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ON THE DIFFERENTIABILITY ISSUE OF THE DRIFT-DIFFUSION EQUATION WITH NONLOCAL LÉVY-TYPE DIFFUSION

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We investigate the differentiability property of the drift-diffusion equation with nonlocal Lévy-type diffusion at either supercritical- or critical-type cases. Under the suitable conditions on the velocity field and the forcing term in terms of the spatial Hölder regularity, and for the initial data without regularity assumption, we show the a priori differentiability estimates for any positive time. If additionally the velocity field is divergence-free, we also prove that the vanishing viscosity weak solution is differentiable with some Hölder continuous derivatives for any positive time.

1. Introduction

We consider the following drift-diffusion equation with nonlocal diffusion:

$$\begin{cases} \partial_t\theta + (u\cdot\nabla)\theta + \mathcal{L}\theta = f & \text{in } \mathbb{R}^d\times\mathbb{R}^+, \\ \theta(x,0) = \theta_0(x) & \text{on } \mathbb{R}^d, \end{cases} \tag{1-1}$$

where θ is a scalar function, u is a velocity vector field of $\mathbb{R}^d$ and f is a scalar function as the forcing term. The nonlocal diffusion operator $\mathcal{L}$ is given by

$$\mathcal{L}g(x) = \text{p. v.}\int_{\mathbb{R}^d} (g(x) - g(x+y))K(y)\,\mathrm{d}y, \tag{1-2}$$

where the symmetric kernel function $K(y) = K(-y)$ defined on $\mathbb{R}^d\setminus\{0\}$ satisfies

$$\int_{\mathbb{R}^d} \min\{1, |y|^2\}|K(y)|\,\mathrm{d}y \le c_1, \tag{1-3}$$

and there exist two constants $\alpha\in(0,1]$ and $\sigma\in[0,\alpha)$ such that

$$\frac{c_2^{-1}}{|y|^{d+\alpha-\sigma}} \le K(y) \le \frac{c_2}{|y|^{d+\alpha}} \quad \text{for all } 0<|y|\le 1, \tag{1-4}$$

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with $c_1 > 0$ and $c_2 \geq 1$ two absolute constants. In the sequel we also consider the special case that K satisfies the nonnegative condition

$$K(y) \geq 0 \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}. \tag{1-5}$$

By taking the Fourier transform on $\mathcal{L}$, we get

$$\widehat{\mathcal{L}\theta}(\xi) = A(\xi)\hat{\theta}(\xi),$$

where the symbol $A(\xi)$ is given by the Lévy–Khinchin formula

$$A(\xi) = \text{p. v.}\int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi))K(x)\,\mathrm{d}x. \tag{1-6}$$

The nonlocal diffusion operator $\mathcal{L}$ defined by (1-2) with the symmetric kernel K satisfying (1-3)–(1-4) corresponds to the stable-type Lévy operator, which is the infinitesimal generator of the stable-type Lévy process (see [Chen et al. 2015; Sato 1999]). If $\sigma = 0$, the operator $\mathcal{L}$ is referred to as the stable-like Lévy operator, and in recent years many deep works have been devoted to studying various regularity problems concerning this diffusion operator (one can see [Komatsu 1995; Husseini and Kassmann 2007; Kassmann 2009; Caffarelli et al. 2011; Caffarelli and Silvestre 2011; Maekawa and Miura 2013; Dabkowski et al. 2014]). The typical example of the stable-like Lévy operator is the fractional Laplacian operator $|D|^\alpha := (-\Delta)^{\alpha/2}$ ($\alpha \in]0, 2[$), which has the following expression formula:

$$|D|^\alpha\theta(x) = c_{d,\alpha}\,\text{p. v.}\int_{\mathbb{R}^d} \frac{\theta(x) - \theta(x+y)}{|y|^{d+\alpha}}\,\mathrm{d}y, \tag{1-7}$$

with $c_{d,\alpha} > 0$ some absolute constant. The operator $\mathcal{L} = |D|^\alpha := (-\Delta)^{\alpha/2}$ ($\alpha \in (0, 2)$) is the infinitesimal generator of the symmetric stable Lévy process (see [Sato 1999]), and recently has been intensely considered in many theoretical problems. If $\sigma \neq 0$, the stable-type Lévy operator can contain more general diffusion operators. An important class is the following multiplier operators $\mathcal{L} = A(D) = A(|D|)$ defined by

$$\mathcal{L} = \frac{|D|^\alpha}{(\log(\lambda + |D|))^\mu}, \quad (\alpha \in (0, 1],\ \mu \geq 0,\ \lambda > 0), \tag{1-8}$$

and one can refer to [Dabkowski et al. 2014, Lemmas 5.1–5.2] for more details on the assumptions on $A(\xi)$ so that the kernel K satisfies (1-3)–(1-4) (the condition (1-5) can also be satisfied under some additional assumption on $A(\xi)$, see [Hmidi 2011; Miao and Xue 2015]). Recently, the logarithmic diffusion operator defined by (1-8) in many systems has attracted a lot of attention and has been variously studied (e.g., [Tao 2009; Hmidi 2011; Chae and Wu 2012; Dabkowski et al. 2014; Miao and Xue 2015]). One can also refer to [Chen et al. 2015, Example 4.2] for other important classes of stable-type Lévy operators.

Recalling that for the drift-diffusion equation (1-1) with $\mathcal{L} = |D|^\alpha$, we conventionally call the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ supercritical, critical and subcritical cases, respectively. Thus the operator $\mathcal{L}$ defined by (1-2) under the kernel conditions (1-3)–(1-4) can be viewed as the critical- and supercritical-type cases and they are the main concern in this paper.

For the drift-diffusion equation (1-1) with the fractional Laplacian operator $\mathcal{L} = |D|^\alpha$, if the velocity field is divergence-free, the $C^{1,\gamma}$-regularity improvement of weak solutions was obtained by Constantin and Wu [2008] by using the Bony's paradifferential calculus. Partially motivated by that work, without the divergence-free restriction on the velocity, Silvestre [2012b] considered the supercritical and critical cases ($\alpha \in (0, 1]$), and proved the interior $C^{1,\gamma}$-regularity of the solution provided that u and f belong to $L^\infty_t C^{1-\alpha+\gamma}_x$ ($\gamma \in (0, \alpha)$), more precisely, the author showed the following regularity estimate:

$$\|\theta\|_{L^\infty([-1/2,0];C^{1,\gamma}(B_{1/2}))} \leq C(\|u\|_{L^\infty([-1,0]\times\mathbb{R}^d)} + \|f\|_{L^\infty([-1,0];C^{1-\alpha+\gamma}(B_1))}), \tag{1-9}$$

where $C > 0$ depends only on d, α and $\|u\|_{L^\infty([-1,0];C^{1-\alpha+\gamma})}$. The proof is by a locally approximate procedure where an extension derived in [Caffarelli and Silvestre 2007] plays a key role. For the drift-diffusion equation (1-1) with more general diffusion operator, so far there are not many such differentiability results. We here only mention a related work [Chen et al. 2015], where the authors considered the backward drift-diffusion equation

$$\partial_t\theta + u \cdot \nabla\theta - (\mathcal{L} + \lambda)\theta = f, \quad \theta|_{t=1}(x) = 0, \quad \lambda \geq 0, \tag{1-10}$$

with $\mathcal{L}$ defined by (1-2)–(1-4) (in fact slightly more general Lévy operator $\mathcal{L}$ considered there), and by applying a purely probabilistic method, the authors proved the $C^{1,\gamma}$-regularity of a continuous solution $\theta : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ under the conditions that u and f are C^δ_x-Hölder continuous ($\delta \in (1-\alpha+\sigma, 1)$) for each time.

If we slightly lower the regularity index in the assumption of u and f, the solution of the equations (1-1)–(1-2) may in general not have such a differentiable regularity. For the drift-diffusion equation (1-1) with $\mathcal{L} = |D|^\alpha$, Silvestre [2012a] proved that if $u \in L^\infty_t \dot{C}^{1-\alpha}_x$ for $\alpha \in (0, 1)$ and $u \in L^\infty_{t,x}$ for $\alpha = 1$, and if $f \in L^\infty_{t,x}$, then the bounded solution becomes Hölder continuous for any positive time. For the drift-diffusion equation (1-1) with stable-like Lévy operator $\mathcal{L}$, and under the divergence-free condition of u, we refer to [Chamorro and Menozzi 2016] for a similar improvement to Hölder continuous solutions (see also [Maekawa and Miura 2013] for a related result). Note that the condition $u \in L^\infty_t \dot{C}^{1-\alpha}$ is invariant under the scaling transformation $u(x, t) \mapsto \lambda^{\alpha-1}u(\lambda^\alpha t, \lambda x)$ for all $\lambda > 0$. If we further weaken the regularity condition on u in the supercritical case, the solution of (1-1)–(1-2) may not even be continuous, indeed, as proved by Silvestre, Vicol and Zlatoš in [Silvestre et al. 2013], there is a divergence-free drift $u \in L^\infty_t C^\delta_x$ with

$\delta < 1-\alpha$ so that the solution of the equation (1-1) with $\mathcal{L} = |D|^\alpha$ and $f = 0$ forms a discontinuity starting from smooth initial data.

In this paper, we are concerned with the differentiability property of the system (1-1)–(1-2), and if the velocity field u is divergence free, we consider the differentiability of weak solutions, which is derived by passing to a limit of the approximate system, while if u is not divergence free, we only consider the a priori differentiability estimate. We impose no regularity assumption on the nonzero initial data, and we generalize the result of Silvestre [2012b] for more general stable-type Lévy operators.

Our first result states that if the velocity field is divergence-free, then the differentiability of the vanishing viscosity weak solution can be achieved for the equations (1-1)–(1-2) under conditions (1-3)–(1-4) and suitable assumptions.

Theorem 1.1. *Let the symmetric kernel $K(y) = K(-y)$ of the diffusion operator $\mathcal{L}$ satisfy* (1-3)–(1-4), *and the velocity field u be divergence-free. Assume that for any given $T > 0$, the drift u, the force f and the initial data θ_0 satisfy*

$$u \in L^\infty([0,T], C^\delta(\mathbb{R}^d)) \quad \textit{for some } \delta \in (1-\alpha+\sigma, 1), \tag{1-11}$$

and

$$f \in L^\infty([0,T]; B^\delta_{p,\infty} \cap B^\delta_{\infty,\infty}(\mathbb{R}^d)), \quad \theta_0 \in L^p(\mathbb{R}^d) \quad \textit{for some } p \in [2,\infty). \tag{1-12}$$

Then there exists a weak solution $\theta \in L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^p([0,T]; B^{\alpha-\sigma/p}_{p,p}(\mathbb{R}^d))$ which satisfies the drift-diffusion equation (1-1)–(1-2) *in the distributional sense* (*see* (3-52) *below*). *Moreover, $\theta \in L^\infty((0,T], C^{1,\gamma}(\mathbb{R}^d))$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$, which precisely satisfies that for every $t' \in (0,T)$,*

$$\|\theta\|_{L^\infty([t',T];C^{1,\gamma}(\mathbb{R}^d))} \le C t'^{-(\gamma+1+d/p)/(\alpha-\sigma)} (\|\theta_0\|_{L^p} + \|f\|_{L^\infty_T(B^\delta_{p,\infty}\cap B^\delta_{\infty,\infty})}), \tag{1-13}$$

with the constant C depending only on $T, \alpha, \sigma, d, \delta$ and $\|u\|_{L^\infty_T \dot{C}^\delta}$.

For our second result, we do not necessarily impose the divergence-free property of the velocity field, and we mainly focus on the a priori differentiability estimates of the drift-diffusion equation (1-1)–(1-2) under the conditions (1-3)–(1-5), or in other words, we concentrate on the uniform-in-ϵ differentiability estimates of the following ϵ-regularized drift-diffusion equation under (1-3)–(1-5)

$$\partial_t\theta + u_\epsilon \cdot \nabla\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_\epsilon, \quad \theta|_{t=0} = \theta_{0,\epsilon} = \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}(0)}), \tag{1-14}$$

where $u_\epsilon = \phi_\epsilon * u$, $f_\epsilon = \phi_\epsilon * f$, $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$ and ϕ is the standard mollifier. The result is as follows.

Theorem 1.2. *Let the kernel $K(y) = K(-y)$ of the diffusion operator $\mathcal{L}$ satisfy the conditions* (1-3)–(1-5). *Let $\theta_0 \in C_0(\mathbb{R}^d)$, with $C_0(\mathbb{R}^d)$ the space of continuous* *functions which decay to zero at infinity. Suppose that for any given $T > 0$, the drift u and the external force f satisfy*

$$u \in L^\infty([0,T]; C^\delta(\mathbb{R}^d)) \quad \text{and} \quad f \in L^\infty([0,T]; C^\delta \cap L^2(\mathbb{R}^d)), \tag{1-15}$$

for some $\delta \in (1-\alpha+\sigma, 1)$, then the solutions $\theta^{(\epsilon)}$ of the regularized drift-diffusion equation (1-14) *uniformly-in-ϵ belong to*

$$L^\infty([0,T]; C_0(\mathbb{R}^d)) \cap L^\infty((0,T], C^{1,\gamma}(\mathbb{R}^d)) \quad \text{for any } \gamma \in (0, \delta+\alpha-\sigma-1).$$

More precisely, for any $t' \in (0,T)$, we have

$$\|\theta^{(\epsilon)}\|_{L^\infty([t',T];C^{1,\gamma}(\mathbb{R}^d))} \le C t'^{-(\gamma+1)/(\alpha-\sigma)} (\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \tag{1-16}$$

where C is a positive constant depending only on α, σ, d, δ and $\|u\|_{L^\infty_T \dot{C}^\delta}$ and is independent of ϵ.

Theorems 1.1 and 1.2 (and Remark 1.3 below) can be applied to the regularity problem of the (weak) solution of various nonlinear drift-diffusion equations, and one can refer to the recent work [Miao and Xue 2015] for some direct applications.

The method in showing Theorems 1.1 and 1.2 is consistent with the method of paradifferential calculus used in [Constantin and Wu 2008], but is mostly in a different style; and by choosing some time function as a weight and developing the technique of weighted estimates (where Lemma 3.4 is of great use), we find that the process used here is efficient and is not sensitive to the divergence-free condition of u so that we can get rid of such an assumption in Theorem 1.2 (noticing that the method in [Constantin and Wu 2008] does not extend to the drift-diffusion equations (1-1)–(1-2) without the divergence-free property of u). We use the L^p ($p \in [2,\infty)$) framework in Theorem 1.1 and the L^∞ framework in proving Theorem 1.2, and the key diffusion effect of the Lévy-type diffusion operator (for high frequency part) is derived in Lemma 3.2 and Lemma 4.2 respectively. The iterative argument also plays an important role in the proof of both theorems, and we can see clearly how the regularity index of the solution improves step by step.

We want to point out that our approach in this paper is purely analytic, and does not use the probabilistic representations of solutions. Note also that the approach of [Silvestre 2012b] is not adopted here, and it seems rather hard (if not impossible) to extend the method of that work for the drift-diffusion equation with more general diffusion operators.

Remark 1.3 (On higher regularity). If the assumptions (1-11)–(1-12) and (1-15) hold for any $\delta > 1-\alpha+\sigma$ by removing the restriction $\delta < 1$, then by following the deduction in Subsections 3B and 4B, we infer that for the cases studied in Theorems 1.1 and 1.2, we a priori have

$$\theta \in L^\infty((0,T]; C^{[\delta+\alpha-\sigma]-1,\gamma}) \quad \text{for all } \gamma \in (0,1)$$

if $\delta+\alpha-\sigma \in \mathbb{N}^+$, and

$$\theta \in L^\infty((0,T];C^{[\delta+\alpha-\sigma],\gamma}) \quad \text{for all } \gamma \in (0,\delta+\alpha-\sigma-[\delta+\alpha-\sigma])$$

if $\delta+\alpha-\sigma \notin \mathbb{N}^+$.

As a consequence of the above result, and if $f=0$ and $u=\mathcal{P}\theta$ in the equation (1-1) with $\mathcal{P}$ composed of zero-order pseudodifferential operators, e.g., the dissipative SQG equation which recently has been intensely considered (see [Caffarelli and Vasseur 2010; Chen et al. 2007; Constantin and Vicol 2012; Córdoba and Córdoba 2004; Dabkowski et al. 2014; Kiselev and Nazarov 2009; Kiselev et al. 2007; Wang and Zhang 2011]):

$$\text{(1-17)} \quad \partial_t\theta+u\cdot\nabla\theta+\mathcal{L}\theta=0, \quad u=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \quad \theta(x,0)=\theta_0(x), \quad x\in\mathbb{R}^2,$$

with $\mathcal{R}_i$ $(i=1,2)$ the usual Riesz transform, we can deduce that under the assumptions of Theorems 1.1 and 1.2 with the condition on u replaced by

$$\theta \in L^\infty([0,T],C^\delta(\mathbb{R}^d)) \quad \text{for some } \delta \in (1-\alpha+\sigma,1),$$

then the corresponding weak solution θ further belongs to $C^\infty((0,T]\times\mathbb{R}^d)$. Indeed, after obtaining the bound of $\|\theta\|_{L^\infty C^{1,\gamma}}$ (and $\|\theta\|_{L^\infty B^{1+\gamma+d/\tilde{p}}_{\tilde{p},\infty}}$ with some $\tilde{p}<\infty$ in Theorem 1.1) for any $\gamma\in(0,\delta+\alpha-\sigma-1)$, from the Calderón–Zygmund theorem, we get $\nabla u \in L^\infty\dot{C}^\gamma$, which further leads to

$$\theta \in L^\infty C^{[1+\gamma+\alpha-\sigma]-1,\gamma'} \quad \text{for all } \gamma' \in (0,1)$$

if $1+\gamma+\alpha-\sigma\in\mathbb{N}$, and

$$\theta \in L^\infty C^{[1+\gamma+\alpha-\sigma],\gamma'} \quad \text{for all } \gamma' \in (0,\gamma+\alpha-\sigma-[\gamma+\alpha-\sigma])$$

if $1+\gamma+\alpha-\sigma\notin\mathbb{N}^+$, (in Theorem 1.1 we in fact obtain a stronger estimate on θ in terms of L^p-based Besov spaces); noting that the regularity index can be arbitrarily close to $\delta+2(\alpha-\sigma)$ by suitably choosing γ and γ', thus by the bootstrapping method, we can iteratively improve the regularity index to any large number and finally conclude the C^∞_x-smoothness of the solution. The C^∞-smoothness in $t\in(0,T]$ can be derived from equation (1-1) and Lemma 2.2.

Remark 1.4. In Theorem 1.2, if the velocity field u is divergence-free, and $\theta_0 \in L^2\cap L^\infty(\mathbb{R}^d)$, and (1-15) is also assumed, then there exists a weak solution θ to the drift-diffusion equation (1-1)–(1-2) which satisfies (1-16) with θ in place of $\theta^{(\epsilon)}$. But if the velocity field is not divergence-free, and under the assumptions of Theorem 1.2, it is not so clear for the authors to pass to the limit $\epsilon\to 0$ in equation (1-14) to obtain the weak solution of the drift-diffusion equations (1-1)–(1-2). Despite that, we believe that the uniform-in-ϵ differentiability estimate (1-16) is meaningful and may have its various applications.

Remark 1.5. By examining the proof of both theorems, the upper bound in (1-4) does not play an essential role in the proof of (1-13) and (1-16), which indeed can be relaxed to larger numbers. But we here include the upper bound in (1-4) is to restrict ourselves to the critical and supercritical type cases.

Remark 1.6. In Theorem 1.2, the condition on f in (1-15) can be replaced by $f \in L^\infty([0, T]; C_0^\delta(\mathbb{R}^d))$ with $C_0^\delta(\mathbb{R}^d)$ the closure of Schwartz class under the norm of Hölder space $C^\delta(\mathbb{R}^d)$, and the same uniform estimate (1-16) holds true for a suitable approximate system of the equations (1-1)–(1-2).

The outline of the paper is as follows. In Section 2, we present some preliminary knowledge on Bony's paradifferential calculus and the Besov spaces, and give a useful lemma on the stable-type Lévy operator $\mathcal{L}$. Section 3 is dedicated to the proof of Theorem 1.1, and we first show several useful auxiliary lemmas, then we prove the key a priori estimate (1-13) in the Section 3B, and then we sketch the proof of the existence part and conclude the theorem. We show Theorem 1.2 in Section 4, and the proof is also divided into three parts: the auxiliary lemmas, the a priori estimates and the uniform-in-ϵ differentiability estimates for the regularized system (1-14), which are treated in the subsections 4A – 4C respectively.

2. Preliminaries

In this preliminary section, we shall gather some notations used in this paper, collect some basic facts on Bony's paradifferential calculus and Besov spaces, and show a useful lemma on the considered Lévy operator $\mathcal{L}$.

Throughout this paper, C stands for a constant which may be different from line to line. The notation $X \lesssim Y$ means that $X \leq CY$, and $X \approx Y$ implies that $X \lesssim Y$ and $Y \lesssim X$ simultaneously. Denote $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class of rapidly decreasing smooth functions, $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ the quotient space of tempered distributions modulo polynomials. We use $\hat{g}$ of $\mathcal{F}(g)$ to denote the Fourier transform of a tempered distribution, that is, $\hat{g}(\xi) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(x)\,\mathrm{d}x$. For a number $a \in \mathbb{R}$, denote by $[a]$ the integer part of a.

Now we recall the so-called Littlewood–Paley operators and their elementary properties. Let (χ, φ) be a couple of smooth functions taking values on $[0, 1]$ such that $\chi \in C_c^\infty(\mathbb{R}^d)$ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$, $\varphi \in C_0^\infty(\mathbb{R}^d)$ is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and satisfies that (see [Bahouri et al. 2011])

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1, \text{for all } \xi \in \mathbb{R}^d, \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \text{for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

For every $u \in S'(\mathbb{R}^d)$, we define the nonhomogeneous Littlewood–Paley operators as follows:

$$\Delta_{-1} f = \chi(D)u; \quad \Delta_j f = \varphi(2^{-j} D) f, \quad S_j f = \sum_{-1 \le k \le j-1} \Delta_k u \quad \text{for all } j \in \mathbb{N}.$$

And the homogeneous Littlewood–Paley operators can be defined as follows:

$$\dot{\Delta}_j f := \varphi(2^{-j} D) f; \quad \dot{S}_j f := \sum_{k \in \mathbb{Z}, k \le j-1} \dot{\Delta}_k f \quad \text{for all } j \in \mathbb{Z}.$$

Also, we denote

$$\widetilde{\Delta}_j f := \Delta_{j-1} f + \Delta_j f + \Delta_{j+1} f.$$

It is clear to see that, for any f and g belonging to $S'(\mathbb{R}^d)$, from the property of the frequency supports, we have

$$\Delta_j \Delta_l f \equiv 0, \quad |j-l| \ge 2 \quad \text{and} \quad \Delta_k (S_{l-1} g \Delta_l g) \equiv 0 \quad |k-l| \ge 5.$$

Now we introduce the definition of Besov spaces. Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$. Then the inhomogeneous Besov space $B^s_{p,r}$ is defined as

$$B^s_{p,r} := \{ f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{B^s_{p,r}} := \|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j \ge -1}\|_{\ell^r} < \infty \},$$

and the homogeneous space $\dot{B}^s_{p,r}$ is given by

$$\dot{B}^s_{p,r} := \{ f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d); \|f\|_{\dot{B}^s_{p,r}} := \|\{2^{js} \|\dot{\Delta}_j f\|_{L^p}\}_{j \in \mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty \}.$$

For any noninteger $s > 0$, the Hölder space $C^s = C^{[s], s-[s]}$ is equivalent to $B^s_{\infty,\infty}$ with $\|f\|_{C^s} \approx \|f\|_{B^s_{\infty,\infty}}$.

Bernstein's inequality plays an important role in the analysis involving Besov spaces.

Lemma 2.1 (see [Bahouri et al. 2011]). *Let $f \in L^a$, $1 \le a \le b \le \infty$. Then for every $(k, j) \in \mathbb{N}^2$, there exists a constant $C > 0$ independent of j such that*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_j f\|_{L^b} \le C 2^{j(k+d/a-d/b)} \|S_j f\|_{L^a},$$

and

$$C^{-1} 2^{jk} \|\Delta_j f\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^a} \le C 2^{jk} \|\Delta_j f\|_{L^a}.$$

The following lemma concerning the Lévy operator $\mathcal{L}$ is useful in the proof of the existence parts.

Lemma 2.2. *Let the operator $\mathcal{L}$ be defined by* (1-2) *with the symmetric kernel $K(y) = K(-y)$ under the conditions* (1-3)–(1-4).

(1) *Assume that* $g \in C^{1,\gamma}(\mathbb{R}^d)$, $\gamma > 0$. *Then we have* $\mathcal{L}g \in L^\infty(\mathbb{R}^d)$ *with* $\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}$.

(2) *Assume that* $h \in \mathcal{S}(\mathbb{R}^d)$ *and* $h_j(x) = 2^{jd}h(2^j x)$, $j \in \mathbb{N}$. *Then we have*

$$\|\mathcal{L}h_j\|_{L^1(\mathbb{R}^d)} \le C2^{j\alpha}. \tag{2-1}$$

(3) *Assume that* $g \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$. *Then* $\|\mathcal{L}\Delta_j g\|_{L^p} \le C2^{j\alpha}\|\widetilde{\Delta}_j g\|_{L^p}$ *for every* $j \in \mathbb{N}$ *and* $\|\mathcal{L}\Delta_{-1}g\|_{L^p} \le C\|g\|_{L^p}$.

Proof of Lemma 2.2. (1) If $\alpha \in (0,1)$, it follows from equation (1-2) that

$$\begin{aligned}\mathcal{L}g(x) &= \mathrm{p.\,v.}\int_{\mathbb{R}^d}(g(x)-g(x+y))K(y)\,\mathrm{d}y \\ &= \mathrm{p.\,v.}\int_{|y|\le 1}(g(x)-g(x+y))K(y)\,\mathrm{d}y \\ &\qquad\qquad + \int_{|y|\ge 1}(g(x)-g(x+y))K(y)\,\mathrm{d}y.\end{aligned}$$

By virtue of inequality (1-3), one has

$$\left|\mathrm{p.v.}\int_{|y|\ge 1}(g(x)-g(x+y))K(y)\mathrm{d}y\right| \le C\|g\|_{L^\infty}\int_{|y|\ge 1}|K(y)|\mathrm{d}y \le C\|g\|_{L^\infty}. \tag{2-2}$$

Thanks to the upper bound of (1-4), we have

$$\begin{aligned}\left|\mathrm{p.v.}\int_{|y|\le 1}(g(x)-g(x+y))K(y)\mathrm{d}y\right| &= \left|\int_{|y|\le 1}\int_0^1 y\cdot(\nabla g)(x+sy)K(y)\mathrm{d}s\,\mathrm{d}y\right| \\ &\le C\|\nabla g\|_{L^\infty}\int_{|y|\le 1}|y||K(y)|\mathrm{d}y \\ &\le C\|\nabla g\|_{L^\infty}\int_{|y|\le 1}|y|\frac{c_2}{|y|^{d+\alpha}}\,\mathrm{d}y \\ &\le C\|\nabla g\|_{L^\infty}.\end{aligned}$$

Hence for the case $\alpha \in (0,1)$, we get

$$\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C(\|g\|_{L^\infty(\mathbb{R}^d)} + \|\nabla g\|_{L^\infty(\mathbb{R}^d)}).$$

If $\alpha = 1$, similarly as above, and by adopting the following equivalent formula of $\mathcal{L}g$ (from the symmetric condition $K(y) = K(-y)$)

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d}(g(x) + y\cdot\nabla g(x)1_{\{|y|\le 1\}} - g(x+y))K(y)\,\mathrm{d}y, \tag{2-3}$$

we can prove that

$$\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}.$$

Both in the cases $\alpha \in (0,1)$ and $\alpha = 1$, we conclude $\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}$.

(2) If $\alpha \in (0, 1)$, from (1-2), (1-4) and the Fubini theorem, we see that

$$\begin{aligned}\|\mathcal{L}h_j\|_{L^1(\mathbb{R}^d)} &\le c_2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|h_j(x)-h_j(x+y)|}{|y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d}\int_{|y|\ge 1} |h_j(x)-h_j(x+y)||K(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\le C\int_{\mathbb{R}^d} \frac{\|h_j(x)-h_j(x+y)\|_{L^1_x}}{|y|^{d+\alpha}}\,\mathrm{d}y + C\|h_j\|_{L^1_x}\int_{|y|\ge 1}|K(y)|\,\mathrm{d}y \\ &\le C\|h_j\|_{\dot B^{\alpha}_{1,1}} + C\|h_j\|_{L^1} \le C2^{j\alpha},\end{aligned}$$

where in the last line we used the characterization of homogeneous Besov spaces (see [Bahouri et al. 2011, Theorem 2.36])

$$\|g\|_{\dot B^s_{p,r}} \approx \left\| \frac{\|g(x)-g(x+y)\|_{L^p}}{|y|^{\alpha}} \right\|_{L^r(\mathbb{R}^d, dy/|y|^d)} \qquad \text{for all } s\in(0,1),\ (p,r)\in[1,\infty]^2.$$

If $\alpha = 1$, we use the following equivalent formula for $\mathcal{L}g$:

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d} (g(x) + y\cdot\nabla g(x)1_{\{|y|\le\epsilon\}} - g(x+y))K(y)\,\mathrm{d}y, \tag{2-4}$$

with $\epsilon > 0$. Thus by choosing $\epsilon = 2^{-j}$, we get

$$\begin{aligned}\|\mathcal{L}h_j\|_{L^1} &\le c_2\int_{\mathbb{R}^d}\int_{|y|\le 2^{-j}} \frac{|h_j(x)+y\cdot\nabla h_j(x)-h_j(x+y)|}{|y|^{d+1}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + c_2\int_{\mathbb{R}^d}\int_{2^{-j}\le|y|\le 1}\frac{|h_j(x)-h_j(x+y)|}{|y|^{d+1}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d}\int_{|y|\ge 1}|h_j(x)-h_j(x+y)||K(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\le C\|\nabla^2 h_j\|_{L^1}\int_{|y|\le 2^{-j}}\frac{1}{|y|^{d-1}}\,\mathrm{d}y \\ &\qquad + C\|h_j\|_{L^1}\left(\int_{|y|\ge 2^{-j}}\frac{1}{|y|^{d+1}}\,\mathrm{d}y + \int_{|y|\ge 1}|K(y)|\,\mathrm{d}y\right) \\ &\le C2^j.\end{aligned}$$

Hence (2-1) follows for every $\alpha \in (0, 1]$.

(3) Denoting $h := \mathcal{F}^{-1}(\varphi)$, $\tilde h := \mathcal{F}^{-1}(\chi)$, we have $\Delta_j g = h_j * g = (2^{jd}h(2^j\cdot)) * g$ ($j\in\mathbb{N}$) and $\Delta_{-1}g = \tilde h * g$. By virtue of the facts that $\Delta_j\widetilde{\Delta}_j = \Delta_j$ ($j\in\mathbb{N}$) and $\mathcal{L}(f*g) = (\mathcal{L}f)*g$, and thanks to the statement (2), we infer that

$$\|\mathcal{L}\Delta_j g\|_{L^p} = \|\mathcal{L}\Delta_j\widetilde{\Delta}_j g\|_{L^p} = \|(\mathcal{L}h_j)*(\widetilde{\Delta}_j g)\|_{L^p} \le C2^{j\alpha}\|\widetilde{\Delta}_j g\|_{L^p}$$

for all $j\in\mathbb{N}$, and $\|\mathcal{L}\Delta_{-1}g\|_{L^p} = \|(\mathcal{L}\tilde h)*g\|_{L^p} \le C\|g\|_{L^p}$. ☐

3. Proof of Theorem 1.1

3A. ***Auxiliary lemmas.*** In this section we introduce some useful auxiliary lemmas. The first lemma is about the pointwise lower bound estimate of the Fourier symbol of the operator $\mathcal{L}$.

Lemma 3.1. *Let the operator $\mathcal{L}$ be defined by* (1-2) *with the kernel $K(y) = K(-y)$ satisfying* (1-3)–(1-4)*. Then the associated symbol $A(\xi)$ given by* (1-6) *satisfies*

$$A(\xi) \geq C^{-1}|\xi|^{\alpha-\sigma} - C, \tag{3-1}$$

where $\alpha \in]0, 1]$, $\sigma \in [0, \alpha[$ and $C = C(d, \alpha, \sigma)$ is a positive constant.

Proof of Lemma 3.1. Recalling that one has (see Equation (3.219) of [Jacob 2005])

$$|\xi|^\alpha = c_{d,\alpha}\, \mathrm{p.\,v.}\!\int_{\mathbb{R}^d} (1 - \cos(y \cdot \xi)) \frac{1}{|y|^{d+\alpha}}\, \mathrm{d}y \quad \text{for all } \alpha \in]0, 2[, \tag{3-2}$$

and by virtue of (1-3)–(1-4), we get

$$\begin{aligned} A(\xi) &= \mathrm{p.\,v.}\!\int_{\mathbb{R}^d} (1 - \cos(y \cdot \xi)) K(y)\, \mathrm{d}y \\ &\geq c_2^{-1} \int_{0<|y|\leq 1} (1 - \cos(y \cdot \xi)) \frac{1}{|y|^{d+\alpha-\sigma}}\, \mathrm{d}y - \int_{|y|\geq 1} |K(y)|\, \mathrm{d}y \\ &\geq c_2^{-1}\biggl(c_{d,\alpha}^{-1}|\xi|^{\alpha-\sigma} - \int_{|y|\geq 1} \frac{1}{|y|^{d+\alpha-\sigma}}\, \mathrm{d}y \biggr) - c_1 \\ &\geq c_2^{-1} c_{d,\alpha}^{-1} |\xi|^{\alpha-\sigma} - C_{d,\alpha,\sigma} - c_1, \end{aligned}$$

which corresponds to (3-1). □

Next we derive the following lower bound estimates of some quantities involving the Lévy operator $\mathcal{L}$ given by (1-2).

Lemma 3.2. *Let $p \geq 2$ and the kernel function $K(y) = K(-y)$ satisfy the conditions* (1-3)–(1-4)*, then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} |\theta(x)|^{p-2}\theta(x)\mathcal{L}\theta(x)\mathrm{d}x \geq C \int_{\mathbb{R}^d} (|D|^{\frac{\alpha-\sigma}{2}} |\theta(x)|^{\frac{p}{2}})^2 \mathrm{d}x - \widetilde{C} \int_{\mathbb{R}^d} |\theta(x)|^p\, \mathrm{d}x, \tag{3-3}$$

and for every $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} \mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\, \mathrm{d}x \geq c 2^{j(\alpha-\sigma)} \|\Delta_j\theta\|_{L^p}^p - \widetilde{C} \|\Delta_j\theta\|_{L^p}^p, \tag{3-4}$$

where the constants $c, C > 0$, $\widetilde{C} \geq 0$ depend only on the coefficients p, α, σ, d.

Proof of Lemma 3.2. First we claim that the following estimate holds true:

$$\begin{aligned}|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x)\\ \geq 2/p(\mathcal{L}|\theta|^{p/2})(x) - 2\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y.\end{aligned}\tag{3-5}$$

Indeed, according to (1-2) and the following estimate deduced from Young's inequality

$$|\theta(x)|^{p/2-2}\theta(x)\theta(y) \leq |\theta(x)|^{p/2-1}|\theta(y)| \leq \frac{p-2}{p}|\theta(x)|^{p/2} + \frac{2}{p}|\theta(y)|^{p/2},\tag{3-6}$$

we have

$$\begin{aligned}&|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x)\\ &= \mathrm{p.\,v.}\int_{\mathbb{R}^d}(|\theta(x)|^{p/2} - |\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y\\ &= \mathrm{p.\,v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2} - |\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y\\ &\quad + \int_{|x-y|\geq 1}(|\theta(x)|^{p/2} - |\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y\\ &\geq \mathrm{p.\,v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2} - |\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y\\ &\quad - \frac{2p-2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y.\end{aligned}\tag{3-7}$$

Due to the positivity property of $K(y)$ on $0<|y|\leq 1$ and the inequality (3-6) again, we see that

$$\begin{aligned}&\mathrm{p.v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2} - |\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\mathrm{d}y\\ &\geq \mathrm{p.v.}\int_{|x-y|\leq 1}\Big(|\theta(x)|^{p/2} - \Big(\frac{p-2}{p}|\theta(x)|^{p/2} + 2/p|\theta(y)|^{p/2}\Big)\Big)K(x-y)\mathrm{d}y\\ &= \frac{2}{p}\mathrm{p.v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2} - |\theta(y)|^{p/2})K(x-y)\mathrm{d}y\\ &= \frac{2}{p}(\mathcal{L}|\theta|^{p/2})(x) - \frac{2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2} - |\theta(y)|^{p/2})K(x-y)\mathrm{d}y\\ &\geq \frac{2}{p}(\mathcal{L}|\theta|^{p/2})(x) - \frac{2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2} + |\theta(y)|^{p/2})|K(x-y)|\mathrm{d}y.\end{aligned}\tag{3-8}$$

Gathering the above estimates leads to (3-5).

As a consequence of (3-5), we get

$$
\begin{aligned}
\text{(3-9)}\quad & \int_{\mathbb{R}^d} |\theta(x)|^{p-2}\theta(x)\mathcal{L}\theta(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} |\theta(x)|^{p/2}|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x)\,\mathrm{d}x \\
&\geq \frac{2}{p}\int_{\mathbb{R}^d} |\theta(x)|^{p/2}(\mathcal{L}|\theta|^{p/2})(x)\,\mathrm{d}x \\
&\quad - 2\int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&:= N_1 + N_2.
\end{aligned}
$$

In view of the Plancherel theorem and the estimate (3-1) concerning the symbol of $\mathcal{L}$, this leads to

$$
\begin{aligned}
N_1 &= \frac{2}{p}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)A(\xi)\overline{\widehat{|\theta|^{p/2}}(\xi)}\,\mathrm{d}\xi \\
&\geq \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} |\xi|^{\alpha-\sigma}\widehat{|\theta|^{p/2}}(\xi)\overline{\widehat{|\theta|^{p/2}}(\xi)}\,\mathrm{d}\xi - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)\overline{\widehat{|\theta|^{p/2}}(\xi)}\,\mathrm{d}\xi \\
&= \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} (|\xi|^{(\alpha-\sigma)/2}\widehat{|\theta|^{p/2}}(\xi))^2\,\mathrm{d}\xi - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)\overline{\widehat{|\theta|^{p/2}}(\xi)}\,\mathrm{d}\xi \\
&= \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} (|D|^{(\alpha-\sigma)/2}|\theta(x)|^{p/2})^2\,\mathrm{d}x - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} |\theta(x)|^p\,\mathrm{d}x.
\end{aligned}
$$

The Young inequality and the condition (1-3) ensure that

$$
\begin{aligned}
-\frac{N_2}{2} &\leq \int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1}|\theta(x)|^{p/2}|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1}|\theta(y)|^{p/2}|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} |\theta(x)|^{p}\int_{|x-y|\geq 1}|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad + \|\theta\|_{L^p}^{p/2}\left\|\int_{\mathbb{R}^d} |\theta(y)|^{p/2}|K(x-y)|1_{\{|x-y|\geq 1\}}\,\mathrm{d}y\right\|_{L^2_x} \\
&\leq \|\theta\|_{L^p}^{p}\int_{|x|\geq 1}|K(x)|\,\mathrm{d}x + \|\theta\|_{L^p}^{p/2}\||\theta(x)|^{p/2}\|_{L^2_x}\int_{|x|\geq 1}|K(x)|\,\mathrm{d}x \\
&\leq 2c_1\|\theta\|_{L^p}^{p}.
\end{aligned}
$$

Inserting the estimates of N_1 and N_2 into (3-9) yields the desired estimate (3-3). Recalling the following inequality (see [Chen et al. 2007, Proposition 3.1]),

$$
\||D|^{\beta}(|\Delta_j\theta|^{p/2})\|_{L^2}^2 \geq \tilde{c}2^{j\beta}\|\Delta_j\theta\|_{L^p}^{p} \quad \text{for every } \beta\in(0,2],\ p\in[2,\infty),\ j\in\mathbb{N},
$$

with a constant $\tilde{c}>0$ independent of j, then the estimate (3-4) follows by combining the above lower bound estimate with (3-3). We thus conclude Lemma 3.2. □

Now we can show the key a priori L^p-estimate of the drift-diffusion equations (1-1)–(1-2).

Lemma 3.3. *Let u be a smooth **divergence-free** vector field and f be a smooth forcing term. Assume that θ is a smooth solution for the drift-diffusion equations* (1-1)–(1-2) *under the assumptions* (1-3)–(1-4) *with $\theta_0 \in L^p(\mathbb{R}^d)$. Then for any $T > 0$,*

$$\max_{0\le t\le T} \|\theta(t)\|_{L^p}^p + \int_0^T \|\theta(\tau)\|^p_{\dot B^{(\alpha-\sigma)/p}_{p,p}}\, d\tau \le e^{C'T}\left(\|\theta_0\|^p_{L^p} + \int_0^T \|f(t)\|^p_{L^p}\,\mathrm{d}t\right), \tag{3-10}$$

with $C' \ge 0$ depending only on p, α, σ, d.

Proof of Lemma 3.3. Multiplying both sides of (1-1) by $|\theta|^{p-2}\theta(x)$ and integrating over the spatial variable, we use the divergence-free condition of u and Hölder's inequality to get

$$\frac{1}{p}\frac{d}{dt}\|\theta\|^p_{L^p} + \int_{\mathbb{R}^d} \mathcal{L}\theta(x)(|\theta|^{p-2}\theta)(x)\,\mathrm{d}x \le \|f\|_{L^p}\|\theta\|^{p-1}_{L^p}.$$

Thanks to (3-3), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}\theta(|\theta|^{p-2}\theta)\,\mathrm{d}x &\ge C\int_{\mathbb{R}^d} (|D|^{(\alpha-\sigma)/2}|\theta(x)|^{p/2})^2\,\mathrm{d}x - \widetilde{C}\int_{\mathbb{R}^d}|\theta(x)|^p\,\mathrm{d}x \\ &\ge C\|\theta\|^p_{\dot B^{(\alpha-\sigma)/p}_{p,p}} - \widetilde{C}\|\theta\|^p_{L^p},\end{aligned}$$

where in the last line we have used the following inequality (see [Chamorro and Lemarié-Rieusset 2012, Theorem 2] or [Chamorro and Menozzi 2016, Theorem 5])

$$\int_{\mathbb{R}^d} (|D|^\gamma|\theta(x)|^{p/2})^2\,\mathrm{d}x \ge c\|\theta\|^p_{\dot B^{\gamma/p}_{p,p}} \quad \text{for all } \gamma \in (0,1).$$

We thus obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|^p_{L^p} + C\|\theta\|^p_{\dot B^{(\alpha-\sigma)/p}_{p,p}} - \widetilde{C}\|\theta\|^p_{L^p} \le \|f\|_{L^p}\|\theta\|^{p-1}_{L^p},$$

which directly implies

$$\frac{d}{dt}\|\theta(t)\|^p_{L^p} + \|\theta\|^p_{\dot B^{(\alpha-\sigma)/p}_{p,p}} \le C\|\theta(t)\|^p_{L^p} + C\|f(t)\|^p_{L^p}.$$

Grönwall's inequality guarantees the desired inequality (3-10). □

The final lemma is concerned with a (time function) weighted estimate, which plays a key role in proving our main results.

Lemma 3.4. *Let $\lambda > 0$ and $0 < \mu < 1$. Then for any $t > 0$, there exists a constant C_μ depending only on μ such that*

$$\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\,\mathrm{d}\tau \le C_\mu \lambda^{-1} t^{-\mu}. \tag{3-11}$$

In particular, for any $t > t_0 \geq 0$, we have

$$\int_{t_0}^{t} e^{-(t-\tau)2^{(\alpha-\sigma)j}} (\tau - t_0)^{-\mu}\, \mathrm{d}\tau = \int_0^{t-t_0} e^{-(t-t_0-\tau)2^{(\alpha-\sigma)j}} \tau^{-\mu}\, \mathrm{d}\tau \leq C_\mu 2^{-(\alpha-\sigma)j}(t-t_0)^{-\mu}. \tag{3-12}$$

Proof of Lemma 3.4. First, by changing of the variable $(t-\tau)\lambda = s$, one deduces

$$\begin{aligned}\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\mathrm{d}\tau &= \lambda^{-1}\int_0^{t\lambda} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s \\ &= \lambda^{-1}\bigg(\int_0^{t\lambda/2} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s + \int_{t\lambda/2}^{t\lambda} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s\bigg) \\ &:= \lambda^{-1}(B_1+B_2).\end{aligned}$$

For the first term B_1, noting that $t - s/\lambda \geq \frac{1}{2}t$ for all $0 \leq s \leq t\lambda/2$, we directly get

$$B_1 \leq 2^\mu t^{-\mu}\int_0^{t\lambda/2} e^{-s}\,\mathrm{d}s \leq 2^\mu t^{-\mu}\int_0^\infty e^{-s}\,\mathrm{d}s \leq 2^\mu t^{-\mu}.$$

For the second term B_2, by changing of the variable $t - s/\lambda = s'$ and using the fact $t\lambda e^{-t\lambda/2} \leq C_0$, we deduce that

$$\begin{aligned}B_2 &\leq e^{-t\lambda/2}\int_{t\lambda/2}^{t\lambda}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s \\ &= t^{1-\mu}\lambda e^{-t\lambda/2}\int_0^{1/2}(s')^{-\mu}\,\mathrm{d}s' = \frac{2^{\mu-1}}{1-\mu}t^{-\mu}(t\lambda e^{-t\lambda/2}) \leq \frac{C_0 2^{\mu-1}}{1-\mu}t^{-\mu}.\end{aligned}$$

Combining the above two estimates, we obtain

$$\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\,\mathrm{d}\tau \leq \Big(2^\mu + \frac{C_0 2^{\mu-1}}{1-\mu}\Big)\lambda^{-1}t^{-\mu} = C_\mu\lambda^{-1}t^{-\mu},$$

which corresponds to (3-11). □

3B. ***A priori estimates.*** In this subsection, we assume θ is a smooth solution for the drift-diffusion equations (1-1)–(1-2) with smooth u and f. We shall show the estimate (1-13) and the proof consists of four steps.

Step 1: the estimation of $\|\theta\|_{L^\infty([t_0,T];B^{s_0}_{p,\infty})}$ for any $s_0 \in (0, \alpha-\sigma)$ and $t_0 \in (0, T)$.

By applying the dyadic operator Δ_j ($j \in \mathbb{N}$, $j \geq 4$) to the equation of θ in (1-1), we get

$$\partial_t\Delta_j\theta + u\cdot\nabla\Delta_j\theta + \mathcal{L}\Delta_j\theta = -[\Delta_j, u\cdot\nabla]\theta + \Delta_j f, \tag{3-13}$$

where $[A, B] = AB - BA$ denotes the commutator of two operators A and B. Bony's paraproduct decomposition leads to

$$
\begin{aligned}
(3\text{-}14)\quad -[\Delta_j, u\cdot\nabla]\theta &= -\sum_{|k-j|\le 4}[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta \\
&\quad - \sum_{|k-j|\le 4}\big(\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_k u\cdot\nabla\Delta_j S_{k-1}\theta\big) \\
&\quad - \sum_{k\ge j-2}\big(\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta) - \Delta_k u\cdot\nabla\Delta_j\widetilde{\Delta}_k\theta\big) \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Multiplying both sides of the equation (4-8) with $|\Delta_j\theta|^{p-2}\Delta_j\theta$ and integrating on the spatial variable over $\mathbb{R}^d$, we use the divergence-free property of u and the Hölder inequality to get

$$
\begin{aligned}
(3\text{-}15)\quad \frac{1}{p}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p + \int_{\mathbb{R}^d}\mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\,\mathrm{d}x& \\
\le (\|\Delta_j f\|_{L^p} + \|I_1\|_{L^p} + \|I_2\|_{L^p} + \|I_3\|_{L^p})\|\Delta_j\theta\|_{L^p}^{p-1}&.
\end{aligned}
$$

According to (3-4) in Lemma 3.2, we see that

$$
(3\text{-}16)\quad \int_{\mathbb{R}^d}\mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\,\mathrm{d}x \ge c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p}^p - C_1\|\Delta_j\theta\|_{L^p}^p,
$$

where c and C_1 are constants depending on p, α, σ, d. Inserting (3-16) into (3-15) and dividing $\|\Delta_j\theta\|_{L^p}^{p-1}$ lead to

$$
\begin{aligned}
(3\text{-}17)\quad \frac{d}{dt}\|\Delta_j\theta\|_{L^p} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p}& \\
\le C_1\|\Delta_j\theta\|_{L^p} + \|\Delta_j f\|_{L^p} + \|I_1\|_{L^p} + \|I_2\|_{L^p} + \|I_3\|_{L^p}&.
\end{aligned}
$$

For $\|I_1\|_{L^p}$, noting that I_1 can be expressed as

$$
(3\text{-}18)\quad I_1 = -\sum_{|k-j|\le 4}\int_{\mathbb{R}^d} h_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)\,\mathrm{d}y,
$$

where $h_j(x) = 2^{jd}(\mathcal{F}^{-1}\varphi)(2^j x)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ is the test function introduced in Section 2, thus from Hölder's inequality, Bernstein's inequality and Young's inequality, one has

$$\begin{aligned}\|I_1\|_{L^p} &\le \sum_{|k-j|\le 4}\left\|\int_{\mathbb{R}^d} h_j(x-y)(S_{k-1}u(y)-S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)\,\mathrm{d}y\right\|_{L^p_x}\\ &\le C\sum_{|k-j|\le 4}\left\|\int_{\mathbb{R}^d}|h_j(x-y)|\|u\|_{\dot C^\delta}|x-y|^\delta|\nabla\Delta_k\theta(y)|\,\mathrm{d}y\right\|_{L^p_x}\\ &\le C\|u\|_{\dot C^\delta}\int_{\mathbb{R}^d}|h_j(x)||x|^\delta\,\mathrm{d}x\sum_{|k-j|\le 4}\|\nabla\Delta_k\theta\|_{L^p}\\ &\le C2^{-j\delta}\|u\|_{\dot C^\delta}\sum_{|k-j|\le 4}2^k\|\Delta_k\theta\|_{L^p}. \end{aligned}\tag{3-19}$$

By virtue of Hölder's inequality and Bernstein's inequality again, we see that

$$\begin{aligned}\|I_2\|_{L^p} &\le \sum_{|k-j|\le 4}\|\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta)\|_{L^p}+\sum_{|k-j|\le 4}\|\Delta_k u\cdot\nabla S_{k-1}\Delta_j\theta\|_{L^p}\\ &\le C\sum_{|k-j|\le 4}\|\Delta_k u\|_{L^\infty}\|\nabla S_{k-1}\theta\|_{L^p}+C\sum_{|k-j|\le 4}\|\Delta_k u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^p}\\ &\le C2^{-j\delta}\sum_{|k-j|\le 4}2^{k\delta}\|\Delta_k u\|_{L^\infty}\left(\sum_{l\le j}2^l\|\Delta_l\theta\|_{L^p}\right)\\ &\le C2^{-j\delta}\|u\|_{\dot C^\delta}\left(\sum_{k\le j}2^k\|\Delta_k\theta\|_{L^p}\right), \end{aligned}\tag{3-20}$$

and by using the divergence-free property of u, we get

$$\begin{aligned}\|I_3\|_{L^p} &\le \sum_{k\ge j-2}\|\nabla\cdot\Delta_j(\Delta_k u\widetilde{\Delta}_k\theta)\|_{L^p}+\sum_{k\ge j-2}\|\Delta_k u\cdot\nabla\widetilde{\Delta}_k\Delta_j\theta\|_{L^p}\\ &\le C\sum_{k\ge j-2}2^j\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k\theta\|_{L^p}\\ &\le C2^j\sum_{k\ge j-2}2^{k\delta}\|\Delta_k u\|_{L^\infty}2^{-k\delta}\|\widetilde{\Delta}_k\theta\|_{L^p}\\ &\le C\|u\|_{\dot C^\delta}2^j\left(\sum_{k\ge j-2}2^{-k\delta}\|\Delta_k\theta\|_{L^p}\right). \end{aligned}\tag{3-21}$$

Gathering the above estimates leads to

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}+c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p} \le C_1\|\Delta_j\theta\|_{L^p}&+\|\Delta_j f\|_{L^p}\\ &+C\|u\|_{\dot C^\delta}2^{-j\delta}\sum_{k\le j+4}2^k\|\Delta_k\theta\|_{L^p}\\ &+C\|u\|_{\dot C^\delta}2^j\sum_{k\ge j-3}2^{-k\delta}\|\Delta_k\theta\|_{L^p}.\end{aligned}$$

Let $j_0 \in \mathbb{N}$ be a number chosen later (see (3-32)) which satisfies $(c/2)2^{j_0(\alpha-\sigma)} \geq C_1$, or more precisely,

$$j_0 \geq \left[\frac{1}{\alpha-\sigma}\log_2\left(\frac{2C_1}{c}\right)\right]+1. \tag{3-22}$$

We infer that for all $j \geq j_0$,

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}+\frac{c}{2}2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p} &\leq \|\Delta_j f\|_{L^p}+C\|u\|_{\dot{C}^\delta}2^{-j\delta}\sum_{k\leq j+4}2^k\|\Delta_k\theta\|_{L^p}\\ &\quad+C\|u\|_{\dot{C}^\delta}2^j\sum_{k\geq j-3}2^{-k\delta}\|\Delta_k\theta\|_{L^p}\\ &:= \|\Delta_j f\|_{L^p}+H_j^1+H_j^2.\end{aligned} \tag{3-23}$$

Thus Grönwall's inequality yields that for every $j \geq j_0 \geq 4$ and $t \geq 0$,

$$\begin{aligned}\|\Delta_j\theta(t)\|_{L^p} &\leq e^{-(c/2)t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^p}\\ &\quad+\int_0^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^p}(\tau)+H_j^1(\tau)+H_j^2(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{3-24}$$

According to Lemma 3.3, we also have the L^p-estimate for equation (1-1):

$$\|\theta(t)\|_{L^p} \leq e^{Ct}\left(\|\theta_0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}\,\mathrm{d}t\right). \tag{3-25}$$

Observing that for all $t > 0$, $j \in \mathbb{N}$ and $s \in (0,\alpha-\sigma)$,

$$\begin{aligned}2^{js}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^p} &\leq t^{-\frac{s}{\alpha-\sigma}}((t2^{j(\alpha-\sigma)})^{\frac{s}{\alpha-\sigma}}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}})\|\Delta_j\theta_0\|_{L^p}\\ &\leq C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^p},\end{aligned} \tag{3-26}$$

thus collecting (3-24), (3-25) and (3-26) leads to

$$\begin{aligned}\|\theta(t)\|_{B^s_{p,\infty}} &\leq \sup_{j\leq j_0}2^{js}\|\Delta_j\theta(t)\|_{L^p}+\sup_{j\geq j_0}2^{js}\|\Delta_j\theta(t)\|_{L^p}\\ &\leq C2^{j_0 s}e^{Ct}(\|\theta_0\|_{L^p}+\|f\|_{L^1_tL^p})+C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^p}\\ &\quad+\sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}(\|\Delta_j f\|_{L^p}(\tau)+H_j^1(\tau)+H_j^2(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{3-27}$$

For the term containing $\|\Delta_j f\|_{L^p}$, we infer that for every $s \in (0, \alpha-\sigma+\delta)$,

$$\begin{aligned}
(3\text{-}28)\quad & \sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}\|\Delta_j f\|_{L^p}(\tau)\,\mathrm{d}\tau \\
&\leq C\sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s-\delta)}\|f(\tau)\|_{\dot B^\delta_{p,\infty}}\,\mathrm{d}\tau \\
&\leq C\|f\|_{L^\infty_t\dot B^\delta_{p,\infty}}\sup_{j\geq j_0}2^{j(s-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\,\mathrm{d}\tau \\
&\leq C\|f\|_{L^\infty_t\dot B^\delta_{p,\infty}}\sup_{j\geq j_0}2^{j(s-\alpha+\sigma-\delta)} \\
&\leq C\|f\|_{L^\infty_t\dot B^\delta_{p,\infty}}.
\end{aligned}$$

For the term including H_j^1 in (3-27), thanks to (3-12) in Lemma 3.4, we deduce that for every $s \in (0, \alpha-\sigma)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$\begin{aligned}
(3\text{-}29)\quad & \sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}H_j^1(\tau)\mathrm{d}\tau \\
&= C\sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot C^\delta}2^{j(s-\delta)}\Bigl(\sum_{k\leq j+4}2^k\|\Delta_k\theta(\tau)\|_{L^p}\Bigr)\mathrm{d}\tau \\
&\leq C\|u\|_{L^\infty_t\dot C^\delta}\sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s-\delta)}\Bigl(\sum_{k\leq j+4}2^{k(1-s)}\|\theta(\tau)\|_{B^s_{p,\infty}}\Bigr)\mathrm{d}\tau \\
&\leq C\|u\|_{L^\infty_t\dot C^\delta}\Bigl(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Bigr) \\
&\qquad\qquad\times\sup_{j\geq j_0}2^{j(1-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\tau^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\leq C\|u\|_{L^\infty_t\dot C^\delta}\Bigl(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Bigr)t^{-\frac{s}{\alpha-\sigma}}\sup_{j\geq j_0}2^{j(1-\delta-\alpha+\sigma)} \\
&\leq Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(1-\alpha+\sigma))}\|u\|_{L^\infty_t\dot C^\delta}\Bigl(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Bigr).
\end{aligned}$$

For the term including H_j^2 in (3-27), by using (3-12) again, we similarly get that for all $s \in (0, \alpha-\sigma)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$
\begin{aligned}
(3\text{-}30)\quad & \sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}H_j^2(\tau)\,\mathrm{d}\tau\\
&= C\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot C^\delta}2^{j(s+1)}\Bigl(\sum_{k\ge j-3}2^{-k\delta}\|\Delta_k\theta(\tau)\|_{L^p}\Bigr)\mathrm{d}\tau\\
&\le C\|u\|_{L^\infty_t\dot C^\delta}\sup_{j\ge j_0}2^{j(s+1)}\Bigl(\sum_{k\ge j-3}2^{-k(\delta+s)}\Bigr)\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^s_{p,\infty}}\mathrm{d}\tau\\
&\le Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(1-\alpha+\sigma))}\|u\|_{L^\infty_t\dot C^\delta}\Bigl(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Bigr).
\end{aligned}
$$

Plugging the estimates (3-28), (3-29), (3-30) into (3-27) yields that for any $0<s<\alpha-\sigma$ and $0<t\le T$,

$$
\begin{aligned}
(3\text{-}31)\quad t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}} \le{}& CT^{s/(\alpha-\sigma)}e^{CT}2^{j_0}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})\\
&+C_{\alpha,\sigma,s}\|\theta_0\|_{L^p}+CT^{s/(\alpha-\sigma)}\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}\\
&+\frac{C\|u\|_{L^\infty_T\dot C^\delta}}{2^{j_0(\delta-(1-\alpha+\sigma))}}\Bigl(\sup_{t\in(0,T]}t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}}\Bigr).
\end{aligned}
$$

Now, by choosing $j_0\in\mathbb{N}$ such that $C2^{j_0(1-\alpha+\sigma-\delta)}\|u\|_{L^\infty_T\dot C^\delta}\le\frac12$ and (3-22) holds, or more precisely,

$$
(3\text{-}32)\qquad j_0=\max\biggl\{\biggl[\frac{\log_2(2C\|u\|_{L^\infty_T\dot C^\delta})}{\delta-(1-\alpha+\sigma)}\biggr],\biggl[\frac{\log_2(C_1/c)}{\alpha-\sigma}\biggr],4\biggr\}+1,
$$

we have that for all $0<s<\alpha-\sigma$,

$$
\begin{aligned}
(3\text{-}33)\quad &\sup_{t\in(0,T]}(t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}})\\
&\qquad\le C(T+1)(e^{CT}2^{j_0s}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})+\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}),
\end{aligned}
$$

which implies that for arbitrarily small $t_0\in(0,T)$ and every $s_0\in(0,\alpha-\sigma)$,

$$
\begin{aligned}
(3\text{-}34)\quad &\sup_{t\in[t_0,T]}\|\theta(t)\|_{B^{s_0}_{p,\infty}}\\
&\qquad\le Ct_0^{-s_0/(\alpha-\sigma)}(T+1)(e^{CT}2^{j_0s}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})+\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}),
\end{aligned}
$$

where j_0 is given by (3-32).

Step 2: the estimation of $\|\theta\|_{L^\infty([t_1,T];B^{s_0+s_1}_{p,\infty})}$ for every $s_0,s_1\in(0,\alpha-\sigma)$ and $t_1\in(t_0,T)$.

For every $j \geq j_0$ with $j_0 \in \mathbb{N}$ satisfying (3-22) chosen later (j_0 may be different from that number in Step 1), adapting the Grönwall inequality to (3-23) over the time interval $[t_0, t]$ (for $t > t_0 > 0$) yields

$$\|\Delta_j \theta(t)\|_{L^p} \leq e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}} \|\Delta_j \theta(t_0)\|_{L^p} + \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} (\|\Delta_j f\|_{L^p}(\tau) + H_j^1(\tau) + H_j^2(\tau))\, \mathrm{d}\tau. \tag{3-35}$$

Noting that for $j \in \mathbb{N}$, $s_0 \in (0, \alpha - \sigma)$ and every $s \in (0, \alpha - \sigma)$,

$$e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} \|\Delta_j \theta(t_0)\|_{L^p} \leq e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}} 2^{js} \|\theta(t_0)\|_{B^{s_0}_{p,\infty}} \leq C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}} \|\theta(t_0)\|_{B^{s_0}_{p,\infty}}, \tag{3-36}$$

thus we get that for all $t \geq t_0 > 0$,

$$\begin{aligned} \|\theta(t)\|_{B^{s_0+s}_{p,\infty}} &\leq \sup_{j \leq j_0} 2^{j(s_0+s)} \|\Delta_j \theta(t)\|_{L^p} + \sup_{j \geq j_0} 2^{j(s_0+s)} \|\Delta_j \theta(t)\|_{L^p} \\ &\leq C 2^{j_0(s_0+s)} e^{Ct} (\|\theta_0\|_{L^p} + \|f\|_{L^1_t L^p}) + C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}} \|\theta(t_0)\|_{B^{s_0}_{p,\infty}} \\ &\quad + \sup_{j \geq j_0} \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} (\|\Delta_j f\|_{L^p}(\tau) + H_j^1(\tau) + H_j^2(\tau))\, \mathrm{d}\tau. \end{aligned} \tag{3-37}$$

For the term containing $\|\Delta_j f\|_{L^p}$, similarly as obtaining (3-28), we get that for every $s \in (0, \alpha - \sigma)$ and $s_0 + s < \delta + \alpha - \sigma$,

$$\begin{aligned} \sup_{j \geq j_0} \int_{t_0}^{t} e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} \|\Delta_j f\|_{L^p}\, \mathrm{d}\tau &\leq C \sup_{j \geq j_0} \int_{t_0}^{t} e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s-\delta)} \|f(\tau)\|_{\dot{B}^{\delta}_{p,\infty}}\, \mathrm{d}\tau \\ &\leq C \|f\|_{L^\infty_t \dot{B}^{\delta}_{p,\infty}} \sup_{j \geq j_0} 2^{j(s_0+s-\delta)} \int_{t_0}^{t} e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}\, \mathrm{d}\tau \\ &\leq C \|f\|_{L^\infty_t \dot{B}^{\delta}_{p,\infty}} \sup_{j \geq j_0} 2^{j(s_0+s-\alpha+\sigma-\delta)} \\ &\leq C \|f\|_{L^\infty_t \dot{B}^{\delta}_{p,\infty}}. \end{aligned} \tag{3-38}$$

For the term including H_j^1 in (3-37), by arguing as (3-29), we deduce that for every $s \in (0, \alpha-\sigma)$ and $s_0+s \le 1$,

$$
\begin{aligned}
(3\text{-}39)\quad & \sup_{j\ge j_0}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} H_j^1(\tau)\,\mathrm{d}\tau \\
&= C\sup_{j\ge j_0}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} \|u(\tau)\|_{\dot{C}^\delta} 2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4} 2^k\|\Delta_k\theta(\tau)\|_{L^p}\Big)\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta} \\
&\qquad \sup_{j\ge j_0}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4} 2^{k(1-s-s_0)}\Big)\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big) \\
&\qquad\qquad \sup_{j\ge j_0}(2^{j(1-\delta)}j)\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s}_{p,\infty}}\Big) \\
&\qquad\qquad (t-t_0)^{-\frac{s}{\alpha-\sigma}}\sup_{j\ge j_0}(2^{-j(\delta-(1-\alpha+\sigma))}j) \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s}_{p,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_0\frac{\delta-(1-\alpha+\sigma)}{2}},
\end{aligned}
$$

and for $1 < s_0+s < \delta+\alpha-\sigma$,

$$
\begin{aligned}
(3\text{-}40)\quad & \sup_{j\ge j_0}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} H_j^1(\tau)\,\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\sup_{j\ge j_0}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4} 2^{k(1-s-s_0)}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big) \\
&\qquad\qquad \sup_{j\ge j_0}2^{j(s_0+s-\delta)}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big) \\
&\qquad\qquad (t-t_0)^{-\frac{s}{\alpha-\sigma}}\sup_{j\ge j_0}(2^{-j(\delta-(s_0+s-\alpha+\sigma))}) \\
&\le C\|u\|_{L_t^\infty \dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(s_0+s-\alpha+\sigma))}.
\end{aligned}
$$

For the term including H_j^2 in (3-37), by using (3-12) again, we estimate similarly as (3-30) to get that for all $s \in (0, \alpha - \sigma)$,

$$\begin{aligned}
\text{(3-41)}\quad & \sup_{j\geq j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} H_j^2(\tau)\,\mathrm{d}\tau \\
&= C\sup_{j\geq j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} \|u(\tau)\|_{\dot C^\delta} 2^{j(s_0+s+1)}\Big(\sum_{k\geq j-3} 2^{-k\delta}\|\Delta_k\theta(\tau)\|_{L^p}\Big)\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty \dot C^\delta}\sup_{j\geq j_0} 2^{j(s_0+s+1)} \\
&\qquad\qquad \Big(\sum_{k\geq j-3} 2^{-k(\delta+s_0+s)}\Big)\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} \|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\,\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty \dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big) \\
&\qquad\qquad \sup_{j\geq j_0} 2^{j(1-\delta)}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty \dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}} 2^{-j_0(\delta-(1-\alpha+\sigma))}.
\end{aligned}$$

Inserting the estimates (3-38)–(3-41) into (3-37), we obtain that for every $t \in (t_0, T]$, $s \in (0, \alpha - \sigma)$ and $s_0 + s < \delta + \alpha - \sigma$,

$$\begin{aligned}
(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{p,\infty}} &\leq CT^{\frac{s}{\alpha-\sigma}} e^{CT}(\|\theta_0\|_{L^p} + \|f\|_{L^1_T L^p}) 2^{j_0(s_0+s)} \\
&\qquad + C_{\alpha,\sigma,s}\|\theta(t_0)\|_{B^{s_0}_{p,\infty}} + CT^{\frac{s}{\alpha-\sigma}}\|f\|_{L_t^\infty \dot B^\delta_{p,\infty}} \\
+ &\begin{cases} \frac{C\|u\|_{L_T^\infty \dot C^\delta}}{2^{j_0(\delta-(1-\alpha+\sigma))/2}}(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{p,\infty}}), & \text{if } s_0+s\leq 1, \\ \frac{C\|u\|_{L_T^\infty \dot C^\delta}}{2^{j_0(\delta-(s_0+s-\alpha+\sigma))}}(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{p,\infty}}), & \text{if } 1 < s_0+s < \delta+\alpha-\sigma. \end{cases}
\end{aligned}$$

Hence by selecting $j_0 \in \mathbb{N}$ as

$$\text{(3-42)}\qquad j_0 = \max\bigg\{\bigg[\frac{2\log_2(2C\|u\|_{L_T^\infty \dot C^\delta})}{\delta-(1-\alpha+\sigma)}\bigg], \bigg[\frac{\log_2(2C_1/c)}{\alpha-\sigma}\bigg], 4\bigg\} + 1$$

if $s_0 + s \leq 1$, and

$$\text{(3-43)}\qquad \max\bigg\{\bigg[\frac{\log_2(2C\|u\|_{L_T^\infty \dot C^\delta})}{\delta-(s_0+s-\alpha+\sigma)}\bigg], \bigg[\frac{\log_2(2C_1/c)}{\alpha-\sigma}\bigg], 4\bigg\} + 1$$

if $1 < s_0+s < \delta+\alpha-\sigma$, we find that for all $s \in (0,\alpha-\sigma)$ and $s_0+s < \delta+\alpha-\sigma$,

$$\sup_{t\in(t_0,T]} ((t-t_0)^{s/(\alpha-\sigma)} \|\theta(t)\|_{B^{s_0+s}_{p,\infty}}) \\ \le C(T+1)(\|\theta_0\|_{L^p} + \|f\|_{L^1_T L^p}) 2^{j_0(s_0+s)} \\ + C\|\theta(t_0)\|_{B^{s_0}_{p,\infty}} + C(T+1)\|f\|_{L^\infty_T \dot{B}^\delta_{p,\infty}},$$

which ensures that for any $t_1 \in (t_0,T)$ and every $s_0, s_1 \in (0,\alpha-\sigma)$ satisfying $s_0+s_1 < \delta+\alpha-\sigma$,

$$\sup_{t\in[t_1,T]} \|\theta(t)\|_{B^{s_0+s_1}_{p,\infty}} \\ \le C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}} \big((T+1)e^{CT}\big(\|\theta_0\|_{L^p}+\|f\|_{L^1_T L^p}\big)2^{j_0(s_0+s_1)} + \|\theta(t_0)\|_{B^{s_0}_{p,\infty}}\big) \\ + C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot{B}^\delta_{p,\infty}}, \tag{3-44}$$

where j_0 is given by (3-42)–(3-43).

Step 3: the estimation of $\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma})}$ for some $\gamma > 0$ and any $\tilde{t} \in (0,T)$.

If $\alpha-\sigma \in \big(\frac{1}{2},1\big)$, we can choose appropriate indexes $s_0, s_1 \in (0,\alpha-\sigma)$ so that $1 < s_0+s_1 < \delta+\alpha-\sigma$, more precisely, denoting by

$$\nu_1 := \min\Big\{\frac{2(\alpha-\sigma)-1}{2}, \frac{\delta+\alpha-\sigma-1}{2}\Big\},$$

s_0+s_1 can be chosen so that $s_0+s_1 = 1+\nu_1$, thus in view of (3-44), we obtain that

$$\sup_{t\in[t_1,T]} \|\theta(t)\|_{B^{1+\nu_1}_{p,\infty}} \le C < \infty. \tag{3-45}$$

If $p > d/\nu_1$, then from the Besov embedding $B^{1+\nu_1}_{p,d} \hookrightarrow B^{1+\nu_1-d/p}_{\infty,\infty}$, we get the bound of $\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma})}$ with $\tilde{t} = t_1$ and $\gamma = \nu_1 - d/p > 0$. If $p \le d/\nu_1$, and we have the embedding $B^{1+\nu_1}_{p,\infty} \hookrightarrow L^{p_1}$ with some $p_1 > d/\nu_1$, by repeating the above Step 1 and Step 2 with p_1 in place of p, we can obtain the estimate of $\|\theta\|_{L^\infty([t_1^1,T];B^{1+\nu_1}_{p_1,\infty})}$ with any $t_1^1 \in (t_1,T)$, which implies the bound of $\|\theta\|_{L^\infty([t_1^1,T];C^{1,\gamma})}$ with $\gamma = \nu_1 - d/p_1$. Otherwise, for $p \le d/\nu_1$ and p_1 satisfying $d/p_1 = d/p - (1+\nu_1)$ is such that $p_1 \in (p, d/\nu_1]$, as above we can obtain the bound of $\|\theta\|_{L^\infty([t_1^1,T];B^{1+\nu_1}_{p_1,\infty})}$ with any $t_1^1 \in (t_1,T)$, then if the embedding $B^{1+\nu_1}_{p_1,\infty} \hookrightarrow L^{p_2}$ with some $p_2 > d/\nu_1$, we can repeat the above Step 1 and Step 2 to conclude the proof, while if p_2 satisfying $d/p_2 = d/p_1 - (1+\nu_1) = d/p - 2(1+\nu_1)$ is still such that $p_2 \in (p_1, d/\nu_1]$, we can iterate the above steps for several times, say m times, to find some number $p_{m+1} > d/\nu_1$ and obtain the bound of $\|\theta\|_{L^\infty([t_1^{m+1},T];B^{1+\nu_1}_{p_{m+1},\infty})}$ with $t_1^{m+1} \in (t_1^m, T)$ any chosen, which further implies the bound of $\|\theta\|_{L^\infty([t_1^{m+1},T];C^{1,\gamma})}$ with $\gamma = 1+\nu_1 - d/p_{m+1}$.

For $\alpha-\sigma \in \big(0,\frac{1}{2}\big]$, we need to iterate the above procedure in Step 2 more times. Assume that for some small number $t_k > 0$, $k \in \mathbb{N}$, we already have a finite bound on $\|\theta(t_k)\|_{B^{s_0+s_1+\cdots+s_k}_{p,\infty}}$ with $s_0, s_1, \dots, s_k \in (0, \alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_k \le 1$. Then by arguing as (3-44), we deduce that for any $t_{k+1} > t_k$, $s_{k+1} \in (0, \alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$,

$$\begin{aligned}\text{(3-46)} \quad &\sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{p,\infty}} \\ &\le C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}\big((T+1)(\|\theta_0\|_{L^p}+\|f\|_{L^1_T L^p})2^{j_0(\sum_{i=0}^{k+1}s_i)}+\|\theta(t_k)\|_{B^{\sum_{i=0}^{k}s_i}_{p,\infty}}\big) \\ &\qquad + C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot B^\delta_{p,\infty}},\end{aligned}$$

where j_0 is also given by (3-42)–(3-43) with s_0+s_1 replaced by $s_0+s_1+\cdots+s_{k+1}$. Hence if $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, $k \in \mathbb{N}^+$, we can select appropriate numbers $s_0, s_1, \dots, s_{k+1} \in (0, \alpha-\sigma)$ so that $1 < s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$, or, more precisely, $s_0+s_1+\cdots+s_{k+1} = 1+\nu_{k+1}$, with

$$\nu_{k+1} := \min\Big\{\frac{(k+2)(\alpha-\sigma)-1}{2}, \frac{\delta+\alpha-\sigma-1}{2}\Big\},$$

and by repeating Step 2 in the above manner for $(k+1)$-times, we obtain

$$\text{(3-47)} \qquad \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{1+\nu_{k+1}}_{p,\infty}} \le C < \infty.$$

The following deduction is similar to that stated below (3-45). If $p > d/\nu_{k+1}$, then from $B^{1+\nu_{k+1}}_{p,\infty} \hookrightarrow B^{1+\nu_{k+1}-d/p}_{\infty,\infty}$, we naturally get the estimate of $\|\theta\|_{L^\infty([t_{k+1},T];C^{1,\gamma})}$ with $\gamma = 1+\nu_{k+1}-d/p$. Otherwise, there exists a unique number $m \in \mathbb{N}$ so that

$$\text{(3-48)} \qquad \frac{d}{p} - m(1+\nu_{k+1}) \ge \nu_{k+1}, \quad \text{and} \quad \frac{d}{p} - (m+1)(1+\nu_{k+1}) < \nu_{k+1},$$

and by denoting $p_j \in [p, \infty)$ by

$$\frac{d}{p_j} = \frac{d}{p} - j(1+\nu_{k+1}), \qquad j = 0, 1, 2, \dots, m,$$

we see that $p = p_0 < p_1 < \cdots < p_m \le d/\nu_{k+1}$, thus by repeating the above process in obtaining (3-47) with p_j replaced by p_{j+1} iteratively ($j = 0, 1, \dots, m-1$), we have the bound of $\|\theta\|_{L^\infty([t^m_{k+1},T];B^{1+\nu_{k+1}}_{p_m,\infty})}$ with any $t^m_{k+1} \in (0, T)$ (with the convention $t^0_i := t_i$ for $i = 0, 1, \dots, k+1$), which ensures that there is some $p_{m+1} > d/\nu_{k+1}$ so that $\|\theta\|_{L^\infty([t^m_{k+1},T];L^{p_{m+1}})}$ is bounded, and then iterating the above process once again leads to the estimate of $\|\theta\|_{L^\infty([t^{m+1}_{k+1},T];B^{1+\nu_{k+1}}_{p_{m+1},\infty})}$ with any $t^{m+1}_{k+1} \in (t^m_{k+1}, T)$ and moreover implies that for $1+\gamma = 1+\nu_{k+1}-d/p_{m+1} = (m+2)(1+\nu_{k+1})-d/p$,

$$\begin{aligned}&\|\theta\|_{L^\infty([t^{m+1}_{k+1},T];C^{1,\gamma})} \approx \|\theta\|_{L^\infty([t^{m+1}_{k+1},T];B^{1+\gamma}_{\infty,\infty})} \\ \text{(3-49)} \qquad &\qquad \le C\Big(\prod_{j=0}^{m+1}\prod_{i=0}^{k}(t^j_{i+1}-t^j_i)^{-\frac{s_{i+1}}{\alpha-\sigma}}(t^j_0-t^{j-1}_{k+1})^{-\frac{s_0}{\alpha-\sigma}}\Big) \\ &\qquad\qquad (\|\theta_0\|_{L^p}+\|f\|_{L^\infty_T(B^\delta_{p,\infty}\cap B^\delta_{\infty,\infty})}),\end{aligned}$$

where $t_i^0 := t_i$ for $i = 0, 1, \dots, k+1$, $t_{k+1}^{-1} := 0$, $C > 0$ is a constant depending only on $p, \alpha, \sigma, \delta, d, T$ and $\|u\|_{L_T^\infty \dot{C}^\delta}$.

Therefore, for every $\alpha \in (0, 1]$, $\sigma \in [0, \alpha)$, $p \in [2, \infty)$, and for any $\tilde{t} \in (0, T)$, there is some $k \in \mathbb{N}$ so that $\alpha - \sigma \in (1/(k+2), 1/(k+1)]$, and there is some number $m \in \mathbb{N}$ so that (3-48) holds, and thus we can choose

$$t_i^j = \frac{j(k+2)+i+1}{(k+2)(m+2)}\tilde{t} \quad \text{for } i = 0, 1, \dots, k+1,\ j = 0, 1, 2, \dots, m+1,$$

and appropriate $s_0, s_1, \dots, s_{k+1} \in (0, \alpha - \sigma)$ such that $s_0 + s_1 + \cdots + s_{k+1} = 1 + \nu_{k+1}$. Then we use (3-49) to get that for some $\gamma > 0$,

$$\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma}(\mathbb{R}^d))} \le C\tilde{t}^{-\frac{\gamma+1+d/p}{\alpha-\sigma}} (\|\theta_0\|_{L^p} + \|f\|_{L_T^\infty(B_{p,\infty}^\delta \cap B_{\infty,\infty}^\delta)}), \tag{3-50}$$

with the constant C depending only on p, α, σ, δ, T, d and $\|u\|_{L_T^\infty \dot{C}^\delta}$.

Step 4: the estimation of $\|\theta\|_{L^\infty([t',T];C^{1,\gamma})}$ for any $\gamma \in (0, \delta + \alpha - \sigma - 1)$ and any $t' \in (0, T)$.

This is achieved by pursuing the above iteration process more times. In fact, for any $\gamma \in (0, \delta+\alpha-\sigma-1)$, there exists some $\tilde{p} < \infty$ so that $\gamma + d/\tilde{p} < \delta+\alpha-\sigma-1$, and according to the above Step 3, we may suppose that there is already a bound of $\|\theta\|_{L^\infty([t'/2,T];B_{\tilde{p},\infty}^{s'})}$ with some $1 < s' < 1+\gamma+d/\tilde{p}$, but by repeating the deduction in Steps 1–2 for several times and due to the increment of regularity index s in each time belonging to $(0, \alpha-\sigma)$, we can derive an upper bound of $\|\theta\|_{L^\infty([t',T];B_{\tilde{p},\infty}^{1+\gamma+d/\tilde{p}})}$, which also satisfies (3-50) with t' in place of $\tilde{t}$.

3C. ***The existence part.*** We consider the approximate system

$$\begin{cases} \partial_t\theta + (u_\epsilon \cdot \nabla)\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_\epsilon, \\ u_\epsilon := \phi_\epsilon * u, \quad f_\epsilon = \phi_\epsilon * f, \quad \theta|_{t=0} = \theta_{0,\epsilon} := \phi_\epsilon * \theta_0, \end{cases} \tag{3-51}$$

where $\phi_\epsilon(x) = \epsilon^{-d}\phi(\epsilon^{-1}x)$ for all $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a test function supported on the ball $B_1(0)$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{1/2}(0)$ and $\int_{\mathbb{R}^d} \phi \,\mathrm{d}x = 1$.

Due to that for all $s \ge 0$, $\|\theta_{0,\epsilon}\|_{B_{p,2}^s(\mathbb{R}^d)} \lesssim_\epsilon \|\theta_0\|_{L^p(\mathbb{R}^d)}$ and $\|u_\epsilon\|_{L_T^\infty C^s(\mathbb{R}^d)} \lesssim_\epsilon \|u\|_{L_T^\infty C^\delta}$ and $\|f_\epsilon\|_{L_T^\infty B_{p,2}^s} \lesssim_\epsilon \|f\|_{L_T^\infty B_{p,\infty}^\delta}$, by using a classical procedure (the operator $\mathcal{L}$ can be treated as Lemma 3.2), we obtain a smooth approximate solution $\theta^{(\epsilon)} \in C([0, T]; B_{p,2}^s(\mathbb{R}^d)) \cap C^1([0, T]; C_b^\infty(\mathbb{R}^d))$, $s > d/p + 1$ for the system (3-51).

Notice that we have the following uniform-in-ϵ estimates that $\|\theta_{0,\epsilon}\|_{L^p} \le \|\theta_0\|_{L^p}$, $\|u_\epsilon\|_{L_T^\infty C^\delta} \le \|u\|_{L_T^\infty C^\delta}$ and $\|f_\epsilon\|_{L_T^\infty(B_{p,\infty}^\delta \cap B_{\infty,\infty}^\delta)} \le \|f\|_{L_T^\infty(B_{p,\infty}^\delta \cap B_{\infty,\infty}^\delta)}$. According to Lemma 3.3, we infer that the solutions $\theta^{(\epsilon)}$ uniformly-in-ϵ belong to the space $L^\infty([0, T]; L^p(\mathbb{R}^d)) \cap L^p([0, T]; B_{p,p}^{(\alpha-\sigma)/p}(\mathbb{R}^d))$. From the system (3-51), we also claim that $\partial_t\theta^{(\epsilon)} \in L^p([0, T]; B_{p,p}^{-2}(\mathbb{R}^d))$ uniformly in $\epsilon > 0$. Indeed, it is derived from the following uniform-in-ϵ estimates:

$$\|f_\epsilon\|_{L^p_T B^{-2}_{p,p}} \le C\|f_\epsilon\|_{L^p_T L^p} \le C\|f\|_{L^p_T L^p} \le CT^{1/p}\|f\|_{L^\infty_T L^p},$$

and (thanks to Lemma 2.2(3))

$$\begin{aligned}\|\mathcal{L}\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} &\le C\|\mathcal{L}\Delta_{-1}\theta^{(\epsilon)}\|_{L^p_T L^p} + \sum_{j\in\mathbb{N}} 2^{-2j}\|\mathcal{L}\Delta_j\theta^{(\epsilon)}\|_{L^p_T L^p} \\ &\le C\|\theta^{(\epsilon)}\|_{L^p_T L^p} + C\sum_{j\in\mathbb{N}} 2^{-2j}2^{j\alpha}\|\theta^{(\epsilon)}\|_{L^p_T L^p} \\ &\le CT^{1/p}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p},\end{aligned}$$

and $\|\Delta\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} \le C\|\theta^{(\epsilon)}\|_{L^p_T B^0_{p,p}} \le CT^{1/p}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p}$, and

$$\begin{aligned}\|(u_\epsilon\cdot\nabla)\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} &\le C\|u_\epsilon\theta^{(\epsilon)}\|_{L^p_T B^0_{p,\infty}} \le CT^{1/p}\|u_\epsilon\|_{L^\infty_T L^\infty}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p} \\ &\le CT^{1/p}\|u\|_{L^\infty_T L^\infty}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p}.\end{aligned}$$

Since the embedding $B^{\alpha-\sigma/p}_{p,p} \hookrightarrow L^p$ is locally compact, the classical Aubin–Lions lemma (see, e.g., [Constantin and Foias 1988, Lemma 8.4]) ensures the strong convergence of $\theta^{(\epsilon)}$ (up to the subsequence, still denoting by $\theta^{(\epsilon)}$) to θ in $L^p_T L^p_{\mathrm{loc}}$. From Fatou's lemma, we get $\theta \in L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^p([0,T]; B^{\alpha-\sigma/p}_{p,p}(\mathbb{R}^d))$. Noticing also that from $u \in L^\infty_T C^\delta$ we have $u_\epsilon \to u$ in $L^\infty_T L^\infty$ as $\epsilon \to 0$, by using Hölder's inequality, it is not hard to check that for any test function $\varphi \in C^\infty_c(\mathbb{R}^d \times [0,T])$ (assuming $\operatorname{supp}\varphi \subseteq \mathcal{O}\times[0,T]$ with a compact set $\mathcal{O} \subseteq \mathbb{R}^d$),

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^d}\theta^{(\epsilon)}u_\epsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d}\theta u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad\le \left|\int_0^T\int_{\mathbb{R}^d}(\theta^{(\epsilon)}-\theta)u_\epsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^T\int_{\mathbb{R}^d}\theta(u_\epsilon-u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad\le C\|\theta^{(\epsilon)}-\theta\|_{L^p_T L^p(\mathcal{O})}\|u_\epsilon\|_{L^\infty_T L^\infty}\|\nabla\varphi\|_{L^{p/(p-1)}_{t,x}} \\ &\qquad\qquad + \|u_\epsilon-u\|_{L^\infty_T L^\infty}\|\theta\|_{L^\infty_T L^p}\|\nabla\varphi\|_{L^1_T L^{p/(p-1)}} \\ &\qquad\le C\|\theta^{(\epsilon)}-\theta\|_{L^p_T L^p(\mathcal{O})}\|u\|_{L^\infty_T L^\infty} + C\|u_\epsilon-u\|_{L^\infty_T L^\infty}\|\theta\|_{L^\infty_T L^p} \\ &\qquad\to 0 \quad \text{as } \epsilon\to 0.\end{aligned}$$

By passing ϵ to 0 in (3-51), from $f_\epsilon \to f$ in $L^\infty_T L^p$ and $\theta_{0,\epsilon} \to \theta_0$ in L^p, we deduce that θ is a distributional solution of (1-1) such that for every $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T])$

$$\begin{aligned}(3\text{-}52)\quad \int_{\mathbb{R}^d} \theta(x,t)\varphi(x,t)\,\mathrm{d}x - \int_{\mathbb{R}^d} \theta_0(x)\varphi(x,0)\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^d} \theta(x,\tau)\partial_\tau\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^t\int_{\mathbb{R}^d} (u\theta)(x,\tau)\nabla\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ - \int_0^t\int_{\mathbb{R}^d} \theta(x,\tau)\mathcal{L}^*\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ + \int_0^t\int_{\mathbb{R}^d} f(x,\tau)\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Moreover, from Lemma 3.3, the weak solution θ also satisfies

$$(3\text{-}53)\qquad \max_{0\le t\le T} \|\theta(t)\|_{L^p} \le e^{C'T}(\|\theta_0\|_{L^p} + \|f\|_{L^p_T L^p}),$$

with some constant $C' = C'(p, \alpha, \sigma, d)$. Moreover, by repeating the process in Section 3B for the approximate system (3-51) and using the Fatou lemma, we get

$$\theta \in C((0, T]; C^{1,\gamma}(\mathbb{R}^d))$$

for any $\gamma \in (0, \delta + \alpha - 1 - \sigma)$. Therefore, we conclude Theorem 1.1.

4. Proof of Theorem 1.2

4A. *Auxiliary lemmas.* Before proceeding with the main proof, we introduce several auxiliary lemmas. First is the maximum principle for the drift-diffusion equations (1-1)–(1-2).

Lemma 4.1. *Let the vector field u and the forcing term f be smooth. Assume that*

$$\theta \in L^\infty([0, T]; H^s(\mathbb{R}^d))$$

($s > d/2+1$) *is a smooth solution for the drift-diffusion equations* (1-1)–(1-2) *under the assumptions of K* (1-3)–(1-5)*. Then we have*

$$(4\text{-}1)\qquad \max_{0\le t\le T} \|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} + \int_0^T \|f(t)\|_{L^\infty}\,\mathrm{d}t.$$

Proof of Lemma 4.1. Thanks to the nonnegative condition (1-5), the proof is similar to [Córdoba and Córdoba 2004, Theorem 4.1]. We here sketch the proof for the sake of completeness. Since $\theta(\cdot\,, t) \in H^s$ with $s > d/2 + 1$ for any $0 \le t \le T$, there exists a point $x_t \in \mathbb{R}^d$ where $|\theta|$ attains its maximum value;with no loss of generality we set

$$\theta(x_t, t) = \|\theta(t)\|_{L^\infty}.$$

It should be noted that $\nabla_x \theta(x_t, t) = 0$ and due to $K(y) \geq 0$ we find

$$(\mathcal{L}\theta)(x_t, t) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} (\theta(x_t, t) - \theta(x_t + y, t))K(y)dy \geq 0.$$

We thus get

$$\frac{d}{dt}\|\theta(t)\|_{L^\infty} \leq \partial_t \theta(x_t, t) \leq \|f(t)\|_{L^\infty} \quad \text{for all } 0 \leq t \leq T.$$

Integrating in time yields the desired estimate (4-1). □

The second is the maximum principle with diffusion effect for the following frequency localized drift-diffusion equation

$$\partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta + \mathcal{L}\Delta_j \theta = g, \quad j \in \mathbb{N}, \tag{4-2}$$

where the operator $\mathcal{L}$ defined by (1-2) with the symmetric kernel K satisfying (1-3)–(1-5).

Lemma 4.2. *Assume that u and f are suitably smooth functions, and θ is a smooth solution to the equation* (4-2) *satisfying $\Delta_j\theta \in C_0(\mathbb{R}^d)$ for all $t > 0$ and $j \in \mathbb{N}$. Then there exist absolute positive constants c and C depending only on α, σ, d such that*

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \leq C\|\Delta_j\theta\|_{L^\infty} + \|g\|_{L^\infty}. \tag{4-3}$$

Proof of Lemma 4.2. Denote by $\theta_j := \Delta_j\theta$, and from $\theta_j(t) \in C_0(\mathbb{R}^d)$ for $j \in \mathbb{N}$, there exists a point $x_{t,j} \in \mathbb{R}^d$ such that $|\theta_j(t, x_{t,j})| = \|\theta_j\|_{L^\infty} > 0$. Without loss of generality, we assume $\theta_j(t, x_{t,j}) = \|\theta_j\|_{L^\infty} > 0$ (otherwise, we consider the equation of $-\theta_j$ and replace θ_j by $-\theta_j$ in the following deduction). Now by using (1-2), (1-5), (1-7) and the fact $\theta(t, x_{t,j}) - \theta(t, x_{t,j} + y) \geq 0$, we get

$$\begin{aligned}
\mathcal{L}\theta_j(x_{t,j}) &= \mathrm{p.\,v.}\int_{\mathbb{R}^d} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y \\
&= \mathrm{p.\,v.}\int_{|y|\leq 1} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y \\
&\qquad\qquad + \int_{|y|>1} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y
\end{aligned}$$

which then gives

$$\begin{aligned}
\mathcal{L}\theta_j(x_{t,j}) = &\geq c_2^{-1}\,\text{p. v.}\!\int_{|y|\leq 1}\frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\qquad + \int_{|y|>1}(\theta_j(x_{t,j})-\theta_j(x_{t,j}+y))K(y)\,\mathrm{d}y \\
&\geq c_2^{-1}\,\text{p. v.}\!\int_{\mathbb{R}^d}\frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\qquad - c_2^{-1}\int_{|y|>1}\frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\geq c_2^{-1}c_{d,\alpha}^{-1}|D|^{\alpha-\sigma}\theta_j(x_{t,j}) - 2c_2^{-1}\|\theta_j\|_{L^\infty}\int_{|y|>1}\frac{1}{|y|^{d+\alpha-\sigma}}\mathrm{d}y \\
&\geq c_2^{-1}c_{d,\alpha}^{-1}|D|^{\alpha-\sigma}\theta_j(x_{t,j}) - C\|\theta_j\|_{L^\infty}.
\end{aligned} \tag{4-4}$$

According to [Wang and Zhang 2011, Lemma 3.4], we have

$$|D|^{\alpha-\sigma}\theta_j(x_{t,j}) \geq \tilde{c}2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty}, \tag{4-5}$$

with some generic constant $\tilde{c} > 0$. Inserting (4-5) into (4-4) yields

$$\mathcal{L}\theta_j(x_{t,j}) \geq c2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty} - C\|\theta_j\|_{L^\infty}. \tag{4-6}$$

Hence, by arguing as Lemma 3.2 of the same work and using the fact $\nabla\theta_j(t, x_{t,j}) = 0$, we get

$$\begin{aligned}
\frac{d}{dt}\|\theta_j\|_{L^\infty} &\leq \partial_t\theta_j(t,x_{t,j}) \\
&= -u(t,x_{t,j})\cdot\nabla\theta_j(t,x_{t,j}) - \mathcal{L}\theta_j(t,x_{t,j}) + g(t,x_{t,j}) \\
&\leq -c2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty} + C\|\theta_j\|_{L^\infty} + \|g\|_{L^\infty},
\end{aligned} \tag{4-7}$$

which finishes the proof of (4-3). □

4B. ***A priori estimates.*** In this subsection, we assume θ is a smooth solution with suitable spatial decay for the drift-diffusion equations (1-1)–(1-2) with sufficiently smooth u and f. We intend to show the key a priori differentiability estimate. The proof is divided into four steps.

Step 1: the estimation of $\|\theta\|_{L^\infty([t_0,T];C^s(\mathbb{R}^d))}$ for any $s \in (1-\delta, \alpha-\sigma)$ and $t_0 \in (0, T)$.

For every $j \in \mathbb{N}$ and $j \geq 4$, applying the inhomogeneous dyadic operator Δ_j to the equation (1-1), we get

$$\partial_t\Delta_j\theta + u\cdot\nabla\Delta_j\theta + \mathcal{L}\Delta_j\theta = \Delta_j f - [\Delta_j, u\cdot\nabla]\theta = \Delta_j f + I_1 + I_2 + I_3, \tag{4-8}$$

where I_1–I_3 defined by (3-14) are the Bony's decomposition of the commutator term $-[\Delta_j, u\cdot\nabla]\theta$. Taking advantage of Lemma 4.2 in the frequency localized equation (3-13), we get

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \le C_1\|\Delta_j\theta\|_{L^\infty} + \|I_1\|_{L^\infty} + \|I_2\|_{L^\infty} + \|I_3\|_{L^\infty} + \|\Delta_j f\|_{L^\infty}. \tag{4-9}$$

Similarly to the derivation of (3-19) and (3-20), we see that

$$\|I_1\|_{L^\infty} \le C2^{-j\delta}\|u\|_{\dot{C}^\delta} \sum_{|k-j|\le 4} 2^k\|\Delta_k\theta\|_{L^\infty}, \tag{4-10}$$

and

$$\|I_2\|_{L^\infty} \le C2^{-j\delta}\|u\|_{\dot{C}^\delta}\Big(\sum_{k\le j} 2^k\|\Delta_k\theta\|_{L^\infty}\Big), \tag{4-11}$$

and for $\|I_3\|_{L^\infty}$, by virtue of Hölder's inequality and Bernstein's inequality, we find

$$\begin{aligned}\|I_3\|_{L^\infty} &\le \sum_{k\ge j-2}\|\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta)\|_{L^\infty} + \sum_{k\ge j-2}\|\Delta_k u\cdot\nabla\widetilde{\Delta}_k\Delta_j\theta\|_{L^\infty} \\ &\le C\sum_{k\ge j-2}\|\Delta_k u\|_{L^\infty}2^k\|\widetilde{\Delta}_k\theta\|_{L^\infty} \\ &\le C\sum_{k\ge j-2}2^{k(1-\delta)}2^{k\delta}\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k\theta\|_{L^\infty} \\ &\le C\|u\|_{\dot{C}^\delta}\Big(\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty}\Big).\end{aligned} \tag{4-12}$$

Inserting the upper estimates (4-10)–(4-12) into (4-9), we have

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} &\le C_2\|\Delta_j\theta\|_{L^\infty} + \|\Delta_j f\|_{L^\infty} + C\|u\|_{\dot{C}^\delta}2^{-j\delta}\sum_{k\le j+4}2^k\|\Delta_k\theta\|_{L^\infty} \\ &\quad + C\|u\|_{\dot{C}^\delta}\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty}.\end{aligned} \tag{4-13}$$

In particular, by some $j_1 \in \mathbb{N}$ chosen later (see (4-24)) so that $c2^{j_1(\alpha-\sigma)} \ge 2C_2$, or more precisely

$$j_1 \ge \Big[\frac{1}{\alpha-\sigma}\log_2\Big(\frac{2C_2}{c}\Big)\Big] + 1, \tag{4-14}$$

we see that for $j \geq j_1$,

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty}+\frac{c}{2}2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \leq \|\Delta_j f\|_{L^\infty}+C\|u\|_{\dot{C}^\delta}2^{-j\delta}\sum_{k\leq j+4}2^k\|\Delta_k\theta\|_{L^\infty} + C\|u\|_{\dot{C}^\delta}\sum_{k\geq j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty} := \|\Delta_j f\|_{L^\infty}+F_j^1+F_j^2. \tag{4-15}$$

Consequently, Grönwall's inequality guarantees that for every $j \geq j_1$ and $t \geq 0$,

$$\|\Delta_j\theta(t)\|_{L^\infty} \leq e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^\infty} + \int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^\infty}(\tau)+F_j^1(\tau)+F_j^2(\tau))\,\mathrm{d}\tau. \tag{4-16}$$

On the other hand, we have the classical maximum principle (4-1) for (1-1):

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}+\int_0^t\|f(\tau)\|_{L^\infty}\,\mathrm{d}t. \tag{4-17}$$

By arguing as (3-26), we get that for all $t > 0$, $j \in \mathbb{N}$ and $s \in (0, \alpha-\sigma)$,

$$2^{js}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^\infty} \leq C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^\infty}, \tag{4-18}$$

we gather (4-16) and (4-17) to obtain

$$\begin{aligned}\|\theta(t)\|_{C^s} &\approx \|\theta(t)\|_{B^s_{\infty,\infty}} \\ &\leq \sup_{j\leq j_1}2^{js}\|\Delta_j\theta(t)\|_{L^\infty}+\sup_{j\geq j_1}2^{js}\|\Delta_j\theta(t)\|_{L^\infty} \\ &\leq C2^{j_1 s}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_tL^\infty})+C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^\infty} \\ &\quad+\sup_{j\geq j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}(\|\Delta_j f\|_{L^\infty}(\tau)+F_j^1(\tau)+F_j^2(\tau))\mathrm{d}\tau.\end{aligned} \tag{4-19}$$

For the term containing $\|\Delta_j f\|_{L^\infty}$ and F_j^1, in a similar way as obtaining (3-28) and (3-29), we obtain that for every $s \in (0, \alpha-\sigma+\delta)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$\sup_{j\geq j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}\|\Delta_j f\|_{L^\infty}(\tau)\,\mathrm{d}\tau \leq C\|f\|_{L^\infty_t\dot{C}^\delta}\sup_{j\geq j_1}2^{j(s-\alpha+\sigma-\delta)} \leq C\|f\|_{L^\infty_t\dot{C}^\delta}, \tag{4-20}$$

and

$$\sup_{j\geq j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}F_j^1(\tau)\,\mathrm{d}\tau \leq Ct^{-\frac{s}{\alpha-\sigma}}2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big). \tag{4-21}$$

For the term including F_j^2 in (4-19), by using (3-12) again, we similarly get that for all $s\in(1-\delta,\alpha-\sigma)$ and $\delta\in(1-\alpha+\sigma,1)$,

$$\begin{aligned}\sup_{j\geq j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}F_j^2(\tau)\,\mathrm{d}\tau &= C\sup_{j\geq j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot{C}^\delta}2^{js}\Big(\sum_{k\geq j-3}2^{k(1-\delta)}\|\Delta_k\theta(\tau)\|_{L^\infty}\Big)\mathrm{d}\tau\\ &\leq C\|u\|_{L_t^\infty\dot{C}^\delta}\sup_{j\geq j_1}2^{js}\Big(\sum_{k\geq j-3}2^{k(1-\delta-s)}\Big)\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\,\mathrm{d}\tau\\ &\leq C\|u\|_{L^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big)\\ &\qquad\qquad \sup_{j\geq j_1}2^{j(1-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\tau^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau\\ &\leq Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_1(\delta-(1-\alpha+\sigma))}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big).\end{aligned} \tag{4-22}$$

Inserting the estimates (4-20), (4-21), (4-22) into (4-19) yields that for any $1-\delta<s<\alpha-\sigma$ and $0<t\leq T$,

$$\begin{aligned} t^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^s_{\infty,\infty}} &\leq Ct^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L_t^1L^\infty})2^{j_1s}+C_{\alpha,\sigma,s}\|\theta_0\|_{L^\infty}Ct^{\frac{s}{\alpha-\sigma}}\|f\|_{L_t^\infty\dot{C}^\delta}\\ &\qquad +C2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big)\\ &\leq CT^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L_T^1L^\infty})2^{j_1s}+C_{\alpha,\sigma,s}\|\theta_0\|_{L^\infty}+CT^{\frac{s}{\alpha-\sigma}}\|f\|_{L_T^\infty\dot{C}^\delta}\\ &\qquad +C2^{-j_1(\delta-(1-\alpha+\sigma))}\|u\|_{L_T^\infty\dot{C}^\delta}\Big(\sup_{t\in(0,T]}t^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^s_{\infty,\infty}}\Big).\end{aligned} \tag{4-23}$$

Since $1-\alpha+\sigma-\delta>0$, by further choosing j_1 such that $C2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_T^\infty\dot{C}^\delta}\leq\frac{1}{2}$ and (4-14) holds, or more precisely,

$$j_1=\max\bigg\{\bigg[\frac{\log_2 2C\|u\|_{L_T^\infty\dot{C}^\delta}}{\delta-(1-\alpha+\sigma)}\bigg],\bigg[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\bigg],4\bigg\}+1, \tag{4-24}$$

we have that for all $1-\delta<s<\alpha-\sigma$,

$$\sup_{t\in(0,T]}(t^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^s_{\infty,\infty}}) \le C(T+1)(2^{j_1 s}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})+\|f\|_{L^\infty_T\dot{C}^\delta}), \tag{4-25}$$

which implies that for arbitrarily small $t_0\in(0,T)$ and every $s_0\in(1-\delta,\alpha-\sigma)$,

$$\sup_{t\in[t_0,T]}\|\theta(t)\|_{B^{s_0}_{\infty,\infty}}\le Ct_0^{-\frac{s_0}{\alpha-\sigma}}(T+1)(2^{j_1 s}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})+\|f\|_{L^\infty_T\dot{C}^\delta}), \tag{4-26}$$

with j_1 given by (4-24).

Step 2: the estimation of $\|\theta\|_{L^\infty([t_1,T];B^{s_0+s_1}_{\infty,\infty})}$ for $s_0\in(1-\delta,\alpha-\sigma)$, $s_1\in(0,\alpha-\sigma)$ and any $t_1\in(t_0,T)$.

For every $j\ge j_1$ with $j_1\in\mathbb{N}$ satisfying (4-14) chosen later (j_1 is slightly different from that number in Step 1), applying the Grönwall inequality to (4-15) over the time interval $[t_0,t]$ (for $t>t_0>0$) gives

$$\|\Delta_j\theta(t)\|_{L^\infty}\le e^{-(c/2)(t-t_0)2^{j(\alpha-\sigma)}}\|\Delta_j\theta(t_0)\|_{L^\infty} + \int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^\infty}+F_j^1+F_j^2)(\tau)\,d\tau. \tag{4-27}$$

Noticing that for $j\in\mathbb{N}$, $s_0\in(1-\delta,\alpha-\sigma)$ and all $s\in(0,\alpha-\sigma)$,

$$\begin{aligned} e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}\|\Delta_j\theta(t_0)\|_{L^\infty} &\le e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}}2^{js}\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}\\ &\le C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}}\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}, \end{aligned} \tag{4-28}$$

by arguing as (4-19) we obtain that for all $t\ge t_0>0$,

$$\begin{aligned} \|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}} &\le \sup_{j\le j_1}2^{j(s_0+s)}\|\Delta_j\theta(t)\|_{L^\infty}+\sup_{j\ge j_1}2^{j(s_0+s)}\|\Delta_j\theta(t)\|_{L^\infty}\\ &\le C2^{j_1(s_0+s)}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_tL^\infty})+C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}}\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}\\ &\quad+\sup_{j\ge j_1}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}(\|\Delta_j f\|_{L^\infty}(\tau)+F_j^1(\tau)+F_j^2(\tau))\,d\tau. \end{aligned} \tag{4-29}$$

In a similar fashion as the estimation of (3-38), (3-39)–(3-40), we find that for every $s\in(0,\alpha-\sigma)$ and $s_0+s<\delta+\alpha-\sigma$,

$$\sup_{j\ge j_1}\int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}\|\Delta_j f\|_{L^\infty}(\tau)\,d\tau\le C\|f\|_{L^\infty_t\dot{B}^\delta_{\infty,\infty}}, \tag{4-30}$$

and

$$\sup_{j\ge j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^1(\tau)\,\mathrm{d}\tau \le \frac{C\|u\|_{L_t^\infty\dot{C}^\delta}}{2^{j_1(\delta-(1-\alpha+\sigma))/2}}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}} \tag{4-31}$$

if $0<s_0+s\le 1$, and

$$\sup_{j\ge j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^1(\tau)\,\mathrm{d}\tau \le \frac{C\|u\|_{L_t^\infty\dot{C}^\delta}}{2^{j_1(\delta-(s_0+s-\alpha+\sigma))}}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}} \tag{4-32}$$

if $1<s_0+s<\delta+\alpha-\sigma$. For the term including F_j^2 in (4-29), by using (3-12) again and the fact that $s_0\in(1-\delta,\alpha-\sigma)$, we get that for all $s\in(0,\alpha-\sigma)$,

$$\begin{aligned}
&\sup_{j\ge j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^2(\tau)\mathrm{d}\tau \\
&= C\sup_{j\ge j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot{C}^\delta}2^{j(s_0+s)}\Big(\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta(\tau)\|_{L^\infty}\Big)\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty\dot{C}^\delta}\sup_{j\ge j_1}2^{j(s_0+s)}\Big(\sum_{k\ge j-3}2^{k(1-\delta-s_0-s)}\Big)\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\Big) \\
&\qquad\qquad \sup_{j\ge j_1}2^{j(1-\delta)}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_1(\delta-(1-\alpha+\sigma))}.
\end{aligned} \tag{4-33}$$

Plugging the estimates (4-30)–(4-33) into (4-29), and in a similar way as obtaining (4-23), we have that for every $t\in(t_0,T]$, $s\in(0,\alpha-\sigma)$ and $s_0+s<\delta+\alpha-\sigma$,

$$\begin{aligned}
(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}} &\le CT^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s)} \\
&\quad + C_{\alpha,\sigma,s}\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}} + CT^{\frac{s}{\alpha-\sigma}}\|f\|_{L_t^\infty\dot{C}^\delta} \\
&\quad + \text{additional term}
\end{aligned} \tag{4-34}$$

where the additional term is given by

$$\frac{C\|u\|_{L_T^\infty\dot{C}^\delta}}{2^{j_1(\delta-(1-\alpha+\sigma))/2}}\Big(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}}\Big)$$

if $s_0+s\le 1$, and

$$\frac{C\|u\|_{L^\infty_T\dot{C}^\delta}}{2^{j_1(\delta+\alpha-\sigma-(s_0+s))}}\Big(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}}\Big)$$

if $1<s_0+s<\delta+\alpha-\sigma$. Hence we choose $j_1\in\mathbb{N}$ as

$$j_1=\max\bigg\{\bigg[\frac{2\log_2 2C\|u\|_{L^\infty_T\dot{C}^\delta}}{\delta-(1-\alpha+\sigma)}\bigg],\Big[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\Big],4\bigg\}+1 \tag{4-35}$$

if $s_0+s\le 1$, and

$$j_1=\max\bigg\{\bigg[\frac{\log_2 2C\|u\|_{L^\infty_T\dot{C}^\delta}}{\delta+\alpha-\sigma-(s_0+s)}\bigg],\Big[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\Big],4\bigg\}+1 \tag{4-36}$$

if $1<s_0+s<\delta+\alpha-\sigma$. We thus find that for all $s\in(0,\alpha-\sigma)$ and $s_0+s<\delta+\alpha-\sigma$,

$$\begin{aligned}\sup_{t\in(t_0,T]}((t-t_0)^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}})&\le C(T+1)(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s)}\\&\quad+C\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}+C(T+1)\|f\|_{L^\infty_T\dot{C}^\delta},\end{aligned}\tag{4-37}$$

which specially guarantees that for any $t_1>t_0>0$ (which may be arbitrarily close to t_0) and every $s_0\in(1-\delta,\alpha-\sigma)$, $s_1\in(0,\alpha-\sigma)$ satisfying $s_0+s_1<\delta+\alpha-\sigma$,

$$\begin{aligned}&\sup_{t\in[t_1,T]}\|\theta(t)\|_{B^{s_0+s_1}_{\infty,\infty}}\\&\quad\le C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}((T+1)(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s_1)}+\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}})\\&\qquad+C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T\dot{C}^\delta},\end{aligned}\tag{4-38}$$

with j_1 given by (4-35)–(4-36).

Step 3:the estimation of $\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma})}$ for some $\gamma>0$ and any $\tilde{t}\in(0,T)$.

If $\alpha-\sigma\in\left(\frac{1}{2},1\right)$, we can select appropriate $s_0\in(1-\delta,\alpha-\sigma)$, $s_1\in(0,\alpha-\sigma)$ so that $1<s_0+s_1<\delta+\alpha-\sigma$, thus from (4-38) we obtain that for $\gamma=s_0+s_1-1>0$,

$$\sup_{t\in[t_1,T]}\|\theta(t)\|_{C^{1,\gamma}}\approx\sup_{t\in[t_1,T]}\|\theta(t)\|_{B^{s_0+s_1}_{\infty,\infty}}\le C,$$

with C the bound on the right-hand side of (4-38).

For the remained scope $\alpha-\sigma\in(0,\frac{1}{2}]$, we have to iterate the above procedure in Step 2 for more times. Assume that for some small number $t_k>0$, $k\in\mathbb{N}$, we have a finite bound on $\|\theta(t_k)\|_{B^{s_0+s_1+\cdots+s_k}_{\infty,\infty}}$ with $s_0\in(1-\delta,\alpha-\sigma)$, $s_1,\ldots,s_k\in(0,\alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_k\le 1$, then by arguing as (4-38), we infer that for any

$t_{k+1} > t_k,\ s_{k+1} \in (1-\delta, \alpha-\sigma)$ satisfying $s_0 + s_1 + \cdots + s_{k+1} < \delta + \alpha - \sigma$,

$$\begin{aligned} \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{\infty,\infty}} \le C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}} \\ \big((T+1)(\|\theta_0\|_{L^\infty} + \|f\|_{L^1_T L^\infty})2^{j_1(\sum_{i=0}^{k+1} s_i)} + \|\theta(t_k)\|_{B^{\sum_{i=0}^k s_i}_{\infty,\infty}}\big) \\ + C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot C^\delta}, \end{aligned} \tag{4-39}$$

where j_1 is also given by (4-35)–(4-36) with s_0+s_1 replaced by $s_0+s_1+\cdots+s_{k+1}$. Hence if $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, $k \in \mathbb{N}^+$, we can choose appropriate numbers $s_0, s_1, \dots, s_{k+1} \in (1-\delta, \alpha-\sigma)$ so that $1 < s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$, and by repeating the above process for $(k+1)$-times, we deduce that for $\gamma = s_0+s_1+\cdots+s_{k+1}-1 > 0$,

$$\begin{aligned} \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{C^{1,\gamma}} &\approx \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{\infty,\infty}} \\ &\le C\bigg(\prod_{i=0}^{k}(t_{i+1}-t_i)^{-\frac{s_{i+1}}{\alpha-\sigma}} t_0^{-\frac{s_0}{\alpha-\sigma}}\bigg)(\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \end{aligned} \tag{4-40}$$

with C a finite constant depending on α, σ, δ, T, d and $\|u\|_{L^\infty_T \dot C^\delta}$.

Hence for every $\alpha \in (0,1]$, $\sigma \in [0,\alpha)$, and for any $\tilde t \in (0,T)$, there is some $k \in \mathbb{N}$ so that $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, and we can choose $t_i = (i+1)/(k+2)\tilde t$ for $i = 0, 1, \dots, k+1$ and appropriate numbers $s_0 \in (1-\delta, \alpha-\sigma)$, $s_1, \dots, s_{k+1} \in (0, \alpha-\sigma)$ such that $1 < s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$, thus from (4-40) we deduce that for some $\gamma > 0$,

$$\|\theta\|_{L^\infty([\tilde t,T];C^{1,\gamma}(\mathbb{R}^d))} \le C\tilde t^{-(\gamma+1)/(\alpha-\sigma)}(\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \tag{4-41}$$

with the constant C depending only on α, σ, δ, T, d and $\|u\|_{L^\infty_T \dot C^\delta}$.

Step 4: the estimation of $\|\theta\|_{L^\infty([\tilde t,T];C^{1,\gamma})}$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$ and any $t' \in (0,T)$.

After obtaining the estimate of $\|\theta\|_{L^\infty([t'/2,T];B^{\tilde s}_{\infty,\infty})}$ with some $1 < \tilde s < 1+\gamma$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$, we can repeat the deduction in Steps 1–2 for several times and due to the increment of regularity index s at each time belonging to $(0, \alpha-\sigma)$, we can derive an upper bound of $\|\theta\|_{L^\infty([t',T];B^{1+\gamma}_{\infty,\infty})}$ by establishing (4-41) with t' in place of $\tilde t$.

4C. ***Uniform-in-ϵ differentiability estimates of the regularized system.*** We consider the approximate system

$$\begin{cases} \partial_t\theta + (u_\epsilon\cdot\nabla)\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_\epsilon, \\ u_\epsilon := \phi_\epsilon * u, \quad f_\epsilon := \phi_\epsilon * f, \\ \theta|_{t=0} = \theta_{0,\epsilon} := \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}(0)}). \end{cases} \tag{4-42}$$

Here $1_\Omega(x)$ is the standard indicator function on the set Ω and $\phi_\epsilon = \epsilon^{-d}\phi(\epsilon^{-1}x) \in C_c^\infty(\mathbb{R}^d)$ is the function introduced in Section 3C.

Due to $\theta_0 \in C_0(\mathbb{R}^d)$, we see that $\theta_{0,\epsilon} = \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}(0)})$ is smooth for every $\epsilon > 0$, and $\|\theta_{0,\epsilon}\|_{H^s(\mathbb{R}^d)} \lesssim_\epsilon \|\theta_0\|_{L^\infty(\mathbb{R}^d)}$ for all $s \ge 0$. Similarly from $u \in L^\infty([0,T]; C^\delta(\mathbb{R}^d))$ and $f(t) \in C^\delta \cap L^2(\mathbb{R}^d)$ for every $t \in [0,T]$, we get $u_\epsilon \in L^\infty([0,T]; C^s(\mathbb{R}^d))$ for all $s \ge \delta$ and $f_\epsilon \in L^\infty([0,T]; H^s(\mathbb{R}^d))$ for all $s \ge 0$. Hence, for every $\epsilon > 0$, by the classical method (e.g., [Miao and Xue 2015, Proposition 7.1]), we obtain an approximate solution $\theta^{(\epsilon)} \in C([0,T]; H^s(\mathbb{R}^d)) \cap C^1((0,T]; C_b^\infty(\mathbb{R}^d))$, $s > d/2+1$ for the system (4-42).

Since we have the uniform-in-ϵ estimates that $\|\theta_{0,\epsilon}\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$, $\|u_\epsilon\|_{L^\infty_T C^\delta} \le \|u\|_{L^\infty_T C^\delta}$ and $\|f_\epsilon\|_{L^\infty_T C^\delta} \le \|f\|_{L^\infty_T C^\delta}$, we consider the equation of $\theta^{(\epsilon)}$ and by arguing as (4-41) and Step 4 in the above subsection, we can derive the uniform-in-ϵ estimate of $\|\theta^{(\epsilon)}\|_{L^\infty([t',T]; C^{1,\gamma}(\mathbb{R}^d))}$ with any $\gamma \in (0, \delta+\alpha-\sigma-1)$ and $t' \in (0,T)$.

Therefore, we have finished the proof of Theorem 1.2.

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