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Note on the well-posedness of a slightly supercritical surface quasi-geostrophic equation

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ABSTRACT

For the following slightly supercritical surface quasi-geostrophic equation

$$\partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = 0, \quad u = \nabla^\perp|D|^{\beta-2}m(D)\theta,\ \beta\in\,]0,1],$$

where $m\in C^\infty(\mathbb{R}^2\setminus\{0\})$ is a radial non-decreasing positive function which roughly has a logarithmic growth near infinity, we apply the method of nonlocal maximum principle to show the global well-posedness of smooth solutions.

1. Introduction

In this article we focus on the following generalized surface quasi-geostrophic (abbr. SQG) equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ u = \nabla^\perp|D|^{\beta-2}m(D)\theta, & \\ \theta(0,x) = \theta_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\beta\in\,]0,1]$, $|D|^\beta \triangleq (-\Delta)^{\beta/2}$ and $m(D)$ is defined via the Fourier transform

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$$m(D)f(x) \triangleq \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\zeta} m(\zeta)\hat{f}(\zeta)\,\mathrm{d}\zeta,$$

with $m(\zeta)=m(|\zeta|)$ a radial non-decreasing function satisfying the following conditions

(i) $m\in C^\infty(\mathbb{R}^2\setminus\{0\})$ and $m>0$ for all $\zeta\neq 0$;
(ii) m obeys that for some universal number $\alpha\in\,]0,1[$,

$$|\zeta|m'\big(|\zeta|\big)\leqslant \alpha m\big(|\zeta|\big),\quad \forall|\zeta|>0; \tag{1.2}$$

(iii) m is of the Mikhlin–Hörmander type, that is, a constant $C>0$ can be found so that

$$\big|\partial_\zeta^k m(\zeta)\big|\leqslant C|\zeta|^{-k}m(\zeta),\quad \forall k\in\{1,2,3,4\},\ \forall\zeta\neq 0. \tag{1.3}$$

The system (1.1) is deeply related to the well-known dissipative SQG equation which arises from the geostrophic study of strongly rotating fluids (cf. [6]), with its form as follows

$$\begin{cases}\partial_t\theta+u\cdot\nabla\theta+|D|^\beta\theta=0,\\ u=\nabla^\perp|D|^{-1}\theta,\quad \theta|_{t=0}=\theta_0,\end{cases} \tag{1.4}$$

where $\beta\in\,]0,2]$ and the cases $\beta>1$, $\beta=1$ and $\beta<1$ are called the subcritical, critical and supercritical cases respectively. Indeed, if we set $m(\zeta)=|\zeta|^{1-\beta}$ ($\beta\in\,]0,1]$), then m satisfies the conditions (i)–(iii), and system (1.1) just corresponds to the critical and supercritical SQG equation.

The SQG equation has recently been intensely studied from mathematical view (cf. a long list of references in [4]), partially due to its simple form and its analogy with the 3D Navier–Stokes/Euler equations. Up to now, the subcritical and critical cases have been in a satisfactory situation. It has been known since [18] and [7] that the SQG equation at the subcritical case $\beta>1$ has the global smooth solution for suitable initial data. For the subtle critical case $\beta=1$, the global regularity issue was independently solved by [14] and [2] almost at the same time. Kiselev et al. [14] proved the global well-posedness for the periodic smooth data from developing an original method, which may be called a method of nonlocal maximum principle, the idea of which is to show that a family of suitable moduli of continuity (for the solution) are preserved by the evolution. While from a totally different direction, Caffarelli et al. [2] established the global regularity of weak solutions by deeply exploiting the DeGiorgi's method. We also refer to [15] and [9] for another two delicate and still quite different proofs of the same issue. However, at the supercritical case whether solutions (for large data) remain globally regular or not is a remarkable open problem. There are only some partial results, for instance: the conditional regularity (e.g. [8]), and the eventual regularity of weak solutions (cf. [10, 13]) and so on.

When $m\equiv 1$ and $\beta\in\,]0,1[$, Eq. (1.1) is referred to as the modified critical SQG equation, for which Constantin et al. [5] introduced it and they used the method of [2] to show the global regularity of the weak solutions. The method of nonlocal maximum principle can also be applied, and this was detailed in [17]. If we set $m=|\zeta|^\alpha$ with $\alpha\in\,]0,1[$, Eq. (1.1) reduces to a class of generalized supercritical SQG equation, and the issue of global regularity also remains open.

For Eq. (1.1) with $\beta=1$ and nontrivial multiplier m, Dabkowski et al. [11] first made an advance and they improved the method of nonlocal maximum principle to prove the global existence of smooth solutions, for the smooth increasing function m that grows slower than $\log\log|\zeta|$ near infinity (i.e., $\lim_{|\zeta|\to\infty} m(\zeta)/\log\log(|\zeta|)=0$; note that a more natural condition of type (1.5) is also discussed in an unpublished version of [11]).

In this note we are inspired by [11] to consider the slightly supercritical SQG equation (1.1) with $\beta\in\,]0,1]$, and by combining the idea of [11] with a refined version of nonlocal maximum principle in [17] (which roots in [13]), we show the global well-posedness of smooth solutions for Eq. (1.1) with a logarithmic multiplier (e.g. $m(\zeta)=\log(2e^{1/\beta}+|\zeta|)$). Precisely, our main result reads as follows.

Theorem 1.1. *Let $\beta \in]0,1]$, $\theta_0 \in H^s(\mathbb{R}^2)$ with $s > 2$. Assume that $\alpha \in]0,\beta[$ in the condition* (ii)*, and $m(\zeta) = m(|\zeta|)$ is a radial non-decreasing function satisfying the conditions* (i)–(iii) *and the following growth condition that*

$$\int_e^\infty \frac{1}{rm(r)}\,\mathrm{d}r = \infty. \tag{1.5}$$

Then the slightly supercritical SQG equation (1.1) *admits a unique global solution $\theta \in C([0,\infty[; H^s(\mathbb{R}^2)) \cap C^\infty(]0,\infty[\times\mathbb{R}^2)$.*

The proof of Theorem 1.1 depends on the elegant method of nonlocal maximum principle (cf. [14] and [11]). The method is to construct a family of suitable moduli of continuity $\{\omega_\lambda\}_{\lambda\geqslant 1}$ (for definition see below) with respect to Eq. (1.1), so that if the initial data θ_0 strictly has some modulus ω_λ, this ω_λ is also strictly obeyed by the solution $\theta(t,x)$ for all time $t > 0$. We always choose ω_λ to be Lipshitzian at zero, thus the preservation of ω_λ implies the uniform bound for Lipshitz norm of the solution, which is a sufficient condition for global regularity. Another key property is that

$$\lim_{\lambda\to\infty} \omega_\lambda(\xi) = \infty, \quad \text{for } \xi > 0, \tag{1.6}$$

as long as one intends to control all the initial data using the family $\{\omega_\lambda\}$ (otherwise, see Remark 1.2). For the critical case $m \equiv 1$ and $\beta = 1$, as [14] shows, one can simply choose that $\omega_\lambda(\xi) = \omega(\lambda\xi)$, and ω is an unbounded function with a double logarithmic growth near infinity. A crucial observation of [11] is that the unboundedness growth can be traded so that the nonlocal maximum principle can be adapted to Eq. (1.1) with slightly rougher velocity, where the family of moduli $\{\omega_\lambda\}$ is a more complicated family satisfying (1.6). However, we know from [17] that the growth condition one can afford in the critical case $\{m \equiv 1,\ \beta = 1\}$ is indeed a logarithm, by relying on the refined estimates for the drift term and dissipation term in the possible breakdown scenario (cf. Section 2), hence it strongly motivates us to slightly improve the result of [11]. We also show that a similar result extends to the slightly supercritical case for $\beta \in]0,1[$.

Remark 1.2. If $m(\zeta) = |\zeta|^{1-\beta}$ with $\beta \in]1/2,1[$, m satisfies the conditions (i), (ii) with $\alpha = 1-\beta \in]0,\beta[$, (iii), and it clearly does not obey (1.5). Hence the family of moduli (3.7) can only control the initial data of limited size, more precisely, in order to ensure that (3.10) holds for sufficiently large λ, we need that

$$\|\nabla\theta_0\|_{L^\infty}^{1-\beta}\|\theta_0\|_{L^\infty}^{\beta} \leqslant c \tag{1.7}$$

where $c > 0$ is a small number depending only on β. Moreover, by modifying the argument in Theorem 1.1, the above criterion also holds for all $\beta \in]0,1[$, and we sketch the proof in Appendix A. Note that this recovers the result in [19] for the supercritical SQG equation.

Remark 1.3. If the dissipation term $|D|^\beta\theta$ is replaced by the general term $|D|^\gamma\theta$ in the generalized SQG equation (1.1) with $\beta \in]0,1]$ and $\alpha \in]0,1[$, then for $\gamma \geqslant \alpha+\beta$, we can also show the global result for this system. We can use the energy method to treat the case $\gamma > \alpha+\beta$, and we can appeal to the nonlocal maximum principle method for the case $\gamma = \alpha+\beta$.

Remark 1.4. For the case $\beta \in]0,1[$, if we rely on the usual version of nonlocal maximum principle in [14] (i.e., $\Psi_\beta^\perp$ is not taken into account), Theorem 1.1 can also be obtained, provided that the scope of α is replaced by $\alpha \in]0,\min\{\beta,1-\beta\}[$ in the condition (ii). By this, Theorem 1.1 can extend to Eq. (1.1) with more general velocity, e.g. the slightly supercritical porous media equation as a generalized model of [3].

The paper is organized as follows. In Section 2, we provide several lemmas related with modulus of continuity, which play a key role in the main proof. Then we dedicate to the proof of Theorem 1.1 in Section 3. In Appendix A, we show some auxiliary lemmas.

Throughout this paper, C stands for a constant which may be different from line to line, and sometimes we use $X \lesssim Y$ instead of $X \leqslant CY$. We denote $B_r(x)$ the disk of $\mathbb{R}^2$ centered at x with radius r.

2. Modulus of continuity

In this section we compile some facts related to the modulus of continuity.

Definition 2.1. A function $\omega : [0,\infty[\mapsto [0,\infty[$ is called a modulus of continuity (abbr. MOC) if ω is continuous on $[0,\infty[$, increasing, concave, and piecewise C^2 with one-sided derivatives (maybe infinite at $\xi = 0$). We call that a function f has (or obeys) the modulus of continuity ω if $|f(x) - f(y)| \leqslant \omega(|x-y|)$ for all x, y. We say f strictly obeys the modulus of continuity ω if the inequality is strict for all $x \neq y$.

We first consider a special act of the dissipation operator $|D|^\beta$ on a function having MOC.

Lemma 2.2. *Let $\beta \in]0,1]$, $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function which has the MOC ω but does not strictly have the MOC ω, and there exist two separate points $x, y \in \mathbb{R}^2$ such that $f(x) - f(y) = \omega(\xi)$ with $\xi = |x-y|$. Then,*

(1) *for $x_0 \triangleq (\xi/2, 0)$ and $y_0 \triangleq (-\xi/2, 0)$, there exist a unique rotation transform ρ and a unique vector $a \in \mathbb{R}^2$ so that $x = \rho x_0 - a$ and $y = \rho y_0 - a$;*
(2) *we have*

$$\big[-|D|^\beta f\big](x) - \big[-|D|^\beta f\big](y) \leqslant \Psi_\beta(\xi) + \Psi_\beta^\perp(\xi), \tag{2.1}$$

where

$$\Psi_\beta(\xi) \triangleq B \int_0^{\frac{\xi}{2}} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+\beta}} \, \mathrm{d}\eta + B \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+\beta}} \, \mathrm{d}\eta, \tag{2.2}$$

and

$$\Psi_\beta^\perp(\xi) \triangleq -C_0 \iint_{B^+_{r_0\xi}(x_0)} \frac{2\omega(2\eta) - \tilde f(\eta,\mu) + \tilde f(-\eta,\mu) - \tilde f(\eta,-\mu) + \tilde f(-\eta,-\mu)}{|x_0 - (\eta,\mu)|^{2+\beta}} \, \mathrm{d}\eta \, \mathrm{d}\mu, \tag{2.3}$$

with $\tilde f(z) \triangleq f(\rho z - a)$ for $z \in \mathbb{R}^2$ and

$$B^+_{r_0\xi}(x_0) \triangleq \big\{(\eta,\mu) \in \mathbb{R}^2;\ \big|x_0 - (\eta,\mu)\big| \leqslant r_0\xi,\ \mu \geqslant 0\big\}.$$

At above B, C_0 and r_0 are positive constants depending only on β.

The proof is essentially parallel with that of [13, Lemma 5.1] or [16, Lemma 5.5], and for convenience we sketch it in Appendix A. Note that both terms Ψ_β and $\Psi_\beta^\perp$ are negative.

Next we consider the acting of the operator $\nabla^\perp|D|^{\beta-2}m(D)$ occurring in the velocity term on the function having MOC.

Lemma 2.3. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a smooth function which obeys the MOC ω but does not strictly obey the MOC ω, and there exist two separate points $x, y\in\mathbb{R}^2$ so that $f(x)-f(y)=\omega(\xi)$ with $\xi=|x-y|$. Suppose that $u=\nabla^\perp|D|^{\beta-2}m(D)f$ ($\beta\in\,]0,1]$) with $m(\zeta)=m(|\zeta|)$ a non-decreasing function satisfying conditions* (i)–(iii)*. Then the following statements hold true.*

(1) *We have*

$$\big|u(x)-u(y)\big|\leqslant\widetilde{\Omega}_\beta(\xi),$$

with

$$\widetilde{\Omega}_\beta(\xi)=A_1\left(\int_0^\xi\frac{\omega(\eta)m(\eta^{-1})}{\eta^\beta}\,\mathrm{d}\eta+\xi\int_\xi^\infty\frac{\omega(\eta)m(\eta^{-1})}{\eta^{1+\beta}}\,\mathrm{d}\eta\right),$$

where A_1 is a positive constant depending only on the function m and β.

(2) *Denote $\ell\triangleq\frac{x-y}{|x-y|}$, we have*

$$\big|\big(u(x)-u(y)\big)\cdot\ell\big|\leqslant\Omega_\beta(\xi)\tag{2.4}$$

with

$$\Omega_\beta(\xi)=A\left(-\xi m\big(\xi^{-1}\big)\Psi_\beta^\perp(\xi)+\xi^{1-\beta}m\big(\xi^{-1}\big)\omega(\xi)+\xi\int_\xi^\infty\frac{\omega(\eta)m(\eta^{-1})}{\eta^{1+\beta}}\,\mathrm{d}\eta\right),\tag{2.5}$$

where $\Psi_\beta^\perp$ is defined by (2.3) *and A is a positive constant depending only on m,α,β,C_0.*

Proof of Lemma 2.3. (1) Is similar to [11, Lemma 2.4] and we omit the details. We only treat (2). By virtue of Lemma A.2 below, we find that

$$u(x)-u(y)=\text{p.v.}\int_{\mathbb{R}^2}\frac{(x-z)^\perp}{|x-z|}H_\beta(x-z)f(z)\,\mathrm{d}z-\text{p.v.}\int_{\mathbb{R}^2}\frac{(y-z)^\perp}{|y-z|}H_\beta(y-z)f(z)\,\mathrm{d}z,$$

where H_β is a radial scalar function satisfying (A.4). We split into several cases. For the difference

$$\left|\int_{|x-z|\geqslant2\xi}\frac{(x-z)^\perp}{|x-z|}H_\beta(x-z)f(z)\,\mathrm{d}z-\int_{|y-z|\geqslant2\xi}\frac{(y-z)^\perp}{|y-z|}H_\beta(y-z)f(z)\,\mathrm{d}z\right|,$$

paralleling as treating the corresponding part in (1), we infer that it is bounded by

$$C\xi\int_\xi^\infty\frac{\omega(\eta)m(\eta^{-1})}{\eta^{1+\beta}}\,\mathrm{d}\eta+C\xi^{1-\beta}\omega(\xi)m\big(\xi^{-1}\big).$$

Then, recalling that $r_0 \in]0,1[$ is the fixed number occurred in Lemma 2.2 and the estimate (A.1) that $m((r_0\xi)^{-1}) \leqslant r_0^{-\alpha} m(\xi^{-1})$, we get

$$\begin{aligned}
&\left| \int\limits_{r_0\xi \leqslant |x-z| \leqslant 2\xi} \frac{(x-z)^\perp}{|x-z|} H_\beta(x-z) f(z)\,\mathrm{d}z \right| \\
&\quad = \left| \int\limits_{r_0\xi \leqslant |x-z| \leqslant 2\xi} \frac{(x-z)^\perp}{|x-z|} H_\beta(x-z)\big(f(z)-f(x)\big)\,\mathrm{d}z \right| \\
&\quad \leqslant C \int\limits_{r_0\xi}^{2\xi} \frac{\omega(r) m(r^{-1})}{r^\beta}\,\mathrm{d}r \leqslant C\xi^{1-\beta} m\big(\xi^{-1}\big)\omega(\xi).
\end{aligned}$$

A similar estimate holds for the corresponding integral with replacing x by y.

Now we consider the contribution of the “dangerous” part, i.e., the integral over $B_{r_0\xi}(x)$ and $B_{r_0\xi}(y)$, and we shall really work on the weak form $|(u(x)-u(y))\cdot \ell|$. By following the same token from Lemma 2.2 and denoting $e_1 \triangleq (1,0)$, we have

$$\begin{aligned}
&\left| \int\limits_{B_{r_0\xi}(x)} \frac{(x-z)^\perp \cdot \ell}{|x-z|} H_\beta(x-z) f(z)\,\mathrm{d}z - \int\limits_{B_{r_0\xi}(y)} \frac{(y-z)^\perp \cdot \ell}{|y-z|} H_\beta(y-z) f(z)\,\mathrm{d}z \right| \\
&= \left| \int\limits_{B_{r_0\xi}(x_0)} \frac{(x_0-z)^\perp \cdot e_1}{|x_0-z|} H_\beta(x_0-z) \tilde f(z)\,\mathrm{d}z - \int\limits_{B_{r_0\xi}(y_0)} \frac{(y_0-z)^\perp \cdot e_1}{|y_0-z|} H_\beta(y_0-z) \tilde f(z)\,\mathrm{d}z \right| \\
&= \left| \iint\limits_{B_{r_0\xi}(x_0)} \frac{H_\beta(|x_0-(\eta,\mu)|)\,\mu}{|x_0-(\eta,\mu)|} \tilde f(\eta,\mu)\,\mathrm{d}\eta\,\mathrm{d}\mu - \iint\limits_{B_{r_0\xi}(y_0)} \frac{H_\beta(|y_0-(\eta,\mu)|)\,\mu}{|y_0-(\eta,\mu)|} \tilde f(\eta,\mu)\,\mathrm{d}\eta\,\mathrm{d}\mu \right| \\
&= \left| \iint\limits_{B_{r_0\xi}(x_0)} \frac{H_\beta(|x_0-(\eta,\mu)|)\,\mu}{|x_0-(\eta,\mu)|} \big(\tilde f(\eta,\mu) - \tilde f(-\eta,\mu)\big)\,\mathrm{d}\eta\,\mathrm{d}\mu \right| \\
&= \left| \iint\limits_{B^+_{r_0\xi}(x_0)} \frac{H_\beta(|x_0-(\eta,\mu)|)\,\mu}{|x_0-(\eta,\mu)|} \big(\tilde f(\eta,\mu) - \tilde f(-\eta,\mu) - \tilde f(\eta,-\mu) + \tilde f(-\eta,-\mu)\big)\,\mathrm{d}\eta\,\mathrm{d}\mu \right| \\
&\leqslant C \iint\limits_{B^+_{r_0\xi}(x_0)} \frac{m(|x_0-(\eta,\mu)|^{-1})\,\mu}{|x_0-(\eta,\mu)|^{2+\beta}} \big|\tilde f(\eta,\mu) - \tilde f(-\eta,\mu) - \tilde f(\eta,-\mu) + \tilde f(-\eta,-\mu)\big|\,\mathrm{d}\eta\,\mathrm{d}\mu. \qquad (2.6)
\end{aligned}$$

We claim that the last expression is bounded from above by

$$-(C/C_0)\xi m\big(\xi^{-1}\big)\Psi_\beta^\perp(\xi).$$

Indeed, we first notice that

$$\begin{aligned}
&\big|\tilde f(\eta,\mu) - \tilde f(-\eta,\mu) - \tilde f(\eta,-\mu) + \tilde f(-\eta,-\mu)\big| \\
&\quad \leqslant 2\omega(2\eta) - \tilde f(\eta,\mu) + \tilde f(-\eta,\mu) - \tilde f(\eta,-\mu) + \tilde f(-\eta,-\mu),
\end{aligned}$$

which can deduced from two obvious estimates $\tilde f(\eta,\mu) - \tilde f(-\eta,\mu) \leqslant \omega(2\eta)$ and $\tilde f(\eta,-\mu) - \tilde f(-\eta,-\mu) \leqslant \omega(2\eta)$. Second, from (A.1) below, we deduce that for all $(\eta,\mu) \in B^+_{r_0\xi}(x_0)$,

$$m\big(\big|x_0-(\eta,\mu)\big|^{-1}\big)\mu \leqslant m\big((r_0\xi)^{-1}\big)(r_0\xi)^{\alpha}\big|x_0-(\eta,\mu)\big|^{-\alpha}\mu$$
$$\leqslant m\big(\xi^{-1}\big)\xi^{\alpha}\big|x_0-(\eta,\mu)\big|^{1-\alpha} \leqslant \xi m\big(\xi^{-1}\big).$$

Therefore, gathering the above estimates yields (2.4). □

As a corollary, we obtain the contribution from the drift term at a special scenario involving MOC.

Corollary 2.4. *Under the assumption of Lemma* 2.3*, we have*

$$\big|u\cdot\nabla f(x) - u\cdot\nabla f(y)\big| \leqslant \Omega_\beta(\xi)\omega'(\xi),$$

with Ω_β given by (2.5)*.*

Proof of Corollary 2.4. Denote $\ell = \frac{x-y}{|x-y|}$, and ν be any unit vector perpendicular to ℓ. As shown in [13, Proposition 2.4], we know that under the scenario described in the assumption,

$$\partial_\ell f(x) = \partial_\ell f(y) = \omega'(\xi), \qquad \partial_\nu f(x) = \partial_\nu f(y) = 0.$$

Hence from (2.4), we get

$$\big|u\cdot\nabla f(x) - u\cdot\nabla f(y)\big| = \big|\big(u(x)-u(y)\big)\cdot\ell\big|\omega'(\xi) \leqslant \Omega_\beta(\xi)\omega'(\xi). \qquad \square$$

3. Proof of Theorem 1.1

We first have the following local existence result for the system (1.1).

Proposition 3.1. *Let $\beta \in\,]0,1]$, $m(\zeta) = m(|\zeta|)$ be a non-decreasing function satisfying conditions* (i)–(iii) *and $\theta_0 \in H^s(\mathbb{R}^2)$, $s > 2$. Then there exists a positive number T depending only on $\beta, m, \|\theta_0\|_{H^s}$ so that the system* (1.1) *generates a unique solution $\theta \in C([0,T]; H^s(\mathbb{R}^2)) \cap C^\infty(]0,T]\times\mathbb{R}^2)$. Moreover, the time T can be continued beyond if one has $\int_0^T \|\nabla\theta(t)\|_{L^\infty}^{\beta+2\alpha/(1-\alpha)}\,\mathrm{d}t < \infty$.*

The proof is quite similar to that in [16,17] (although the direct consequence is only the case $\alpha \in\,]0, 1-\beta/2[$ in condition (ii)), since from Lemma A.1 we know that $m(|\zeta|) \leqslant C\max\{|\zeta|^\alpha, 1\}m(1)$. We shall sketch the main points of the proof in Appendix A.

We shall follow the scheme in [14] to show that the evolution of the system (1.1) will preserve a family of suitable moduli of continuity. The following key lemma states the possible breakdown scenario.

Lemma 3.2. *Let $\theta \in C([0,T^*[; H^s) \cap C^\infty(]0,T^*[\times\mathbb{R}^2)$ be the maximal lifespan solution constructed in Proposition* 3.1*, and ω be a modulus of continuity satisfying $\omega(0) = 0$, $\omega'(0+) < \infty$ and $\omega''(0+) = -\infty$. Assume that initially θ_0 strictly obeys the MOC ω. Let $T_* > 0$ be the first time that $\theta(t)$ do not strictly have the MOC ω, then there exists two points $x \neq y \in \mathbb{R}^2$ so that*

$$\theta(T_*,x) - \theta(T_*,y) = \omega\big(|x-y|\big). \tag{3.1}$$

The proof is by now classical, cf. [14,16,17].

Under this scenario, if we can show that

$$f'(T_*) < 0, \quad \text{with } f(t) = \theta(t,x) - \theta(t,y), \tag{3.2}$$

then it clearly contradicts the definition of T_*, and this further implies that $\nabla\theta \in L^\infty([0,T^*[\times\mathbb{R}^2)$. Combining it with the continuation criterion in Proposition 3.1 yields $T^* = \infty$. Hence, the target is to show (3.2). Since $\theta(t,x)$ ($\forall t \in]0,T^*[$) solves Eq. (1.1) in the classical sense, we have

$$f'(T_*) = -u\cdot\nabla\theta(T_*,x) + u\cdot\nabla\theta(T_*,y) - |D|^\beta\theta(T_*,x) + |D|^\beta\theta(T_*,y).$$

According to Lemma 2.2 and Corollary 2.4, to prove (3.2), it suffices to prove that

$$\Omega_\beta(\xi)\omega'(\xi) + \Psi_\beta(\xi) + \Psi_\beta^\perp(\xi) < 0, \quad \text{for all } \xi = |x-y| > 0, \tag{3.3}$$

where Ω_β, Ψ_β, $\Psi_\beta^\perp$ are defined by (2.5), (2.2) and (2.3), respectively.

Next, motivated by [11,14], we shall construct a family of appropriate moduli of continuity adapted to the slightly supercritical equation. Let $\kappa \in]0,1[$ be a fixed number with its value chosen later, then for every $\lambda \geqslant 1$, we define $\delta = \delta(\lambda)$ to be the unique solution of the equation

$$\lambda\delta m\big(\delta^{-1}\big) = \kappa. \tag{3.4}$$

This can be seen from Lemma A.1 that the function $g(r) = rm(r^{-1})$ is strictly increasing on $]0,\infty[$. It is also obvious to find that $\delta(\lambda)$ is a strictly decreasing function about λ and that

$$\delta(\lambda) \to 0+, \quad \text{as } \lambda \to \infty \tag{3.5}$$

due to $g(0+) = 0$. Then with $\delta(\lambda)$ ($\lambda \geqslant 1$) at our disposal, we define a continuous function ω_λ satisfying that for $\beta = 1$,

$$\begin{cases} \omega_\lambda(0) = 0, \\ \omega_\lambda'(\xi) = \lambda - \dfrac{\lambda^2}{6\kappa}\xi m\big(\xi^{-1}\big)\Big(3 + (1-\alpha)\log\dfrac{\delta(\lambda)}{\xi}\Big), & \forall \xi \in \big]0,\delta(\lambda)\big], \\ \omega_\lambda'(\xi) = \dfrac{\gamma}{3\xi m(\xi^{-1})}, & \forall \xi \in \big]\delta(\lambda),\infty\big[, \end{cases} \tag{3.6}$$

and that for $\beta \in]0,1[$,

$$\begin{cases} \omega_\lambda(0) = 0, \\ \omega_\lambda'(\xi) = \lambda - \dfrac{\lambda^2}{2\kappa}\delta(\lambda)^{1-\beta}\xi^\beta m\big(\xi^{-1}\big), & \forall \xi \in \big]0,\delta(\lambda)\big], \\ \omega_\lambda'(\xi) = \dfrac{\gamma}{3\xi m(\xi^{-1})}, & \forall \xi \in \big]\delta(\lambda),\infty\big[, \end{cases} \tag{3.7}$$

where α is the universal number occurring in (1.2) and $\kappa = \kappa(\alpha,\beta,m,A)$, $\gamma = \gamma(\kappa,\alpha,\beta,A)$ are two sufficiently small positive constants to be chosen later.

Now we justify that ω_λ is truly a suitable MOC satisfying $\omega_\lambda'(0+) < \infty$ and $\omega_\lambda''(0+) = -\infty$. Since $g(r) = rm(r^{-1})$ is strictly increasing on $]0,\infty[$, if we assume $\kappa \leqslant m(1)$, then we have $\delta(\lambda) \leqslant \delta(1) \leqslant 1$, thus by (A.1) it is clear to see that

$$0 \leqslant \begin{cases} \xi m(\xi^{-1})(1+\log\xi^{-1}) \leqslant m(1)\,\xi^{1-\alpha}(1+\log\xi^{-1}) \to 0, & \text{as } \xi \to 0+,\ \beta = 1, \\ \xi^{\beta} m(\xi^{-1}) \leqslant m(1)\xi^{\beta-\alpha} \to 0, & \text{as } \xi \to 0+,\ \beta \in]0,1[, \end{cases}$$

and hence $\omega_\lambda'(0+) = \lambda$. Next we consider the concavity property. For $\xi \in]0, \delta(\lambda)[$, from (1.2) we have that if $\beta = 1$,

$$\begin{aligned} \omega_\lambda''(\xi) &= -\frac{\lambda^2}{6\kappa}\left(\left(m\left(\xi^{-1}\right) - \xi^{-1}m'\left(\xi^{-1}\right)\right)\left(3 + (1-\alpha)\log\frac{\delta(\lambda)}{\xi}\right) - (1-\alpha)m\left(\xi^{-1}\right)\right) \\ &\leqslant -\frac{\lambda^2}{6\kappa}\left((1-\alpha)m\left(\xi^{-1}\right)\left(3 + (1-\alpha)\log\frac{\delta(\lambda)}{\xi}\right) - (1-\alpha)m\left(\xi^{-1}\right)\right) \\ &\leqslant -\frac{(1-\alpha)\lambda^2}{12\kappa} m\left(\xi^{-1}\right)\left(3 + (1-\alpha)\log\frac{\delta(\lambda)}{\xi}\right), \end{aligned} \tag{3.8}$$

and if $\beta \in]0,1[$,

$$\begin{aligned} \omega_\lambda''(\xi) &= -\frac{\lambda^2}{2\kappa}\delta(\lambda)^{1-\beta}\xi^{\beta-1}\left(\beta m\left(\xi^{-1}\right) - \xi^{-1}m'\left(\xi^{-1}\right)\right) \\ &\leqslant -\frac{(\beta-\alpha)\lambda^2}{2\kappa}\delta(\lambda)^{1-\beta}\xi^{\beta-1}m\left(\xi^{-1}\right). \end{aligned} \tag{3.9}$$

Notice that from $\lim_{\xi\to 0+} m(\xi^{-1}) \geqslant m(1)$, these formulae imply that

$$\omega_\lambda''(0+) \leqslant \begin{cases} -\frac{(1-\alpha)\lambda^2}{12\kappa} m(1) \lim_{\xi\to 0+}(3 + (1-\alpha)\log\frac{\delta(\lambda)}{\xi}) = -\infty, & \text{for } \beta = 1, \\ -\frac{(\beta-\alpha)\lambda^2}{2\kappa}\delta(\lambda)^{1-\beta} m(1) \lim_{\xi\to 0+} \xi^{\beta-1} = -\infty, & \text{for } \beta \in]0,1[. \end{cases}$$

While for $\xi \in]\delta(\lambda), \infty[$, we get

$$\omega_\lambda''(\xi) = -\frac{\gamma}{3\xi^2(m(\xi^{-1}))^2}\left(m\left(\xi^{-1}\right) - \xi^{-1}m'\left(\xi^{-1}\right)\right) \leqslant -\frac{(1-\alpha)\gamma}{3\xi^2 m(\xi^{-1})}.$$

For $\xi = \delta(\lambda)$, from (3.4) we know that for all $\beta \in]0,1]$,

$$\omega_\lambda'\left(\delta(\lambda)-\right) = \lambda - \frac{\lambda^2}{2\kappa}\delta(\lambda)m\left(\delta(\lambda)^{-1}\right) = \frac{\lambda}{2},$$

and

$$\omega_\lambda'\left(\delta(\lambda)+\right) = \frac{\gamma}{3\delta(\lambda)m(\delta(\lambda)^{-1})} = \frac{\gamma\lambda}{3\kappa} \leqslant \frac{\lambda}{3}$$

where we have assumed that $\gamma < \kappa$. Hence ω_λ is concave on $]0,\infty[$. Finally, due to the concavity of ω_λ and the fact that $\omega_\lambda'(\xi) > 0$ for all $\xi \in]\delta(\lambda), \infty[$, we know that ω_λ is strictly increasing on $]0,\infty[$.

Next we show that for every $\theta_0 \in H^s(\mathbb{R}^2)$ with $s > 2$, θ_0 strictly obeys the modulus of continuity ω_λ for sufficiently large λ. Similarly as in [11, Section 3.2], due to the monotonicity and concavity of the MOC, it suffices to choose ω_λ so that

$$2\|\theta_0\|_{L^\infty} < \omega_\lambda\left(\frac{2\|\theta_0\|_{L^\infty}}{\|\nabla\theta_0\|_{L^\infty}}\right). \tag{3.10}$$

Observe that for each $c_0 > 0$, from (3.5) we have $c_0 > \delta(\lambda)$ for λ large enough, and that

$$\omega_\lambda(c_0) \geqslant \int_{\delta(\lambda)}^{c_0} \frac{\gamma}{3\xi m(\xi^{-1})}\,\mathrm{d}\xi = \int_{1/c_0}^{1/\delta(\lambda)} \frac{\gamma}{3rm(r)}\,\mathrm{d}r \to \infty, \quad \text{as } \lambda \to \infty,$$

where the convergence followed from the growth condition (1.5). Hence, if we choose λ sufficiently large, (3.10) can be guaranteed by each $\theta_0 \in H^s$ with $s > 2$, and this leads to the desired result.

Based on the above discussion, we know that for the MOC ω_λ with some sufficiently large λ, the assumptions in Lemma 3.2 are satisfied. Thus to get rid of the possible breakdown scenario, it reduces to prove that (3.3) holds for this MOC ω_λ, that is, we only need to show that for all $\xi > 0$,

$$A\left(-\Psi^{\perp}_{\beta,\lambda}(\xi) + \frac{\omega_\lambda(\xi)}{\xi^\beta} + \int_\xi^\infty \frac{\omega_\lambda(\eta)}{\eta^{1+\beta}}\,\mathrm{d}\eta\right)\xi m\big(\xi^{-1}\big)\omega'_\lambda(\xi) + \Psi^{\perp}_{\beta,\lambda}(\xi) + \Psi_{\beta,\lambda}(\xi) < 0, \tag{3.11}$$

where $\Psi^{\perp}_{\beta,\lambda} \leqslant 0$ is defined by (2.3) adapted to ω_λ and

$$\begin{aligned}\Psi_{\beta,\lambda}(\xi) = {} & B\int_0^{\frac{\xi}{2}} \frac{\omega_\lambda(\xi+2\eta)+\omega_\lambda(\xi-2\eta)-2\omega_\lambda(\xi)}{\eta^{1+\beta}}\,\mathrm{d}\eta \\ & + B\int_{\frac{\xi}{2}}^\infty \frac{\omega_\lambda(2\eta+\xi)-\omega_\lambda(2\eta-\xi)-2\omega_\lambda(\xi)}{\eta^{1+\beta}}\,\mathrm{d}\eta.\end{aligned}$$

We shall divide into two cases to check it in the sequel.

Case 1. $0 < \xi \leqslant \delta(\lambda)$, $\beta \in\,]0, 1]$.

Since for all $\eta > 0$, $\omega'_\lambda(\eta) < \omega'_\lambda(0+) = \lambda$, we know that $\omega_\lambda(\eta) \leqslant \lambda\eta$, and thus $\frac{\omega_\lambda(\xi)}{\xi^\beta} \leqslant \lambda\xi^{1-\beta} \leqslant \lambda\delta(\lambda)^{1-\beta}$, and

$$\int_\xi^{\delta(\lambda)} \frac{\omega_\lambda(\eta)}{\eta^{1+\beta}}\,\mathrm{d}\eta \leqslant \lambda\int_\xi^{\delta(\lambda)} \frac{1}{\eta^\beta}\,\mathrm{d}\eta \leqslant \begin{cases} \lambda\log(\frac{\delta(\lambda)}{\xi}), & \text{for } \beta = 1, \\ \frac{\lambda}{1-\beta}\delta(\lambda)^{1-\beta}, & \text{for } \beta \in\,]0,1[, \end{cases}$$

where we have used the fact $\delta(\lambda) \leqslant 1$. According to the integration by parts and the fact that the function $r^{-\alpha}m(r)$ is non-increasing on $]0, \infty[$ (cf. Lemma A.1), we get that for $\beta \in\,]0, 1]$,

$$\begin{aligned}\int_{\delta(\lambda)}^\infty \frac{\omega_\lambda(\eta)}{\eta^{1+\beta}}\,\mathrm{d}\eta &= \frac{1}{\beta}\frac{\omega_\lambda(\delta(\lambda))}{\delta(\lambda)^\beta} + \frac{1}{\beta}\int_{\delta(\lambda)}^\infty \frac{\gamma}{3\eta^{1+\beta}m(\eta^{-1})}\,\mathrm{d}\eta \\ &\leqslant \frac{\lambda}{\beta}\delta(\lambda)^{1-\beta} + \frac{\gamma}{3\beta\delta(\lambda)^\alpha m(\delta(\lambda)^{-1})}\int_{\delta(\lambda)}^\infty \frac{1}{\eta^{1+\beta-\alpha}}\,\mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\lambda}{\beta}\delta(\lambda)^{1-\beta} + \frac{\gamma}{3\beta(\beta-\alpha)\delta(\lambda)^{\beta} m(\delta(\lambda)^{-1})} \\
&\leqslant \frac{\lambda}{\beta}\delta(\lambda)^{1-\beta} + \frac{\gamma\lambda}{3\beta(\beta-\alpha)\kappa}\delta(\lambda)^{1-\beta} \leqslant \frac{2\lambda}{\beta}\delta(\lambda)^{1-\beta},
\end{aligned} \tag{3.12}
$$

where in the last line we have used (3.4) and the assumption $\gamma \leqslant 3(\beta-\alpha)\kappa$. Due to $\delta(\lambda) \leqslant \delta(1) \leqslant 1$ (if $\kappa \leqslant m(1)$) and $\delta(1) \leqslant \kappa/m(\delta(1)^{-1}) \leqslant \kappa/m(1)$, from (A.1), we also see that

$$
\xi m(\xi^{-1}) \leqslant m(1)\xi^{1-\alpha} \leqslant m(1)\delta(1)^{1-\alpha} \leqslant m(1)^{\alpha}\kappa^{1-\alpha}.
$$

Hence, collecting the above estimates, we obtain that the positive contribution from the drift term is bounded by

$$
\begin{cases}
-Am(1)^{\alpha}\kappa^{1-\alpha}\Psi^{\perp}_{\beta,\lambda} + A\lambda^2\xi m(\xi^{-1})\left(3+\log\left(\dfrac{\delta(\lambda)}{\xi}\right)\right), & \text{for } \beta = 1, \\
-Am(1)^{\alpha}\kappa^{1-\alpha}\Psi^{\perp}_{\beta,\lambda} + \dfrac{2A}{\beta(1-\beta)}\lambda^2\delta(\lambda)^{1-\beta}\xi m(\xi^{-1}), & \text{for } \beta \in]0,1[.
\end{cases}
$$

For the contribution from the dissipation term, by virtue of the Taylor formula and (3.8), (3.9), we infer that

$$
\Psi_{\beta,\lambda}(\xi) \leqslant B\int_0^{\frac{\xi}{2}} \frac{\omega''_{\lambda}(\xi)2\eta^2}{\eta^{1+\beta}}\,\mathrm{d}\eta \leqslant
\begin{cases}
-\frac{B(1-\alpha)^2}{12\kappa}\lambda^2\xi m(\xi^{-1})(3+\log\frac{\delta(\lambda)}{\xi}), & \text{for } \beta = 1, \\
-\frac{B(\beta-\alpha)}{8\kappa}\lambda^2\delta(\lambda)^{1-\beta}\xi m(\xi^{-1}), & \text{for } \beta \in]0,1[.
\end{cases}
$$

Therefore, if $\kappa = \kappa(\alpha,\beta,m,A,B)$ is chosen small enough, we have

$$
Am(1)^{\alpha}\kappa^{1-\alpha} < 1, \quad \text{and} \quad
\begin{cases}
A < \dfrac{B(1-\alpha)^2}{12\kappa}, & \text{for } \beta = 1, \\
\dfrac{2A}{\beta(1-\beta)} < \dfrac{B(\beta-\alpha)}{8\kappa}, & \text{for } \beta \in]0,1[,
\end{cases}
$$

and they further ensure that (3.11) holds.

Case 2. $\delta(\lambda) < \xi < \infty$, $\beta \in]0,1]$.

Similarly as obtaining (3.12), we have

$$
\begin{aligned}
\int_{\xi}^{\infty} \frac{\omega_{\lambda}(\eta)}{\eta^{1+\beta}}\,\mathrm{d}\eta &= \frac{\omega_{\lambda}(\xi)}{\beta\xi^{\beta}} + \frac{1}{\beta}\int_{\xi}^{\infty} \frac{\gamma}{3\eta^{1+\beta}m(\eta^{-1})}\,\mathrm{d}\eta \\
&\leqslant \frac{\omega_{\lambda}(\xi)}{\beta\xi^{\beta}} + \frac{\gamma}{3\beta\xi^{\alpha}m(\xi^{-1})}\int_{\xi}^{\infty}\frac{1}{\eta^{1+\beta-\alpha}}\,\mathrm{d}\eta \\
&= \frac{\omega_{\lambda}(\xi)}{\beta\xi^{\beta}} + \frac{\gamma}{3\beta(\beta-\alpha)\xi^{\beta}m(\xi^{-1})}.
\end{aligned}
$$

We claim that for all $\beta \in]0,1]$, if we choose γ sufficiently small, we have

$$\frac{\gamma}{3m(\xi^{-1})} \leqslant \bar{C}\omega_\lambda(\xi), \quad \text{for all } \xi > \delta(\lambda), \tag{3.13}$$

with $\bar{C} > 1$ is a number fixed later. For $\xi \in]\delta(\lambda), \frac{\bar{C}}{\bar{C}-1}\delta(\lambda)]$, thanks to (3.4) and the fact that $m(\delta(\lambda)^{-1}) \leqslant (\bar{C}/(\bar{C}-1))^\alpha m(\xi^{-1})$, we get

$$\omega_\lambda(\xi) \geqslant \omega_\lambda\big(\delta(\lambda)\big) \geqslant \frac{\lambda\delta(\lambda)}{2} \geqslant \frac{\kappa}{2m(\delta(\lambda)^{-1})} \geqslant \frac{(\bar{C}-1)^\alpha\kappa}{2\bar{C}^\alpha m(\xi^{-1})} \geqslant \frac{\gamma}{3m(\xi^{-1})}$$

where we have used the assumption $\gamma \leqslant \frac{3}{2}\bar{C}^{-\alpha}(\bar{C}-1)^\alpha\kappa$ and the following estimate

$$\omega\big(\delta(\lambda)\big) = \int_0^{\delta(\lambda)} \omega'_\lambda(\eta)\,\mathrm{d}\eta \geqslant \omega'_\lambda\big(\delta(\lambda)\big)\delta(\lambda) = \frac{\lambda}{2}\delta(\lambda).$$

For $\xi \in]\frac{\bar{C}}{\bar{C}-1}\delta(\lambda), \infty[$, by virtue of Lemma A.1 and the fact that $\delta(\lambda) \leqslant (1-1/\bar{C})\xi$, we see that

$$\begin{aligned}
\omega_\lambda(\xi) &\geqslant \int_{\delta(\lambda)}^{\xi} \frac{\gamma}{3\eta m(\eta^{-1})}\,\mathrm{d}\eta \geqslant \frac{\gamma}{3\xi^\alpha m(\xi^{-1})}\int_{\delta(\lambda)}^{\xi} \frac{1}{\eta^{1-\alpha}}\,\mathrm{d}\eta \\
&\geqslant \frac{\gamma}{3\xi^\alpha m(\xi^{-1})}\int_{(1-1/\bar{C})\xi}^{\xi} \frac{1}{\eta^{1-\alpha}}\,\mathrm{d}\eta \geqslant \bar{C}^{-1}\frac{\gamma}{3m(\xi^{-1})}.
\end{aligned}$$

Thus the assertion (3.13) follows. Hence, recalling that $\xi m(\xi^{-1})\omega'_\lambda(\xi) = \frac{\gamma}{3}$, we know that the contribution from the drift term is bounded by

$$-\frac{A\gamma}{3}\Psi^{\perp}_{\beta,\lambda}(\xi) + \frac{A\gamma}{3}\left(\frac{2}{\beta} + \frac{\bar{C}}{\beta(\beta-\alpha)}\right)\frac{\omega_\lambda(\xi)}{\xi^\beta}.$$

On the other hand, from (3.13), we get

$$\begin{aligned}
\omega_\lambda(2\xi) &= \omega_\lambda(\xi) + \int_\xi^{2\xi} \frac{\gamma}{3\eta\, m(\eta^{-1})}\,\mathrm{d}\eta \\
&\leqslant \omega_\lambda(\xi) + \frac{1}{\xi^\alpha m(\xi^{-1})}\int_\xi^{2\xi} \frac{\gamma}{3\eta^{1-\alpha}}\,\mathrm{d}\eta \\
&\leqslant \omega_\lambda(\xi) + \frac{2^\alpha - 1}{\alpha}\frac{\gamma}{3m(\xi^{-1})} \\
&\leqslant (1 + c_\alpha\bar{C})\omega_\lambda(\xi),
\end{aligned}$$

where $c_\alpha \triangleq \frac{2^\alpha-1}{\alpha}$. Note that $c_\alpha \in]\log 2, 1[$ for $\alpha \in]0,1[$, thus by choosing $\bar{C} = \frac{1+c_\alpha}{2c_\alpha} > 1$, we see $\omega_\lambda(2\xi) \leqslant (\frac{3}{2} + \frac{c_\alpha}{2})\omega_\lambda(\xi)$. Combining it with the fact that $\omega_\lambda(2\eta+\xi) - \omega_\lambda(2\eta-\xi) \leqslant \omega_\lambda(2\xi)$, we find

$$\Psi_{\beta,\lambda}(\xi) \leqslant -B \int_{\frac{\xi}{2}}^{\infty} \frac{(1/2 - c_\alpha/2)\omega_\lambda(\xi)}{\eta^{1+\beta}}\,\mathrm{d}\eta = -\frac{2^{\beta-1}B(1-c_\alpha)}{\beta}\frac{\omega_\lambda(\xi)}{\xi^\beta}.$$

Hence, to guarantee that (3.11) is true, it suffices to choose $\gamma = \gamma(\kappa,\alpha,\beta,A,B)$ small enough so that

$$\frac{A\gamma}{3} < 1 \quad \text{and} \quad \frac{A\gamma}{3}\left(\frac{2}{\beta} + \frac{\bar{C}}{\beta(\beta-\alpha)}\right) < \frac{2^{\beta-1}B(1-c_\alpha)}{\beta}.$$

Therefore, for the suitable MOC (3.6) and (3.7) with some sufficiently large λ and some sufficiently small κ, γ, the breakdown scenario (3.1) cannot happen, and Theorem 1.1 is proved.

Appendix A

For the function satisfying (1.2), it has the following useful property.

Lemma A.1. *Assume that $m(r)$ $(r > 0)$ is a smooth non-decreasing positive function satisfying* (1.2) *with $\alpha \in]0,1[$. Then we have that*

$$m(r) \leqslant (r/b)^\alpha m(b), \quad \forall r \geqslant b > 0, \tag{A.1}$$

and the function $r\,m(r^{-1})$ is a strictly increasing function for all $r > 0$.

Proof of Lemma A.1. Considering $g_1(r) = r^{-\alpha}m(r)$ for all $r > 0$, from (1.2), we have

$$g_1'(r) = r^{-\alpha-1}\big(rm'(r) - \alpha m(r)\big) \leqslant 0,$$

hence (A.1) follows. Similarly, denoting $g_2(r) = r\ m(r^{-1})$ for $r > 0$, we have

$$g_2'(r) = m\big(r^{-1}\big) - r^{-1}m'\big(r^{-1}\big) \geqslant (1-\alpha)m\big(r^{-1}\big) > 0,$$

thus $g_2(r)$ is a strictly increasing function on $]0,\infty[$. □

Then we show the criterion (1.7) for the supercritical SQG equation with $\beta \in]0,1[$.

Justifying criterion (1.7) for $0 < \beta < 1$. We also use the nonlocal maximum principle method. Let $\gamma, \kappa \in]0,1[$ be two numbers fixed later, then for $\lambda \geqslant 1$, define $\delta(\lambda) = (\kappa/\lambda)^{1/\beta}$, and a continuous function ω_λ as

$$\begin{cases} \omega_\lambda(0) = 0, \\ \omega_\lambda'(\xi) = \lambda - \dfrac{\lambda^2}{6\kappa}\xi^\beta\left(3 + \beta\log\dfrac{\delta(\lambda)}{\xi}\right), & \forall \xi \in \big]0, \delta(\lambda)\big], \\ \omega_\lambda'(\xi) = \dfrac{\gamma}{3\xi^\beta}, & \forall \xi \in \big]\delta(\lambda), \infty\big[. \end{cases} \tag{A.2}$$

It can be seen that ω_λ is indeed a MOC satisfying that $\omega_\lambda'(0+) = \lambda$ and $\omega_\lambda''(0+) = -\infty$. Moreover, if θ_0 strictly obeys the MOC ω_λ, then in order to show that ω_λ is also strictly preserved by $\theta(t)$, it suffices to prove that for $\xi > 0$,

$$A\left(-\xi^{\beta}\Psi^{\perp}_{\beta,\lambda}(\xi)+\omega_\lambda(\xi)+\xi\int\limits_{\xi}^{\infty}\frac{\omega_\lambda(\eta)}{\eta^2}\,\mathrm{d}\eta\right)\omega'_\lambda(\xi)+\Psi^{\perp}_{\beta,\lambda}(\xi)+\Psi_{\beta,\lambda}(\xi)<0, \tag{A.3}$$

where $\Psi^{\perp}_{\beta,\lambda}(\xi)$, $\Psi_{\beta,\lambda}(\xi)$ are respectively given by (2.3), (2.2) adapted to MOC ω_λ. For the suitable MOC (A.2) with some sufficiently large λ and some sufficiently small $\kappa=\kappa(\beta)$, $\gamma=\gamma(\beta)$, in a similar way as obtaining (3.11), we can show that (A.3) holds true for all $\xi>0$, and here we omit the details. Therefore, if the condition (1.7) is satisfied for a small positive number $c=c(\beta)$, this implies that θ_0 strictly has the MOC ω_λ, and it further leads to the global result. □

The next lemma concerns the kernel estimates of the operator occurred in the expression of u.

Lemma A.2. *Let $\beta\in\,]0,1]$, $K_{\beta,i}(x)$ be the kernel of the operator $\partial_i|D|^{\beta-2}m(D)$ $(i=1,2)$ with $m(\zeta)=m(|\zeta|)$ a non-decreasing function satisfying the conditions* (i)–(iii)*. Then we have that for every $x\neq 0\in\mathbb{R}^2$,*

$$K_{\beta,i}(x)=\frac{x_i}{|x|}H_\beta(x),\quad \text{with } \big|H_\beta(x)\big|\leqslant C|x|^{-1-\beta}m\big(|x|^{-1}\big), \tag{A.4}$$

and

$$\big|\nabla K_{\beta,i}(x)\big|\leqslant C|x|^{-2-\beta}m\big(|x|^{-1}\big). \tag{A.5}$$

Proof of Lemma A.2. The proof is similar to that of [11, Lemma 4.1] with suitable modifications. Given ψ a smooth radial function supported on $\{|x|\leqslant 1\}$ and satisfies $\psi\equiv 1$ on $\{|x|\leqslant 1/2\}$. Let $\psi_R(\cdot)\triangleq\psi(\frac{\cdot}{R})$ for $R>0$, and $L_\beta(x)$ be the radial kernel function of the operator $|D|^{\beta-2}m(D)$, then for some $R>0$ to be chosen later, we have that

$$\begin{aligned}
L_\beta(x)&=C\int\limits_{\mathbb{R}^2}e^{ix\cdot\zeta}|\zeta|^{\beta-2}m\big(|\zeta|\big)\,\mathrm{d}\zeta=C\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}|\zeta|^{\beta-2}m\big(|\zeta|\big)\,\mathrm{d}\zeta\\
&=C\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}\psi_R(\zeta)|\zeta|^{\beta-2}m(\zeta)\,\mathrm{d}\zeta+C\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}\big(1-\psi_R(\zeta)\big)|\zeta|^{\beta-2}m(\zeta)\,\mathrm{d}\zeta\\
&=C\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}\psi_R(\zeta)|\zeta|^{\beta-2}m(\zeta)\,\mathrm{d}\zeta+C|x|^{-4}\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}\,\partial^4_{\zeta_1}\big(\big(1-\psi_R(\zeta)\big)|\zeta|^{\beta-2}m(\zeta)\big)\,\mathrm{d}\zeta
\end{aligned}$$

where in the last line we have used the integration by parts. Thus we see that $K_{\beta,i}(x)=\partial_i L_\beta(x)=(x_i/|x|)H_\beta(x)$ with

$$\begin{aligned}
H_\beta(x)&=C\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}i\zeta_1\psi_R(\zeta)|\zeta|^{\beta-2}m(\zeta)\,\mathrm{d}\zeta+C|x|^{-4}\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}i\zeta_1\partial^4_{\zeta_1}\big(\big(1-\psi_R(\zeta)\big)|\zeta|^{\beta-2}m(\zeta)\big)\,\mathrm{d}\zeta\\
&\quad+C|x|^{-5}\int\limits_{\mathbb{R}^2}e^{i|x|\zeta_1}\partial^4_{\zeta_1}\big(\big(1-\psi_R(\zeta)\big)|\zeta|^{\beta-2}m(\zeta)\big)\,\mathrm{d}\zeta\\
&\triangleq \mathrm{I}_\beta(x)+\mathrm{II}_\beta(x)+\mathrm{III}_\beta(x).
\end{aligned}$$

For $\mathrm{I}_\beta(x)$, it is obvious to find that

$$\left|\mathrm{I}_\beta(x)\right| \leqslant C \int\limits_{B_R(0)} |\zeta|^{\beta-1} m(\zeta)\,\mathrm{d}\zeta \leqslant C R^{\beta+1} m(R).$$

For $\mathrm{II}_\beta(x)$, by (1.3) and Lemma A.1, we obtain

$$\begin{aligned}\left|\mathrm{II}_\beta(x)\right| &\leqslant C|x|^{-4} \int\limits_{|\zeta|\geqslant R/2} |\zeta|^{\beta-5} m\big(|\zeta|\big)\,\mathrm{d}\zeta \\ &\leqslant C|x|^{-4} R^{-\alpha} m(R) \int\limits_{|\zeta|\geqslant R/2} |\zeta|^{\beta+\alpha-5}\,\mathrm{d}\zeta \\ &\leqslant C|x|^{-4} R^{\beta-3} m(R).\end{aligned}$$

For the last term, similarly as above, we get

$$\left|\mathrm{III}_\beta(x)\right| \leqslant C|x|^{-5} R^{\beta-4} m(R).$$

Hence, choosing $R = |x|^{-1}$ leads to the desired estimate (A.4). The bound (A.5) can be obtained in the same fashion. $\square$

Now we sketch the proof of Proposition 3.1. Before that, we recall some notations from Littlewood–Paley theory. Choose two nonnegative radial functions $\chi, \varphi \in C^\infty(\mathbb{R}^2)$ (cf. [1]) which are supported respectively in the disk $\{\zeta \in \mathbb{R}^2\colon |\zeta| \leqslant \frac{4}{3}\}$ and the shell $\{\zeta \in \mathbb{R}^2\colon \frac{3}{4} \leqslant |\zeta| \leqslant \frac{8}{3}\}$ such that

$$\chi(\zeta) + \sum_{q\geqslant 0} \varphi\big(2^{-q}\zeta\big) = 1, \quad \forall \zeta \in \mathbb{R}^2.$$

Then for $f \in \mathcal{S}'(\mathbb{R}^2)$, we define the nonhomogeneous Littlewood–Paley operators

$$\Delta_{-1} f := \chi(D) f; \qquad \Delta_q f := \varphi\big(2^{-q} D\big) f, \quad \forall q \in \mathbb{N}.$$

We also have $H^s = B^s_{2,2}$ with

$$B^s_{2,2} \triangleq \left\{ f \in \mathcal{S}'\big(\mathbb{R}^2\big);\ \|f\|^2_{B^s_{2,2}} \triangleq \sum_{q\geqslant -1} 2^{2qs} \|\Delta_q f\|^2_{L^2} < \infty \right\}.$$

Proof of Proposition 3.1. We here focus on the *a priori* estimates, and we claim that for the smooth solution θ to the system (1.1),

$$\big\|\theta(t)\big\|^2_{H^s} + \|\theta\|^2_{L^2_t H^{s+\beta/2}} \leqslant C\|\theta_0\|^2_{H^s} + C \int_0^t \Big(\big\|\theta(\tau)\big\|^{2/(1-\alpha)}_{\dot{C}^{\alpha+(1-\alpha)\beta/2}} + 1\Big) \big\|\theta(\tau)\big\|^2_{H^s}\,\mathrm{d}\tau, \tag{A.6}$$

where $\dot{C}^r$ ($r \in]0,1[$) is the usual homogeneous Hölder space. Indeed, for $q \in \mathbb{N}$, applying Δ_q to Eq. (1.1) yields

$$\partial_t \Delta_q \theta + S_{q+1} u \cdot \nabla \Delta_q \theta + |D|^\beta \Delta_q \theta = F_q(u, \theta),$$

with $F_q(u,\theta) = S_{q+1} u \cdot \nabla \Delta_q \theta - \Delta_q(u \cdot \nabla\theta)$. By the energy method, we find

$$\left\| \Delta_q \theta(t) \right\|_{L^2}^2 + c 2^{q\beta} \| \Delta_q \theta \|_{L_t^2 L^2}^2 \leqslant \| \Delta_q \theta_0 \|_{L^2}^2 + C \int_0^t \left(2^{-q\beta/2} \left\| F_q(u,\theta)(\tau) \right\|_{L^2} \right)^2 \mathrm{d}\tau .$$

Similarly as obtaining (6.2) in [16], we infer that for all $\alpha \in]0,1[$,

$$\begin{aligned} &2^{-q\beta/2} \left\| F_q(u,\theta) \right\|_{L^2} \\ &\quad \leqslant C \left(\|\theta\|_{\dot{C}^{\alpha+(1-\alpha)\beta/2}}^2 + \|\theta\|_{L^2}^2 \right) \left(\sum_{k \geqslant q-4} 2^{(q-k)(1-\beta/2)} 2^{\beta\alpha k/2} \| \Delta_k \theta \|_{L^2} + \sum_{|k-q| \leqslant 4} 2^{\beta\alpha k/2} \| \Delta_k \theta \|_{L^2} \right), \end{aligned}$$

where we have used $\| \Delta_{-1} u \|_{L^\infty} \lesssim \| \Delta_{-1} m(D) \theta \|_{L^2} \lesssim m(1) \|\theta\|_{L^2}$, and

$$\left\| m(D) \Delta_k \theta \right\|_{L^\infty} \leqslant C m(1) 2^{k\alpha} \| \Delta_k \theta \|_{L^\infty}, \quad \forall k \in \mathbb{N},$$

which is from Lemma A.1 and the Bernstein-type inequality (cf. [1, Lemma 2.2]). Thus we get

$$\begin{aligned} &\sum_{q \in \mathbb{N}} 2^{2qs} \left\| \Delta_q \theta(t) \right\|_{L^2}^2 + c \sum_{q \in \mathbb{N}} 2^{2q(s+\beta/2)} \| \Delta_q \theta \|_{L_t^2 L^2}^2 \\ &\quad \leqslant \sum_{q \in \mathbb{N}} 2^{2qs} \| \Delta_q \theta_0 \|_{L^2}^2 + C \int_0^t \left\| \theta(\tau) \right\|_{\dot{C}^{\alpha+(1-\alpha)\beta/2} \cap L^2}^2 \left\| \theta(\tau) \right\|_{B_{2,2}^{s+\beta\alpha/2}}^2 \mathrm{d}\tau . \end{aligned}$$

For the low frequency part, it is not hard to see that

$$\left\| \Delta_{-1} \theta(t) \right\|_{L^2}^2 \leqslant \| \Delta_{-1} \theta_0 \|_{L^2}^2 + C \int_0^t \left\| \theta(\tau) \right\|_{H^\alpha}^2 \left\| \theta(\tau) \right\|_{L^2}^2 \mathrm{d}\tau .$$

Combining the above two estimates, and from $\| \cdot \|_{H^s} \approx \| \cdot \|_{B_{2,2}^s}$, we obtain

$$\left\| \theta(t) \right\|_{H^s}^2 + \|\theta\|_{L_t^2 H^{s+\beta/2}}^2 \leqslant C \|\theta_0\|_{H^s}^2 + C \int_0^t \left\| \theta(\tau) \right\|_{\dot{C}^{\alpha+(1-\alpha)\beta/2} \cap L^2}^2 \left\| \theta(\tau) \right\|_{H^{s+\beta\alpha/2}}^2 \mathrm{d}\tau .$$

The interpolation inequality and the Young inequality lead to

$$\begin{aligned}\big\|\theta(t)\big\|_{H^s}^2+\|\theta\|_{L_t^2H^{s+\beta/2}}^2&\leqslant C\|\theta_0\|_{H^s}^2+C\int_0^t\big\|\theta(\tau)\big\|_{\dot C^{\alpha+(1-\alpha)\beta/2}\cap L^2}^2\big\|\theta(\tau)\big\|_{H^s}^{2(1-\alpha)}\big\|\theta(\tau)\big\|_{H^{s+\beta/2}}^{2\alpha}\,\mathrm{d}\tau\\&\leqslant C\|\theta_0\|_{H^s}^2+C\int_0^t\big\|\theta(\tau)\big\|_{\dot C^{\alpha+(1-\alpha)\beta/2}\cap L^2}^{2/(1-\alpha)}\big\|\theta(\tau)\big\|_{H^s}^2\,\mathrm{d}\tau+\frac12\|\theta\|_{L_t^2H^{s+\beta/2}}^2.\end{aligned}$$

This implies (A.6). Furthermore, since $\|\theta(t)\|_{L^2\cap L^\infty}\leqslant\|\theta_0\|_{L^2\cap L^\infty}$ for all $t\in\,]0,T^*[$, and by interpolation, we also get

$$\big\|\theta(t)\big\|_{H^s}^2+\|\theta\|_{L_t^2H^{s+\beta/2}}^2\leqslant C\|\theta_0\|_{H^s}^2+C_1\int_0^t\big(\big\|\nabla\theta(\tau)\big\|_{L^\infty}^{(2\alpha+\beta-\alpha\beta)/(1-\alpha)}+1\big)\big\|\theta(\tau)\big\|_{H^s}^2\,\mathrm{d}\tau,$$

where C_1 depends on $\|\theta_0\|_{L^2\cap L^\infty}$. The remaining parts are essentially the same with those in [16,17], and we omit the details. At last, we remark that using the slightly different program in [12, Theorem 3.1], one can similarly prove Proposition 3.1. □

Finally we ketch the proof of Lemma 2.2.

Proof of Lemma 2.2. We only consider (2). Denoting by $\mathcal{P}_{h,n}^\beta(x)$ the n-dimensional kernel of the operator $e^{-h|D|^\beta}$, then from [13] we know that for every $\beta\in\,]0,2[$,

$$\mathcal{P}_{h,n}^\beta(x)>0,\qquad\int\mathcal{P}_{h,n}^\beta=1,\qquad\frac{c_\beta h}{(|x|^2+h^{2/\beta})^{(n+\beta)/2}}\leqslant\mathcal{P}_{h,n}^\beta(x)\leqslant\frac{C_\beta h}{(|x|^2+h^{2/\beta})^{(n+\beta)/2}}.\tag{A.7}$$

Thus we have

$$\begin{aligned}-|D|^\beta f(x)+|D|^\beta f(y)&=\lim_{h\to0}\frac1h\big(\mathcal{P}_{h,2}^\beta*f(x)-\mathcal{P}_{h,2}^\beta*f(y)-f(x)+f(y)\big)\\&=\lim_{h\to0}\frac1h\big(\mathcal{P}_{h,2}^\beta*f(x)-\mathcal{P}_{h,2}^\beta*f(y)-\omega(\xi)\big).\end{aligned}$$

We see that

$$\begin{aligned}&\mathcal{P}_{h,2}^\beta*f(x)-\mathcal{P}_{h,2}^\beta*f(y)\\&\quad=\int_{\mathbb{R}^2}\big(\mathcal{P}_{h,2}^\beta(\rho x_0-a-z)-\mathcal{P}_{h,2}^\beta(\rho y_0-a-z)\big)f(z)\,\mathrm{d}z\\&\quad=\iint_{\mathbb{R}^2}\big(\mathcal{P}_{h,2}^\beta(\xi/2-\eta,\mu)-\mathcal{P}_{h,2}^\beta(-\xi/2-\eta,\mu)\big)\tilde f(\eta,\mu)\,\mathrm{d}\eta\,\mathrm{d}\mu\\&\quad=\int_{\mathbb{R}}\mathrm{d}\mu\int_0^\infty\big(\mathcal{P}_{h,2}^\beta(\xi/2-\eta,\mu)-\mathcal{P}_{h,2}^\beta(\xi/2+\eta,\mu)\big)\big(\tilde f(\eta,\mu)-\tilde f(-\eta,\mu)\big)\,\mathrm{d}\eta\\&\quad=\int_{\mathbb{R}}\mathrm{d}\mu\int_0^\infty\big(\mathcal{P}_{h,2}^\beta(\xi/2-\eta,\mu)-\mathcal{P}_{h,2}^\beta(\xi/2+\eta,\mu)\big)\omega(2\eta)\,\mathrm{d}\eta\end{aligned}$$

$$+\int_{\mathbb{R}} \mathrm{d}\mu \int_0^\infty \left(\mathcal{P}_{h,2}^\beta\left(\frac{\xi}{2}-\eta,\mu\right)-\mathcal{P}_{h,2}^\beta\left(\frac{\xi}{2}+\eta,\mu\right)\right)\big(\tilde f(\eta,\mu)-\tilde f(-\eta,\mu)-\omega(2\eta)\big)\,\mathrm{d}\eta$$
$$\triangleq \Psi_{\beta,h}(\xi)+\Psi_{\beta,h}^\perp(\xi).$$

Similarly as that in [13], we get

$$\lim_{h\to 0}\frac{1}{h}\big(\Psi_{\beta,h}(\xi)-\omega(\xi)\big)=\Psi_\beta(\xi).$$

Now we estimate $\lim_{h\to 0}\frac{1}{h}\Psi_{\beta,h}^\perp(\xi)\triangleq\Psi_\beta^\perp(\xi)$. From (A.7) and the fact that h is arbitrarily small, there exists a small number $r_0\in\,]0,\frac{1}{4}[$ depending only on β such that for every $z\in B_{r_0\xi}(x_0)\subset\mathbb{R}^2$, we have

$$\mathcal{P}_{h,2}^\beta(x_0-z)-\mathcal{P}_{h,2}^\beta(y_0-z)\geqslant\frac{1}{2}\mathcal{P}_{h,2}^\beta(x_0-z).$$

Thus we deduce that

$$\begin{aligned}
\Psi_\beta^\perp(\xi)&=\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\mathrm{d}\mu\int_0^\infty\big(\mathcal{P}_{h,2}^\beta(\xi/2-\eta,\mu)-\mathcal{P}_{h,2}^\beta(\xi/2+\eta,\mu)\big)\big(\tilde f(\eta,\mu)-\tilde f(-\eta,\mu)-\omega(2\eta)\big)\,\mathrm{d}\eta\\
&\leqslant\lim_{h\to 0}\frac{1}{2h}\iint_{B_{r_0\xi}(x_0)}\mathcal{P}_{h,2}^\beta(\xi/2-\eta,\mu)\big(\tilde f(\eta,\mu)-\tilde f(-\eta,\mu)-\omega(2\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}\mu\\
&\leqslant\frac{c_\beta}{2}\iint_{B_{r_0\xi}(x_0)}\frac{\tilde f(\eta,\mu)-\tilde f(-\eta,\mu)-\omega(2\eta)}{|x_0-(\eta,\mu)|^{2+\beta}}\,\mathrm{d}\eta\,\mathrm{d}\mu\\
&=-\frac{c_\beta}{2}\iint_{B_{r_0\xi}^+(x_0)}\frac{2\omega(2\eta)-\tilde f(\eta,\mu)+\tilde f(-\eta,\mu)-\tilde f(\eta,-\mu)+\tilde f(-\eta,-\mu)}{|x_0-(\eta,\mu)|^{2+\beta}}\,\mathrm{d}\eta\,\mathrm{d}\mu.
\end{aligned}\tag{A.8}$$

Note that although the denominator of (A.8) contains the non-integrable singularity, the whole integral is still absolutely integrable due to the cancellation in the numerator (cf. [17, Lemma 5.5]). □

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