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Yudovich type solution for the 2D inviscid Boussinesq system with critical and supercritical dissipation

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Abstract

In this paper we consider the Yudovich type solution of the 2D inviscid Boussinesq system with critical and supercritical dissipation. For the critical case, we show that the system admits a global and unique Yudovich type solution; for the supercritical case, we prove the local and unique existence of Yudovich type solution, and the global result under a smallness condition of θ_0. We also give a refined blowup criterion in the supercritical case.

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1. Introduction

In this paper we address the Cauchy problem of the following two-dimensional Boussinesq system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu |D|^\alpha u + \nabla P = \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\beta \theta = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \qquad \theta|_{t=0} = \theta_0, \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$, $\nu, \kappa \geqslant 0$, $(\alpha, \beta) \in]0, 2]^2$, $e_2 = (0, 1)$ the canonical vector, and $|D|^\alpha = (-\Delta)^{\alpha/2}$ is defined by the Fourier transform $\widehat{|D|^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$. The unknowns are the velocity vector field $u = (u_1, u_2) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, the scalar pressure $P : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$, and the scalar quantity $\theta : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ which denotes the density field in the context of geostrophic fluids or the temperature field in the thermal convection. Boussinesq systems are widely used to model the geophysical flows such as atmospheric fronts and oceanic circulation, and also play an important role in the study of Rayleigh–Bénard convection (cf. [4,26]). Besides, the 2D Boussinesq system and its fractional generation have the mathematical significance: they are the two-dimensional models which retain the key vortex-stretching mechanism as the 3D Navier–Stokes/Euler equations; indeed, as pointed out in many literatures (e.g. [24]), the totally inviscid Boussinesq case (1.1) (i.e. $\nu = \kappa = 0$) shares a deep formal analog with the 3D axisymmetric Euler system with swirl.

Due to the physical background and mathematical relevance, recently there have been intense works concerned on the Boussinesq systems (e.g. [2,3,5,6,11–17,19,20,25,31] and references therein), and here for our purpose we only recall the notable works about the 2D inviscid Boussinesq system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\beta \theta = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \qquad \theta|_{t=0} = \theta_0. \end{cases} \tag{1.2}$$

For $\kappa = 0$, this is the most difficult case for mathematical study; indeed, since the vorticity $\omega = curl\, u = \partial_1 u_2 - \partial_2 u_1$ solves

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta, \tag{1.3}$$

thus to control the key L^∞-norm of ω, we need that $\int_0^T \|\partial_1 \theta\|_{L^\infty} \mathrm{d}t < \infty$, but no such *a priori* bound is known. Up to now, we know the local well-posedness in various functional frameworks and the blowup criteria (cf. [6,23,11]), and lower bound for the life-span of the solution (cf. [11]). Note that in all these settings, the velocity field is at least Lipschitzian. For the dissipative cases $\kappa > 0$, $\beta > 0$, one can resort to the dissipation effect to gain some benefit. For the Laplacian dissipation case (i.e. $\kappa > 0$, $\beta = 2$), Chae [5] proved the global well-posedness of the smooth solution for (1.2) with $(u_0, \theta_0) \in H^s \times H^s$, $s > 2$. Later, Hmidi and Keraani [16] proved the global result for (1.2) with the rough data that $u_0 \in B^{1+2/p}_{p,1}$ ($p \in]2, \infty]$) and θ_0 belongs to a suitable Lebesgue space, which in some sense extended the work of Vishik [27] on the 2D Euler system. Moreover, as a natural extention of Yudovich [32] on the 2D Euler system to treat the non-Lipschitzian velocity field, Danchin and Paicu [13] showed that (1.2) has a global unique solution for the Yudovich type data, that is, the initial velocity u_0 has finite energy and bounded vorticity and θ_0 belongs to $L^2 \cap B^{-1}_{\infty,1}$ (this additional assumption on $B^{-1}_{\infty,1}$ is indeed optimal); note that, this global result in fact holds true for a more general system called the 2D inviscid Bénard system. For $\beta \in]1,2[$, Hmidi and Zerguine [18] followed the idea of [16] to show the global well-posedness of (1.2) with the rough data; Wu and the second author [30] proved the global unique solution of (1.2) (also the 2D inviscid Bénard system) for the Yudovich type data which naturally generalized the conditions in [13], (see [29] for a similar result and other issues). We point out that in view of the maximal regularity estimate for the equation of θ used in [18], *the case $\beta = 1$, $\beta > 1$ and $\beta < 1$ can be called as the critical, subcritical and supercritical case respectively*. For the subtle critical case $\beta = 1$, by deeply developing the structures of the coupling system about (θ, ω), Hmidi, Keraani, and Rousset [17] proved the global result for (1.2) with the rough data $u_0 \in B^1_{\infty,1} \cap \dot{W}^{1,p}$ and $\theta_0 \in B^0_{\infty,1} \cap L^p$ ($p \in]2,\infty[$). On the other hand, if the dissipation term $|D|^\beta\theta$ in the equation of θ is replaced by the partial horizontal dissipation $\partial_1^2\theta$ or vertical dissipation $\partial_2^2\theta$ in the Boussinesq system (1.2), we refer the readers to the interesting works [14,22,3].

In this paper, partially continuing the works [32,13,30], we are devoted to treat the Yudovich type solution for the 2D inviscid Boussinesq system (1.2) with critical and supercritical dissipation. In the sequel we assume $\kappa = 1$ for simplicity. Our main result reads as follows.

Theorem 1.1. *Let $\nu = 0$, $\kappa = 1$, $\beta \in]0,1]$. Assume that $p \in [2,\infty[$, $\theta_0 \in L^2(\mathbb{R}^2) \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}$ and $u_0 \in L^2(\mathbb{R}^2)$ is a divergence-free vector field with the vorticity $\omega_0 = \partial_1 u_{0,2} - \partial_2 u_{0,1}$ belonging to $L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then*

$$\begin{cases} \text{if } \beta = 1, & \text{for every } T > 0, \\ \text{if } \beta \in]0,1[, & \text{for some } T > 0 \text{ depending only on } \beta,\ \|\omega_0\|_{L^\infty \cap L^p},\ \|\theta_0\|_{B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}}, \end{cases}$$

the 2D inviscid Boussinesq system (1.2) has a unique solution (u, ω, θ) on $[0,T]$ which satisfies

$$u \in C^{0,1}\big([0,T]; L^2(\mathbb{R}^2)\big), \qquad \omega \in L^\infty\big([0,T]; L^p \cap L^\infty\big), \quad \text{and} \tag{1.4}$$

$$\theta \in C\big([0,T]; L^2 \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}\big) \cap L^1\big([0,T]; B^1_{\infty,1}\big). \tag{1.5}$$

Besides, for the supercritical regime $\beta \in]0,1[$, θ_0 is additionally such that $\theta_0 \in L^{p_0} \cap L^{p_1}$ with $p_0 \in [1, \frac{2p}{2+p\beta}[$ and $p_1 \in [2, \frac{2}{\beta}[$, then the above constant T can be arbitrarily large provided that

$$\|\theta_0\|_{L^{p_0} \cap L^{p_1} \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}} \leqslant c_0 \tag{1.6}$$

for some positive constant c_0 depending only on $\|\omega_0\|_{L^\infty \cap L^p}$ and β.

Remark 1.2. The smallness condition (1.6) in the supercritical case is independent of T, and there is no smallness restriction on the velocity field. Note that for $\theta = \theta_0 \equiv 0$, the inviscid

Boussinesq system (1.2) reduces to the 2D Euler system, thus the global result under the condition (1.6) can be viewed as a perturbation result of the Yudovich solution for 2D Euler system. But for the totally inviscid case $\{\nu = \kappa = 0\}$, so far it remains open to obtain an analogous result.

Remark 1.3. If we control the term $\|\nabla S_3 u\|_{L^\infty}$ in (4.11) by $\|u\|_{L^2}$ instead of $\|\omega\|_{L^p}$, we can also get the similar results as those stated in Theorem 1.1. We just note that in the supercritical case, the time $T > 0$ will depend only on $\{\beta, \|u_0\|_{L^2}, \|\omega_0\|_{L^\infty}, \|\theta_0\|_{L^2\cap B^{1-\beta}_{\infty,1}}\}$, and if additionally $\theta_0 \in L^{p_0}$ with $p_0 \in [1, \frac{2}{1+\beta}[$ (so that (4.5) holds), then T can be arbitrarily large as long as $\|\theta_0\|_{L^{p_0}\cap B^{1-\beta}_{\infty,1}} \leqslant c_0$ with c_0 a small constant depending only on β, $\|u_0\|_{L^2}$ and $\|\omega_0\|_{L^\infty}$.

We also have the following refined blowup criterion in the supercritical case.

Proposition 1.4. *Let $\beta \in]0,1[$, $T^* > 0$ be the maximal existence time of the Yudovich type solution (u,ω,θ) constructed in Theorem 1.1. If $T^* < \infty$, then we necessarily have*

$$\int_0^{T^*} \|\theta(t)\|_{B^{1-\beta}_{\infty,2}}\,\mathrm{d}t = \infty. \tag{1.7}$$

Remark 1.5. If we neglect the convection term in the velocity equation of the Boussinesq system (1.2), then the system reduces to the following system

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = 0, \\ u = u_0 + \int_0^t \mathbb{P}(\theta e_2)(\tau)\,\mathrm{d}\tau, \\ u|_{t=0} = u_0, \qquad \operatorname{div} u_0 = 0, \qquad \theta|_{t=0} = \theta_0, \end{cases} \tag{1.8}$$

where $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$ is the Leray operator which maps into the divergence-free field. Although (1.8) has a simpler form than (1.2), it is still not clear to show the global regularity of the solution at the case $\beta \in]0,1[$. Indeed, for the transport–diffusion equation

$$\partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = 0, \quad \beta \in]0,1[,$$

a typical regularity criterion is as follows (e.g. [8,9]): one needs that $u \in L^\infty_T C^{1-\beta}$ to ensure that L^∞-solution is Hölder continuous on $]0,T]$, while one needs $u \in L^\infty_T C^\gamma$ ($\gamma > 1-\beta$) to guarantee that Hölder continuous solution is smooth on $]0,T]$. A direct consequence of this criterion is that we *a priori* need

$$\sup_{t\in]0,T]} \left\| \int_0^t \mathbb{P}(\theta e_2)\,\mathrm{d}\tau \right\|_{C^\gamma} \leqslant C \int_0^T \|\theta(t)\|_{C^\gamma}\,\mathrm{d}t < \infty, \quad \gamma > 1-\beta, \tag{1.9}$$

to ensure the smoothness of the solution up to T. Note that, though they are in the different situations, the criterion (1.7) (at least formally) is slightly better than the criterion (1.9).

Compared with [30,29] (where the critical case was unsolved), the new ingredient in the proof of Theorem 1.1 is to apply the frequency-localized maximum principle Lemma 2.3, which sufficiently develops the dissipation effect of transport–diffusion equation in the point-wise sense; indeed, combining it with the generalized Bernstein inequality and the commutator estimate (3.1), we can prove the *a priori* bound that for every $p \in [2, \infty[$, $s \leqslant 1$ and $r \geqslant 0$,

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)} \lesssim \|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\left(\sum_{q\in\mathbb{N}} 2^{q(s-\beta)}(q+1)^{r+1}\|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)}\right) + \text{l.o.t.}$$

By iteratively using the above estimate (the times depending on β), we can estimate $\|\omega\|_{L^\infty_t(L^\infty\cap L^p)} + \|\theta\|_{L^1_t(B^1_{\infty,1}\cap B^1_{p,1})}$ by a polynomial $F(\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}) := \sum_{0\leqslant k\leqslant k_0} c_k(t) \times \|\omega\|^k_{L^\infty_t(L^\infty\cap L^p)}$ with $c_k(t)$ depending on the initial data and t (see (4.26)–(4.27)), and from which we obtain the local results. Especially, by virtue of Lemma 2.2, with another suitable assumption on θ_0, we can show that $c_k(t) \leqslant C_k$ independent of t, which leads to the global result under the smallness condition. For the proof of Proposition 1.4 and its analogy in the critical case, we adopt the elegant method of [17] to use the hidden structure; more precisely, let $\mathcal{R}_\beta = \partial_1|D|^{-\beta}$ and $\Gamma = \omega + \mathcal{R}_\beta\theta$, we find

$$\begin{aligned}\partial_t\Gamma + u\cdot\nabla\Gamma &= -\mathcal{R}_\beta(u\cdot\nabla\theta) + u\cdot\nabla\mathcal{R}_\beta\theta\\ &= -[\mathcal{R}_\beta, u\cdot\nabla]\theta;\end{aligned}$$

then thanks to the commutator estimates involving $\mathcal{R}_\beta$, the regularization estimate of θ and the relation $\omega = \Gamma - \mathcal{R}_\beta\theta$, we manage to establish the expected blowup criterion. The global regularity in the critical case is a consequence of this criterion and an *a priori* estimate in [17].

The paper is organized as follows. Section 2 presents some preparatory results. In Section 3, we show some auxiliary commutator estimates. We prove Theorem 1.1 and Proposition 1.4 in Section 4 and Subsection 4.4 respectively.

2. Preliminary

In this preparatory section, we introduce some common notations and the definition of Besov spaces, and compile some useful lemmas.

Throughout this paper the following notations will be used.

⋄ C (or C_0) stands for a positive constant which may be different from line to line. $X \lesssim Y$ means that there exists a positive harmless constant C such that $X \leqslant CY$. We use the sub-indices (like C_s or $\lesssim_s$) to indicate the parameter dependence of the constant C.

⋄ $\mathcal{D}(\mathbb{R}^d) = C^\infty_c(\mathbb{R}^d)$ denotes the space of test functions, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class, and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

⋄ We use $\mathcal{F}f$ or $\widehat{f}$ to denote the Fourier transform of a tempered distribution f, and $\mathcal{F}^{-1}f$ denotes its Fourier inverse transform.

⋄ For A, B two operators, denote by $[A, B]$ the commutator operator $AB - BA$.

In order to define Besov space, we need the following dyadic partition of unity (cf. [1]). Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{D}(\mathbb{R}^d)$ which are supported respectively in the ball $\{\xi \in \mathbb{R}^d\colon |\xi| \leqslant \frac{4}{3}\}$ and the shell $\{\xi \in \mathbb{R}^d\colon \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{q \geqslant 0} \varphi\big(2^{-q}\xi\big) = 1, \quad \forall \xi \in \mathbb{R}^d.$$

Let $h = \mathcal{F}^{-1}(\varphi)$, $\tilde{h} = \mathcal{F}^{-1}(\chi)$, then for $f \in \mathcal{S}'(\mathbb{R}^d)$, we define the Littlewood–Paley operators

$$\Delta_{-1} f = \chi(D) f = \int_{\mathbb{R}^d} \tilde{h}(y) f(x-y)\,\mathrm{d}y; \qquad \Delta_q f = 0, \quad \text{for } q \leqslant -2;$$

$$\Delta_q f = \varphi\big(2^{-q}D\big) f = 2^{qd} \int_{\mathbb{R}^d} h\big(2^q y\big) f(x-y)\,\mathrm{d}y, \quad \text{for } q \in \mathbb{N};$$

$$S_q f = \chi\big(2^{-q}D\big) f = \sum_{-1 \leqslant j \leqslant q-1} \Delta_j f = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}\big(2^q y\big) f(x-y)\,\mathrm{d}y, \quad \text{for } q \in \mathbb{N}. \tag{2.1}$$

With the choice of χ, φ, it is obvious to see that

$$\Delta_q \Delta_j f = 0, \quad \text{for } |q-j| \geqslant 2;$$
$$\Delta_q(S_{j-1} f \Delta_j g) = 0, \quad \text{for } |q-j| \geqslant 5.$$

Now we give the definition of Besov spaces. Let $(p,r) \in [1,\infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space $B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{B^s_{p,r}} < \infty$ with

$$\|f\|_{B^s_{p,r}} = \begin{cases} \sup_{q \geqslant -1} 2^{qs} \|\Delta_q f\|_{L^p}, & \text{for } r = \infty, \\ (\sum_{q \geqslant -1} 2^{qsr} \|\Delta_q f\|^r_{L^p})^{1/r}, & \text{for } r \in [1, \infty[. \end{cases}$$

We also introduce two kinds of space–time Besov spaces. The first one is the classical space–time Besov space $L^\rho([0,T], B^s_{p,r})$, abbreviated by $L^\rho_T B^s_{p,r}$, which is defined in the usual way. The second one is the Chemin–Lerner mixed space–time Besov space $\widetilde{L}^\rho([0,T], B^s_{p,r})$, abbreviated by $\widetilde{L}^\rho_T B^s_{p,r}$, which is the set of tempered distribution f satisfying

$$\|f\|_{\widetilde{L}^\rho_T B^s_{p,r}} = \big\| \big\{2^{qs} \|\Delta_q f\|_{L^\rho_T L^p}\big\}_{q \geqslant -1} \big\|_{\ell^r} < \infty.$$

Due to Minkowski's inequality, we immediately obtain

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r}, \quad \text{if } r \geqslant \rho, \quad \text{and} \quad \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \quad \text{if } \rho \geqslant r.$$

Bernstein's inequality is fundamental in the analysis involving Besov spaces (cf. [1]).

Lemma 2.1. *Let $f \in L^a$, $1 \leqslant a \leqslant b \leqslant \infty$. Then for every $(k,q) \in \mathbb{N}^2$ there exists a constant $C > 0$ such that*

$$\sup_{|\alpha|=k} \left\| \partial^\alpha S_q f \right\|_{L^b} \leqslant C 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))} \|S_q f\|_{L^a},$$

$$C^{-1} 2^{qk} \|\Delta_q f\|_{L^a} \leqslant \sup_{|\alpha|=k} \left\| \partial^\alpha \Delta_q f \right\|_{L^a} \leqslant C 2^{qk} \|\Delta_q f\|_{L^a}.$$

The L^p-estimate for the transport(–diffusion) equation is very useful.

Lemma 2.2. *Let u be a smooth divergence-free vector field in $\mathbb{R}^d$ $(d \geqslant 2)$ and θ be a smooth solution of the following transport(–diffusion) equation*

$$\partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\alpha \theta = f, \quad \operatorname{div} u = 0, \quad \theta|_{t=0} = \theta_0, \quad \alpha \in]0,2[, \tag{2.2}$$

with $\kappa \geqslant 0$. Then the following statements hold.

(1) *For every $p \in [1,\infty]$ and $t \in \mathbb{R}^+$, we have*

$$\left\| \theta(t) \right\|_{L^p} \leqslant \|\theta_0\|_{L^p} + \int_0^t \left\| f(\tau) \right\|_{L^p} \mathrm{d}\tau. \tag{2.3}$$

(2) *If $\kappa > 0$ and $f \equiv 0$ in Eq. (2.2), then for every $p \in [1,\infty[$, $r \in [1,p]$ and $t \in \mathbb{R}^+$, we have*

$$\left\| \theta(t) \right\|_{L^p} \leqslant \frac{C \|\theta_0\|_{L^p \cap L^r}}{(1+t)^{\frac{1}{\alpha}(\frac{d}{r}-\frac{d}{p})}}, \tag{2.4}$$

where C is positive constant depending only on p, r, α, d, κ.
Besides, if $p = \infty$ and $r \in [2,\infty]$, (2.4) also holds true.

Proof of Lemma 2.2. The proof of (2.3) is classical, see [10]. For (2.4), although some typical cases may have occurred in the literatures, we here present the details of the proof for completeness. For $p = \infty$ and $r \in [2,\infty]$, this corresponds to [30, Prop. 3.1]. For $p \in [2,\infty[$ and $r < p$ (the case $r = p$ reduces to (2.3)), by taking the inner product of $|\theta|^{p-2}\theta$ with the homogeneous transport–diffusion equation (2.2) and using the divergence-free condition, we see that

$$\begin{aligned} 0 &\geqslant \frac{1}{p} \frac{d}{dt} \left\| \theta(t) \right\|_{L^p}^p + \kappa \int_{\mathbb{R}^d} |D|^\alpha \theta(t,x) \, |\theta|^{p-2} \theta(t,x) \, \mathrm{d}x \\ &\geqslant \frac{1}{p} \frac{d}{dt} \left\| \theta(t) \right\|_{L^p}^p + \frac{2\kappa}{p} \left\| |D|^{\frac{\alpha}{2}} \left(|\theta|^{\frac{p}{2}} \right) \right\|_{L^2}^2, \end{aligned}$$

where in the last line we also have used [21, Lem. 3.3]. The Sobolev embedding $\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ leads to

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + C_1\|\theta(t)\|_{L^{\frac{pd}{d-\alpha}}}^p \leqslant 0.$$

By virtue of interpolation and (2.3), we find

$$\|\theta(t)\|_{L^p} \leqslant C\|\theta(t)\|_{L^r}^{\delta}\|\theta(t)\|_{L^{\frac{pd}{d-\alpha}}}^{1-\delta} \leqslant C\|\theta_0\|_{L^r}^{\delta}\|\theta(t)\|_{L^{\frac{pd}{d-\alpha}}}^{1-\delta}$$

with $\delta = \frac{r\alpha}{(p-r)d+r\alpha} \in]0,1[$. Thus

$$\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + C_2\frac{\|\theta(t)\|_{L^p}^{p/(1-\delta)}}{\|\theta_0\|_{L^r}^{(\delta p)/(1-\delta)}} \leqslant 0,$$

that is,

$$\frac{d}{dt}\|\theta(t)\|_{L^p} \leqslant -C_2\frac{\|\theta(t)\|_{L^p}^{1+\delta p/(1-\delta)}}{\|\theta_0\|_{L^r}^{(\delta p)/(1-\delta)}}.$$

Direct computation yields

$$\|\theta(t)\|_{L^p} \leqslant C_3\frac{\|\theta_0\|_{L^p\cap L^r}}{(1+t)^{(1-\delta)/(\delta p)}} = C_3\frac{\|\theta_0\|_{L^p\cap L^r}}{(1+t)^{\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{p})}}.$$

For $p \in [1,2[$, (2.4) can be achieved by interpolation. □

The following localized maximum principle plays a key role in the main proof (cf. [28, Thm. 3.3]).

Lemma 2.3. *Let θ, u, f be smooth solutions with $\Delta_q\theta(t) \in C_0(\mathbb{R}^2)$ for $t > 0$ and $q \in \mathbb{N}$, and let $\Delta_q\theta$ be a solution of the following transport–diffusion equation*

$$\partial_t\Delta_q\theta + u\cdot\nabla\Delta_q\theta + |D|^\alpha\Delta_q\theta = f, \quad \alpha \in]0,2[.$$

Then there exists an absolute positive constant c independent of θ, u, f, q such that for a.e. $t > 0$,

$$\frac{d}{dt}\|\Delta_q\theta(t)\|_{L^\infty} + c2^{q\alpha}\|\Delta_q\theta(t)\|_{L^\infty} \leqslant \|f(t)\|_{L^\infty}.$$

The next lemma is useful in dealing with the commutator terms (cf. [17]).

Lemma 2.4. *Let $p \in [1,\infty]$, $m \geqslant p$, $\bar{m} = \frac{m}{m-1}$ be the dual number. Then,*

$$\|h*(fg) - f(h*g)\|_{L^p} \leqslant \|xh\|_{L^{\bar{m}}}\|\nabla f\|_{L^p}\|g\|_{L^m}. \tag{2.5}$$

3. Commutator estimates

In this section we show some commutator estimates.

Lemma 3.1. *Assume that $u = (u_1, \dots, u_d)$ is a smooth divergence-free vector field of $\mathbb{R}^d$ $(d \geqslant 2)$ with its vorticity ω, and θ is a smooth function.*

(1) *Let $s \in]-1, 1[$, $r \geqslant 0$, $p \in [1, \infty]$, then*

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \big\|[\Delta_q, u\cdot\nabla]\theta\big\|_{L^p} \lesssim_{r,s} \big(\|\nabla S_3 u\|_{L^\infty} + \|\omega\|_{L^\infty}\big)\Big(\|S_3\theta\|_{L^p} + \sum_{q\in\mathbb{N}} 2^{qs}(q+1)^{r+1}\|\Delta_q\theta\|_{L^p}\Big), \tag{3.1}$$

where S_3 is the low-frequency cut-off operator.

(2) *Let $s \in]-1, 1[$, $p \in [1, \infty]$, $r \in [1, \infty]$, then*

$$\Big\|\big\{2^{qs}\big\|[\Delta_q, u\cdot\nabla]\theta\big\|_{L^p}\big\}_{q\in\mathbb{N}}\Big\|_{\ell^r} \lesssim_s \|\nabla u\|_{L^p}\|\theta\|_{B^s_{\infty,r}}. \tag{3.2}$$

Proof of Lemma 3.1. The Bony's decomposition yields

$$\begin{aligned}
[\Delta_q, u\cdot\nabla]\theta &= \sum_{j\in\mathbb{N}}[\Delta_q, S_{j-1}u\cdot\nabla]\Delta_j\theta + \sum_{j\in\mathbb{N}}[\Delta_q, \Delta_j u\cdot\nabla]S_{j-1}\theta \\
&\quad + \sum_{j\geqslant -1}[\Delta_q, \Delta_j u\cdot\nabla]\widetilde{\Delta}_j\theta \\
&:= \mathrm{I}_q + \mathrm{II}_q + \mathrm{III}_q,
\end{aligned}$$

with $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

(1) First we prove (3.1). For I_q, by the expression (2.1) of dyadic operator $\Delta_q = h_q * = 2^{2q}h(2^q\cdot)*$ with $h = \mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^2)$ and the mean value theorem, we find

$$\begin{aligned}
\|\mathrm{I}_q\|_{L^p} &\leqslant C_0 \sum_{j\in\mathbb{N},|j-q|\leqslant 4} \|xh_q\|_{L^1}\|\nabla S_{j-1}u\|_{L^\infty} 2^j\|\Delta_j\theta\|_{L^p} \\
&\leqslant C_0 \sum_{j\in\mathbb{N},|j-q|\leqslant 4} \big(\|u\|_{L^2} + (j+1)\|\omega\|_{L^\infty}\big)\|\Delta_j\theta\|_{L^p},
\end{aligned}$$

where we have used the following estimate

$$\begin{aligned}
\|\nabla S_{j-1}u\|_{L^\infty} &\leqslant \|\nabla\Delta_{-1}u\|_{L^\infty} + \sum_{k\in\mathbb{N},k\leqslant j-2}\|\nabla\Delta_k u\|_{L^\infty} \\
&\lesssim \|\nabla\Delta_{-1}u\|_{L^\infty} + (j+1)\|\omega\|_{L^\infty}.
\end{aligned}$$

Thus

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \|\mathrm{I}_q\|_{L^p} \leqslant C_s \|\nabla\Delta_{-1}u\|_{L^\infty} \sum_{j\in\mathbb{N}} 2^{js}(j+1)^r \|\Delta_j\theta\|_{L^p} + C_s\|\omega\|_{L^\infty} \sum_{j\in\mathbb{N}} 2^{js}(j+1)^{r+1}\|\Delta_j\theta\|_{L^p}.$$

For II_q, we directly get

$$\begin{aligned}
\|\mathrm{II}_q\|_{L^p} &\leqslant \sum_{j\in\mathbb{N},|j-q|\leqslant 4} \big\|\Delta_q(\Delta_j u\cdot\nabla S_{j-1}\theta)\big\|_{L^p} + \sum_{j\in\mathbb{N},j\geqslant q} \|\Delta_j u\cdot\nabla\Delta_q S_{j-1}\theta\|_{L^p} \\
&\leqslant C_0 \sum_{j\in\mathbb{N},|j-q|\leqslant 4} \|\Delta_j u\|_{L^\infty}\|\nabla S_{j-1}\theta\|_{L^p} + C_0 \sum_{j\in\mathbb{N},j\geqslant q} \|\Delta_j u\|_{L^\infty} 2^q \|\Delta_q\theta\|_{L^p} \\
&\leqslant C_0\|\omega\|_{L^\infty} 2^{-q}\|\nabla S_{q+3}\theta\|_{L^p} + C_0\|\omega\|_{L^\infty}\|\Delta_q\theta\|_{L^p}.
\end{aligned}$$

Noting that for $s<1$,

$$\begin{aligned}
&\sum_{q\in\mathbb{N}} 2^{q(s-1)}(q+1)^r\|\nabla S_{q+3}\theta\|_{L^p} \\
&\quad\leqslant \sum_{q\in\mathbb{N}} 2^{q(s-1)}(q+1)^r \Big(\|\Delta_{-1}\theta\|_{L^p} + \sum_{k\in\mathbb{N},k\leqslant q+2} 2^k\|\Delta_k\theta\|_{L^p}\Big) \\
&\quad\leqslant C_{r,s}\|\Delta_{-1}\theta\|_{L^p} + \sum_{k\in\mathbb{N}} 2^k\|\Delta_k\theta\|_{L^p}\Big(\sum_{q\in\mathbb{N},q\geqslant k-2} 2^{q(s-1)}(q+1)^r\Big) \\
&\quad\leqslant C_{r,s}\|\Delta_{-1}\theta\|_{L^p} + C_{r,s}\sum_{k\in\mathbb{N}} 2^k\|\Delta_k\theta\|_{L^p} 2^{k(s-1)/2}(k+1)^r\Big(\sum_{q\geqslant k-2} 2^{q(s-1)/2}\Big) \\
&\quad\leqslant C_{r,s}\|\Delta_{-1}\theta\|_{L^p} + C_{r,s}\sum_{k\in\mathbb{N}} 2^{ks}(k+1)^r\|\Delta_k\theta\|_{L^p},
\end{aligned}$$

hence,

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r\|\mathrm{II}_q\|_{L^p} \leqslant C_{r,s}\|\Delta_{-1}\theta\|_{L^p}\|\omega\|_{L^\infty} + C_{r,s}\|\omega\|_{L^\infty}\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r\|\Delta_q\theta\|_{L^p}.$$

We further decompose III_q as follows

$$\begin{aligned}
\mathrm{III}_q &= \sum_{j=-1,0} [\Delta_q, \Delta_j u\cdot\nabla]\widetilde{\Delta}_j\theta + \sum_{j\geqslant 1}\nabla\cdot\Delta_q(\Delta_j u\widetilde{\Delta}_j\theta) - \sum_{j\geqslant 1}\Delta_j u\cdot\nabla\Delta_q\widetilde{\Delta}_j\theta \\
&= \mathrm{III}_q^1 + \mathrm{III}_q^2 + \mathrm{III}_q^3.
\end{aligned} \tag{3.3}$$

Since $\mathrm{III}_q^1 = 0$ for all $q\geqslant 4$, then similarly as the treating of $\|\mathrm{I}_q\|_{L^p}$, we obtain that for $0\leqslant q\leqslant 3$,

$$\big\|\mathrm{III}_q^1\big\|_{L^p} \leqslant C_0\sum_{j=-1,0}\|xh_q\|_{L^1}\|\nabla\Delta_j u\|_{L^\infty}\|\nabla\widetilde{\Delta}_j\theta\|_{L^p} \leqslant C_0\|\nabla S_3 u\|_{L^\infty}\|S_3\theta\|_{L^p},$$

hence,

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \left\|\mathrm{III}_q^1\right\|_{L^p} = \sum_{0\leqslant q\leqslant 3} 2^{qs}(q+1)^r \left\|\mathrm{III}_q^1\right\|_{L^p} \lesssim_{r,s} \|\nabla S_3 u\|_{L^\infty} \|S_3\theta\|_{L^p}.$$

For III_q^2, we infer that

$$\left\|\mathrm{III}_q^2\right\|_{L^p} \leqslant C_0 \sum_{j\geqslant 1, j\geqslant q-3} 2^q \|\Delta_j u\|_{L^\infty} \|\widetilde{\Delta}_j\theta\|_{L^p} \leqslant C_0 \|\omega\|_{L^\infty} 2^q \sum_{j\in\mathbb{N}, j\geqslant q-3} 2^{-j} \|\Delta_j\theta\|_{L^p},$$

thus for $s > -1$,

$$\begin{aligned}
\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \left\|\mathrm{III}_q^2\right\|_{L^p} &\leqslant C_0 \|\omega\|_{L^\infty} \sum_{q\in\mathbb{N}} \sum_{j\in\mathbb{N}, j\geqslant q-3} 2^{q(s+1)}(q+1)^r 2^{-j} \|\Delta_j\theta\|_{L^p} \\
&\leqslant C_0 \|\omega\|_{L^\infty} \sum_{j\in\mathbb{N}} 2^{-j} \|\Delta_j\theta\|_{L^p} \left(\sum_{q\leqslant j+3} 2^{q(s+1)}(q+1)^r \right) \\
&\leqslant C_{r,s} \|\omega\|_{L^\infty} \sum_{j\in\mathbb{N}} 2^{js}(j+1)^r \|\Delta_j\theta\|_{L^p}.
\end{aligned}$$

It is easy to see that

$$\left\|\mathrm{III}_q^3\right\|_{L^p} \leqslant C_0 \sum_{j\geqslant 1, |j-q|\leqslant 2} \|\Delta_j u\|_{L^\infty} 2^q \|\Delta_q \widetilde{\Delta}_j\theta\|_{L^p} \leqslant C_0 \|\omega\|_{L^\infty} \|\Delta_q\theta\|_{L^p},$$

thus

$$\sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \left\|\mathrm{III}_q^3\right\|_{L^p} \leqslant C_0 \|\omega\|_{L^\infty} \sum_{q\in\mathbb{N}} 2^{qs}(q+1)^r \|\Delta_q\theta\|_{L^p}.$$

Gathering the upper estimates yields (3.1).

(2) The proof of (3.2) is more or less standard, mainly relying on the above Bony decomposition and Lemma 2.4, and we omit the details. □

Next we consider the commutator estimates involving the operator $\mathcal{R}_\alpha := \partial_1|D|^{-\alpha} = |D|^{1-\alpha}\mathcal{R}_1$ for every $\alpha \in]0,1]$ (noting that $\mathcal{R}_1$ is the usual Riesz transform).

Lemma 3.2. *Assume that $u = (u_1, \ldots, u_d)$ is a smooth divergence-free vector field of $\mathbb{R}^d$ ($d \geqslant 2$) with its vorticity ω, and θ is a smooth function. Let $s \in]-1, \alpha[$, $p \in [2,\infty]$, $r \in [1,\infty]$, then*

$$\left\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\right\|_{B^s_{p,r}} \lesssim_\alpha \|\nabla u\|_{L^p} \left(\|\theta\|_{B^{s+1-\alpha}_{\infty,r}} + \|\theta\|_{L^2}\right). \tag{3.4}$$

Besides, when $p = \infty$, we also have for $\sigma \in]\frac{d}{\alpha-s}, \infty[$,

$$\left\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\right\|_{B^s_{\infty,r}} \lesssim_{\alpha,\sigma} \|\nabla u\|_{L^\sigma} \left(\|\theta\|_{B^{s+1-\alpha+d/\sigma}_{\infty,r}} + \|\theta\|_{L^2}\right). \tag{3.5}$$

Before showing this lemma, we first recall some useful properties of $\mathcal{R}_\alpha$ (cf. [17,25])

Lemma 3.3. *Let* $0<\alpha\leqslant 1$, $q\in\mathbb{N}$.

(1) *Let* $\chi\in\mathcal{S}(\mathbb{R}^d)$. *Then for every* $(s,p)\in\,]\alpha-1,\infty[\,\times[1,\infty]$, *the operator* $|D|^s\chi(2^{-q}D)\mathcal{R}_\alpha$ *is bounded in* L^p *with the norm*

$$\big\||D|^s\chi\big(2^{-q}|D|\big)\mathcal{R}_\alpha\big\|_{\mathcal{L}(L^p)}\lesssim 2^{q(s+1-\alpha)}.$$

In particular, the kernel $K(x)$ *of* $|D|^s\chi(D)\mathcal{R}_\alpha$ *satisfies that*

$$K(x)\leqslant\frac{C}{(1+|x|)^{d+s+1-\alpha}},\quad \forall x\in\mathbb{R}^d.$$

(2) *Let* $\mathcal{C}$ *be a ring. Then there exists* $\phi\in\mathcal{S}(\mathbb{R}^d)$ *whose spectrum does not meet the origin such that*

$$\mathcal{R}_\alpha f=2^{q(d+1-\alpha)}\phi\big(2^q\cdot\big)*f$$

for every f *whose Fourier variable supported on* $2^q\mathcal{C}$.

Now we are devoted to prove Lemma 3.2.

Proof of Lemma 3.2. Note that (3.4) is essentially the same as [25, Eq. (3.4)], thus here we omit the details for (3.4) and only focus on (3.5). Once again using Bony's decomposition yields

$$\begin{aligned}[\mathcal{R}_\alpha,u\cdot\nabla]\theta&=\sum_{j\in\mathbb{N}}[\mathcal{R}_\alpha,S_{j-1}u\cdot\nabla]\Delta_j\theta+\sum_{j\in\mathbb{N}}[\mathcal{R}_\alpha,\Delta_ju\cdot\nabla]S_{j-1}\theta+\sum_{j\geqslant-1}[\mathcal{R}_\alpha,\Delta_ju\cdot\nabla]\widetilde{\Delta}_j\theta\\&:=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

For I, since the Fourier transform of $S_{j-1}u\Delta_j\theta$ ($j\in\mathbb{N}$) is supported in a ring of size 2^j, then from Lemma 3.3-(2) and (2.5), we have for every $q\geqslant-1$,

$$\begin{aligned}\|\Delta_q\mathrm{I}\|_{L^\infty}&\lesssim\sum_{|j-q|\leqslant4}\big\|[\phi_j*,S_{j-1}u\cdot\nabla]\Delta_j\theta\big\|_{L^\infty}\\&\lesssim\sum_{|j-q|\leqslant4}\|x\phi_j\|_{L^1}\|\nabla S_{j-1}u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^\infty}\\&\lesssim\sum_{|j-q|\leqslant4}2^{-j\alpha}2^{jd/\sigma}\|\nabla u\|_{L^\sigma}2^j\|\Delta_j\theta\|_{L^\infty}\\&\lesssim c_j2^{-js}\|\nabla u\|_{L^\sigma}\|\theta\|_{B^{s+1-\alpha+d/\sigma}_{\infty,r}},\end{aligned}$$

where $\phi_j(x)=2^{j(d+1-\alpha)}\phi(2^jx)$ with $\phi\in\mathcal{S}(\mathbb{R}^d)$ and $(c_j)_{j\geqslant-1}$ with $\|c_j\|_{\ell^r}=1$. Thus

$$\|\mathrm{I}\|_{B^s_{\infty,r}}\lesssim\|\nabla u\|_{L^\sigma}\|\theta\|_{B^{s+1-\alpha+d/\sigma}_{\infty,r}}.$$

For II, as above we have

$$\begin{aligned}\|\Delta_q \mathrm{II}\|_{L^\infty} &\lesssim \sum_{|j-q|\leqslant 4, q\in\mathbb{N}} \big\|[\phi_j *, \Delta_j u\cdot\nabla]S_{j-1}\theta\big\|_{L^\infty} \\ &\lesssim \sum_{|j-q|\leqslant 4} 2^{-j\alpha}2^{jd/\sigma}\|\nabla\Delta_j u\|_{L^\sigma}\|\nabla S_{j-1}\theta\|_{L^\infty} \\ &\lesssim \|\nabla u\|_{L^\sigma}2^{-qs}\sum_{-1\leqslant k\leqslant q+2} 2^{-(k-q)(s-\alpha+d/\sigma)}2^{k(s+1-\alpha+d/\sigma)}\|\Delta_k\theta\|_{L^\infty}.\end{aligned}$$

Thus by using discrete convolution inequality, we obtain for every $s<\alpha-d/\sigma$,

$$\|\mathrm{II}\|_{B^s_{\infty,r}} \lesssim \|\nabla u\|_{L^\sigma}\|\theta\|_{B^{s+1-\alpha+d/\sigma}_{\infty,r}}.$$

For III, we further write

$$\mathrm{III} = \sum_{j\geqslant 0}\mathrm{div}[\mathcal{R}_\alpha, \Delta_j u]\widetilde{\Delta}_j\theta + \sum_{1\leqslant i\leqslant d}[\partial_i\mathcal{R}_\alpha, \Delta_{-1}u_i]\widetilde{\Delta}_{-1}\theta = \mathrm{III}^1 + \mathrm{III}^2.$$

By Bernstein's inequality and Lemma 3.3-(1), we treat the term III^1 as follows

$$\begin{aligned}\big\|\Delta_q\mathrm{III}^1\big\|_{L^\infty} &\leqslant \sum_{j\in\mathbb{N}, j\geqslant q-3}\big\|\Delta_q\,\mathrm{div}\,\mathcal{R}_\alpha(\Delta_j u\widetilde{\Delta}_j\theta)\big\|_{L^\infty} + \sum_{j\in\mathbb{N}, j\geqslant q-3}\big\|\Delta_q\,\mathrm{div}(\Delta_j u\mathcal{R}_\alpha\widetilde{\Delta}_j\theta)\big\|_{L^\infty} \\ &\lesssim \sum_{j\in\mathbb{N}, j\geqslant q-3}\big(2^{q(2-\alpha)}+2^q2^{j(1-\alpha)}\big)2^{-j}2^{jd/\sigma}\|\Delta_j\nabla u\|_{L^\sigma}\|\widetilde{\Delta}_j\theta\|_{L^\infty} \\ &\lesssim \|\nabla u\|_{L^\sigma}2^{-qs}\sum_{j\geqslant q-4}\big(2^{(q-j)(s+2-\alpha)}+2^{(q-j)(s+1)}\big)2^{j(s+1-\alpha+d/\sigma)}\|\Delta_j\theta\|_{L^\infty}.\end{aligned}$$

Thus for every $s>-1$,

$$\big\|\mathrm{III}^1\big\|_{B^s_{\infty,r}} \lesssim \|\nabla u\|_{L^\sigma}\|\theta\|_{B^{s+1-\alpha+d/\sigma}_{\infty,r}}.$$

For the second term III^2, from the spectral localization property, there exists $\chi'\in\mathcal{S}(\mathbb{R}^d)$ such that

$$\mathrm{III}^2 = \sum_{1\leqslant i\leqslant d}\big[\partial_i\mathcal{R}_\alpha\chi'(D), \Delta_{-1}u_i\big]\widetilde{\Delta}_{-1}\theta,$$

where according to Lemma 3.3 we know that $\partial_i\mathcal{R}_\alpha\chi'(D)$ is a convolution operator with kernel h' satisfying

$$\big|h'(x)\big| \leqslant C\big(1+|x|\big)^{-d-2+\alpha}, \quad \forall x\in\mathbb{R}^d.$$

Thus from the fact $\Delta_q\mathrm{III}^2=0$ for every $q\geqslant 3$ and Lemma 2.4 with $m=\max\{\sigma,2\}$ (we can choose $m=\infty$ for $\alpha\in\,]0,1[$), we have

$$\begin{aligned}\|\mathrm{III}^2\|_{B^s_{\infty,r}} &\lesssim \big\|[h'*, \Delta_{-1}u]\widetilde{\Delta}_{-1}\theta\big\|_{L^\sigma} \\ &\lesssim \|xh'\|_{L^{\bar m}}\|\nabla\Delta_{-1}u\|_{L^\sigma}\|\widetilde{\Delta}_{-1}\theta\|_{L^m} \\ &\lesssim \|\nabla u\|_{L^\sigma}\|\theta\|_{L^2}.\end{aligned}$$

This ends the proof of estimate (3.5). □

4. Proof of Theorem 1.1

The outline of the proof is as follows: first we show some key *a priori* estimates, next based on these estimates we prove the existence result, and then we treat the uniqueness issue in Subsection 4.3, later we establish the crucial refined blowup criterion, and by using it we prove the global regularity of the critical case in the last subsection.

4.1. A priori *estimates*

In this subsection, we *a priori* assume that the solution (u,θ) is a smooth solution to the inviscid Boussinesq system (1.2) (with suitable spatial decay near infinity).

First we consider the energy estimate and $L^{\widetilde{p}}$-estimate. From the L^2-estimate of the equation

$$\partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = 0,$$

we have

$$\|\theta\|^2_{L^\infty_t L^2} + \|\theta\|^2_{L^2_t \dot{H}^{\beta/2}} \leqslant \|\theta_0\|^2_{L^2}.$$

Moreover, thanks to Lemma 2.2, we also obtain that for every $\widetilde{p}\in[1,\infty]$ and $t\in\mathbb{R}^+$,

$$\|\theta\|_{L^\infty_t L^{\widetilde{p}}} \leqslant \|\theta_0\|_{L^{\widetilde{p}}}, \tag{4.1}$$

and for $p\in[2,\infty[$ and general $p_0<p$ and $2\leqslant p_1<\infty$,

$$\big\|\theta(t)\big\|_{L^p} \leqslant \frac{C_\beta\|\theta_0\|_{L^p\cap L^{p_0}}}{(1+t)^{\frac{2}{\beta}(\frac{1}{p_0}-\frac{1}{p})}} \quad \text{and} \quad \big\|\theta(t)\big\|_{L^\infty} \leqslant \frac{C_\beta\|\theta_0\|_{L^{p_1}\cap L^\infty}}{(1+t)^{2/(\beta p_1)}}.$$

In particular, if $\beta\in]0,1[$ and $\theta_0\in L^{p_0}\cap L^{p_1}\cap L^\infty$ with $p_0\in[1,\frac{2p}{2+p\beta}[$ and $p_1\in[2,\frac{2}{\beta}[$, the above estimates ensure that

$$\|\theta\|_{L^\rho_t L^\infty} \leqslant C_\beta\|\theta_0\|_{L^{p_1}\cap L^\infty}, \quad \text{for all } \rho\in[1,\infty], \tag{4.2}$$

and

$$\|\theta\|_{L^\rho_t L^p} \leqslant C_\beta\|\theta_0\|_{L^p\cap L^{p_0}}, \quad \text{for all } \rho\in[1,\infty]. \tag{4.3}$$

For the velocity equation, by taking a scalar product with u, we see that

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_{L^2}^2=\int_{\mathbb{R}^2} u_2(t,x)\theta(t,x)\,\mathrm{d}x \leqslant \left\|u(t)\right\|_{L^2}\left\|\theta(t)\right\|_{L^2}.$$

Thus dividing both sides by $\|u(t)\|_{L^2}$ and integrating in time lead to

$$\|u\|_{L_t^\infty L^2} \leqslant \|u_0\|_{L^2}+\|\theta\|_{L_t^1 L^2}.$$

In general, we get

$$\|u\|_{L_t^\infty L^2} \leqslant \|u_0\|_{L^2}+t\|\theta_0\|_{L^2}, \tag{4.4}$$

but under the condition that $\beta \in]0,1[$ and $\theta_0 \in L^{p_0}\cap L^2$ with $p_0 \in [1,\frac{2}{1+\beta}[$, we also get

$$\|u\|_{L_t^\infty L^2} \leqslant \|u_0\|_{L^2}+C_\beta\|\theta_0\|_{L^2\cap L^{p_0}}. \tag{4.5}$$

Next we try to get the key *a priori* estimate of $\|\omega(t)\|_{L^\infty}$. From the maximum principle of the vorticity equation

$$\partial_t\omega+u\cdot\nabla\omega=\partial_1\theta,$$

we have

$$\left\|\omega(t)\right\|_{L^\infty} \leqslant \|\omega_0\|_{L^\infty}+\int_0^t\left\|\partial_1\theta(\tau)\right\|_{L^\infty}\mathrm{d}\tau \leqslant \|\omega_0\|_{L^\infty}+C_0\|\theta\|_{L_t^1 B_{\infty,1}^1}. \tag{4.6}$$

The high-low frequency decomposition leads to

$$\|\theta\|_{L_t^1 B_{\infty,1}^1}=2^{-1}\|\Delta_{-1}\theta\|_{L_t^1 L^\infty}+\sum_{q\in\mathbb{N}}2^q\|\Delta_q\theta\|_{L_t^1 L^\infty} \leqslant \frac{1}{2}\|\theta\|_{L_t^1 L^\infty}+\sum_{q\in\mathbb{N}}2^q\|\Delta_q\theta\|_{L_t^1 L^\infty}.$$

For every $q\in\mathbb{N}$, applying Lemma 2.3 to the frequency localized equation

$$\partial_t\Delta_q\theta+u\cdot\nabla\Delta_q\theta+|D|^\beta\Delta_q\theta=-[\Delta_q,u\cdot\nabla]\theta, \tag{4.7}$$

we know there exists a positive constant c independent of q,θ so that

$$\frac{d}{dt}\|\Delta_q\theta\|_{L^\infty}+c2^{q\beta}\|\Delta_q\theta\|_{L^\infty} \leqslant \left\|[\Delta_q,u\cdot\nabla]\theta\right\|_{L^\infty}.$$

Gronwall inequality yields

$$\left\|\Delta_q\theta(t)\right\|_{L^\infty} \leqslant e^{-c2^{q\beta}t}\|\Delta_q\theta_0\|_{L^\infty}+\int_0^t e^{-c2^{q\beta}(t-\tau)}\left\|[\Delta_q,u\cdot\nabla]\theta(\tau)\right\|_{L^\infty}\mathrm{d}\tau.$$

Thus for every $\rho \in [1, \infty]$,

$$2^{q\beta} \|\Delta_q \theta\|_{L_t^1 L^\infty} + 2^{q\beta/\rho} \|\Delta_q \theta\|_{L_t^\rho L^\infty} \leqslant C_0 \|\Delta_q \theta_0\|_{L^\infty} + C_0 \big\| [\Delta_q, u \cdot \nabla]\theta \big\|_{L_t^1 L^\infty}. \tag{4.8}$$

Hence we find

$$\begin{aligned} \|\theta\|_{L_t^1 B^1_{\infty,1}} + \|\theta\|_{\widetilde{L}_t^\rho B^{1-\beta+\beta/\rho}_{\infty,1}} &\leqslant C_0 \|\theta\|_{(L_t^1 L^\infty) \cap (L_t^\rho L^\infty)} + C_0 \|\theta_0\|_{B^{1-\beta}_{\infty,1}} \\ &\quad + C_0 \sum_{q \in \mathbb{N}} 2^{q(1-\beta)} \big\| [\Delta_q, u \cdot \nabla]\theta \big\|_{L_t^1 L^\infty}, \end{aligned} \tag{4.9}$$

and for some $\sigma \in]1, \infty[$ chosen later,

$$\begin{aligned} \|\theta\|_{L_t^1 B^1_{\infty,1}} + \|\theta\|_{\widetilde{L}_t^\rho B^{1-\beta+\beta/\rho}_{\infty,1}} &\leqslant C_0 \|\theta\|_{(L_t^1 L^\infty) \cap (L_t^\rho L^\infty)} + C_0 \|\theta_0\|_{B^{1-\beta}_{\infty,1}} \\ &\quad + C_0 t^{1-1/\sigma} \sum_{q \in \mathbb{N}} 2^{q(1-\beta)} \big\| [\Delta_q, u \cdot \nabla]\theta \big\|_{L_t^\sigma L^\infty}. \end{aligned} \tag{4.10}$$

By virtue of Lemma 3.1 and Calderón–Zygmund theorem, we infer that for $p \in [2, \infty[$,

$$\begin{aligned} &\sum_{q \in \mathbb{N}} 2^{q(1-\beta)} \big\| [\Delta_q, u \cdot \nabla]\theta \big\|_{L^\infty} \\ &\quad \lesssim_\beta \big(\|\nabla S_3 u\|_{L^\infty} + \|\omega\|_{L^\infty} \big) \bigg(\|S_3 \theta\|_{L^\infty} + \sum_{q \in \mathbb{N}} 2^{q(1-\beta)} (q+1) \|\Delta_q \theta\|_{L^\infty} \bigg) \\ &\quad \lesssim_\beta \big(\|\omega\|_{L^p} + \|\omega\|_{L^\infty} \big) \bigg(\|\theta\|_{L^\infty} + \sum_{q \in \mathbb{N}} 2^{q(1-\beta)} (q+1) \|\Delta_q \theta\|_{L^\infty} \bigg). \end{aligned} \tag{4.11}$$

Thus we also need to control $\|\omega\|_{L_t^\infty L^p}$. From L^p-estimate of the transport equation, we know that

$$\|\omega\|_{L_t^\infty L^p} \leqslant \|\omega_0\|_{L^p} + \|\partial_1 \theta\|_{L_t^1 L^p} \leqslant \|\omega_0\|_{L^p} + C_0 \|\theta\|_{L_t^1 B^1_{p,1}}. \tag{4.12}$$

High-low frequency decomposition ensures that

$$\|\theta\|_{L_t^1 B^1_{p,1}} = 2^{-1} \|\Delta_{-1}\theta\|_{L_t^1 L^p} + \sum_{q \in \mathbb{N}} 2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leqslant C_0 \|\theta\|_{L_t^1 L^p} + \sum_{q \in \mathbb{N}} 2^q \|\Delta_q \theta\|_{L_t^1 L^p}.$$

For the frequency-localized equation (4.7), mainly using the following generalized Bernstein inequality (cf. [7])

$$\int_{\mathbb{R}^2} |D|^\beta \Delta_q \theta(x) \, |\Delta_q \theta|^{p-2} \Delta_q \theta(x) \, \mathrm{d}x \geqslant c 2^{q\beta} \|\Delta_q \theta\|_{L^p}^p$$

with $c > 0$ independent of q and θ, we can obtain an estimate analogous to (4.8) that for every $\rho \in [1, \infty]$,

$$2^{q\beta}\|\Delta_q\theta\|_{L^1_t L^p} + 2^{q\beta/\rho}\|\Delta_q\theta\|_{L^\rho_t L^p} \leqslant C_0\|\Delta_q\theta_0\|_{L^p} + C_0\big\|[\Delta_q, u\cdot\nabla]\theta\big\|_{L^1_t L^p}. \tag{4.13}$$

This implies that we can get the estimates similar to (4.9) and (4.10) by replacing the corresponding index ∞ by p. Thanks to Lemma 3.1 again, we also infer that

$$\begin{aligned}&\sum_{q\in\mathbb{N}} 2^{q(1-\beta)}\big\|[\Delta_q, u\cdot\nabla]\theta\big\|_{L^p} \\ &\quad\lesssim_{\beta,p} \big(\|\omega\|_{L^p} + \|\omega\|_{L^\infty}\big)\bigg(\|\theta\|_{L^p} + \sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L^p}\bigg).\end{aligned}$$

Collecting the upper estimates, and denoting

$$a(t) := C_0\|\theta\|_{L^1_t(L^\infty\cap L^p)} + C_0\|\theta\|_{L^\rho_t(L^\infty\cap L^p)} + C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}, \tag{4.14}$$

we get

$$\begin{aligned}&\|\theta\|_{L^1_t(B^1_{\infty,1}\cap B^1_{p,1})} + \|\theta\|_{\widetilde{L}^\rho_t(B^{1-\beta+\beta/\rho}_{\infty,1}\cap B^{1-\beta+\beta/\rho}_{p,1})} \\ &\quad\leqslant a(t) + C_\beta\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\bigg(\|\theta\|_{L^1_t(L^\infty\cap L^p)} + \sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)}\bigg),\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\|\theta\|_{L^1_t(B^1_{\infty,1}\cap B^1_{p,1})} + \|\theta\|_{\widetilde{L}^\rho_t(B^{1-\beta+\beta/\rho}_{\infty,1}\cap B^{1-\beta+\beta/\rho}_{p,1})} \\ &\quad\leqslant a(t) + C_\beta t^{1-1/\sigma}\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\bigg(\|\theta\|_{L^\sigma_t(L^\infty\cap L^p)} + \sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L^\sigma_t(L^\infty\cap L^p)}\bigg).\end{aligned}$$

From (4.8), (4.13) and Lemma 3.1, we see that

$$\begin{aligned}&\sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)} \\ &\quad\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} + C_0\sum_{q\in\mathbb{N}} 2^{q(1-2\beta)}(q+1)\big\|[\Delta_q, u\cdot\nabla]\theta\big\|_{L^1_t(L^\infty\cap L^p)} \\ &\quad\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} \\ &\qquad + C_\beta\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\bigg(\|\theta\|_{L^1_t(L^\infty\cap L^p)} + \sum_{q\in\mathbb{N}} 2^{q(1-2\beta)}(q+1)^2\|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)}\bigg),\end{aligned}$$

and

$$\sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L_t^\sigma(L^\infty\cap L^p)} \leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} + C_\beta\|\omega\|_{L_t^\infty(L^\infty\cap L^p)} \times \Big(\|\theta\|_{L_t^1(L^\infty\cap L^p)} + \sum_{q\in\mathbb{N}} 2^{q(1-\beta-\beta/\sigma)}(q+1)^2\|\Delta_q\theta\|_{L_t^1(L^\infty\cap L^p)}\Big).$$

Denoting by

$$\begin{cases} A(t) := \|\omega\|_{L_t^\infty L^p} + \|\omega\|_{L_t^\infty L^\infty}, \\ \Theta(t) := \|\theta\|_{L_t^1(B^1_{\infty,1}\cap B^1_{p,1})} + \|\theta\|_{\widetilde{L}_t^\rho(B^{1-\beta+\beta/\rho}_{\infty,1}\cap B^{1-\beta+\beta/\rho}_{p,1})}, \quad \text{for } \rho\in[1,\infty], \end{cases} \tag{4.16}$$

we have

$$\Theta(t) \leqslant a(t) + C_\beta A(t)\Big(a(t) + C_\beta A(t)\Big(a(t) + \sum_{q\in\mathbb{N}} 2^{q(1-2\beta)}(q+1)^2\|\Delta_q\theta\|_{L_t^1(L^\infty\cap L^p)}\Big)\Big), \tag{4.17}$$

and

$$\Theta(t) \leqslant a(t) + C_\beta t^{1-1/\sigma}A(t) \times \Big(a(t) + C_\beta A(t)\Big(a(t) + \sum_{q\in\mathbb{N}} 2^{q(1-\beta-\beta/\sigma)}(q+1)^2\|\Delta_q\theta\|_{L_t^1(L^\infty\cap L^p)}\Big)\Big). \tag{4.18}$$

We first consider the case $\beta\in\,]1/2,1]$. Then we can choose $\sigma = 2\beta \in\,]1,2]$, and from (4.1), we find

$$\begin{aligned}\Theta(t) &\leqslant a(t) + C_\beta A(t)\Big(a(t) + C_\beta A(t)\Big(a(t) + \|\theta\|_{L_t^1(L^\infty\cap L^p)}\sum_{q\in\mathbb{N}} 2^{q(1-2\beta)}(q+1)^2\Big)\Big) \\ &\leqslant a(t) + C_\beta A(t)\big(a(t) + C_\beta a(t)A(t)\big),\end{aligned}$$

and

$$\begin{aligned}\Theta(t) &\leqslant a(t) + C_\beta t^{1-1/\sigma}A(t)\Big(a(t) + C_\beta A(t)\Big(a(t) + \|\theta\|_{L_t^1(L^\infty\cap L^p)}\sum_{q\in\mathbb{N}} 2^{q(1/2-\beta)}(q+1)^2\Big)\Big) \\ &\leqslant a(t) + C_\beta t^{1-1/(2\beta)}A(t)\big(a(t) + C_\beta a(t)A(t)\big).\end{aligned}$$

Combining the upper two estimates with (4.6), (4.12), and by setting

$$\widetilde{a}(t) := \|\omega_0\|_{L^\infty\cap L^p} + C_0 a(t),$$

we have

$$A(t)+\Theta(t)\leqslant\widetilde{a}(t)+C_{\beta,1}a(t)A(t)\big(1+C_{\beta,2}A(t)\big),\tag{4.19}$$

and

$$A(t)+\Theta(t)\leqslant\widetilde{a}(t)+t^{1-1/(2\beta)}C_{\beta,1}a(t)A(t)\big(1+C_{\beta,2}A(t)\big).\tag{4.20}$$

For every $T>0$, according to (4.19), as long as $a(T)$ small enough such that

$$C_{\beta,1}a(T)\big(1+C_{\beta,2}2\widetilde{a}(T)\big)<1/2,$$

we have

$$A(T)+\Theta(T)\leqslant 2\widetilde{a}(T).$$

Generally, from (4.1), we have

$$a(T)\leqslant C_0(1+T)\|\theta_0\|_{L^\infty\cap L^p}+C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}\leqslant C_0(1+T)\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}},\tag{4.21}$$

and

$$\widetilde{a}(T)\leqslant\|\omega_0\|_{L^\infty\cap L^p}+C_0a(T)\leqslant\|\omega_0\|_{L^\infty\cap L^p}+C_0,\tag{4.22}$$

where in the above inequality we have used the assumption $(1+T)\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}\leqslant 1$ (without loss of generality), and the notation $A_0:=\|\omega_0\|_{L^\infty\cap L^p}$, thus under the condition that

$$\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}<\frac{1}{2C_0(T+1)C_{\beta,1}(1+2C_{\beta,2}(A_0+C_0))},$$

we have

$$A(T)+\Theta(T)\leqslant 2(A_0+C_0).\tag{4.23}$$

Moreover, in the supercritical regime $\beta\in]0,1[$ and for $\theta_0\in L^{p_0}\cap L^{p_1}\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}$ with $p_0\in]1,\frac{2p}{2+p\beta}[$ and $p_1\in[2,\frac{2}{\beta}[$, according to (4.2) and (4.3), we deduce that

$$a(T)\leqslant C_\beta\|\theta_0\|_{L^{p_0}\cap L^{p_1}\cap L^\infty}+C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}\leqslant C_\beta\|\theta_0\|_{L^{p_0}\cap L^{p_1}\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}},$$

and

$$\widetilde{a}(T)\leqslant\|\omega_0\|_{L^\infty\cap L^p}+C_0a(T)\leqslant A_0+C_\beta,$$

thus under the condition

$$\|\theta_0\|_{L^{p_0}\cap L^{p_1}\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}} < \min\left\{\frac{1}{2C_\beta C_{\beta,1}(1+2C_{\beta,2}(A_0+C_\beta))}, 1\right\},$$

we have

$$A(T)+\Theta(T)\leqslant 2(A_0+C_\beta). \tag{4.24}$$

On the other hand, we can show the short-time estimate for the large data. Based on (4.20), and noting that for $t\leqslant 1$,

$$a(t)\leqslant C_0\Theta_0 \quad \text{and} \quad \widetilde{a}(t)\leqslant A_0+C_0\Theta_0 \quad \text{with } \Theta_0:=\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}},$$

then for t small enough ($t\leqslant 1$ with no loss of generality) such that

$$t^{1-1/(2\beta)}C_{\beta,1}C_0\Theta_0\big(1+C_{\beta,2}(2A_0+2C_0\Theta_0)\big)<1/2,$$

we have

$$A(t)+\Theta(t)\leqslant 2A_0+2C_0\Theta_0. \tag{4.25}$$

Next we consider the general case $\beta\in]0,1]$. Then there exists a unique integer $k\in\mathbb{Z}^+$ so that $\beta\in]\frac{1}{k+1},\frac{1}{k}]$, and we can choose σ in (4.10) to be $\sigma=\frac{2}{1/\beta-(k-1)}\in]1,2]$. Similarly as obtaining (4.17) and (4.18), by iteration, we deduce that

$$\begin{aligned}\Theta(t)&\leqslant a(t)+C_{\beta,1}a(t)A(t)+\cdots+C_{\beta,k}a(t)A(t)^k\\&\quad+C_{\beta,k+1}A(t)^{k+1}\left(a(t)+\sum_{q\in\mathbb{N}}2^{q(1-(k+1)\beta)}(q+1)^{k+1}\|\Delta_q\theta\|_{L^1_tL^\infty}\right)\\&\leqslant a(t)+C_{\beta,1}a(t)A(t)+\cdots+C_{\beta,k+1}A(t)^{k+1}a(t),\end{aligned}$$

and

$$\begin{aligned}\Theta(t)&\leqslant a(t)+t^{\frac{(k+1)\beta-1}{2\beta}}\Bigg\{C_{\beta,1}a(t)A(t)+\cdots+C_{\beta,k}a(t)A(t)^k\\&\quad+C_{\beta,k+1}A(t)^{k+1}\left(a(t)+\sum_{q\in\mathbb{N}}2^{q\frac{1-(k+1)\beta}{2}}(q+1)^{k+1}\|\Delta_q\theta\|_{L^1_tL^\infty}\right)\Bigg\}\\&\leqslant a(t)+t^{\frac{(k+1)\beta-1}{2\beta}}\big(C_{\beta,1}a(t)A(t)+\cdots+C_{\beta,k+1}A(t)^{k+1}a(t)\big).\end{aligned}$$

Hence,

$$A(t)+\Theta(t)\leqslant\widetilde{a}(t)+C_0a(t)A(t)\big(C_{\beta,1}+\cdots+C_{\beta,k+1}A(t)^k\big), \tag{4.26}$$

and

$$A(t)+\Theta(t)\leqslant\widetilde{a}(t)+C_0t^{\frac{(k+1)\beta-1}{2\beta}}a(t)A(t)\big(C_{\beta,1}+\cdots+C_{\beta,k+1}A(t)^k\big). \tag{4.27}$$

Therefore, similarly as obtaining (4.23), (4.24) and (4.25), from the continuity method, the following *a priori* estimates hold:

(1) For every $T>0$, provided that

$$\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} < \min\left\{\frac{1}{2C_0(T+1)(C_{\beta,1}+\cdots+C_{\beta,k+1}(2A_0+2C_0)^k)}, \frac{1}{1+T}\right\}, \tag{4.28}$$

we have

$$A(T)+\Theta(T)\leqslant 2(A_0+C_0).$$

Besides, for $\beta\in\,]0,1[$ and $\theta_0\in L^{p_0}\cap L^{p_1}\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}$ with $p_0\in[1,\frac{2}{1+\beta}[$ and $p_1\cap[2,\frac{2}{\beta}[$, under the condition that

$$\|\theta_0\|_{L^{p_0}\cap L^{p_1}\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}} < \min\left\{\frac{1}{2C_\beta C_0(C_{\beta,1}+\cdots+C_{\beta,k+1}(2A_0+2C_\beta)^k)}, 1\right\}, \tag{4.29}$$

we have

$$A(T)+\Theta(T)\leqslant 2(A_0+C_\beta). \tag{4.30}$$

(2) For $t\leqslant 1$ small enough such that

$$t<T_0=\left(\frac{1}{2C_0\Theta_0(C_{\beta,1}+\cdots+C_{\beta,k+1}(2A_0+2C_0\Theta_0)^k)}\right)^{\frac{2\beta}{(k+1)\beta-1}}, \tag{4.31}$$

we have

$$A(t)+\Theta(t)\leqslant 2A_0+2C_0\Theta_0.$$

Here for both cases (1)–(2), k is the positive integer so that $\beta\in\,]\frac{1}{k+1},\frac{1}{k}]$, $A(t)$ and $\Theta(t)$ are introduced in (4.16), $A_0:=\|u_0\|_{L^2}+\|\omega_0\|_{L^\infty}$, $\Theta_0:=\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}}$.

Next, we derive the *a priori* upper bound on the quantity $\|\theta\|_{\widetilde{L}^\infty_t L^2}+\|\theta\|_{\widetilde{L}^1_t H^\beta}$ which will be useful in the continuity-in-time issue. We start with (4.13) for $p=2$, that is, for every $q\in\mathbb{N}$,

$$2^{q\beta}\|\Delta_q\theta\|_{L^1_tL^2}+\|\Delta_q\theta\|_{L^\infty_tL^2}\leqslant C_0\|\Delta_q\theta_0\|_{L^2}+C_0\int_0^t\big\|[\Delta_q,u\cdot\nabla]\theta(\tau)\big\|_{L^2}\,d\tau.$$

By virtue of (3.1), we have

$$\begin{aligned}\sum_{q\in\mathbb{N}}2^{q\beta/2}\|\Delta_q\theta\|_{L_t^1L^2}&\leqslant C_0\sum_{q\in\mathbb{N}}2^{-q\beta/2}\|\Delta_q\theta_0\|_{L^2}+C_0\sum_{q\in\mathbb{N}}2^{-q\beta/2}\big\|[\Delta_q,u\cdot\nabla]\theta\big\|_{L_t^1L^2}\\&\leqslant C_0\|\theta_0\|_{L^2}+C_\beta A(t)\bigg(\|\theta\|_{L_t^1L^2}+\sum_{q\in\mathbb{N}}2^{-q\beta/2}(q+1)\|\Delta_q\theta\|_{L_t^1L^2}\bigg)\\&\leqslant C_0\|\theta_0\|_{L^2}+C_\beta\|\theta_0\|_{L^2}tA(t).\end{aligned}$$

Using (3.1) again, we see that

$$\begin{aligned}\sum_{q\in\mathbb{N}}\big\|[\Delta_q,u\cdot\nabla]\theta\big\|_{L_t^1L^2}&\leqslant C_0A(t)\bigg(\|\theta\|_{L_t^1L^2}+\sum_{q\in\mathbb{N}}(q+1)\|\Delta_q\theta\|_{L_t^1L^2}\bigg)\\&\leqslant C_0A(t)\bigg(t\|\theta_0\|_{L^2}+C_\beta\sum_{q\in\mathbb{N}}2^{q\beta/2}\|\Delta_q\theta\|_{L_t^1L^2}\bigg)\\&\leqslant C_\beta(1+t)\|\theta_0\|_{L^2}A(t)\big(1+A(t)\big).\end{aligned}$$

Hence from the above estimates and the embedding $\ell^1\hookrightarrow\ell^2$, we get

$$\begin{aligned}\|\theta\|_{\widetilde{L}_t^\infty L^2}+\|\theta\|_{\widetilde{L}_t^1H^\beta}&\leqslant\|\Delta_{-1}\theta\|_{L_t^1L^2}+\|\Delta_{-1}\theta\|_{L_t^\infty L^2}\\&\quad+\bigg(\sum_{q\in\mathbb{N}}\big(\|\Delta_q\theta\|_{L_t^\infty L^2}+2^{q\beta}\|\Delta_q\theta\|_{L_t^1L^2}\big)^2\bigg)^{1/2}\\&\leqslant C_0(1+t)\|\theta_0\|_{L^2}+\sum_{q\in\mathbb{N}}\big\|[\Delta_q,u\cdot\nabla]\theta\big\|_{L_t^1L^2}\\&\leqslant C_\beta(1+t)\|\theta_0\|_{L^2}\big(1+A(t)\big)^2.\end{aligned}$$

4.2. Existence

We construct the approximate solutions $(u^n,\theta^n)_{n\in\mathbb{N}}$ as follows

$$\begin{cases}\partial_tu^n+u^n\cdot\nabla u^n+\nabla P^n=\theta^ne_2,\\\partial_t\theta^n+u^n\cdot\nabla\theta^n+|D|^\beta\theta^n=0,\\\operatorname{div}u^n=0,\qquad u^n\big|_{t=0}=S_nu_0,\qquad\theta^n\big|_{t=0}=S_n\theta_0,\end{cases}\tag{4.32}$$

where S_n is the low-frequency cut-off operator defined in (2.1). Since for every $n\in\mathbb{N}$, $S_nu_0\in H^s(\mathbb{R}^2)$, $S_n\theta_0\in H^s(\mathbb{R}^2)$ for all $s\geqslant0$, then from the standard theory (cf. [1]) there exists a local smooth solution (u^n,θ^n) (satisfying the spatial decay) to the approximate system. Thanks to Theorem 1.1 of [11], we know the following blowup criterion: if the maximal existence time $T_n^*<\infty$, then we necessarily need that $\int_0^{T_n^*}\|\nabla\theta^n(t)\|_{L^\infty}\,dt=\infty$; in other words, if for some T such that $\int_0^T\|\nabla\theta^n(t)\|_{L^\infty}\,dt<\infty$, then the time T can be proceeded forward. Thus from the following estimates that $\|S_n\theta_0\|_{L^2\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}}\leqslant\|\theta_0\|_{L^2\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}}$, $\|S_nu_0\|_{L^2}\leqslant\|u_0\|_{L^2}$, $\|S_n\omega_0\|_{L^p\cap L^\infty}\leqslant\|\omega_0\|_{L^p\cap L^\infty}$ for some $p\in[2,\infty[$ and every $n\in\mathbb{N}$, we can get a uniform local time T_0 defined by (4.31) so that for every $t<T_0$,

$$\begin{cases} u^n \in L^\infty\big([0,t]; L^2(\mathbb{R}^2)\big), \qquad \omega^n \in L^\infty\big([0,t]; L^p \cap L^\infty\big), \\ \theta^n \in \widetilde{L}^\infty\big([0,t]; L^2 \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}\big) \\ \qquad \cap L^2\big([0,t]; H^{\beta/2}\big) \cap L^1\big([0,t]; B^1_{\infty,1}\big), \end{cases} \quad \text{uniformly in } n. \tag{4.33}$$

Note that by the blowup criterion, we at least have $T_n^* \geqslant T_0$ for every $n \in \mathbb{N}$. In a similar way as the treating in [25], we can further obtain that for every $t < T_0$,

$$\partial_t u^n \in L^\infty\big([0,t]; L^2(\mathbb{R}^2)\big), \qquad \partial_t \theta^n \in L^2\big([0,t]; H^{\beta/2-1}(\mathbb{R}^2)\big), \quad \text{uniformly in } n. \tag{4.34}$$

Hence from the uniform estimates (4.33)–(4.34) and the Aubin–Lions compactness theorem, we can extract a suitable subsequence of the solution sequence $(u^n, \theta^n)_{n\in\mathbb{N}}$ so that it converges strongly in $L^\infty([0, T_0[; H^{\beta/2-1}_{\mathrm{loc}})$ to some function (u, θ), which moreover satisfies

$$\begin{cases} u \in C^{0,1}\big([0, T_0[; L^2(\mathbb{R}^2)\big), \qquad \omega \in L^\infty\big([0, T_0[; L^p \cap L^\infty\big), \\ \theta \in \widetilde{L}^\infty\big([0, T_0[; L^2 \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1}\big) \cap L^2\big([0, T_0[; H^{\beta/2}\big) \cap L^1\big([0, T_0[; B^1_{\infty,1}\big). \end{cases} \tag{4.35}$$

Then it is clear to pass to the limit in (4.32), and (u, θ) solves the original Boussinesq system (1.2) in the sense of distribution. Meanwhile, we can show that $\theta \in C([0, T_0[; L^2 \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1})$. Indeed, this is a standard procedure, since we have $\theta \in \widetilde{L}^\infty([0, T_0[; L^2 \cap B^{1-\beta}_{p,1} \cap B^{1-\beta}_{\infty,1})$ (cf. [7]). Moreover, if additionally $\theta_0 \in L^r(\mathbb{R}^2)$ with $r \in [1, 2[$, we also have $\theta \in L^\infty([0, T_0]; L^r(\mathbb{R}^2))$.[1]

Besides, we can prove the following global result. For any $T > 0$, if (4.28) or (4.29) holds, then (4.33)–(4.34) are satisfied for all $[0, T]$ and $T_n^* > T$ for every $n \in \mathbb{N}$. By passing to the limit, we know that the limiting function (u, θ) is a solution to (1.2) for all $[0, T]$ and satisfies the desired estimates.

4.3. Uniqueness

Now we sketch the proof of uniqueness part (see also [13,30]). Let $(u^{(1)}, \theta^{(1)}, P^{(1)})$ and $(u^{(2)}, \theta^{(2)}, P^{(2)})$ satisfying (1.4)–(1.5) be two Yudovich type solutions to the Boussinesq system (1.2) with the same initial data (u_0, θ_0). Denote $\delta u = u^{(1)} - u^{(2)}$, $\delta\theta = \theta^{(1)} - \theta^{(2)}$ and $\delta P = P^{(1)} - P^{(2)}$, then the difference system writes

$$\begin{cases} \partial_t \delta u + u^{(1)} \cdot \nabla \delta u + \nabla \delta P = \delta\theta e_2 - \delta u \cdot \nabla u^{(2)}, \\ \partial_t \delta\theta + u^{(1)} \cdot \nabla \delta\theta + |D|^\beta \delta\theta = -\delta u \cdot \nabla \theta^{(2)}, \\ \operatorname{div} u^{(1)} = \operatorname{div} \delta u = 0, \qquad (\delta u, \delta\theta)|_{t=0} = 0. \end{cases}$$

From the usual energy method we have that for every $\widetilde{p} \in [p, \infty[$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta u(t)\|_{L^2}^2 &\leqslant \|\delta\theta(t)\|_{L^2}\|\delta u(t)\|_{L^2} + \|\delta u(t)\|^2_{L^{2\widetilde{p}/(\widetilde{p}-1)}}\|\nabla u^{(2)}(t)\|_{L^{\widetilde{p}}} \\ &\leqslant \|\delta\theta(t)\|_{L^2}\|\delta u(t)\|_{L^2} + C_0 \widetilde{p}\|\delta u(t)\|_{L^2}^{2(1-1/\widetilde{p})}\|\delta u(t)\|_{L^\infty}^{2/\widetilde{p}}\|\omega^{(2)}(t)\|_{L^p\cap L^\infty}, \end{aligned} \tag{4.36}$$

[1] But it is not clear to obtain $\theta \in C([0, T_0[; L^r(\mathbb{R}^2))$ $(r \in [1, 2[)$.

and

$$\frac{1}{2}\frac{d}{dt}\left\|\delta\theta(t)\right\|_{L^2}^2 \leqslant \|\delta u\|_{L^2}\left\|\nabla\theta^{(2)}(t)\right\|_{L^\infty}\left\|\delta\theta(t)\right\|_{L^2},$$

where in the last line of (4.36) we have used interpolation and the Calderón–Zygmund theorem that

$$\sup_{\widetilde{p}\in[p,\infty[}\frac{\|\nabla u^{(2)}\|_{L^{\widetilde{p}}}}{\widetilde{p}} \lesssim \sup_{\widetilde{p}\in[p,\infty[}\left\|\omega^{(2)}\right\|_{L^{\widetilde{p}}} \lesssim \left\|\omega^{(2)}\right\|_{L^p\cap L^\infty}.$$

Let $\epsilon > 0$ be a small number, and set $\mathcal{X}_\epsilon(t) = \sqrt{\epsilon^2 + \|\delta u\|_{L^2}^2 + \|\delta\theta(t)\|_{L^2}^2}$, then we get

$$\frac{d}{dt}\mathcal{X}_\epsilon(t) \leqslant C_0\widetilde{p}\left\|\delta u(t)\right\|_{L^\infty}^{2/\widetilde{p}}\left\|\omega^{(2)}(t)\right\|_{L^p\cap L^\infty}\mathcal{X}_\epsilon(t)^{1-2/\widetilde{p}} + \left(1 + \left\|\nabla\theta^{(2)}(t)\right\|_{L^\infty}\right)\mathcal{X}_\epsilon(t).$$

By a direct computation, we infer that

$$\mathcal{X}_\epsilon(t) \leqslant e^{t+\|\nabla\theta^{(2)}\|_{L^1_t L^\infty}}\left(\epsilon^{2/\widetilde{p}} + C_0\int_0^t \left\|\delta u(\tau)\right\|_{L^\infty}^{2/\widetilde{p}}\left\|\omega^{(2)}(\tau)\right\|_{L^p\cap L^\infty}\mathrm{d}\tau\right)^{\frac{\widetilde{p}}{2}}.$$

Passing ϵ to 0 yields

$$\left\|\delta u(t)\right\|_{L^2}^2 + \left\|\delta\theta(t)\right\|_{L^2}^2 \leqslant e^{2t+2\|\nabla\theta^{(2)}\|_{L^1_t L^\infty}}\|\delta u\|_{L^\infty_t L^\infty}^2\left(C_0 t\left\|\omega^{(2)}(\tau)\right\|_{L^\infty_t(L^p\cap L^\infty)}\right)^{\widetilde{p}}.$$

Since $\theta^{(2)} \in L^1([0,T]; B^1_{\infty,1})$, $\delta u \in L^\infty([0,T]; L^\infty)$ and $\omega^{(2)} \in L^\infty([0,T]; L^p \cap L^\infty)$, by choosing $t > 0$ small enough, say t_0, we have $C_0 t_0\|\omega_1\|_{L^\infty_{t_0}(L^p\cap L^\infty)} \leqslant \frac{1}{2}$. Then letting $\widetilde{p}$ tend to ∞, we deduce $(\delta u, \delta\theta) \equiv 0$ on $[0, t_0]$. Since $(\delta u, \delta\theta) \in C([0,T]; L^2(\mathbb{R}^2))$, from a connectivity argument we can show the uniqueness on $[0,T]$.

4.4. Refined blowup criterion

Let $\beta \in]0,1]$, (u,ω,θ) be the Yudovich type solution on $[0,T^*[$ to the Boussinesq system (1.2) constructed in Subsection 4.2, with T^* the maximal existence time. If $T^* < \infty$, our target is to prove the following blowup criterion

$$\int_0^{T^*}\left\|\theta(t)\right\|_{B^{1-\beta}_{\infty,2}}\mathrm{d}t = \infty. \tag{4.37}$$

Note that from the existence part, we have already known a natural blowup criterion that

$$\|\omega\|_{L^\infty([0,T^*[;L^\infty\cap L^p)} + \|\theta\|_{L^\infty([0,T^*[;B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1})} = \infty.$$

Thus if we assume that for any $T < T^*$,

$$B(T) := \int_0^T \|\theta(t)\|_{B^{1-\beta}_{\infty,2}} \, \mathrm{d}t < \infty, \tag{4.38}$$

it suffices to show the upper bound of $\|\omega\|_{L^\infty_T(L^\infty \cap L^p)} + \|\theta\|_{L^\infty_T(B^{1-\beta}_{\infty,1} \cap B^{1-\beta}_{p,1})}$.

Based on (4.38), we first prove that we can derive the upper bound of $\|\omega\|_{L^\infty_T L^{\widetilde{p}}} + \|\theta\|_{\widetilde{L}^1_t B^1_{\widetilde{p},2}}$ for every $\widetilde{p} \in [p, \infty[$ ($p \in [2, \infty[$ is the fixed number in Theorem 1.1). Here we adopt an idea of [17,25] to apply the structure of the following coupling system of (ω, θ)

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta, \\ \partial_t \theta + u \cdot \nabla \theta + |D|^\beta \theta = 0. \end{cases}$$

Denote by $\mathcal{R}_\beta := \partial_1/|D|^\beta = |D|^{1-\beta}\mathcal{R}_1$ ($\beta \in]0, 1]$) with $\mathcal{R}_1$ the usual Riesz transform, then applying the operator $\mathcal{R}_\beta$ to the equation of θ yields

$$\partial_t \mathcal{R}_\beta \theta + u \cdot \nabla \mathcal{R}_\beta \theta + |D|^\beta \mathcal{R}_\beta \theta = -[\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{4.39}$$

Noting that $|D|^\beta \mathcal{R}_\beta \theta = \partial_1 \theta$ and by setting

$$\Gamma := \omega + \mathcal{R}_\beta \theta,$$

we get

$$\partial_t \Gamma + u \cdot \nabla \Gamma = -[\mathcal{R}_\beta, u \cdot \nabla]\theta. \tag{4.40}$$

From the $L^{\widetilde{p}}$-estimate of the transport equation and the continuous embedding $B^0_{\widetilde{p},2} \hookrightarrow L^{\widetilde{p}}$ ($\widetilde{p} \in [2, \infty[$), we have that for every $t \in [0, T]$,

$$\begin{aligned} \|\Gamma(t)\|_{L^{\widetilde{p}}} &\leqslant \|\Gamma_0\|_{L^{\widetilde{p}}} + \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{L^{\widetilde{p}}} \, \mathrm{d}\tau \\ &\leqslant C_0 \|\omega_0\|_{L^{\widetilde{p}}} + C_0 \|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_0 \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{\widetilde{p},2}} \, \mathrm{d}\tau, \end{aligned}$$

where we have used the following estimate that $\|\mathcal{R}_\beta \theta_0\|_{L^{\widetilde{p}}} \lesssim \||D|^{1-\beta}\theta_0\|_{B^0_{\widetilde{p},2}} \lesssim \|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}}$. By virtue of Lemma 3.2 and the Calderón–Zygmund theorem, we see that

$$\begin{aligned} \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{\widetilde{p},2}} &\lesssim_{\beta,\widetilde{p}} \|\nabla u(\tau)\|_{L^{\widetilde{p}}} \big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}} + \|\theta(\tau)\|_{L^2}\big) \\ &\lesssim_{\beta,\widetilde{p}} \|\omega(\tau)\|_{L^{\widetilde{p}}} \big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}} + \|\theta_0\|_{L^2}\big). \end{aligned}$$

Thus

$$\|\Gamma(t)\|_{L^p} \leqslant C_0\|\omega_0\|_{L^{\widetilde{p}}} + C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_{\widetilde{p},\beta}\int_0^t \|\omega(\tau)\|_{L^{\widetilde{p}}}\big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}} + \|\theta_0\|_{L^2}\big)\,\mathrm{d}\tau.$$

Similarly, from the L^p-estimate of Eq. (4.39), we get

$$\begin{aligned}\|\mathcal{R}_\beta\theta(t)\|_{L^{\widetilde{p}}} &\leqslant \|\mathcal{R}_\beta\theta_0\|_{L^{\widetilde{p}}} + \int_0^t \|[\mathcal{R}_\beta, u\cdot\nabla]\theta(\tau)\|_{L^{\widetilde{p}}}\,\mathrm{d}\tau \\ &\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_{p,\beta}\int_0^t \|\omega(\tau)\|_{L^{\widetilde{p}}}\big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}} + \|\theta_0\|_{L^2}\big)\,\mathrm{d}\tau.\end{aligned}$$

Hence, gathering the upper two estimates yields

$$\begin{aligned}\|\omega(t)\|_{L^{\widetilde{p}}} \leqslant \|\Gamma(t)\|_{L^{\widetilde{p}}} + \|\mathcal{R}_\beta\theta(t)\|_{L^{\widetilde{p}}} &\leqslant C_0\|\omega_0\|_{L^{\widetilde{p}}} + C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} \\ &\quad + C_{\widetilde{p},\beta}\int_0^t \|\omega(\tau)\|_{L^{\widetilde{p}}}\big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}} + \|\theta_0\|_{L^2}\big)\,\mathrm{d}\tau.\end{aligned}$$

Gronwall inequality ensures that for every $t \in [0,T]$, $\widetilde{p} \in [p,\infty[$ and $p \in [2,\infty[$,

$$\|\omega(t)\|_{L^{\widetilde{p}}} \leqslant C_0\big(\|\omega_0\|_{L^{\widetilde{p}}} + \|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}}\big)e^{C_{\widetilde{p},\beta}t\|\theta_0\|_{L^2} + C_{\widetilde{p},\beta}B(t)} \tag{4.41}$$

with $B(t)$ given by (4.38). Now we intend to show the higher regularity estimate of θ. For the frequency localized equation

$$\partial_t\Delta_q\theta + u\cdot\nabla\Delta_q\theta + |D|^\beta\Delta_q\theta = -[\Delta_q, u\cdot\nabla]\theta, \quad q\in\mathbb{N},$$

from (4.13), we infer that

$$2^{q\beta}\|\Delta_q\theta\|_{L^1_tL^{\widetilde{p}}} \lesssim \|\Delta_q\theta_0\|_{L^{\widetilde{p}}} + \int_0^t \|[\Delta_q, u\cdot\nabla]\theta(\tau)\|_{L^{\widetilde{p}}}\,\mathrm{d}\tau.$$

Thanks to the Minkowski inequality and (3.2), we obtain

$$\begin{aligned}\Big(\sum_{q\in\mathbb{N}} 2^{2q}\|\Delta_q\theta\|^2_{L^1_tL^{\widetilde{p}}}\Big)^{1/2} &\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_0\int_0^t\Big(\sum_{q\in\mathbb{N}} 2^{2q(1-\beta)}\|[\Delta_q, u\cdot\nabla]\theta(\tau)\|^2_{L^{\widetilde{p}}}\Big)^{1/2}\mathrm{d}\tau \\ &\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_{\widetilde{p},\beta}\int_0^t \|\omega(\tau)\|_{L^{\widetilde{p}}}\|\theta(\tau)\|_{B^{1-\beta}_{\infty,2}}\,\mathrm{d}\tau \\ &\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_{\widetilde{p},\beta}\|\omega\|_{L^\infty_tL^{\widetilde{p}}}B(t).\end{aligned}$$

Hence, from (4.41) we know that for $p \in [2, \infty[$ and every $\widetilde{p} \in [p, \infty[$,

$$\begin{aligned} \|\theta\|_{\widetilde{L}^1_t B^1_{\widetilde{p},2}} &\leqslant C_0 \|\Delta_{-1}\theta\|_{L^1_t L^{\widetilde{p}}} + C_0\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + C_{\widetilde{p},\beta}\|\omega\|_{L^\infty_t L^{\widetilde{p}}} B(t) \\ &\leqslant C_0 t\|\theta_0\|_{L^2} + C_{\widetilde{p},\beta}\big(\|\theta_0\|_{B^{1-\beta}_{\widetilde{p},2}} + \|\omega_0\|_{L^{\widetilde{p}}}\big) e^{C_{\widetilde{p},\beta}\|\theta_0\|_{L^2} t + C_{\widetilde{p},\beta} B(t)}. \end{aligned} \tag{4.42}$$

In particular, by setting $p_2 := \max\{\frac{6}{\beta}, p\}$ and the Besov embedding, we have

$$\begin{aligned} \|\theta\|_{\widetilde{L}^1_t B^{1-2/p_2}_{\infty,2}} \lesssim \|\theta\|_{\widetilde{L}^1_t B^1_{p_2,2}} &\leqslant C_0 t\|\theta_0\|_{L^2} \\ &\quad + C_{p,\beta}\big(\|\theta_0\|_{B^{1-\beta}_{p_2,2}} + \|\omega_0\|_{L^{p_2}}\big) e^{C_{p,\beta}\|\theta_0\|_{L^2} t + C_{p,\beta} B(t)}. \end{aligned} \tag{4.43}$$

Next we are devoted to derive the bound of $\|\omega\|_{L^\infty_T L^\infty}$ and $\|\theta\|_{L^\infty_T (B^{1-\beta}_{\infty,1} \cap B^{1-\beta}_{p,1})}$. From the maximum principle of Eqs. (4.39)–(4.40), we see that

$$\begin{aligned} \big\|\Gamma(t)\big\|_{L^\infty} + \big\|\mathcal{R}_\beta\theta(t)\big\|_{L^\infty} &\leqslant \|\Gamma_0\|_{L^\infty} + \|\mathcal{R}_\beta\theta_0\|_{L^\infty} + 2\int_0^t \big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta(\tau)\big\|_{L^\infty}\,\mathrm{d}\tau \\ &\leqslant \|\omega_0\|_{L^\infty} + C_0\|\theta_0\|_{L^2\cap B^{1-\beta}_{\infty,1}} + C_0\int_0^t \big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta(\tau)\big\|_{B^0_{\infty,1}}\,\mathrm{d}\tau, \end{aligned}$$

where in the second line we have used the estimate $\|\mathcal{R}_\beta\theta_0\|_{L^\infty} \leqslant \|\mathcal{R}_\beta\theta_0\|_{B^0_{\infty,1}} \lesssim \|\theta_0\|_{L^2\cap B^{1-\beta}_{\infty,1}}$ for $\beta \in]0,1]$. According to Lemma 3.2 and the embedding, we treat the commutator term as follows (noting that $p_2 = \max\{\frac{6}{\beta}, p\}$)

$$\begin{aligned} \big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta(\tau)\big\|_{B^0_{\infty,1}} &\lesssim_{p,\beta} \big\|\nabla u(\tau)\big\|_{L^{p_2}}\big(\big\|\theta(\tau)\big\|_{B^{1-\beta+2/p_2}_{\infty,1}} + \big\|\theta(\tau)\big\|_{L^2}\big) \\ &\lesssim_{p,\beta} \big\|\omega(\tau)\big\|_{L^{p_2}}\big(\big\|\theta(\tau)\big\|_{B^{1-4/p_2}_{\infty,1}} + \|\theta_0\|_{L^2}\big). \end{aligned}$$

Hence from (4.41), (4.43) and the Besov embedding $B^{1-2/p_2}_{\infty,2} \hookrightarrow B^{1-4/p_2}_{\infty,1}$, we deduce that

$$\begin{aligned} \big\|\omega(t)\big\|_{L^\infty} &\leqslant \big\|\Gamma(t)\big\|_{L^\infty} + \big\|\mathcal{R}_\beta\theta(t)\big\|_{L^\infty} \\ &\leqslant \|\omega_0\|_{L^\infty} + C_0\|\theta_0\|_{L^2\cap B^{1-\beta}_{\infty,1}} + C_{p,\beta}\|\omega\|_{L^\infty_t L^{p_1}}\big(\|\theta\|_{L^1_t B^{1-4/p_1}_{\infty,1}} + t\|\theta_0\|_{L^2}\big) \\ &\leqslant C_{p,\beta}\big(1 + \|\omega_0\|_{L^p\cap L^\infty} + \|\theta_0\|_{L^2\cap B^{1-\beta}_{p,2}\cap B^{1-\beta}_{\infty,1}}\big)^2 e^{C_{p,\beta} t\|\theta_0\|_{L^2} + C_{p,\beta}B(t)}. \end{aligned}$$

Taking advantage of (4.15) and the fact that $1 - 4/p_2 \geqslant 1 - 2\beta/3 > 1 - \beta$ lead to

$$\begin{aligned} &\|\theta\|_{L^\infty_t B^{1-\beta}_{\infty,1}} + \|\theta\|_{L^\infty_t B^{1-\beta}_{p,1}} \\ &\quad\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} \\ &\qquad + C_\beta\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\Big(\|\theta\|_{L^1_t(L^\infty\cap L^p)} + \sum_{q\in\mathbb{N}} 2^{q(1-\beta)}(q+1)\|\Delta_q\theta\|_{L^1_t(L^\infty\cap L^p)}\Big) \end{aligned}$$

$$\leqslant C_0\|\theta_0\|_{B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1}} + C_{p,\beta}\|\omega\|_{L^\infty_t(L^\infty\cap L^p)}\|\theta\|_{L^1_t(B^{1-4/p_2}_{\infty,1}\cap B^{1-\beta/2}_{p,1})}$$

$$\leqslant C_{p,\beta}\big(1+\|\omega_0\|_{L^p\cap L^\infty}+\|\theta_0\|_{L^2\cap B^{1-\beta}_{p,1}\cap B^{1-\beta}_{\infty,1}}\big)^3 e^{C_{p,\beta}t\|\theta_0\|_{L^2}+C_{p,\beta}B(t)}.$$

Therefore, if $B(T)<\infty$, we get $\|\omega\|_{L^\infty_T(L^\infty\cap L^p)}+\|\theta\|_{L^\infty_T(B^{1-\beta}_{\infty,1}\cap B^{1-\beta}_{p,1})}<\infty$, and this finishes the proof of the blowup criterion (4.37).

4.5. Global regularity in the critical case

For the critical case $\beta=1$, from [17, Prop. 5.2], we *a priori* have the uniform bound that $\|\theta\|_{L^1_t B^0_{\infty,2}}\leqslant C_0(1+t^2)$, $\forall t\in\mathbb{R}^+$, which is obtained only under the condition that $\omega_0\in L^p$, $\theta_0\in L^p\cap L^\infty$ for $p\in\,]2,\infty[$, thus as a consequence of the refined blowup criterion (4.37), we immediately obtain the global result.

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