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Global well-posedness for the 2D inviscid Bénard system with fractional diffusivity and Yudovich's type data

Gang Wu [a], Liutang Xue [b,*]

[a] School of Mathematical Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, PR China
[b] The Graduate School of China Academy of Engineering Physics, P.O. Box 2101, Beijing 100088, PR China

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ABSTRACT

In this paper we consider the Cauchy problem of the two-dimensional inviscid Bénard system with fractional diffusivity. We show that there is a global unique solution to this system with Yudovich's type data.

1. Introduction

In this article we focus on the two-dimensional inviscid Bénard system with fractional diffusivity

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = \theta e_2, \\ \partial_t \theta + u\cdot\nabla\theta + \kappa|D|^\beta\theta = \gamma u_2, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u^0, \quad \theta|_{t=0} = \theta^0, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: wugangmaths@yahoo.com.cn (G. Wu), xue_lt@163.com (L. Xue).

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where $\beta \in]0,2]$, $e_2 \triangleq (0,1)$ is the canonical vector. The unknowns are the velocity field $u=(u^1,u^2)$, the temperature θ and the pressure P. The coefficient $\kappa > 0$ is the thermal diffusivity, $\gamma \geqslant 0$ is a non-dimensional coefficient and the fractional dissipation operator $|D|^\beta$ is defined via the Fourier transform

$$\widehat{|D|^\beta f}(\xi) = |\xi|^\beta \widehat{f}(\xi).$$

The fractional dissipation operator has been used to model many physical phenomena (cf. [14]) in hydrodynamics and molecular biology such as anomalous diffusion in semiconductor growth [24]. We call the system (1.1) the inviscid Bénard system since in the case of $\beta = 2$, it describes the Rayleigh–Bénard convective motion in a heated inviscid fluid (see e.g. [1,7,21,25]). In this case, the forcing term θe_2 in the velocity equation models the acting of the buoyancy force on the fluid motion.

When $\beta = 2$ and $\gamma \geqslant 0$, Danchin and Paicu in [12] proved that the inviscid Bénard system has a global unique solution for Yudovich's type data, that is, the initial data (u^0,θ^0) has finite energy and bounded vorticity and the initial temperature θ^0 is also under a natural additional assumption (precisely, $\theta^0 \in L^2 \cap B^{-1}_{\infty,1}$). The authors also showed the global result in the case of infinite energy velocity field which can admit the vortex-patches-like structures.

When $\gamma = 0$, the system (1.1) is also often referred to as the inviscid Boussinesq system, which is related with many models arising from atmospheric and oceanographic dynamics (cf. [23]). Due to its physical significance and mathematical relevance, there have been intense works studying on the viscous or inviscid Boussinesq system (e.g. [4,5,11–13,15–17,19,22] and the references therein), and here we only recall some noticeable works about the 2D inviscid Boussinesq system. For $\beta = 2$, the global well-posedness of the smooth solution for the system (1.1) was settled independently by Chae [5] and Hou and Li [19] almost at the same time. Moreover, Hmidi et al. [15] proved the global result for this system with rough data, precisely, they required that $u^0 \in B^{1+2/p}_{p,1}$ with $p \in]2,\infty]$ and θ^0 belongs to a suitable Lebesgue space. This result indeed extended the work of Vishik [27] on the 2D incompressible Euler equations. Another further improvement to the less regular data was achieved by [12] mentioned above, as a natural extension of the important work of Yudovich [28]. If the full Laplacian is replaced by the fractional dissipation, for $\beta \in]1,2[$, this corresponds to the subcritical case, and Hmidi et al. [18] followed [15] and showed the global well-posedness of the system (1.1) with rough data, more precisely, $u^0 \in B^{1+2/p}_{p,1}$ with $p \in]1,\infty[$ and $\theta^0 \in B^{1-\beta+2/p}_{p,1} \cap L^r$ with $r \in]\frac{2}{\beta-1},\infty[$. For the subtle critical case $\beta = 1$, Hmidi et al. in [16] proved the global result for the system (1.1) by deeply developing the structures of the coupling system. On the other hand, if the fractional dissipation $|D|^\beta\theta$ is replaced by the partial horizontal dissipation $\partial_1^2\theta$ or vertical dissipation $\partial_2^2\theta$ in the Boussinesq system, we refer the readers to the interesting works [13] and [4].

In this article, we are devoted to continue the works [12,28] to show the global unique solution for the inviscid Bénard system (1.1) with fractional diffusivity and Yudovich's type data. For brevity, we set $\kappa = \gamma = 1$ (noting that $\gamma = 0$ is a simpler case). Our main result reads as follows.

Theorem 1.1. *Let $\kappa = \gamma = 1$, $\beta \in]1,2[$, $\theta^0 \in L^2 \cap B^{1-\beta}_{\infty,1}$ and $u^0 \in L^2(\mathbb{R}^2)$ be a divergence-free vector field. In addition assume that the initial vorticity $\omega^0 \triangleq \partial_1 u^{2,0} - \partial_2 u^{1,0}$ satisfies $\omega^0 \in L^p \cap L^\infty$ with $2 \leqslant p < \infty$. Then the inviscid Bénard system (1.1) generates a unique global solution (u,θ) such that*

$$u \in \mathcal{C}^{0,1}_{\mathrm{loc}}\big(\mathbb{R}^+; L^2(\mathbb{R}^2)\big), \qquad \omega \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^\infty \cap L^p\big) \quad \text{and} \tag{1.2}$$

$$\theta \in \mathcal{C}\big(\mathbb{R}^+; L^2 \cap B^{1-\beta}_{\infty,1}\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+; H^{\beta/2}\big) \cap L^1_{\mathrm{loc}}\big(\mathbb{R}^+; B^1_{\infty,1}\big). \tag{1.3}$$

Compared with [12], our result is more involved in the proof. Indeed, it seems impossible to directly apply the method of [12] to the case $\beta \in]1,2[$: one point is that from the energy estimate we get $\theta \in L^2_T \dot{H}^{\beta/2}$, which only implies that $\partial_1\theta \in L^2_T \dot{H}^{-(1-\beta/2)}$; the second point lies on that when trying to obtain $\omega \in L^\infty_T L^\infty$ and $\theta \in L^1_T B^1_{\infty,1}$, we cannot get a suitable Gronwall-type inequality about

$\|\omega(t)\|_{L^\infty}$ or $\|\theta\|_{L^1_t B^1_{\infty,1}}$ (from (2.6) and a simple computation). Rather, we shall more deeply study the system (1.1) to show the global result; precisely, motivated by [16], we shall consider the coupling system of vorticity ω and temperature θ

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta, \\ \partial_t \theta + u \cdot \nabla \theta + |D|^\beta \theta = u_2, \end{cases}$$

and by introducing $\Gamma = \omega + \mathcal{R}_\beta \theta$ with $\mathcal{R}_\beta = \partial_1 |D|^{-\beta}$, we see that Γ solves the following more treatable equation

$$\partial_t \Gamma + u \cdot \nabla \Gamma = \mathcal{R}_\beta u_2 - [\mathcal{R}_\beta, u \cdot \nabla]\theta,$$

where $[A, B] \triangleq AB - BA$ is the commutator operator, then the desired estimate of ω can be obtained from the information of Γ and $\mathcal{R}_\beta \theta$. In the process, the commutator estimates involving $\mathcal{R}_\beta$ will be encountered and we shall settle them at Proposition 4.2 below; another key point is that we need to derive the L^∞-bound of $\theta(t)$ only from the L^2-information of (u, θ) (which is not necessary in [12]), and we shall exploit the fundamental DeGiorgi–Nash estimate for the transport-diffusion equation with forcing term (cf. Proposition 3.1 below) to reach the target.

We also want to stress one point that if one *a priori* knows that the velocity field is not Lipschitzian and only satisfies $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^\infty)$, it is not clear to propagate the initial regularity for the transport(-diffusion) equation equipped with this velocity field. But if the regularity index is negative, the initial regularity can be almost preserved (with a limited loss of regularity) for the transport-diffusion equation and the smoothing effect can also be obtained. This is shown in Proposition 3.2 below and it plays an important role in the proof.

Remark 1.2. The additional assumption that $\theta^0 \in B^{1-\beta}_{\infty,1}$ is quite natural due to that we always expect that vorticity ω is bounded for all positive time. In fact, noticing that $\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta$, there is no gain of smoothness from this transport equation, thus we have to require that $\partial_1 \theta \in L^1_{\mathrm{loc}}(\mathbb{R}^+; L^\infty)$. Furthermore, from the equation $\partial_t \theta + |D|^\beta \theta = -u \cdot \nabla \theta + u_2$, we at least call for that

$$e^{-t|D|^\beta} \nabla \theta^0 \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+; L^\infty\big),$$

where the semigroup operator $e^{-t|D|^\beta}$ is defined by $\mathcal{F}(e^{-t|D|^\beta} f)(\xi) = e^{-t|\xi|^\beta} \widehat{f}(\xi)$. For $T > 0$ suitably large, by the following characterization of inhomogeneous Besov space in terms of semigroup $e^{-t|D|^\beta}$ (with the proof in Appendix A) that

$$c_{T,\beta} \|f\|_{B^{-\beta}_{\infty,1}} \leqslant \big\|e^{-t|D|^\beta} f\big\|_{L^1_T L^\infty} \leqslant C_{T,\beta} \|f\|_{B^{-\beta}_{\infty,1}}, \tag{1.4}$$

where $c_{T,\beta}, C_{T,\beta}$ are positive constants depending only on T, β, we infer that $\nabla \theta^0 \in B^{-\beta}_{\infty,1}$ and thus $\theta^0 \in B^{1-\beta}_{\infty,1}$.

Remark 1.3. When $\beta = 1$ and $\kappa = \gamma = 1$, the system (1.1) corresponds to the 2D inviscid Bénard system with critical diffusivity, and it is not clear how to show the global regularity in this case. The obstacle we have to overcome first is the lack of the L^∞-information of θ; since Proposition 3.1 do not concern the endpoint case that $\{\beta = 1, p = \infty, q = n = 2\}$. However, if $\gamma = 0$, we do not need to face that problem, and as we know, the global regularity issue for the corresponding critical system has been solved by [16], which calls for $u^0 \in B^1_{\infty,1} \cap \dot{W}^{1,p}$ and $\theta^0 \in L^p \cap B^0_{\infty,1}$ with $p \in]2, \infty[$. Yet for the Yudovich's type data like in Theorem 1.1, the issue of global regularity is still not clear; and it has some difficulty in deriving the *a priori* estimate of $\|\nabla \theta\|_{L^1_t L^\infty}$ (which plays a role in the uniqueness part).

Remark 1.4. By following the scheme of [6,12], Theorem 1.1 can be generalized to the initial velocities which are L^2 perturbations of infinite energy smooth stationary solutions for the 2D incompressible Euler equations, and here we omit the details. Note that in this way the data can admit the vortex-patches-like structures.

The paper is organized as follows. Section 2 is devoted to present some preparatory results on Besov spaces, and give some useful preliminary lemmas. In Section 3, we show some crucial *a priori* estimates about the linear transport-diffusion equation. Section 4 concerns the operator $\mathcal{R}_\beta$, and we treat some commutator estimates involving R_β. We prove our main result in Section 5. At last, we sketch the proof of (1.4) in Appendix A.

2. Preliminaries

In the preparatory section, we introduce some common notations and some basic points about Besov spaces, and compile some auxiliary lemmas.

Throughout this paper the following notations will be used.

- ⋄ The notion $X \lesssim Y$ means that there exists a positive harmless constant C such that $X \leqslant CY$. $X \approx Y$ means that both $X \lesssim Y$ and $Y \lesssim X$ are satisfied.
- ⋄ $\mathcal{D}(\mathbb{R}^n)$ denotes the space of test functions, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class, and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.
- ⋄ We use $\mathcal{F}f$ or $\widehat{f}$ to denote the Fourier transform of a tempered distribution f.

In order to define Besov space we need the following dyadic partition of unity (see e.g. [6]). Choose two nonnegative radial functions χ, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ which are supported respectively in the ball $\{\xi \in \mathbb{R}^n\colon |\xi| \leqslant \frac{4}{3}\}$ and the shell $\{\xi \in \mathbb{R}^n\colon\ 1 \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi\big(2^{-j}\xi\big) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the nonhomogeneous Littlewood–Paley operators

$$\Delta_{-1} f \triangleq \chi(D) f, \qquad \Delta_j f \triangleq \varphi\big(2^{-j}D\big) f, \qquad S_j f \triangleq \sum_{-1\leqslant k\leqslant j-1} \Delta_k f, \quad \forall j \in \mathbb{N}.$$

Now we introduce the definition of Besov spaces. Let $(p,r) \in [1,\infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space

$$B^s_{p,r} \triangleq \big\{f \in \mathcal{S}'\big(\mathbb{R}^n\big);\ \|f\|_{B^s_{p,r}} \triangleq \big\|\big\{2^{js}\|\Delta_j f\|_{L^p}\big\}_{j\geqslant -1}\big\|_{\ell^r} < \infty\big\}.$$

We point out that for all $s \in \mathbb{R}$, $B^s_{2,2} = H^s$.

Next we introduce two kinds of space–time Besov spaces. The first one is the classical space–time Besov space $L^\rho([0,T], B^s_{p,r})$, abbreviated by $L^\rho_T B^s_{p,r}$, which is the set of tempered distribution f such that

$$\|f\|_{L^\rho_T B^s_{p,r}} \triangleq \big\|\big\|\big\{2^{js}\|\Delta_j f\|_{L^p}\big\}_{j\geqslant -1}\big\|_{\ell^r}\big\|_{L^\rho([0,T])} < \infty.$$

The second one is the Chemin–Lerner's mixed space–time Besov space $\widetilde{L}^\rho([0,T], B^s_{p,r})$, abbreviated by $\widetilde{L}^\rho_T B^s_{p,r}$, which is the set of tempered distribution f satisfying

$$\|f\|_{\widetilde{L}^\rho_T B^s_{p,r}} \triangleq \big\|\big\{2^{js}\|\Delta_j f\|_{L^\rho_T L^p}\big\}_{j\geqslant -1}\big\|_{\ell^r} < \infty.$$

Due to Minkowiski's inequality, we immediately obtain

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r}, \quad \text{if } r \geqslant \rho, \quad \text{and} \quad \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \quad \text{if } \rho \geqslant r.$$

Bernstein's inequality is fundamental in the analysis involving Besov spaces (see e.g. [2]).

Lemma 2.1. *Let $f \in L^a$, $1 \leqslant a \leqslant b \leqslant \infty$. Then for every $(k, q) \in \mathbb{N}^2$ there exists a constant $C > 0$ such that*

$$\sup_{|\alpha|=k} \left\| \partial^\alpha S_q f \right\|_{L^b} \leqslant C 2^{q(k+n(\frac{1}{a}-\frac{1}{b}))} \| S_q f \|_{L^a},$$

$$C^{-1} 2^{qk} \| \Delta_q f \|_{L^a} \leqslant \sup_{|\alpha|=k} \left\| \partial^\alpha \Delta_q f \right\|_{L^a} \leqslant C 2^{qk} \| \Delta_q f \|_{L^a}.$$

The following classical L^p-estimate and logarithmic estimate for the transport(-diffusion) equation is shown in [10] and [15,27] respectively.

Proposition 2.2. *Let u be a smooth divergence-free vector field in $\mathbb{R}^n$ and θ be a smooth solution of the transport(-diffusion) equation*

$$\partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\beta \theta = F, \qquad \operatorname{div} u = 0, \quad \theta|_{t=0} = \theta^0, \ \beta \in]0, 2], \tag{2.1}$$

with $\kappa \geqslant 0$. Then,

(1) *for every $p \in [1, \infty]$ and $t \in \mathbb{R}^+$, we have*

$$\left\| \theta(t) \right\|_{L^p} \leqslant \| \theta_0 \|_{L^p} + \int_0^t \left\| F(\tau) \right\|_{L^p} \mathrm{d}\tau; \tag{2.2}$$

(2) *for every $(p, r) \in [1, \infty]^2$ and $t \in \mathbb{R}^+$, we have*

$$\| \theta \|_{\widetilde{L}^\infty_t B^0_{p,r}} \leqslant C \left(1 + \int_0^t \left\| \nabla u(\tau) \right\|_{L^\infty} \mathrm{d}\tau \right) \left(\left\| \theta^0 \right\|_{B^0_{p,r}} + \| F \|_{\widetilde{L}^1_t B^0_{p,r}} \right). \tag{2.3}$$

We also use the following maximal regularity estimate of the linear dissipative equation.

Lemma 2.3. *Let θ be the smooth solution of the linear dissipative equation*

$$\partial_t \theta + |D|^\beta \theta = F, \quad \theta|_{t=0} = \theta^0, \ \beta \in]0, 2].$$

Then for every $s \in \mathbb{R}$, $1 \leqslant \sigma_1 \leqslant \sigma \leqslant \infty$ and $(p, r) \in [1, \infty]^2$, we have

$$\| \theta \|_{\widetilde{L}^\sigma_t B^{s+\beta/\rho}_{p,r}} \leqslant C (1+t)^{1/\sigma} \left(\left\| \theta^0 \right\|_{B^s_{p,r}} + \left(1 + t^{1-1/\sigma_1} \right) \| F \|_{\widetilde{L}^{\sigma_1}_t B^{s-\beta+\beta/\sigma_1}_{p,r}} \right). \tag{2.4}$$

Note that when $\beta=2$, estimate (2.4) has occurred in many references, e.g. [12].

Proof. Duhamel's formula leads to

$$\theta(t,x)=e^{-t|D|^\beta}\theta^0(x)+\int_0^t e^{-(t-\tau)|D|^\beta}F(\tau,x)\,\mathrm{d}\tau.$$

For every $j\in\mathbb{N}$, by virtue of the following estimate (cf. [2, Lemma 2.4] and its generalization) that

$$\big\|\Delta_j e^{-t|D|^\beta}f\big\|_{L^p}\leqslant Ce^{-ct2^{j\beta}}\|\Delta_j f\|_{L^p}, \tag{2.5}$$

with C, c absolute constants independent of j, and from Young's inequality, we find that

$$\|\Delta_j\theta\|_{L^\sigma_t L^p}\leqslant C2^{-j\beta/\sigma}\big(\big\|\Delta_j\theta^0\big\|_{L^p}+2^{-j(\beta-\beta/\sigma_1)}\|\Delta_j F\|_{L^{\sigma_1}_t L^p}\big).$$

For $j=-1$, due to that the semigroup operator $e^{-\kappa t|D|^\beta}$ is bounded on L^p, we deduce that

$$\|\Delta_{-1}\theta\|_{L^\sigma_t L^p}\leqslant t^{1/\sigma}\|\Delta_{-1}\theta\|_{L^\infty_t L^p}\leqslant Ct^{1/\sigma}\big(\big\|\Delta_{-1}\theta^0\big\|_{L^p}+t^{1-1/\sigma_1}\|\Delta_{-1}F\|_{L^{\sigma_1}_t L^p}\big).$$

Collecting the upper two estimates, multiplying both sides by 2^{js} and taking ℓ^r norm over j, we have

$$\|\theta\|_{\widetilde{L}^\sigma_t B^{s+\beta/\sigma}_{p,r}}\leqslant C(1+t)^{1/\sigma}\big(\big\|\theta^0\big\|_{B^s_{p,r}}+(1+t)^{1-1/\sigma_1}\|F\|_{\widetilde{L}^{\sigma_1}_t B^s_{p,r}}\big). \qquad \square$$

The product estimate as follows is useful in the proof.

Lemma 2.4. *Let u be a smooth divergence-free vector field of $\mathbb{R}^n$ and f be a smooth function. Then we have that for every $s\in\,]0,1[$ and $(p,r)\in[1,\infty]^2$,*

$$\|u\cdot\nabla f\|_{B^{-s}_{p,r}}\leqslant C\|u\|_{L^\infty}\|f\|_{B^{1-s}_{p,r}}. \tag{2.6}$$

Proof. Thanks to Bony's decomposition, we get

$$\begin{aligned}u\cdot\nabla f&=\sum_{k\in\mathbb{N}}S_{k-1}u\cdot\nabla\Delta_k f+\sum_{k\in\mathbb{N}}\Delta_k u\cdot\nabla S_{k-1}f+\sum_{k\geqslant-1}\widetilde{\Delta}_k u\cdot\nabla\Delta_k f\\&\triangleq A_1+A_2+A_3,\end{aligned}$$

with $\widetilde{\Delta}_k\triangleq\Delta_{k-1}+\Delta_k+\Delta_{k+1}$. It is easy to see that

$$\|A_1\|_{B^{-s}_{p,r}}\lesssim\big\|\big\{2^{-ks}\|S_{k-1}u\cdot\nabla\Delta_k f\|_{L^p}\big\}_{k\geqslant-1}\big\|_{\ell^r_k}\lesssim\|u\|_{L^\infty}\|f\|_{B^{1-s}_{p,r}}.$$

For A_2 and A_3, by a direct computation we find that for every $s\in\,]0,1[$,

$$\begin{aligned}2^{-js}\|\Delta_j A_2\|_{L^p} &\leqslant \sum_{|k-j|\leqslant 4} 2^{-js}\big\|\Delta_j(\Delta_k u\cdot\nabla S_{k-1}f)\big\|_{L^p}\\ &\lesssim \|u\|_{L^\infty}\sum_{|k-j|\leqslant 4}\sum_{k'\leqslant k-2} 2^{(k'-k)s}2^{k'(1-s)}\|\Delta_{k'}f\|_{L^p}\\ &\lesssim c_j\|u\|_{L^\infty}\|f\|_{B^{1-s}_{p,r}},\end{aligned}$$

and

$$\begin{aligned}2^{-js}\|\Delta_j A_3\|_{L^p} &\leqslant 2^{-js}\sum_{k\geqslant j-3}\big\|\Delta_j\nabla\cdot(\widetilde{\Delta}_k u\Delta_k f)\big\|_{L^p}\\ &\lesssim \|u\|_{L^\infty}\sum_{k\geqslant j-3} 2^{(j-k)(1-s)}2^{k(1-s)}\|\Delta_k f\|_{L^p}\\ &\lesssim c_j\|u\|_{L^\infty}\|f\|_{B^{1-s}_{p,r}},\end{aligned}$$

where $(c_j)_{j\geqslant -1}$ satisfies that $\|(c_j)\|_{\ell^r}=1$, thus

$$\|A_2\|_{B^{-s}_{p,r}}+\|A_3\|_{B^{-s}_{p,r}}\lesssim \|u\|_{L^\infty}\|f\|_{B^{1-s}_{p,r}}.$$

Gathering the upper estimates leads to (2.6). □

The lemma as follows is useful in dealing with the commutator term (see e.g. [16]).

Lemma 2.5. *Let $(p,m)\in[1,\infty]^2$, $p\geqslant\bar{m}$ with $\bar{m}=\frac{m}{m-1}$. Then,*

$$\big\|h\star(fg)-f(h\star g)\big\|_{L^p}\leqslant\|xh\|_{L^{\bar{m}}}\|\nabla f\|_{L^p}\|g\|_{L^m}. \tag{2.7}$$

Finally, we recall the following simple lemma concerning the iterative sequence.

Lemma 2.6. *Let $C>0$, $b>1$, $\epsilon>0$ and the nonnegative sequence $\{B_k\}_{k\in\mathbb{N}}$ satisfy the following recurrence relation*

$$B_{k+1}\leqslant Cb^kB_k^{1+\epsilon},\quad \forall k\in\mathbb{N}.$$

Then if B_0 satisfies that

$$B_0\leqslant C^{-1/\epsilon}b^{-1/\epsilon^2}, \tag{2.8}$$

we have $\lim_{k\to\infty}B_k=0$.

Proof. By induction, we can show that $B_k\leqslant b^{-k/\epsilon}C^{-1/\epsilon}b^{-1/\epsilon^2}$ for every $k\in\mathbb{N}$. □

3. On the transport-diffusion equation with fractional dissipation

In this section we consider some crucial *a priori* estimates of the transport-diffusion equation with fractional dissipation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = f, \quad \beta\in\,]0,2[, \\ \operatorname{div} u = 0, \quad \theta(0,x)=\theta^0(x),\ x\in\mathbb{R}^n. \end{cases} \tag{3.1}$$

First is the fundamental DeGiorgi–Nash estimate.

Proposition 3.1. *Let u be a smooth divergence-free vector field in $\mathbb{R}^n$ ($n\geqslant 2$), and θ be a smooth solution of the transport-diffusion equation (3.1). Suppose that $r\in[2,\infty]$, $p\in\,]1,\infty]$ and $q\in\,]\frac{n}{\beta},\infty]$ such that*

$$\frac{\beta}{p}+\frac{n}{q}<\beta, \tag{3.2}$$

and $\theta^0\in L^r(\mathbb{R}^n)$, $f\in L^p_T L^q$ with $T>0$. Then there exists a $C>0$ depending only on r, p, q, β, n such that for every $t\in\,]0,T]$,

$$\|\theta(t)\|_{L^\infty}\leqslant \frac{C}{t^{\frac{n}{r\beta}}}\|\theta^0\|_{L^r}+C\big(1+T^{\frac{1}{\beta}(\beta-\frac{\beta}{p}-\frac{n}{q})}\big)\|f\|_{L^p_T L^q}. \tag{3.3}$$

Note that when $\beta=2$, a similar result was obtained by Hmidi and Rousset in [17].

Proof. Because of the linearity of Eq. (3.1), we can study separately the following two systems

$$\partial_t\theta+u\cdot\nabla\theta+|D|^\beta\theta=0,\quad \theta|_{t=0}=\theta^0,\ \beta\in\,]0,2[, \tag{3.4}$$

and

$$\partial_t\theta+u\cdot\nabla\theta+|D|^\beta\theta=f,\quad \theta|_{t=0}=0,\ \beta\in\,]0,2[. \tag{3.5}$$

For the homogeneous transport-diffusion equation (3.4), the corresponding estimate in the case of $r=2$ that

$$\|\theta(t)\|_{L^\infty}\leqslant\frac{C_n}{t^{n/(2\beta)}}\|\theta^0\|_{L^2},\quad t>0,$$

has appeared in [3,9] by using DeGiorgi–Nash's iterative method. Interpolating it with the classical maximum principle $\|\theta\|_{L^\infty_T L^\infty}\leqslant\|\theta^0\|_{L^\infty}$ leads to the expected estimate (3.3) concerning θ^0. Noticing that from a limiting process, we only need to require $\theta^0\in L^r$.

Now we consider the system (3.5). First we shall prove that for every $T\in\,]0,1]$, there exists an absolute constant $C>0$ depending only on p, q, n such that

$$\|\theta\|_{L^\infty_T L^\infty}\leqslant C\|f\|_{L^p_T L^q}. \tag{3.6}$$

Observe that the L^q *a priori* estimate is obvious. Since $p>1$, we use Proposition 2.2 and Hölder's inequality to find

$$\|\theta\|_{L^\infty_T L^q}\leqslant\|f\|_{L^1_T L^q}\leqslant\|f\|_{L^p_T L^q}. \tag{3.7}$$

When $q=\infty$, this is just the estimate (3.3) involving f. While for $q<\infty$, we shall use DeGiorgi–Nash's iterative argument to improve the L^q-estimate to an L^∞-estimate. Let Λ be a positive number chosen later, and $\Lambda_k \triangleq \Lambda(1-2^{-k-1})$ for all $k\in\mathbb{N}$. From a pointwise positivity inequality for fractional derivative operator (cf. [10,20]) that for every convex function Φ,

$$\Phi'(\theta)|D|^\beta\theta \geqslant |D|^\beta\Phi(\theta),$$

we know that

$$1_{\{\theta\geqslant\Lambda_k\}}|D|^\beta\theta \geqslant |D|^\beta(\theta-\Lambda_k)_+.$$

Thus

$$\partial_t(\theta-\Lambda_k)_+ + u\cdot\nabla(\theta-\Lambda_k)_+ + |D|^\beta(\theta-\Lambda_k)_+ \leqslant f1_{\{\theta\geqslant\Lambda_k\}},$$

and multiplying this equation with $(\theta-\Lambda_k)_+$, integrating over the spatial variable, we see that for every $s\in\,]0,\frac{\beta}{2}]$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|(\theta-\Lambda_k)_+(t)\big\|_{L^2}^2 + \big\|(\theta-\Lambda_k)_+(t)\big\|_{\dot H^{\frac{\beta}{2}}}^2 &\leqslant \left|\int_{\mathbb{R}^n} f(t,x)(\theta-\Lambda_k)_+(t,x)\,\mathrm{d}x\right| \\ &\leqslant \big\|f(t)1_{\{\theta(t)\geqslant\Lambda_k\}}\big\|_{\dot H^{-s}}\big\|(\theta-\Lambda_k)_+(t)\big\|_{\dot H^s}.\end{aligned}$$

Denoting by

$$U_k \triangleq \big\|(\theta-\Lambda_k)_+\big\|_{L_T^\infty L^2}^2 + \big\|(\theta-\Lambda_k)_+\big\|_{L_T^2\dot H^{\frac{\beta}{2}}}^2,$$

and integrating in the time interval $[0,T]$, we get

$$U_k \leqslant 2\int_0^T \big\|f(t)1_{\{\theta(t)\geqslant\Lambda_k\}}\big\|_{\dot H^{-s}}\big\|(\theta-\Lambda_k)_+(t)\big\|_{\dot H^s}\,\mathrm{d}t.$$

By virtue of the continuous embedding (cf. Corollary 1.39 of [2]) $L^{\frac{2n}{n+2s}}\hookrightarrow \dot H^{-s}$ ($\beta\in\,]0,2[$, $s\in\,]0,\frac{\beta}{2}]$), the Hölder inequality and the interpolation inequality that for every $s\in\,]0,\frac{\beta}{2}]$,

$$\big\|(\theta-\Lambda_k)_+\big\|_{L_T^{\beta/s}\dot H^s}^2 \leqslant C_0U_k,$$

we further obtain

$$\begin{aligned}U_k &\leqslant C_0\big\|f(t)1_{\{\theta(t)\geqslant\Lambda_k\}}\big\|_{L_T^{\beta/(\beta-s)}L^{2n/(n+2s)}}\big\|(\theta-\Lambda_k)_+\big\|_{L_T^{\beta/s}\dot H^s} \\ &\leqslant C_0\big\|f(t)1_{\{\theta(t)\geqslant\Lambda_k\}}\big\|_{L_T^{\beta/(\beta-s)}L^{2n/(n+2s)}}U_k^{1/2}.\end{aligned}$$

The Young inequality yields

$$U_k \leqslant C_0\big\|f(t)1_{\{\theta(t)\geqslant\Lambda_k\}}\big\|_{L_T^{\beta/(\beta-s)}L^{2n/(n+2s)}}^2.$$

If $p \in]2, \infty]$, we can choose $s = \frac{\beta}{2}$, and then $\frac{\beta}{\beta - s} = 2 < p$ and $\frac{2n}{n+2s} = \frac{2n}{n+\beta} < \frac{n}{\beta} < q$. If $p \in]1, 2[$, in order to pick some $s \in]0, \frac{\beta}{2}]$ such that $p > \frac{\beta}{\beta - s}$ and $q > \frac{2n}{n+2s}$, equivalently, $\frac{n}{q} - \frac{n}{2} < s < \beta - \frac{\beta}{p}$, and noting that $\frac{n}{q} < \beta - \frac{\beta}{p} < \frac{\beta}{2} < \frac{n}{2}$, we can choose $s = \frac{\beta}{2} - \frac{\beta}{2p}$. For such s, we further use the Hölder inequality to find that for every $p \in]2, \infty]$,

$$
\begin{aligned}
U_k &\leqslant C_0 \|f\|^2_{L^p_T L^q} \|1_{\{\theta(t) \geqslant \Lambda_k\}}\|^2_{L_T^{\frac{2p}{p-2}} L^{\frac{1}{1/2+\beta/(2n)-1/q}}} \\
&\leqslant C_0 \|f\|^2_{L^p_T L^q} \left(\int_0^T \left| \{\theta(t) \geqslant \Lambda_k\} \right|^{\frac{2p}{p-2}(\frac{1}{2} + \frac{\beta}{2n} - \frac{1}{q})} \, dt \right)^{\frac{p-2}{p}}, \qquad (3.8)
\end{aligned}
$$

and for every $p \in]1, 2[$,

$$
\begin{aligned}
U_k &\leqslant C_0 \left\| f(t) 1_{\{\theta(t) \geqslant \Lambda_k\}} \right\|^2_{L_T^{\frac{2p}{p+1}} L^{\frac{2n}{n+\beta-\beta/p}}} \\
&\leqslant C_0 \|f\|^2_{L^p_T L^q} \|1_{\{\theta(t) \geqslant \Lambda_k\}}\|^2_{L_T^{\frac{2p}{p-1}} L^{\frac{1}{1/2+\beta/(2n)-\beta/(2np)-1/q}}} \\
&\leqslant C_0 \|f\|^2_{L^p_T L^q} \left(\int_0^T \left| \{\theta(t) \geqslant \Lambda_k\} \right|^{\frac{2p}{p-1}(\frac{1}{2} + \frac{\beta}{2n} - \frac{\beta}{2np} - \frac{1}{q})} \, dt \right)^{\frac{p-1}{p}}, \qquad (3.9)
\end{aligned}
$$

where $|\{\theta(t) \geqslant \Lambda_k\}|$ means the Lebesgue measure of the set $\{x\colon\ \theta(t,x) \geqslant \Lambda_k\} \subset \mathbb{R}^n$. Noting that $\theta(t,x) - \Lambda_{k-1} \geqslant \Lambda 2^{-k-1}$ for all $\theta(t,x) \geqslant \Lambda_k$, we have that for every $\delta \geqslant 1$,

$$
1_{\{\theta(t) \geqslant \Lambda_k\}} \leqslant \left(\frac{(\theta(t) - \Lambda_{k-1})_+}{\Lambda 2^{-k-1}} \right)^{\delta},
$$

and

$$
\left| \{\theta(t) \geqslant \Lambda_k\} \right| \leqslant \frac{2^{(k+1)\delta}}{\Lambda^{\delta}} \left\| (\theta - \Lambda_{k-1})_+(t) \right\|^{\delta}_{L^{\delta}}.
$$

Hence, inserting the above estimate into (3.8) and (3.9) leads to that for every $p \in]2, \infty]$,

$$
U_k \leqslant C_0 \|f\|^2_{L^p_T L^q} \left(\frac{2^{k+1}}{\Lambda} \right)^{\delta(1+\frac{\beta}{n}-\frac{2}{q})} \left(\int_0^T \left\| (\theta - \Lambda_{k-1})_+(t) \right\|_{L^{\delta}}^{\delta(1+\frac{\beta}{n}-\frac{2}{q})\frac{p}{p-2}} \, dt \right)^{\frac{p-2}{p}}, \qquad (3.10)
$$

and for every $p \in]1, 2[$,

$$
U_k \leqslant C_0 \|f\|^2_{L^p_T L^q} \left(\frac{2^{k+1}}{\Lambda} \right)^{\delta(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q})} \left(\int_0^T \left\| (\theta - \Lambda_{k-1})_+(t) \right\|_{L^{\delta}}^{\delta(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q})\frac{p}{p-1}} \, dt \right)^{\frac{p-1}{p}}. \qquad (3.11)
$$

Since $\dot{H}^{\frac{\beta}{2}} \hookrightarrow L^{\frac{2n}{n-\beta}}$, and from interpolation and Hölder's inequality, we know that for every $m \in [2, \frac{2n}{n-\beta}]$ and $\frac{\beta}{\sigma} + \frac{n}{m} \geqslant \frac{n}{2}$,

$$
\left\| (\theta - \Lambda_{k-1})_+ \right\|^2_{L^{\sigma}_T L^m} \leqslant C_0 U_{k-1}.
$$

Thus to fit our purpose, we need to choose some $\delta \in [2, \frac{2n}{n-\beta}]$ satisfying that for every $p \in]2, \infty]$

$$\delta\left(1+\frac{\beta}{n}-\frac{2}{q}\right)>2, \qquad \beta\frac{(p-2)/p}{\delta(1+\beta/n-2/q)}+\frac{n}{\delta}\geqslant\frac{n}{2}, \tag{3.12}$$

and for every $p \in]1, 2[$

$$\delta\left(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q}\right)>2, \qquad \beta\frac{(p-1)/p}{\delta(1+\beta/n-\beta/(np)-2/q)}+\frac{n}{\delta}\geqslant\frac{n}{2}. \tag{3.13}$$

The first condition of (3.12) can be satisfied as long as $\frac{2}{1+\beta/n-2/q} < \frac{2n}{n-\beta}$, which is equivalent to $q > \frac{n}{\beta}$, and the second condition of (3.12) is equivalent to

$$\beta\left(1-\frac{2}{p}\right)+n\left(1+\frac{\beta}{n}-\frac{2}{q}\right)=n+2\beta-\frac{2\beta}{p}-\frac{2n}{q}\geqslant\frac{n}{2}\delta\left(1+\frac{\beta}{n}-\frac{2}{q}\right)>n,$$

which can be guaranteed as long as $\frac{\beta}{p}+\frac{n}{q}<\beta$. Similarly, the first condition of (3.13) can be satisfied as long as $\frac{2}{1+\beta/n-\beta/(np)-2/q} < \frac{2n}{n-\beta}$, which is equivalent to $\frac{\beta}{2p}+\frac{n}{q}<\beta$, and the second condition of (3.13) is equivalent to

$$\beta\left(1-\frac{1}{p}\right)+n\left(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q}\right)=n+2\beta-\frac{2\beta}{p}-\frac{2n}{q}\geqslant\frac{n}{2}\delta\left(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q}\right)>n,$$

which can be guaranteed as long as $\frac{\beta}{p}+\frac{n}{q}<\beta$. Therefore, for all $p \in]1, \infty]$ and $q \in]\frac{n}{\beta}, \infty[$ satisfying (3.2), some suitable $\delta \in [2, \frac{2n}{n-\beta}]$ satisfying (3.12) or (3.13) can be chosen. For such δ, we further have

$$U_k \leqslant C_0\|f\|^2_{L^p_T L^q}\left(2^{k+1}/\Lambda\right)^{2\mu}U^{\mu}_{k-1}, \quad k\in\mathbb{Z}^+,$$

where $\mu > 1$ is defined by

$$\mu \triangleq \begin{cases} \frac{\delta}{2}(1+\frac{\beta}{n}-\frac{2}{q}), & \text{if } p\in]2,\infty],\\ \frac{\delta}{2}(1+\frac{\beta}{n}-\frac{\beta}{np}-\frac{2}{q}), & \text{if } p\in]1,2[. \end{cases}$$

We also need to estimate U_0. From (3.8), (3.9), (3.7) and the following Tchebychev's inequality

$$\left|\left\{\theta(t)\geqslant\Lambda_0=\Lambda/2\right\}\right|\leqslant(2/\Lambda)^q\left\|\theta(t)\right\|^q_{L^q}, \quad q<\infty,$$

we obtain that for all $T \in]0, 1]$,

$$U_0\leqslant\|f\|^2_{L^p_T L^q}(2/\Lambda)^{\frac{2q\mu}{\delta}}\|\theta\|^{\frac{2q\mu}{\delta}}_{L^\infty_T L^q}\leqslant\|f\|^{2+\frac{2q\mu}{\delta}}_{L^p_T L^q}(2/\Lambda)^{\frac{2q\mu}{\delta}}.$$

According to Lemma 2.6, we can choose $\Lambda > 0$ satisfying

$$\|f\|^{2+\frac{2q\mu}{\delta}}_{L^p_T L^q}(2/\Lambda)^{\frac{2q\mu}{\delta}}=C_0^{-\frac{1}{\mu-1}}\|f\|^{-\frac{2}{\mu-1}}_{L^p_T L^q}2^{-\frac{2\mu}{\mu-1}}\Lambda^{\frac{2\mu}{\mu-1}}2^{-\frac{2\mu}{(\mu-1)^2}},$$

equivalently,

$$\Lambda = C_{\mu,q,\delta}\|f\|_{L^p_T L^q}, \tag{3.14}$$

so that we have $U_k \to 0$ as $k \to \infty$, which implies $\|(\theta - \Lambda)_+\|_{L^\infty_1 L^2} = 0$. Hence, for a.e. $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\theta(t,x) \leqslant \Lambda.$$

Applying the above deduction to $-\theta$, we also get $\theta(t,x) \geqslant -\Lambda$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}^n$. Clearly, (3.6) follows.

Next we shall use a scaling argument to show the estimate (3.3) involving f for $T \geqslant 1$. For every $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\zeta > 0$, denote by $\tau \triangleq \zeta^{-\beta} t$, $y \triangleq \zeta^{-1} x$ and define $\tilde{\theta}(\tau, y) \triangleq \theta(\zeta^\beta \tau, \zeta y) = \theta(t,x)$. By a direct computation, we see that $\tilde{\theta}(\tau, y)$ also satisfies the transport-diffusion equation with the scaled divergence-free vector field $\tilde{u}$

$$\partial_\tau \tilde{\theta} + \tilde{u} \cdot \nabla_y \tilde{\theta} + |D_y|^\beta \tilde{\theta} = \tilde{f},$$

where $\tilde{f}(\tau, y) \triangleq \zeta^\beta f(\zeta^\beta \tau, \zeta y)$. If we set $\zeta = T^{\frac{1}{\beta}}$, then $(\tau, y) \in [0,1] \times \mathbb{R}^n$, and we can use (3.6) and the variable substitution to find that for every $T \geqslant 1$,

$$\left\|\theta(t)\right\|_{L^\infty_T L^\infty} = \left\|\tilde{\theta}(\tau)\right\|_{L^\infty_1 L^\infty} \leqslant C\|\tilde{f}\|_{L^p_1 L^q} = C T^{\frac{1}{\beta}(\beta - \frac{\beta}{p} - \frac{n}{q})}\|f\|_{L^p_T L^q}.$$

Combining this estimate and (3.6) leads to the estimate (3.3) concerning f. □

The second one is a useful smoothing estimate of the transport-diffusion equation.

Proposition 3.2. *Let u be a smooth divergence-free vector field of $\mathbb{R}^n$ with vorticity ω, and θ be a smooth solution of system* (3.1)*. Then for every $(s,p,\sigma) \in]-\infty, 0[\times [2,\infty[\times [1,\infty]$, we have*

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\sigma_t B^{s+\beta/\sigma}_{p,1}} \lesssim_{\sigma,p,s} & \left\|\theta^0\right\|_{B^s_{p,1}} + \|\theta\|_{L^1_t L^p}\left(\|\omega\|_{L^\infty_t L^\infty} + \|u\|_{L^\infty_t L^2}\right) \\ & + \|f\|_{L^1_t L^p} + \|\Delta_{-1}\theta\|_{L^\sigma_t L^p}.\end{aligned} \tag{3.15}$$

Proof. Denoting by $\theta_q \triangleq \Delta_q \theta$ for all $q \in \mathbb{N}$, and by applying Δ_q to Eq. (3.1) we get

$$\partial_t \theta_q + u \cdot \nabla \theta_q + |D|^\beta \theta_q = -[\Delta_q, u \cdot \nabla]\theta + \Delta_q f.$$

Since θ_q is a real-valued function, we multiply both sides of the upper equation by $|\theta_q|^{p-2}\theta_q$ and integrate in the spatial variable to find that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\left\|\theta_q(t)\right\|^p_{L^p} + \int_{\mathbb{R}^n} |D|^\beta \theta_q(t,x)|\theta_q|^{p-2}\theta_q(t,x)\,\mathrm{d}x \\ &\quad \leqslant \left(\left\|[\Delta_q, u \cdot \nabla]\theta(t)\right\|_{L^p} + \left\|\Delta_q f(t)\right\|_{L^p}\right)\left\|\theta_q(t)\right\|^{p-1}_{L^p}.\end{aligned}$$

According to the generalized Bernstein inequality (cf. [8]), there exists an absolute constant $c > 0$ independent of q such that

$$\int_{\mathbb{R}^n} |D|^\beta \theta_q(t,x)|\theta_q|^{p-2}\theta_q(t,x)\,\mathrm{d}x \geqslant c 2^{q\beta}\|\theta_q\|^p_{L^p}.$$

Hence we have

$$\frac{d}{dt}\|\theta_q(t)\|_{L^p}+c2^{q\beta}\|\theta_q(t)\|_{L^p}\leqslant\|[\Delta_q,u\cdot\nabla]\theta(t)\|_{L^p}+\|\Delta_q f(t)\|_{L^p}.$$

Gronwall's inequality leads to

$$\|\theta_q(t)\|_{L^p}\leqslant e^{-c2^{q\beta}t}\|\theta_q^0\|_{L^p}+\int_0^t e^{-c2^{q\beta}(t-\tau)}\left(\|[\Delta_q,u\cdot\nabla]\theta(\tau)\|_{L^p}+\|\Delta_q f(\tau)\|_{L^p}\right)\mathrm{d}\tau.$$

We further get

$$\|\theta_q\|_{L^\sigma_t L^p}\lesssim 2^{-q\beta/\sigma}\left(\|\theta_q^0\|_{L^p}+\|[\Delta_q,u\cdot\nabla]\theta\|_{L^1_t L^p}+\|\Delta_q f\|_{L^1_t L^p}\right).$$

From the commutator estimate (cf. [15, Proposition 5.4]), we have for every $q\in\mathbb{N}$,

$$\|[\Delta_q,u\cdot\nabla]\theta(\tau)\|_{L^p}\lesssim\left(q\|\omega(\tau)\|_{L^\infty}+\|u(\tau)\|_{L^2}\right)\|\theta(\tau)\|_{L^p},$$

thus we infer that for every $s\in]-\infty,0[$,

$$\begin{aligned}\sum_{q\in\mathbb{N}}2^{q(s+\beta/\sigma)}\|\theta_q\|_{L^\sigma_t L^p}&\lesssim\|\theta^0\|_{B^s_{p,1}}+\left(\sum_{q\in\mathbb{N}}2^{qs}q\right)\|\theta\|_{L^1_t L^p}\left(\|\omega\|_{L^\infty_t L^\infty}+\|u\|_{L^\infty_t L^2}\right)+\|f\|_{L^1_t B^s_{p,1}}\\&\lesssim_s\|\theta^0\|_{B^s_{p,1}}+\|\theta\|_{L^1_t L^p}\left(\|\omega\|_{L^\infty_t L^\infty}+\|u\|_{L^\infty_t L^2}\right)+\|f\|_{L^1_t L^p}.\end{aligned}$$

On the other hand, for the low frequency part we immediately obtain

$$2^{-(s+\beta/\sigma)}\|\Delta_{-1}\theta\|_{L^\sigma_t L^p}\lesssim\|\Delta_{-1}\theta\|_{L^\sigma_t L^p}.$$

Gathering the upper two estimations yields (3.15). □

4. Modified Riesz transform and commutators

First we introduce a pseudo-differential operator $\mathcal{R}_\beta$ defined by

$$\mathcal{R}_\beta\triangleq\partial_1|D|^{-\beta}=|D|^{1-\beta}\mathcal{R},\quad\beta\in]1,2[,$$

where $\mathcal{R}\triangleq\partial_1/|D|$ is the usual Riesz transform. For convenience, we call $\mathcal{R}_\beta$ the modified Riesz transform. Note that we have encountered $\mathcal{R}_\beta$ in [22], but there $\beta\in]0,1[$.

We collect some useful properties of this operator as follows.

Proposition 4.1. *Let $j\in\mathbb{N}$, $\beta\in]1,2[$, $\mathcal{R}_\beta$ be the modified Riesz transform. Then the following statements hold true.*

(1) *For every $1<p<q<\infty$ satisfying $\frac{1}{q}=\frac{1}{p}-\frac{\beta-1}{n}$, $\mathcal{R}_\beta$ is a bounded linear operator which maps L^p to L^q.*

(2) *Let $\chi \in \mathcal{D}(\mathbb{R}^n)$. Then for every $(p,s) \in [1,\infty] \times]\beta-1,\infty[$ and $f \in L^p(\mathbb{R}^n)$,*

$$\big\| |D|^s \chi\big(2^{-j}|D|\big)\mathcal{R}_\beta f \big\|_{L^p} \lesssim 2^{j(s+1-\beta)} \|f\|_{L^p}.$$

Moreover, $|D|^s\chi(|D|)\mathcal{R}_\beta$ is a convolution operator with kernel K satisfying

$$\big|K(x)\big| \lesssim \frac{1}{(1+|x|)^{n+s+1-\beta}}, \quad \forall x \in \mathbb{R}^n.$$

(3) *Let $\mathcal{O}$ be an annulus centered at the origin. Then for every f with spectrum supported on $2^j\mathcal{O}$, there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform supported away from the origin, such that*

$$\mathcal{R}_\beta f = 2^{j(n+1-\beta)}\phi\big(2^j\cdot\big) \star f.$$

Proof. Since $\mathcal{R}_\beta = |D|^{1-\beta}\mathcal{R}$, (1) is a consequence of the Calderón–Zygmund theorem and Hardy–Littlewood–Sobolev's inequality. Due to $|D|^s\mathcal{R}_\beta = |D|^{s+1-\beta}\mathcal{R}$, (2) follows from [16, Proposition 3.1]. (3) can be justified by choosing a suitable bump function. □

Next we consider the crucial commutators involving estimates $\mathcal{R}_\beta$.

Proposition 4.2. *Let $\beta \in]1,2[$, $(p,r) \in [2,\infty] \times [1,\infty]$, u be a smooth divergence-free vector field of $\mathbb{R}^n$ ($n \geqslant 2$) with vorticity ω and θ be a smooth scalar function. Then we have that for every $s \in]\beta-2,\beta[$,*

$$\big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{B^s_{p,r}} \lesssim_{s,\beta} \|\nabla u\|_{L^p}\big(\|\theta\|_{B^{s+1-\beta}_{\infty,r}} + \|\theta\|_{L^2}\big). \tag{4.1}$$

Besides, if $p=\infty$, we also have

$$\big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta\big\|_{B^s_{\infty,r}} \lesssim_{s,\beta} \big(\|\omega\|_{L^\infty} + \|u\|_{L^2}\big)\|\theta\|_{B^{s+(1-\beta)/2}_{\infty,r}} + \|u\|_{L^2}\|\theta\|_{L^2}. \tag{4.2}$$

Proof. By virtue of Bony's decomposition, we have

$$\begin{aligned}[\mathcal{R}_\beta, u\cdot\nabla]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R}_\beta, S_{q-1}u\cdot\nabla]\Delta_q\theta + \sum_{q\in\mathbb{N}}[\mathcal{R}_\beta, \Delta_q u\cdot\nabla]S_{q-1}\theta + \sum_{q\geqslant -1}[\mathcal{R}_\beta, \Delta_q u\cdot\nabla]\widetilde{\Delta}_q\theta \\ &\triangleq \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

For I, from Proposition 4.1(3), Lemma 2.5 and Bernstein's inequality we obtain

$$\begin{aligned}\|\Delta_j \mathrm{I}\|_{L^p} &\lesssim \sum_{|q-j|\leqslant 4}\big\|[\phi_q\star, S_{q-1}u\cdot\nabla]\Delta_q\theta\big\|_{L^p} \\ &\lesssim \sum_{|q-j|\leqslant 4} 2^{-q\beta}\|\nabla S_{q-1}u\|_{L^p}2^q\|\Delta_q\theta\|_{L^\infty} \\ &\lesssim 2^{-js}\|\nabla u\|_{L^p}2^{q(s+1-\beta)}\|\Delta_q\theta\|_{L^\infty},\end{aligned}$$

where $\phi_q(x) \triangleq 2^{q(n+1-\beta)}\phi(2^q x)$ with $\phi \in \mathcal{S}$ and $\|x\phi_q\|_{L^1} \lesssim 2^{-q\beta}$. Thus

$$\|\mathrm{I}\|_{B^s_{p,r}} \lesssim \|\nabla u\|_{L^p}\|\theta\|_{B^{s+1-\beta}_{\infty,r}}.$$

For II, as above we have

$$\begin{aligned}\|\Delta_j \mathrm{II}\|_{L^p} &\lesssim \sum_{|q-j|\leqslant 4,\ q\in\mathbb{N}} \big\|[\phi_q\star,\ \Delta_q u\cdot\nabla]S_{q-1}\theta\big\|_{L^p} \\ &\lesssim \sum_{|q-j|\leqslant 4,\ q\in\mathbb{N}} 2^{-q\beta}\|\nabla\Delta_q u\|_{L^p}\|\nabla S_{q-1}\theta\|_{L^\infty} \\ &\lesssim 2^{-js}\|\nabla u\|_{L^p}\sum_{q'\leqslant j+2} 2^{(j-q')(s-\beta)}2^{q'(s-\beta+1)}\|\Delta_{q'}\theta\|_{L^\infty}.\end{aligned}$$

Thus for every $s<\beta$, the discrete Young inequality ensures that

$$\|\mathrm{II}\|_{B^s_{p,r}} \lesssim \|\nabla u\|_{L^p}\|\theta\|_{B^{s+1-\beta}_{\infty,r}}.$$

Since $\operatorname{div} u=0$, we further write III as follows

$$\begin{aligned}\mathrm{III} &= \sum_{q\geqslant 2}\operatorname{div}\mathcal{R}_\beta(\Delta_q u\widetilde{\Delta}_q\theta)+\sum_{q\geqslant 2}\operatorname{div}(\Delta_q u\mathcal{R}_\beta\widetilde{\Delta}_q\theta)+\sum_{-1\leqslant q\leqslant 1,\ 1\leqslant i\leqslant n}\big[\partial_i\mathcal{R}_\beta,\ \Delta_q u^i\big]\widetilde{\Delta}_q\theta \\ &\triangleq \mathrm{III}^1+\mathrm{III}^2+\mathrm{III}^3.\end{aligned}$$

From Proposition 4.1(2) and Bernstein's inequality, we get

$$\begin{aligned}\big\|\Delta_j\mathrm{III}^1\big\|_{L^p} &\leqslant \sum_{q\geqslant j-4,\ q\geqslant 2}\big\|\operatorname{div}R_\beta\Delta_j(\Delta_q u\widetilde{\Delta}_q\theta)\big\|_{L^p} \\ &\lesssim \sum_{q\geqslant j-4,\ q\geqslant 2} 2^{j(2-\beta)}2^{-q}\|\nabla\Delta_q u\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty} \\ &\lesssim 2^{-js}\|\nabla u\|_{L^p}\sum_{q\geqslant j-5}2^{(j-q)(s+2-\beta)}2^{q(s+1-\beta)}\|\Delta_q\theta\|_{L^\infty}.\end{aligned}$$

Similarly, for III^2 we directly have

$$\begin{aligned}\big\|\Delta_j\mathrm{III}^2\big\|_{L^p} &\leqslant \sum_{q\geqslant j-4,\ q\geqslant 2} 2^j\|\Delta_q u\mathcal{R}_\beta\widetilde{\Delta}_q\theta\|_{L^p} \\ &\lesssim \sum_{q\geqslant j-4,\ q\geqslant 2} 2^j2^{-q}\|\nabla\Delta_q u\|_{L^p}2^{q(1-\beta)}\|\widetilde{\Delta}_q\theta\|_{L^\infty} \\ &\lesssim 2^{-js}\|\nabla u\|_{L^p}\sum_{q\geqslant j-5}2^{(j-q)(s+1)}2^{q(s+1-\beta)}\|\Delta_q\theta\|_{L^\infty}.\end{aligned}$$

Thus the discrete convolution inequality guarantees that for every $s>\beta-2$,

$$\big\|\mathrm{III}^1\big\|_{B^s_{p,r}}+\big\|\mathrm{III}^2\big\|_{B^s_{p,r}} \lesssim \|\nabla u\|_{L^p}\|\theta\|_{B^{s+1-\beta}_{\infty,r}}.$$

For the third term, from the frequency-localization property, there exists $\chi' \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\chi'(\xi) \equiv 1$ for $|\xi| \leqslant 3$ such that

$$\mathrm{III}^3 = \sum_{-1 \leqslant q \leqslant 1,\ 1 \leqslant i \leqslant n} \big[\partial_i R_\beta \chi'(D), \Delta_q u^i\big] \widetilde{\Delta}_q \theta.$$

Proposition 4.1(2) shows that $\partial_i R_\beta \chi'(D)$ is a convolution operator with kernel h' satisfying

$$\big|h'(x)\big| \leqslant \frac{C}{(1+|x|)^{n+2-\beta}}, \quad \forall x \in \mathbb{R}^n.$$

Thus from the fact that $\Delta_j \mathrm{III}^3 = 0$ for every $j \geqslant 5$, and applying Lemma 2.5 with $\bar{m} \in]\frac{n}{n+1-\beta}, 2]$ (equivalently, $m \in [2, \frac{n}{\beta-1}[$), $n \geqslant 2$ and $p \geqslant 2$, we have

$$\begin{aligned}
\big\|\mathrm{III}^3\big\|_{B^s_{p,r}} &\lesssim \sum_{-1 \leqslant q \leqslant 1} \big\|\big[h'\star, \Delta_q u\big] \widetilde{\Delta}_q \theta\big\|_{L^p} \\
&\lesssim \sum_{-1 \leqslant q \leqslant 1} \big\|xh'\big\|_{L^{\bar{m}}} \|\nabla \Delta_q u\|_{L^p} \|\widetilde{\Delta}_q \theta\|_{L^m} \\
&\lesssim \|\nabla u\|_{L^p} \|\theta\|_{L^2}.
\end{aligned}$$

In the above we also have used the fact that $xh' \in L^{\bar{m}}$. This ends the proof of (4.1).

Next we reconsider the case $p = \infty$. Since $\|\Delta_q \nabla u\|_{L^\infty} \approx \|\Delta_q \omega\|_{L^\infty}$ for every $q \in \mathbb{N}$, we directly obtain

$$\big\|\mathrm{II} + \mathrm{III}^1 + \mathrm{III}^2\big\|_{B^s_{\infty,r}} \lesssim \|\omega\|_{L^\infty} \|\theta\|_{B^{s+1-\beta}_{\infty,r}}.$$

Taking advantage of the following fact that for every $q \in \mathbb{N}$

$$\|\nabla S_{q-1} u\|_{L^\infty} \lesssim \|\Delta_{-1} \nabla u\|_{L^\infty} + \sum_{0 \leqslant q' \leqslant q-2} \|\nabla \Delta_{q'} u\|_{L^\infty} \lesssim \|u\|_{L^2} + q\|\omega\|_{L^\infty},$$

we can also estimate I as follows

$$\|\mathrm{I}\|_{B^s_{\infty,r}} \lesssim \|u\|_{L^2} \|\theta\|_{B^{s+1-\beta}_{\infty,r}} + \|\omega\|_{L^\infty} \|\theta\|_{B^{s+\frac{1-\beta}{2}}_{\infty,r}}.$$

For the last reminder term III^3, estimating as above we have for every $m \in [2, \frac{n}{\beta-1}[$

$$\begin{aligned}
\big\|\mathrm{III}^3\big\|_{B^s_{\infty,r}} &\lesssim \sum_{-1 \leqslant q \leqslant 1} \big\|xh'\big\|_{L^{\bar{m}}} \|\nabla \Delta_q u\|_{L^\infty} \|\widetilde{\Delta}_q \theta\|_{L^m} \\
&\lesssim \|u\|_{L^2} \|\theta\|_{L^2}.
\end{aligned}$$

This ends the proof of (4.2). □

5. Proof of Theorem 1.1

The system we consider is as follows

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = \theta e_2, \quad \beta\in]1,2[, \\ \partial_t\theta + u\cdot\nabla\theta + |D|^\beta\theta = u_2, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u^0,\ \theta|_{t=0} = \theta^0. \end{cases} \tag{5.1}$$

The proof's outline is as follows: first we show some key *a priori* estimates, next based on them we prove the existence and continuity-in-time results, and finally we treat the uniqueness issue.

5.1. A priori estimates

Proposition 5.1. *Let (u,θ) be a smooth solution of the system (5.1) with $(u^0,\theta^0)\in L^2\times L^2$. Then we have*

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \|\theta\|^2_{L^2_t\dot H^{\frac{\beta}{2}}} \leqslant \left(\|\theta^0\|_{L^2}^2 + \|u^0\|_{L^2}^2\right)e^{Ct}. \tag{5.2}$$

Proof. Taking scalar product of the first equation of (5.1) with u, multiplying the second equation by θ and integrating over the spatial variable, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2\right) + \|\theta(t)\|^2_{\dot H^{\beta/2}} &= 2\int_{\mathbb{R}^2}\theta(t,x)u_2(t,x)\,\mathrm{d}x \\ &\leqslant 2\|\theta(t)\|_{L^2}\|u(t)\|_{L^2} \\ &\leqslant \|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2. \end{aligned}$$

Gronwall's inequality further leads to (5.2). □

Proposition 5.2. *Let (u,θ) be a smooth solution of system (5.1). If $\theta^0\in L^2\cap B^{1-\beta}_{\infty,1}$, $u^0\in L^2$ and the initial vorticity $\omega^0\in L^\infty$, then we have that for every $t>0$,*

$$\|\theta(t)\|_{L^\infty} \leqslant \frac{C}{t^{1/\beta}}\|\theta^0\|_{L^2} + Ce^{Ct}, \tag{5.3}$$

and for every $t\geqslant 0$,

$$\|\omega(t)\|_{L^\infty} + \|\theta\|_{\widetilde L^\infty_t B^{1-\beta}_{\infty,1}} + \|\theta\|_{L^1_t B^1_{\infty,1}} \leqslant Ce^{\exp\{Ct\}}. \tag{5.4}$$

Proof. (5.3) is a direct consequence of Propositions 3.1 and 5.1, i.e., for every $t>0$,

$$\begin{aligned} \|\theta(t)\|_{L^\infty} &\leqslant \frac{C}{t^{1/\beta}}\|\theta^0\|_{L^2} + \left(1+t^{1-1/\beta}\right)\|u\|_{L^\infty_t L^2} \\ &\leqslant \frac{C}{t^{1/\beta}}\|\theta^0\|_{L^2} + Ce^{Ct}. \end{aligned}$$

Next we treat (5.4). Applying the operator $\mathcal{R}_\beta \triangleq \partial_1|D|^{-\beta}$ to the equation of temperature θ, we have

$$\partial_t\mathcal{R}_\beta\theta + u\cdot\nabla\mathcal{R}_\beta\theta + |D|^\beta\mathcal{R}_\beta\theta = -[\mathcal{R}_\beta, u\cdot\nabla]\theta + \mathcal{R}_\beta u_2. \tag{5.5}$$

We denote by $\Gamma \triangleq \omega + \mathcal{R}_\beta\theta$, and due to $|D|^\beta \mathcal{R}_\beta = \partial_1$, we obtain

$$\partial_t \Gamma + u \cdot \nabla \Gamma = -[\mathcal{R}_\beta, u \cdot \nabla]\theta + \mathcal{R}_\beta u_2. \tag{5.6}$$

From the L^∞ maximal principle of the transport equation (5.6), we find

$$\begin{aligned}\|\Gamma(t)\|_{L^\infty} &\leqslant \|\Gamma^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{L^\infty}\,d\tau + \int_0^t \|\mathcal{R}_\beta u_2(\tau)\|_{L^\infty}\,d\tau \\ &\lesssim \|\omega^0\|_{L^\infty} + \|\theta^0\|_{L^2 \cap B^{1-\beta}_{\infty,1}} + \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{\infty,1}}\,d\tau + \int_0^t \|\mathcal{R}_\beta u(\tau)\|_{B^0_{\infty,1}}\,d\tau,\end{aligned}$$

where we have used the following fact that for all $\beta \in]1, 2[$,

$$\begin{aligned}\|\mathcal{R}_\beta \theta^0\|_{L^\infty} &\lesssim \||D|^{1-\beta}\mathcal{R}\theta^0\|_{B^0_{\infty,1}} \lesssim \||D|^{1-\beta}\mathcal{R}\Delta_{-1}\theta^0\|_{L^{\frac{2}{2-\beta}}} + \sum_{q\geqslant 0} 2^{q(1-\beta)}\|\Delta_q\theta^0\|_{L^\infty} \\ &\lesssim_\beta \|\theta^0\|_{L^2\cap B^{1-\beta}_{\infty,1}}.\end{aligned} \tag{5.7}$$

According to the commutator estimate (4.2) and the continuous embedding $L^\infty \hookrightarrow B^0_{\infty,\infty} \hookrightarrow B^{(1-\beta)/2}_{\infty,1}$ ($\beta > 1$), we get

$$\begin{aligned}\|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{\infty,1}} &\lesssim \big(\|\omega(\tau)\|_{L^\infty} + \|u(\tau)\|_{L^2}\big)\big(\|\theta(\tau)\|_{B^{(1-\beta)/2}_{\infty,1}} + \|\theta(\tau)\|_{L^2}\big) \\ &\lesssim \big(\|\omega(\tau)\|_{L^\infty} + e^{C\tau}\big)\big(\tau^{-\frac{1}{\beta}} + e^{C\tau}\big).\end{aligned} \tag{5.8}$$

By a high–low frequency decomposition, we also find

$$\begin{aligned}\|\mathcal{R}_\beta u(\tau)\|_{B^0_{\infty,1}} &\lesssim \||D|^{1-\beta}\mathcal{R}\Delta_{-1}u(\tau)\|_{L^{\frac{2}{2-\beta}}} + \sum_{q\geqslant 0}\||D|^{1-\beta}\mathcal{R}\Delta_q u(\tau)\|_{L^\infty} \\ &\lesssim \|u(\tau)\|_{L^2} + \sum_{q\geqslant 0} 2^{-\beta}\|\omega(\tau)\|_{L^\infty} \\ &\lesssim_\beta e^{C\tau} + \|\omega(\tau)\|_{L^\infty}.\end{aligned} \tag{5.9}$$

On the other hand, from the equation of $\mathcal{R}_\beta\theta$ (5.5), we also deduce that

$$\|\mathcal{R}_\beta\theta(t)\|_{L^\infty} \lesssim \|\theta^0\|_{L^2\cap B^{1-\beta}_{\infty,1}} + \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{\infty,1}}\,d\tau + \int_0^t \|\mathcal{R}_\beta u(\tau)\|_{B^0_{\infty,1}}\,d\tau.$$

Hence we infer that

$$\begin{aligned}\|\omega(t)\|_{L^\infty} &\leqslant \|\Gamma(t)\|_{L^\infty} + \|\mathcal{R}_\beta\theta(t)\|_{L^\infty} \\ &\lesssim \|\omega^0\|_{L^\infty} + \|\theta^0\|_{L^2\cap B^{1-\beta}_{\infty,1}} + e^{Ct} + \int_0^t \big(\tau^{-\frac{1}{\beta}} + e^{C\tau}\big)\|\omega(\tau)\|_{L^\infty}\,d\tau.\end{aligned}$$

Thus the Gronwall inequality yields that $\|\omega(t)\|_{L^\infty} \leqslant Ce^{\exp\{Ct\}}$.

By interpolation and the Sobolev embedding, we see that for every $q \in]2,\infty[$, $\sigma = (1-\beta)(1-\frac{2}{q}) < 0$ and $\epsilon > 0$,

$$L^2 \cap B^{1-\beta}_{\infty,1} \hookrightarrow B^{\sigma}_{q,\infty} \hookrightarrow B^{\sigma-\epsilon}_{q,1}.$$

Thanks to (3.15), (5.2), (5.3) and the interpolation inequality $\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{1/2}\|\omega\|_{L^\infty}^{1/2}$ (cf. [12]), we further get

$$\begin{aligned}\|\theta\|_{L^1_t B^{\sigma-\epsilon+\beta}_{q,1}} &\lesssim \|\theta^0\|_{B^{\sigma-\epsilon}_{q,1}} + \|\theta\|_{L^1_t L^q}\big(1 + \|\omega\|_{L^\infty_t L^\infty} + \|u\|_{L^\infty_t L^2}\big) + \|u_2\|_{L^1_t L^q} \\ &\lesssim \|\theta^0\|_{L^2\cap B^{1-\beta}_{\infty,1}} + \int_0^t \|\theta(\tau)\|_{L^2\cap L^\infty}\,\mathrm{d}\tau\big(1 + \|\omega\|_{L^\infty_t L^\infty} + \|u\|_{L^\infty_t L^2}\big) + \|u\|_{L^1_t(L^2\cap L^\infty)} \\ &\lesssim e^{\exp\{Ct\}}.\end{aligned}$$

Since $\sigma - \epsilon + \beta - \frac{2}{q} = 1 - \tilde{\epsilon}$ with $\tilde{\epsilon} \triangleq \frac{2}{q}(2-\beta) + \epsilon > 0$, by choosing $q \in]2,\infty[$ large enough and $\epsilon > 0$ small enough, we see that $1-\tilde{\epsilon}$ can be sufficiently close to 1 (to fit our purpose in the sequel, it suffices to choose $\tilde{\epsilon} = (\beta-1)/2$) and

$$\|\theta\|_{L^1_t B^{1-\tilde{\epsilon}}_{\infty,1}} \lesssim \|\theta\|_{L^1_t B^{\sigma-\epsilon+\beta}_{q,1}} \lesssim e^{\exp\{Ct\}}. \tag{5.10}$$

Now we view the equation of θ as a linear dissipative equation with forcing term

$$\partial_t\theta + |D|^\beta\theta = -u\cdot\nabla\theta + u_2, \quad \theta|_{t=0} = \theta^0. \tag{5.11}$$

Taking advantage of Lemma 2.3, we get that for every $\rho \in [1,\infty]$

$$\|\theta\|_{\widetilde{L}^\rho_t B^{1-\beta+\beta/\rho}_{\infty,1}} \leqslant C(1+t)^{1/\rho}\big(\|\theta^0\|_{B^{1-\beta}_{\infty,1}} + \|u\cdot\nabla\theta\|_{L^1_t B^{1-\beta}_{\infty,1}} + \|u\|_{L^1_t B^{1-\beta}_{\infty,1}}\big).$$

From Lemma 2.4, estimate (5.10) (with $0 < \tilde{\epsilon} < \beta - 1$) and interpolation, we infer that

$$\begin{aligned}\|u\cdot\nabla\theta\|_{L^1_t B^{1-\beta}_{\infty,1}} + \|u\|_{L^1_t B^{1-\beta}_{\infty,1}} &\lesssim \|u\|_{L^\infty_t L^\infty}\|\theta\|_{L^1_t B^{2-\beta}_{\infty,1}} + \|u\|_{L^1_t L^\infty} \\ &\lesssim (1+t)\|u\|_{L^\infty_t L^2}^{1/2}\|\omega\|_{L^\infty_t L^\infty}^{1/2}\big(1 + \|\theta\|_{L^1_t B^{1-\tilde{\epsilon}}_{\infty,1}}\big) \\ &\lesssim e^{\exp\{Ct\}}.\end{aligned}$$

Hence for every $\rho \in [1,\infty]$, we have

$$\|\theta\|_{\widetilde{L}^\rho_t B^{1-\beta+\beta/\rho}_{\infty,1}} \lesssim e^{\exp\{Ct\}}. \qquad \square$$

Proposition 5.3. *Let (u,θ) be a smooth solution of system (5.1). If $\theta^0 \in L^2 \cap B^{1-\beta}_{\infty,1}$, $u^0 \in L^2$ and $\omega^0 \in L^p \cap L^\infty$ with $p \geqslant 2$, then we have*

$$\|\omega(t)\|_{L^p} + \|\theta\|_{\widetilde{L}^\infty_t L^2} + \|\theta\|_{\widetilde{L}^1_t H^\beta} \leqslant Ce^{\exp\{Ct\}}. \tag{5.12}$$

Proof. For the case of $p \in [2, \frac{2}{2-\beta}[$, there exists $\epsilon \in]0, \frac{2}{p} - (2-\beta)[$ such that $B^{\beta-1-\epsilon}_{2,1} \hookrightarrow L^p$. Thus, from the equation of the vorticity

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta$$

and Proposition 2.2, we get

$$\begin{aligned} \|\omega(t)\|_{L^p} &\leqslant \|\omega^0\|_{L^p} + \|\partial_1\theta\|_{L^1_t L^p} \\ &\lesssim \|\omega^0\|_{L^p} + \|\partial_1\theta\|_{L^1_t B^{\beta-1-\epsilon}_{2,1}} \lesssim \|\omega^0\|_{L^p} + \|\theta\|_{L^1_t B^{\beta-\epsilon}_{2,1}}. \end{aligned}$$

Applying Proposition 3.2 to the equation of θ, we know that for every $\epsilon > 0$

$$\begin{aligned} \|\theta\|_{L^1_t B^{\beta-\epsilon}_{2,1}} &\lesssim \|\theta^0\|_{B^{-\epsilon}_{2,1}} + \|\theta\|_{L^1_t L^2}\big(1 + \|\omega\|_{L^\infty_t L^\infty} + \|u\|_{L^\infty_t L^2}\big) + \|u\|_{L^1_t L^2} \\ &\lesssim e^{\exp\{Ct\}}. \end{aligned} \tag{5.13}$$

Hence, we have for every $p \in [2, \frac{2}{2-\beta}[$,

$$\|\omega(t)\|_{L^p} \lesssim e^{\exp\{Ct\}}.$$

Besides, according to estimates (2.6), (5.13) (with $\epsilon \in]0, \beta - 1[$) and the Sobolev embedding, we infer that for every $\rho \in [1, \infty]$,

$$\begin{aligned} \|\theta\|_{\widetilde{L}^\rho_t H^{\beta/\rho}} &\lesssim (1+t)^{1/\rho}\big(\|\theta^0\|_{L^2} + \|u \cdot \nabla\theta\|_{\widetilde{L}^1_t L^2} + \|u\|_{\widetilde{L}^1_t L^2}\big) \\ &\lesssim (1+t)^{1/\rho}\big(\|\theta^0\|_{L^2} + \|u\|_{L^\infty_t L^\infty}\|\theta\|_{L^1_t H^1} + \|u\|_{L^1_t L^2}\big) \\ &\lesssim e^{\exp\{Ct\}}. \end{aligned}$$

Next we consider the case of $p \in [\frac{2}{2-\beta}, \infty[$ and we shall use a similar method as treating the L^∞ case. By applying Proposition 2.2 to Eq. (5.6), and from the Besov embedding and interpolation we find that

$$\|\Gamma(t)\|_{L^p} \leqslant \|\omega^0\|_{L^p} + \|\theta^0\|_{L^2 \cap B^{1-\beta}_{\infty,1}} + \int_0^t \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{p,1}}\,\mathrm{d}\tau + \int_0^t \|\mathcal{R}_\beta u(\tau)\|_{B^0_{p,1}}\,\mathrm{d}\tau,$$

where we also have used the following estimate (by Proposition 4.1(1) and (5.7))

$$\|\mathcal{R}_\beta \theta^0\|_{L^p} \lesssim \|\mathcal{R}_\beta \theta^0\|_{L^{\frac{2}{2-\beta}} \cap B^0_{\infty,1}} \lesssim \|\theta^0\|_{L^2 \cap B^{1-\beta}_{\infty,1}}.$$

For the commutator term, from (4.1) we get

$$\begin{aligned} \|[\mathcal{R}_\beta, u \cdot \nabla]\theta(\tau)\|_{B^0_{p,1}} &\lesssim \|\omega(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^{1-\beta}_{\infty,1}} + \|\theta(\tau)\|_{L^2}\big) \\ &\lesssim \|\omega(\tau)\|_{L^p}\big(\tau^{-\frac{1}{\beta}} + e^{C\tau}\big). \end{aligned}$$

Taking advantage of a high–low frequency decomposition, we deduce that

$$
\begin{aligned}
\|\mathcal{R}_\beta u\|_{B^0_{p,1}} &= \big\|\Delta_{-1}|D|^{1-\beta}\mathcal{R}u\big\|_{L^p} + \sum_{q\in\mathbb{N}}\big\|\Delta_q|D|^{1-\beta}\mathcal{R}u\big\|_{L^p} \\
&\lesssim \big\|\Delta_{-1}|D|^{1-\beta}u\big\|_{L^{\frac{2}{2-\beta}}} + \sum_{q\in\mathbb{N}} 2^{-\beta}\|\Delta_q\omega\|_{L^p} \\
&\lesssim \|u\|_{L^2} + \|\omega\|_{L^p}.
\end{aligned}
$$

Noting that

$$
\big\|\mathcal{R}_\beta\theta(t)\big\|_{L^p} \lesssim \big\|\theta^0\big\|_{L^2\cap B^{1-\beta}_{\infty,1}} + \int_0^t \big\|[\mathcal{R}_\beta, u\cdot\nabla]\theta(\tau)\big\|_{B^0_{p,1}}\,\mathrm{d}\tau + \int_0^t \big\|\mathcal{R}_\beta u(\tau)\big\|_{B^0_{p,1}}\,\mathrm{d}\tau,
$$

we infer that

$$
\begin{aligned}
\big\|\omega(t)\big\|_{L^p} &\leqslant \big\|\Gamma(t)\big\|_{L^p} + \big\|\mathcal{R}_\beta\theta(t)\big\|_{L^p} \\
&\lesssim \big\|\omega^0\big\|_{L^p} + \big\|\theta^0\big\|_{L^2\cap B^{1-\beta}_{\infty,1}} + e^{Ct} + \int_0^t \big\|\omega(\tau)\big\|_{L^p}\big(\tau^{-\frac{1}{\beta}} + e^{C\tau}\big)\,\mathrm{d}\tau.
\end{aligned}
$$

Gronwall's inequality further ensures that (5.12) holds for every $p\in[\frac{2}{2-\beta},\infty[$. $\square$

Proposition 5.4. *Let (u,θ) be a smooth solution of system* (5.1)*. If $\theta^0\in L^2\cap B^{1-\beta}_{\infty,1}$ and $u^0\in L^2\cap B^1_{\infty,1}$, then we have*

$$
\|u\|_{L^\infty_t B^1_{\infty,1}} + \|\omega\|_{L^\infty_t B^0_{\infty,1}} \leqslant Ce^{\exp\{\exp\{Ct\}\}}. \tag{5.14}
$$

Proof. By applying logarithmic estimate (2.3) to Eqs. (5.6) and (5.5), we have

$$
\begin{aligned}
&\big\|\Gamma(t)\big\|_{B^0_{\infty,1}} + \big\|\mathcal{R}_\beta\theta(t)\big\|_{B^0_{\infty,1}} \\
&\quad\lesssim \big(1+\|\nabla u\|_{L^1_tL^\infty}\big)\big(\big\|\omega^0\big\|_{B^0_{\infty,1}} + \big\|\theta^0\big\|_{L^2\cap B^{1-\beta}_{\infty,1}} + \big\|[\mathcal{R}_\beta,u\cdot\nabla]\theta\big\|_{L^1_tB^0_{\infty,1}} + \|\mathcal{R}_\beta u_2\|_{L^1_tB^0_{\infty,1}}\big) \\
&\quad\lesssim \big(1+\|\nabla u\|_{L^1_tL^\infty}\big)e^{\exp\{Ct\}}
\end{aligned}
$$

where in the last line we have used (5.8), (5.9) and (5.4). Since by a high–low frequency decomposition and the Calderón–Zygmund theorem, we see that

$$
\big\|\nabla u(t)\big\|_{L^\infty} \leqslant \big\|\nabla\Delta_{-1}u(t)\big\|_{L^\infty} + \sum_{q\geqslant 0}\big\|\Delta_q\nabla u(t)\big\|_{L^\infty} \lesssim \big\|u(t)\big\|_{L^2} + \big\|\omega(t)\big\|_{B^0_{\infty,1}}.
$$

Thus we obtain

$$
\big\|\omega(t)\big\|_{B^0_{\infty,1}} \leqslant \big\|\Gamma(t)\big\|_{B^0_{\infty,1}} + \big\|\mathcal{R}_\beta\theta(t)\big\|_{B^0_{\infty,1}} \lesssim e^{\exp\{Ct\}}\left(1+\int_0^t \big\|\omega(\tau)\big\|_{B^0_{\infty,1}}\,\mathrm{d}\tau\right).
$$

Gronwall's inequality leads to $\|\omega(t)\|_{B^0_{\infty,1}} \lesssim e^{\exp\{\exp\{Ct\}\}}$. Combining this estimate with (5.2), it is obvious to get

$$\|u(t)\|_{B^1_{\infty,1}} \lesssim \|u(t)\|_{L^2} + \|\omega(t)\|_{B^0_{\infty,1}} \lesssim e^{\exp\{\exp\{Ct\}\}}. \qquad \square$$

5.2. Existence

We smooth the data to get the following approximate system,

$$\begin{cases} \partial_t u^{(k)} + u^{(k)} \cdot \nabla u^{(k)} + \nabla P^{(k)} = \theta^{(k)} e_2, \quad k \in \mathbb{N}, \\ \partial_t \theta^{(k)} + u^{(k)} \cdot \nabla \theta^{(k)} + |D|^\beta \theta^{(k)} = u_2^{(k)}, \\ \operatorname{div} u^{(k)} = 0; \quad u^{(k)}\big|_{t=0} = S_k u^0,\ \theta^{(k)}\big|_{t=0} = S_k \theta^0. \end{cases} \tag{5.15}$$

Since $S_k u^0, S_k\theta^0 \in H^s$ for every $s \in \mathbb{R}$, from the classical theory of quasi-linear hyperbolic systems (cf. [2]), the approximate system has a unique smooth solution $(u^{(k)}, \theta^{(k)})$ on $[0, T]$ with some $T > 0$ that may depend on k. We also have a blowup criterion as follows: if the quantity $\|\nabla u^{(k)}\|_{L^1_T L^\infty}$ is finite, the time T can always be continued beyond. Then for every $k \in \mathbb{N}$, the *a priori* estimate (5.14) ensures that the solution $(u^{(k)}, \theta^{(k)})$ is globally and smoothly defined. Since $\|S_k\theta^0\|_{L^2 \cap B^{1-\beta}_{\infty,1}} \leqslant \|\theta^0\|_{L^2 \cap B^{1-\beta}_{\infty,1}}$, $\|S_k u^0\|_{L^2} \leqslant \|u^0\|_{L^2}$ and $\|S_k\omega^0\|_{L^p \cap L^\infty} \leqslant \|\omega^0\|_{L^p \cap L^\infty}$ with $p \in [2, \infty[$, we find that the *a priori* estimates obtained in Propositions 5.1–5.3 for system (5.15) are uniform in k, that is,

$$u^{(k)} \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^2\big(\mathbb{R}^2\big)\big) \quad \text{and} \quad \omega^{(k)} \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^\infty \cap L^p\big) \quad \text{uniformly in } k, \tag{5.16}$$

and

$$\theta^{(k)} \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^2 \cap B^{1-\beta}_{\infty,1}\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+; H^{\beta/2}\big) \cap L^1_{\mathrm{loc}}\big(\mathbb{R}^+; B^1_{\infty,1}\big) \quad \text{uniformly in } k. \tag{5.17}$$

We also need some uniform information about $\partial_t u^{(k)}$ and $\partial_t\theta^{(k)}$ to show the convergence result. First we consider the estimate of $\partial_t u^{(k)}$. Denoting by $\mathcal{P} \triangleq \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$ the Leray projection operator which maps a vector field to the divergence-free vector field, and by applying it to the equation of $u^{(k)}$, we get

$$\partial_t u^{(k)} = \mathcal{P}\big(\theta^{(k)} e_2\big) - \mathcal{P}\big(u^{(k)} \cdot \nabla u^{(k)}\big).$$

From the Calderón–Zygmund theorem, Hölder's inequality and interpolation, we directly obtain

$$\begin{aligned} \big\|\partial_t u^{(k)}\big\|_{L^\infty_t L^2} &\leqslant \big\|\mathcal{P}\big(\theta^{(k)} e_2\big)\big\|_{L^\infty_t L^2} + \big\|\mathcal{P}\big(u^{(k)} \cdot \nabla u^{(k)}\big)\big\|_{L^\infty_t L^2} \\ &\lesssim \big\|\theta^{(k)}\big\|_{L^\infty_t L^2} + \big\|u^{(k)}\big\|_{L^\infty_t L^{2p/(p-2)}} \big\|\nabla u^{(k)}\big\|_{L^\infty_t L^p} \\ &\lesssim_p \big\|\theta^{(k)}\big\|_{L^\infty_t L^2} + \big\|u^{(k)}\big\|^{1-1/p}_{L^\infty_t L^2} \big\|\omega^{(k)}\big\|^{1/p}_{L^\infty_t L^\infty} \big\|\omega^{(k)}\big\|_{L^\infty_t L^p}, \end{aligned}$$

thus the uniform estimates (5.2), (5.4) and (5.12) imply that

$$\partial_t u^{(k)} \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^2\big) \quad \text{uniformly in } k. \tag{5.18}$$

Next we treat $\partial_t\theta^{(k)}$. According to the equation of $\theta^{(k)}$, estimate (2.6) and the Sobolev embedding, we find that

$$\begin{aligned}\|\partial_t\theta^{(k)}\|_{L^2_tH^{-\beta/2}} &\leqslant \|u^{(k)}\cdot\nabla\theta^{(k)}\|_{L^2_tH^{-\beta/2}} + \||D|^\beta\theta^{(k)}\|_{L^2_tH^{-\beta/2}} + \|u_2^{(k)}\|_{L^2_tH^{-\beta/2}}\\ &\lesssim \|u^{(k)}\|_{L^\infty_tL^\infty}\|\theta^{(k)}\|_{L^2_tH^{1-\beta/2}} + \|\theta^{(k)}\|_{L^2_tH^{\beta/2}} + \|u^{(k)}\|_{L^2_tL^2}\\ &\lesssim \|u^{(k)}\|^{1/2}_{L^\infty_tL^2}\|\omega^{(k)}\|^{1/2}_{L^\infty_tL^\infty}\|\theta^{(k)}\|_{L^2_tH^{\beta/2}} + \|\theta^{(k)}\|_{L^2_tH^{\beta/2}} + \|u^{(k)}\|_{L^2_tL^2},\end{aligned}$$

thus combining it with uniform estimates (5.2) and (5.4) yields that

$$\partial_t\theta^{(k)} \in L^2_{\mathrm{loc}}\big(\mathbb{R}^+; H^{-\beta/2}\big) \quad \text{uniformly in } k. \tag{5.19}$$

Therefore, from (5.16)–(5.19) and Aubin–Lions's compactness lemma (cf. [26]), we know that up to an extraction of subsequence, the approximate solution sequence $(u^{(k)},\theta^{(k)})_{k\in\mathbb{N}}$ converges strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{-\beta/2}_{\mathrm{loc}})$ to some function (u,θ), and (u,θ) moreover satisfies that

$$u \in \mathcal{C}^{0,1}_{\mathrm{loc}}\big(\mathbb{R}^+; L^2\big(\mathbb{R}^2\big)\big) \quad \text{and} \quad \omega \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^\infty \cap L^p\big),$$

and

$$\theta \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; L^2 \cap B^{1-\beta}_{\infty,1}\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+; H^{\beta/2}\big) \cap L^1_{\mathrm{loc}}\big(\mathbb{R}^+; B^1_{\infty,1}\big).$$

Interpolating with the uniform bounds (5.16)–(5.17), then it is clear to pass the limit in the approximate system (5.15) and (u,θ) solves the system (5.1) in the sense of distribution. Furthermore, we can show that $\theta \in \mathcal{C}(\mathbb{R}^+; L^2 \cap B^{1-\beta}_{\infty,1})$. Indeed, from (5.4) and (5.12) we know that $\theta \in \widetilde{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2 \cap B^{1-\beta}_{\infty,1})$, thus by a classical deduction (cf. [8]) we can get the desired result. This finishes the existence part of Theorem 1.1.

5.3. Uniqueness

The proof of uniqueness issue is similar to that in [12] with proper modification and we here sketch it. Let (u_i,θ_i,P_i) $(i=1,2)$ satisfying (1.2) and (1.3) be two solutions of the Euler–Bénard system (5.1) with the same initial data. Set $\tilde{u} \triangleq u_1 - u_2$, $\tilde{\theta} \triangleq \theta_1 - \theta_2$ and $\widetilde{P} \triangleq P_1 - P_2$, then the difference system writes

$$\begin{cases}\partial_t\tilde{u} + u_1\cdot\nabla\tilde{u} + \nabla\widetilde{P} = \tilde{\theta}e_2 - \tilde{u}\cdot\nabla u_2,\\ \partial_t\tilde{\theta} + u_1\cdot\nabla\tilde{\theta} + |D|^\beta\tilde{\theta} = \tilde{u}_2 - \tilde{u}\cdot\nabla\theta_2,\\ \operatorname{div} u_1 = \operatorname{div}\tilde{u} = 0, \quad (\tilde{u},\tilde{\theta})\big|_{t=0} = 0.\end{cases}$$

Since $\partial_t\tilde{u} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2)$, from the usual energy method we have that for every $q \in [p,\infty[$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{u}(t)\|^2_{L^2} &\leqslant \|\tilde{\theta}(t)\|_{L^2}\|\tilde{u}_2(t)\|_{L^2} + \|\tilde{u}(t)\|^2_{L^{2q/(q-1)}}\|\nabla u_1(t)\|_{L^q}\\ &\leqslant \|\tilde{\theta}(t)\|_{L^2}\|\tilde{u}(t)\|_{L^2} + Cq\|\tilde{u}(t)\|^{2(q-1)/q}_{L^2}\|\tilde{u}(t)\|^{2/q}_{L^\infty}\|\omega_1(t)\|_{L^p\cap L^\infty},\end{aligned}$$

where in the last line we have used interpolation and the Calderón–Zygmund theorem that

$$\sup_{q\in[p,\infty[}\frac{\|\nabla u_1\|_{L^q}}{q} \lesssim \|\omega_1\|_{L^p\cap L^\infty}.$$

We also see that

$$\partial_t\tilde{\theta} + |D|^\beta\tilde{\theta} = \tilde{u}_2 - \tilde{u}\cdot\nabla\theta_2 - u_1\cdot\nabla\tilde{\theta}, \quad \tilde{\theta}|_{t=0} = 0,$$

and due to that all the right-side terms belong to $L^\rho_{\mathrm{loc}}(\mathbb{R}^+; L^2)$ with $\rho \in [1, \beta[$ (from (5.12)), we have $\partial_t\tilde{\theta} \in L^\rho_{\mathrm{loc}}(\mathbb{R}^+; L^2)$ from Lemma 2.3. Thus from the energy method we find

$$\frac{1}{2}\frac{d}{dt}\big\|\tilde{\theta}(t)\big\|^2_{L^2} \leqslant \big\|\tilde{u}(t)\big\|_{L^2}\big\|\tilde{\theta}(t)\big\|_{L^2} + \|\tilde{u}\|_{L^2}\big\|\nabla\theta_2(t)\big\|_{L^\infty}\big\|\tilde{\theta}(t)\big\|_{L^2}.$$

Let $\epsilon > 0$ be a small number, and denote $\mathcal{X}_\epsilon(t) \triangleq \sqrt{\epsilon^2 + \|\tilde{u}\|^2_{L^2} + \|\tilde{\theta}(t)\|^2_{L^2}}$, then we get

$$\frac{d}{dt}\mathcal{X}_\epsilon(t) \leqslant Cq\big\|\tilde{u}(t)\big\|^{2/q}_{L^\infty}\big\|\omega_1(t)\big\|_{L^p\cap L^\infty}\mathcal{X}_\epsilon(t)^{1-2/q} + \big(1 + \big\|\nabla\theta_2(t)\big\|_{L^\infty}\big)\mathcal{X}_\epsilon(t).$$

By a direct computation, we infer that

$$\mathcal{X}_\epsilon(t) \leqslant e^{t+\|\nabla\theta_2\|_{L^1_t L^\infty}}\left(\epsilon^{2/q} + C\int_0^t \big\|\tilde{u}(\tau)\big\|^{2/q}_{L^\infty}\big\|\omega_1(\tau)\big\|_{L^p\cap L^\infty}\,\mathrm{d}\tau\right)^{\frac{q}{2}}.$$

Passing ϵ to 0, we obtain

$$\big\|\tilde{u}(t)\big\|^2_{L^2} + \big\|\tilde{\theta}(t)\big\|^2_{L^2} \leqslant e^{2t+2\|\nabla\theta_2\|_{L^1_t L^\infty}}\|\tilde{u}\|^2_{L^\infty_t L^\infty}\big(Ct\big\|\omega_1(\tau)\big\|_{L^\infty_t(L^p\cap L^\infty)}\big)^q.$$

Since $\theta_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^+; B^1_{\infty,1})$, $\tilde{u} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^\infty)$ and $\omega_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^p \cap L^\infty)$, by choosing $T > 0$ small enough, we have $CT\|\omega_1\|_{L^\infty_T(L^p\cap L^\infty)} \leqslant \frac{1}{2}$. Then letting q tend to ∞, we deduce $(\tilde{u}, \tilde{\theta}) \equiv 0$ on $[0, T]$. Since $(\tilde{u}, \tilde{\theta}) \in \mathcal{C}(\mathbb{R}^+; L^2)$, from a connectivity argument we can show the uniqueness result on $\mathbb{R}^+$.

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Appendix A

In this section we sketch the proof of (1.4).

Proof of (1.4). We first show that $\|e^{-t|D|^\beta}f\|_{L^1_T L^\infty} \lesssim_{T,\beta} \|f\|_{B^{-\beta}_{\infty,1}}$. From a high–low frequency decomposition and (2.5), we deduce that

$$\begin{aligned}\big\|e^{-t|D|^\beta}f\big\|_{L^1_T L^\infty} &\lesssim \big\|e^{-t|D|^\beta}\Delta_{-1}f\big\|_{L^1_T L^\infty} + \sum_{j\geqslant 0}\big\|\Delta_j e^{-t|D|^\beta}f\big\|_{L^1_T L^\infty} \\ &\lesssim \|\Delta_{-1}f\|_{L^\infty}T + \sum_{j\geqslant 0}\big\|e^{-ct2^{j\beta}}\big\|_{L^1_T}\|\Delta_j f\|_{L^\infty} \\ &\lesssim \big(2^{-\beta}T + 1\big)\|f\|_{B^{-\beta}_{\infty,1}}.\end{aligned}$$

Next we prove the inverse inequality. For $j \in \mathbb{N}$ and $T > 0$, we have

$$\Delta_j f = \int_0^{T_1} \left(\mathrm{Id} - e^{-T_1|D|^\beta} - T_1|D|^\beta e^{-T_1|D|^\beta}\right)^{-1} t|D|^{2\beta} e^{-t|D|^\beta} \Delta_j f \, \mathrm{d}t,$$

where $T_1 \triangleq T/2$. Indeed, by applying the Fourier transform, it follows from $\int_0^{T_1|\xi|^\beta} te^{-t}\,\mathrm{d}t = 1 - e^{-T_1|\xi|^\beta} - T_1|\xi|^\beta e^{-T_1|\xi|^\beta} > 0$. Note that due to $\operatorname{supp}\widehat{\Delta_j f} \subset \{\xi\colon |\xi| \geqslant 1\}$, by choosing T large enough, we get $1 \gg e^{-T_1|\xi|^\beta} + T_1|\xi|^\beta e^{-T_1|\xi|^\beta}$. In a similar way as obtaining (2.5), we infer that for every $j \in \mathbb{N}$,

$$\left\| \left(\mathrm{Id} - e^{-T_1|D|^\beta} - T_1|D|^\beta e^{-T_1|D|^\beta}\right)^{-1} \Delta_j f \right\|_{L^\infty} \lesssim \left(1 - e^{-cT}\right)^{-1} \|\Delta_j f\|_{L^\infty}.$$

Then since $e^{-t|D|^\beta} = e^{-t|D|^\beta/2} e^{-t|D|^\beta/2}$, from Bernstein's inequality and (2.5), we find that

$$\|\Delta_j f\|_{L^\infty} \lesssim \left(1 - e^{-cT}\right)^{-1} \int_0^{T/2} t 2^{2j\beta} e^{-ct2^{j\beta}} \left\| e^{-t|D|^\beta/2} f \right\|_{L^\infty} \mathrm{d}t.$$

Thus

$$\begin{aligned} \sum_{j\in\mathbb{N}} 2^{-j\beta} \|\Delta_j f\|_{L^\infty} &\lesssim \left(1 - e^{-cT}\right)^{-1} \int_0^{T} \left(\sum_{j\in\mathbb{N}} t 2^{j\beta} e^{-ct2^{j\beta}} \right) \left\| e^{-t|D|^\beta} f \right\|_{L^\infty} \mathrm{d}t \\ &\lesssim \left(1 - e^{-cT}\right)^{-1} \left\| e^{-t|D|^\beta} f \right\|_{L^1_T L^\infty}. \end{aligned}$$

While for $j = -1$, according to

$$\Delta_{-1} f = \frac{1}{T} \int_0^T e^{t|D|^\beta} \left(e^{-t|D|^\beta} \Delta_{-1} f\right) \mathrm{d}t,$$

and from Bernstein's inequality and $e^{t|D|^\beta} = \sum_{k\in\mathbb{N}} (t|D|^\beta)^k / k!$, we have

$$\begin{aligned} \|\Delta_{-1} f\|_{L^\infty} &\lesssim \frac{1}{T} \int_0^T e^{Ct} \left\| e^{-t|D|^\beta} \Delta_{-1} f \right\|_{L^\infty} \mathrm{d}t \\ &\lesssim \frac{1}{T} e^{CT} \left\| e^{-t|D|^\beta} f \right\|_{L^1_T L^\infty}. \end{aligned}$$

Hence $\|f\|_{B^{-\beta}_{\infty,1}} \lesssim_{T,\beta} \|e^{-t|D|^\beta} f\|_{L^1_T L^\infty}$. $\square$

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