L\lambda^k)^2}\int_0^{S_0}\int_{|y|\sim\lambda^k L}\left(|V|^3+|P|^{\frac{3}{2}}\right)\mathrm{d}y\mathrm{d}s\lesssim\sum_{k=0}^{\infty}\frac{1}{(L\lambda^k)^2}(L\lambda^k)^{\beta_p}\lesssim\frac{1}{L^{2-\beta_p}}.\end{aligned}$$

From (1.11), the second term on the right-hand side of (3.5) can be treated as

$$\sum_{k=0}^{\infty}\frac{1}{(L\lambda^{k+1})^3}\int_0^{S_0}\int_{|y|\sim\lambda^k L}|V(y,s)|^2\,\mathrm{d}y\mathrm{d}s\lesssim\sum_{k=0}^{\infty}\frac{1}{(L\lambda^k)^3}L\lambda^k\lesssim\frac{1}{L^2}.$$

By plugging the above two estimates into (3.5), we have

$$\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\lesssim\frac{1}{L^{1-\beta_p}}+\frac{1}{L}\lesssim\frac{1}{L^{1/(p-2)}},$$

which implies that $V\equiv 0$ for all $(y,s)\in\mathbb{R}^3\times\mathbb{R}^+$, but this contradicts with the nontriviality assumption of V. Therefore, the desired estimate (3.2) is followed and we prove (1.13).

Proof of lemma 3.1. We only need to treat the integral term in the expression formula (3.6), denoting by $\tilde{P}(y,s)$, and we use the following decomposition

$$\begin{aligned}\tilde{P}(y,s)\ &=\text{p.v.}\int_{|z|\leqslant 2L}K_{ij}(y-z)V_i(z,s)V_j(z,s)\,\mathrm{d}z+\int_{|z|\geqslant 2L}K_{ij}(y-z)V_i(z,s)V_j(z,s)\,\mathrm{d}z\\ &:=\tilde{P}_{1,L}(y,s)+\tilde{P}_{2,L}(y,s).\end{aligned}$$

By the Calderón–Zygmund theorem, we first see that

$$\left\|\left(\int_{|y|\leqslant L}|\tilde{P}_{1,L}(y,s)|^{\frac{p}{2}}\,\mathrm{d}y\right)^{\frac{2}{p}}\right\|_{L_s^{r/2}}\lesssim\left\|\left(\int_{|y|\leqslant 2L}|V(y,s)|^p\,\mathrm{d}y\right)^{1/p}\right\|_{L_s^r}^2\lesssim L^{\frac{2a}{p}}.$$

For $\tilde{P}_{2,L}$, by the dyadic decomposition, Minkowski's inequality and Hölder's inequality we have

$$\begin{aligned}\left\|\left(\int_{|y|\leqslant L}|\tilde{P}_{2,L}(y,s)|^{\frac{p}{2}}\,\mathrm{d}y\right)^{\frac{2}{p}}\right\|_{L_s^{r/2}}&\lesssim\left\|\left(\int_{|y|\leqslant L}\left(\sum_{k=1}^{\infty}\int_{2^kL\leqslant|z|\leqslant 2^{k+1}L}\frac{1}{|y-z|^3}|V(z,s)|^2\,\mathrm{d}z\right)^{\frac{p}{2}}\mathrm{d}y\right)^{\frac{2}{p}}\right\|_{L_s^{r/2}}\\ &\lesssim\sum_{k=1}^{\infty}\left\|\left(\int_{|y|\leqslant L}\left(\int_{|z|\sim 2^kL}\frac{1}{|z|^3}|V(z,s)|^2\,\mathrm{d}z\right)^{\frac{p}{2}}\mathrm{d}y\right)^{\frac{2}{p}}\right\|_{L_s^{r/2}}\\ &\lesssim L^{\frac{6}{p}}\sum_{k=1}^{\infty}(2^kL)^{-6/p}\left\|\left(\int_{|z|\sim 2^kL}|V(z,s)|^p\,\mathrm{d}z\right)^{\frac{2}{p}}\right\|_{L_s^{r/2}}\\ &\lesssim L^{\frac{6}{p}}\sum_{k=1}^{\infty}(2^kL)^{-\frac{2(3-a)}{p}}\lesssim L^{\frac{2a}{p}}.\end{aligned}$$

Hence gathering the above estimates yields (3.7).

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Appendix A. Justifying (1.11) from the local energy inequality (2.20)

We begin with the local energy inequality (2.20), and by setting $l_1 = \lambda$, $l_2 = \lambda L \gg 1$ and using (2.9) and (2.13), we get

$$\begin{aligned}
&\frac{S_0}{\lambda^2 L} \sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y,s)|^2\,\mathrm{d}y + \frac{S_0}{\lambda^2} \sum_{k=1}^{k_1} \frac{1}{\lambda^k} \int_0^{S_0} \int_{|y|\leqslant \lambda^k} |\nabla V(y,s)|^2\,\mathrm{d}y\mathrm{d}s \\
\leqslant\ & \frac{1}{\lambda} \int_0^{S_0} \int_{|y|\leqslant \lambda^2} |V(y,s)|^2\,\mathrm{d}y\mathrm{d}s + \frac{C\lambda S_0}{\lambda-1} \sum_{k=0}^{k_2} \frac{1}{\lambda^{2k+2}} \int_0^{S_0} \int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}} (|V|^3 + |P||V|)\,\mathrm{d}y\mathrm{d}s \\
&+ \frac{C\lambda^2 S_0}{(\lambda-1)^2} \sum_{k=0}^{k_2} \frac{1}{\lambda^{3k+3}} \int_0^{S_0} \int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}} |V(y,s)|^2\,\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where $k_1 = [\log_\lambda(L/\lambda)]$ and $k_2 = [\log_\lambda(\lambda L)]$. By suppressing the dependence on λ, we see that

$$\sup_{s\in[0,S_0]} \int_{|y|\leqslant L} |V(y,s)|^2\,\mathrm{d}y + D(L) \lesssim L + E_1(L) + E_2(L),$$

with $D(L)$, $E_1(L)$ and $E_2(L)$ defined by

$$\begin{aligned}
D(L) &:= \frac{L}{C} \sum_{k=1}^{[\log_\lambda(L/\lambda)]} \frac{1}{\lambda^k} \int_0^{S_0} \int_{|y|\leqslant\lambda^k} |\nabla V(y,s)|^2\,\mathrm{d}y\mathrm{d}s \quad \text{and} \\
E_1(L) &:= L \sum_{k=0}^{[\log_\lambda(\lambda L)]} \frac{1}{\lambda^{2k}} \int_0^{S_0} \int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}} (|V|^3 + |P||V|)\,\mathrm{d}y\mathrm{d}s, \quad \text{and} \\
E_2(L) &:= L \sum_{k=0}^{[\log_\lambda(\lambda L)]} \frac{1}{\lambda^{3k}} \int_0^{S_0} \int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}} |V(y,s)|^2\,\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

For $E_1(L)$, by Hölder's inequality we directly have

$$\begin{aligned}
E_1(L) &\leqslant CL \sum_{k=0}^{[\log_\lambda(\lambda L)]} \frac{1}{\lambda^{2k}} \int_0^{S_0} \int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}} (|V|^3 + |P|^{\frac{3}{2}})\,\mathrm{d}y\mathrm{d}s \\
&\leqslant CL \sum_{k=0}^{[\log_\lambda(\lambda L)]} \frac{1}{\lambda^{2k}} \lambda^{3k(1-3/p)} \int_0^{S_0} \left(\int_{|y|\sim\lambda^k} \left(|V|^p + |P|^{\frac{p}{2}}\right)\mathrm{d}y \right)^{3/p} \mathrm{d}s.
\end{aligned}$$

Thus by using lemma 3.1, we get

$$E_1(L) \leqslant CL \sum_{k=0}^{[\log_\lambda(\lambda L)]} \lambda^{k(1-9/p)} \leqslant \begin{cases} CL[\log_\lambda L], & \text{for } p\in[3,9], \\ CL^{2-9/p}, & \text{for } p\in\,]9,\infty[. \end{cases} \tag{A.1}$$

For $E_2(L)$, Hölder's inequality directly yields

$$\begin{aligned}
E_2(L) &\leqslant CL \sum_{k=0}^{[\log_\lambda(\lambda L)]} \frac{1}{\lambda^{3k}} \lambda^{3k(1-2/p)} \left(\int_0^{S_0} \left(\int_{|y|\sim\lambda^k} |V(y,s)|^p\,\mathrm{d}y \right)^{3/p} \mathrm{d}s \right)^{2/3} \\
&\leqslant CL \sum_{k=0}^{[\log_\lambda(\lambda L)]} \lambda^{-6k/p} \leqslant CL. \qquad\qquad (A.2)
\end{aligned}$$

Hence, by gathering the above estimates we get

$$\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y+D(L)\leqslant\begin{cases}CL[\log_\lambda L], & \text{for } p\in[3,9],\\ CL^{2-9/p}, & \text{for } p\in\,]9,\infty[.\end{cases}\tag{A.3}$$

Next we intend to improve (A.3.) to obtain the desired estimate (1.11). For $p\in\,]9,\infty[$, first by the interpolation inequality we see that

$$\begin{aligned}\int_0^{S_0}\int_{|y|\leqslant L}|V(y,s)|^3\,\mathrm{d}y\mathrm{d}s &\lesssim \int_0^{S_0}\left(\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y\right)^{\beta_p}\left(\int_{|y|\leqslant L}|V(y,s)|^p\,\mathrm{d}y\right)^{\frac{1}{p-2}}\mathrm{d}s\\ &\lesssim L^{(2-9/p)\beta_p},\end{aligned}$$

with $\beta_p := \frac{p-3}{p-2}$; and then using this estimate and lemma 3.1 again yields

$$\begin{aligned}E_1(L) &\leqslant CL\sum_{k=0}^{[\log_\lambda(\lambda L)]}\frac{1}{\lambda^{2k}}\int_0^{S_0}\int_{\lambda^k\leqslant|y|\leqslant\lambda^{k+2}}\left(|V|^3+|P|^{\frac{3}{2}}\right)\mathrm{d}y\mathrm{d}s\\ &\leqslant CL\sum_{k=0}^{[\log_\lambda(\lambda L)]}\frac{1}{\lambda^{2k}}\lambda^{k(2-9/p)\beta_p}\leqslant CL.\end{aligned}$$

The estimate of the term $E_2(L)$ can also be (A.2.), and thus we have that for $p\in\,]9,\infty[$,

$$\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y+D(L)\lesssim L+E_1(L)+E_2(L)\lesssim L.\tag{A.4}$$

For $p\in[3,9]$, from (A.3.) we can choose some fixed number $\epsilon\in\,]0,1[$ so that

$$\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y+D(L)\leqslant C_\epsilon L^{1+\epsilon}.$$

Similarly as obtaining the above estimates in the case of $p\in\,]9,\infty[$, we get

$$\int_0^{S_0}\int_{|y|\leqslant L}|V(y,s)|^3\,\mathrm{d}y\mathrm{d}s\lesssim L^{(1+\epsilon)\beta_p},$$

and

$$E_1(L)\leqslant CL\sum_{k=0}^{[\log_\lambda(\lambda L)]}\frac{1}{\lambda^{2k}}\lambda^{k(1+\epsilon)\beta_p}\leqslant CL,$$

and thus for $p\in[3,9]$,

$$\sup_{s\in[0,S_0]}\int_{|y|\leqslant L}|V(y,s)|^2\,\mathrm{d}y+D(L)\lesssim L.\tag{A.5}$$

Hence (1.11) is ensured by (A.4.), (A.5.) and the following estimate

$$D(L)\gtrsim\int_0^{S_0}\int_{|y|\leqslant\lambda^{-2}L}|\nabla V(y,s)|^2\,\mathrm{d}y\mathrm{d}s.$$

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