Commun. Math. Sci.
Vol. 14, No. 7, pp. 1963–1997

REGULARITY RESULTS FOR THE 2D BOUSSINESQ EQUATIONS WITH CRITICAL OR SUPERCRITICAL DISSIPATION*

JIAHONG WU†, XIAOJING XU‡, LIUTANG XUE§, AND ZHUAN YE¶

Abstract. The incompressible Boussinesq equations serve as an important model in geophysics as well as in the study of Rayleigh–Bénard convection. One generalization is to replace the standard Laplacian operator by a fractional Laplacian operator, namely $(-\Delta)^{\alpha/2}$ in the velocity equation and $(-\Delta)^{\beta/2}$ in the temperature equation. This paper is concerned with the two-dimensional (2D) incompressible Boussinesq equations with critical dissipation ($\alpha+\beta=1$) or supercritical dissipation ($\alpha+\beta<1$). We prove two main results. This first one establishes the global-in-time existence of classical solutions to the critical Boussinesq equations with $\alpha+\beta=1$ and $0.7692\approx\frac{10}{13}<\alpha<1$. The second one proves the eventual regularity of Leray–Hopf type weak solutions to the Boussinesq equations with supercritical dissipation $\alpha+\beta<1$ and $0.7692\approx\frac{10}{13}<\alpha<1$.

Key words. 2D Boussinesq equations, generalized supercritical SQG, global regularity, eventual regularity.

AMS subject classifications. 35Q35, 35B65, 76D03.

1. Introduction

The Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and play an important role in the study of Rayleigh–Bérnard convection (see, e.g., [41, 46] and references therein). This paper is concerned with the following initial-value problem for the 2D incompressible Boussinesq equations with fractional dissipation

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+\Lambda^\alpha u+\nabla p=\theta e_2,\\ \partial_t\theta+(u\cdot\nabla)\theta+\Lambda^\beta\theta=0,\\ \nabla\cdot u=0,\\ u(x,0)=u_0(x),\quad \theta(x,0)=\theta_0(x),\end{cases}\tag{1.1}$$

where $u:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}^2$ denotes the velocity, $p:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}$ the pressure and $\theta:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}$ the temperature, $e_2=(0,1)$, and $\alpha,\beta\in[0,2]$ are real parameters. In addition, $\Lambda:=(-\Delta)^{\frac12}$ denotes the Zygmund operator and the fractional Laplacian operator Λ^α is defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\hat f(\xi).$$

*Received: July 18, 2015; accepted (in revised form): February 5, 2016. Communicated by Alexander Kiselev.
The authors are very grateful to the anonymous referees for their insightful comments and many valuable suggestions, which greatly improved the exposition of the manuscript. Wu was partially supported by NSF grant DMS1209153 and the AT&T Foundation at Oklahoma State University. Xu was partially supported by NSFC (No. 11371059), BNSF (No. 2112023) and by the Fundamental Research Funds for the Central Universities of China. Xue was partially supported by NSFC (No. 11401027) and the Youth Scholars Program of Beijing Normal University.

†Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences Stillwater, OK 74078, USA (jiahong@math.okstate.edu).
‡School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China (xjxu@bnu.edu.cn).
§School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China (xuelt@bnu.edu.cn).
¶Corresponding Author, Department of Mathematics and Statistics, Jiangsu Normal University, 101 Shanghai Road, Xuzhou 221116, Jiangsu, P.R. China (yezhuan815@126.com).

We make the convention that by $\alpha=0$ we mean that there is no dissipation in the velocity equation, and similarly $\beta=0$ means that there is no dissipation in the temperature equation.

The Boussinesq equations are also mathematically significant. The global regularity problem for the 2D Boussinesq equations has attracted considerable attention and progress has been made. The global regularity for the initial value problem (1.1) with $\alpha=2$ and $\beta=0$ or $\beta=2$ and $\alpha=0$ was established by Chae [6] and by Hou and Li [30]. Further progress in this direction was made by Hmidi, Keraani, and Rousset [28, 29], who established the global well-posedness result for the initial value problem (1.1) with $\alpha=1$ and $\beta=0$ or $\beta=1$ and $\alpha=0$. More general critical cases $\alpha+\beta=1$ were examined by by Jiu, Miao, Wu, and Zhang [31], who were able to obtain the global regularity for the general critical case $0.9132\approx\frac{23-\sqrt{145}}{12}<\alpha<1$ and $\beta=1-\alpha$. A very recent work of Stefanov and Wu [50] improved [31] by allowing α to vary in a bigger interval, $0.798103\approx\frac{\sqrt{1777}-23}{24}<\alpha<1$ and $\beta=1-\alpha$. We remark that even in the subcritical ranges, namely $\alpha+\beta>1$, it is not trivial to prove the global regularity of the initial value problem (1.1), as demonstrated by the papers dealing with the subcritical cases (see, e.g., [43, 57, 59–63]).

The global regularity for the Boussinesq equations with supercritical dissipation is currently open. To understand the regularity problem for the supercritical regime, Jiu, Wu and Yang [32] recently studied the regularity of weak solutions to the initial value problem (1.1) when α and β are in the supercritical range. They obtained the eventual regularity for (α,β) in the supercritical range $\alpha+\beta<1$ and $0.9132\approx\frac{23-\sqrt{145}}{12}<\alpha<1$. Finally we mention that many other results on the 2D incompressible Boussinesq equations have been obtained (see, e.g., [1, 2, 5, 9, 22, 34, 39, 40, 55, 56]).

This paper obtains two main results. The first establishes the global regularity for the critical Boussinesq equations with $\alpha+\beta=1$ and $0.7692\approx\frac{10}{13}<\alpha<1$. This result allows α in a slightly bigger range than in [50]. More precisely, the following result holds.

THEOREM 1.1. *Let $0.7692\approx\frac{10}{13}<\alpha<1$ and $\alpha+\beta=1$. Assume that $(u_0,\theta_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ for $s>2$, and $\nabla\cdot u_0=0$, then the system (1.1) admits a unique global solution such that, for any given $T>0$*

$$u\in C([0,T];H^s(\mathbb{R}^2))\cap L^2([0,T];H^{s+\frac{\alpha}{2}}(\mathbb{R}^2)),$$

$$\theta\in C([0,T];H^s(\mathbb{R}^2))\cap L^2([0,T];H^{s+\frac{\beta}{2}}(\mathbb{R}^2)).$$

Let us briefly give the explanation about the improvement in Theorem 1.1 was made possible compared with the previous work [50]. The following three components are made the improvement is possible: the first one is to establish a bound on G in the space $L_t^\infty L_x^m$ (see Lemma 3.2); the second one is to establish a redefined result (see Lemma 2.5); the last one is to make use of the Hölder estimates of the advection fractional-diffusion equation (see lemmas 2.3 and 2.4).

The second main result proves the eventual regularity of weak solutions to the 2D Boussinesq equation (1.1) with α and β in a supercritical regime. This result improves the result of Jiu, Wu, and Yang [32] by allowing α in a bigger range. More precisely, we have the following theorem.

THEOREM 1.2. *Consider the initial value problem (1.1) with $\alpha+\beta<1$. Assume either*

$$\frac{4}{5}<\alpha<1, \qquad \beta<1-\alpha$$

or

$$0.7692 \approx \frac{10}{13} < \alpha \leq \frac{4}{5}, \quad \frac{-3(3\alpha-2)+\sqrt{\alpha^2-204\alpha+164}}{8} < \beta < 1-\alpha.$$

Assume that $(u_0,\theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *for* $s>2$, *and* $\nabla \cdot u_0 = 0$. *Let* (u,θ) *be a global Leray–Hopf weak solution of the initial value problem (1.1). Then there exist* $\widehat{T}>0$ *such that* (u,θ) *is actually a classical solution on* $[\widehat{T},+\infty]$.

The proof of Theorem 1.2 relies on the following key proposition, which assesses the eventual regularity of weak solutions to a generalized surface quasi-geostrophic (SQG) equation.

PROPOSITION 1.1. *Consider the following initial-value problem*

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta + \Lambda^\beta \theta = 0, \\ u = u_G + u_\theta,\ u_\theta = -\nabla^\perp \Lambda^{-2-\alpha} \partial_{x_1} \theta, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.2}$$

with $\alpha>0, \beta>0$ *and* $\alpha+\beta<1$. *Suppose that* u_G *satisfies for any* $T>0$

$$\sup_{0\leq t\leq T} \|u_G(t)\|_{C^{1-\mu}} \leq C(T) < \infty, \qquad 0 \leq \mu < \beta. \tag{1.3}$$

Assume $\theta_0 \in H^s(\mathbb{R}^2)$ *for some* $s>2$ *and let* θ *be a corresponding global weak solution of problem (1.2). Then there exists* $\widehat{T}>0$, *more precisely,*

$$\widehat{T} = C_1 \gamma^{\frac{1-\alpha}{1-\alpha-\beta}} \|\theta_0\|_{L^\infty}^{\frac{\beta}{1-\alpha-\beta}},$$

such that $\theta \in L^\infty([\widehat{T},\infty), C^\gamma(\mathbb{R}^2))$ *for some* $1-\alpha-\beta<\gamma<1-\alpha$,

$$[\theta(t)]_{C^\gamma} \leq C_2 \gamma^{-\frac{\gamma}{1-\alpha-\beta}} \|\theta_0\|_{L^\infty}^{-\frac{\gamma+\alpha+\beta-1}{1-\alpha-\beta}},$$

where C_1 *and* C_2 *are constants depending on* α *and* β, *and* $[\theta(t)]_{C^\gamma}$ *denotes the homogeneous* C^γ*-norm of* θ. *Furthermore,* θ *belongs to* $L^\infty([\widehat{T},\infty) \times C^{1,\zeta}(\mathbb{R}^2))$ *for some* $\zeta>0$.

Proposition 1.1 is proven via the approach of pointwise inequality for fractional Laplacian following [16, 17, 21]. The details are given at the begin of Section 5.

The present paper is structured as follows. Section 2 states some analytic tools and preliminary results to be used in the subsequent sections. The proof of Theorem 1.1 is presented in Section 3. Section 4 establishes some *a priori* estimates for Boussinesq equations (1.1) with $\alpha+\beta<1$. Section 5 is devoted to the proof of Proposition 1.1 and Theorem 1.2. Finally, we state an eventual regularity and a global regularity type result for a generalized surface quasi-geostrophic type equation in the appendix. In addition, the proof of Lemma 2.5 is also provided in the appendix.

2. Preliminaries

This section serves as a preparation. It provides the definition of the Littlewood–Paley decomposition, Besov spaces, and related inequalities, and several estimates to be used in the subsequent sections.

Materials on the Littlewood–Paley theory and Besov spaces can be found in many papers and books (see, e.g., [3, 42, 47]). Let (χ,φ) be a couple of smooth functions with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B}:=\{\xi\in\mathbb{R}^n, |\xi|\le\frac{4}{3}\}$, $\varphi\in C_0^\infty(\mathbb{R}^n)$ in the annulus $\mathcal{C}:=\{\xi\in\mathbb{R}^n, \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ satisfying

$$\chi(\xi)+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\xi)=1,\quad \forall\xi\in\mathbb{R}^n;\qquad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\ \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

For every $u\in S'$ (tempered distributions) we define the non-homogeneous Littlewood–Paley operators as follows:

$$\Delta_j u=0 \text{ for } j\le-2;\quad \Delta_{-1}u=\chi(D)u=\mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi)),$$
$$\Delta_j u=\varphi(2^{-j}D)u=\mathcal{F}^{-1}(\varphi(2^{-j}\xi)\widehat{u}(\xi)),\ \forall j\in\mathbb{N}.$$

Meanwhile, we define the homogeneous dyadic blocks as

$$\dot\Delta_j u=\varphi(2^{-j}D)u=\mathcal{F}^{-1}(\varphi(2^{-j}\xi)\widehat{u}(\xi)),\quad \forall j\in\mathbb{Z}.$$

We now recall the definitions of homogeneous and inhomogeneous Besov spaces defined via the dyadic decomposition.

Definition 2.1. *Let $s\in\mathbb{R}, (p,r)\in[1,+\infty]^2$. The homogeneous Besov space $\dot B^s_{p,r}$ and inhomogeneous Besov space $B^s_{p,r}$ are defined as a space of $f\in S'(\mathbb{R}^n)$ such that*

$$\dot B^s_{p,r}=\{f\in S'(\mathbb{R}^n), \|f\|_{\dot B^s_{p,r}}<\infty\},\quad B^s_{p,r}=\{f\in S'(\mathbb{R}^n), \|f\|_{B^s_{p,r}}<\infty\},$$

where

$$\|f\|_{\dot B^s_{p,r}}=\begin{cases}\Big(\sum_{j\in\mathbb{Z}}2^{jrs}\|\dot\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\in\mathbb{Z}}2^{js}\|\dot\Delta_j f\|_{L^p}, & r=\infty,\end{cases}$$

and

$$\|f\|_{B^s_{p,r}}=\begin{cases}\Big(\sum_{j\ge-1}2^{jrs}\|\Delta_j f\|^r_{L^p}\Big)^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\ge-1}2^{js}\|\Delta_j f\|_{L^p}, & r=\infty.\end{cases}$$

For $s>0, (p,r)\in[1,+\infty]^2$, we have the following fact

$$\|f\|_{B^s_{p,r}}\approx\|f\|_{L^p}+\|f\|_{\dot B^s_{p,r}}.$$

Moreover, Besov space $B^s_{\infty,\infty}$ with $0<s<1$ is equivalent to the classical Hölder space C^s (see, e.g., [3]), namely

$$\|f\|_{B^s_{\infty,\infty}}\approx\|f\|_{C^s},\quad \|f\|_{C^s}:=\|f\|_{L^\infty}+\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^s}.$$

The Besov spaces $\dot B^s_{p,r}$ and $B^s_{p,r}$ with $0<s<1$ and $1\le p,q,r\le\infty$ can be equivalently defined by the norms

$$\|f\|_{\dot B^s_{p,r}}=\Big(\int_{\mathbb{R}^n}\frac{\|f(x+.)-f(.)\|^r_{L^p}}{|x|^{n+sr}}\,dx\Big)^{\frac{1}{r}},$$

$$\|f\|_{B^s_{p,r}}=\|f\|_{L^p}+\Big(\int_{\mathbb{R}^n}\frac{\|f(x+.)-f(.)\|^r_{L^p}}{|x|^{n+sr}}\,dx\Big)^{\frac{1}{r}}.$$

For the case $r=\infty$, the expressions are interpreted in the normal way. The following embedding relations will also be used,

$$B^{s_1}_{p,r_1}\hookrightarrow B^{s_2}_{p,r_2},\ s_1>s_2, \tag{2.1}$$

$$B^s_{p,\min\{p,2\}}\hookrightarrow W^{s,p}\hookrightarrow B^s_{p,\max\{p,2\}},\quad 1<p<\infty,$$

$$B^{s_1}_{p_1,r_1}\hookrightarrow B^{s_2}_{p_2,r_2},\quad s_1-\frac{d}{p_1}=s_2-\frac{d}{p_2},\ 1\le p_1\le p_2\le\infty,\ 1\le r_1\le r_2\le\infty.$$

We want to point out that at some point that the definitions of the spaces are given in $\mathbb{R}^2$ since the definitions are first stated in $\mathbb{R}^n$. We will also need the following commutator estimate.

Lemma 2.1 (see [50]). *Assume that $\frac{1}{2}<\alpha<1$ and $1<p_2,p_3<\infty$, $1<p_1\le\infty$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. Then for $0\le s_1<1-\alpha$ and $s_1+s_2>1-\alpha$, the following holds true:*

$$\Big|\int_{\mathbb{R}^2}F[\mathcal{R}_\alpha,u_G\cdot\nabla]\theta\,dx\Big|\le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|G\|_{L^{p_3}}. \tag{2.2}$$

Similarly, for $0\le s_1<1-\alpha$ and $s_1+s_2>2-2\alpha$, the following holds true:

$$\Big|\int_{\mathbb{R}^2}F[\mathcal{R}_\alpha,u_\theta\cdot\nabla]H\,dx\Big|\le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|H\|_{L^{p_3}}. \tag{2.3}$$

Here and in sequel, $\mathcal{R}_\alpha:=\partial_{x_1}\Lambda^{-\alpha}$, $u_G:=\nabla^\perp\Delta^{-1}G$, $u_\theta:=\nabla^\perp\Delta^{-1}\mathcal{R}_\alpha\theta$, and the standard commutator notation $[A,f]g=A(fg)-f(Ag)$.

The next lemma will be used frequently. For the reader's convenience, a proof for this lemma is also provided.

Lemma 2.2. *Let $2<m<\infty$ and $0<s<1$, then the following holds true:*

$$\|\Lambda^s(|f|^{m-2}f)\|_{L^p}\le C\|f\|_{\dot{B}^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}}, \tag{2.4}$$

$$\big\||f|^{m-2}f\big\|_{W^{s,p}}\le C\|f\|_{B^s_{q,p}}\|f\|^{m-2}_{L^{r(m-2)}}, \tag{2.5}$$

where $p,q,r\in(1,\infty)^3$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$.

Proof. For $0<s<1$, we make use of the following characterization of $\dot{W}^{s,p}$ (see, e.g., [3,47])

$$\|\Lambda^s(|f|^{m-2}f)\|^p_{L^p}\approx\int_{\mathbb{R}^2}\frac{\||f|^{m-2}f(x+.)-|f|^{m-2}f(.)\|^p_{L^p}}{|x|^{2+sp}}\,dx.$$

Applying the simple inequality

$$\Big||a|^{m-2}a-|b|^{m-2}b\Big|\le C(m)|a-b|(|a|^{m-2}+|b|^{m-2})$$

and Hölder's inequality, we have

$$\begin{aligned}\||f|^{m-2}f(x+\cdot)-|f|^{m-2}f(\cdot)\|_{L^p} &\leq C\|f(x+\cdot)-f(\cdot)\|_{L^q}\||f|^{m-2}\|_{L^r}\\ &\leq C\|f(x+\cdot)-f(\cdot)\|_{L^q}\|f\|^{m-2}_{L^{r(m-2)}}.\end{aligned}$$

Thus, it follows from the characterization of Besov space that

$$\begin{aligned}\|\Lambda^s(|f|^{m-2}f)\|^p_{L^p} &\leq C\int_{\mathbb{R}^2}\frac{\|f(x+\cdot)-f(\cdot)\|^p_{L^q}\|f\|^{(m-2)p}_{L^{r(m-2)}}}{|x|^{2+sp}}\,dx\\ &\leq C\|f\|^{(m-2)p}_{L^{r(m-2)}}\int_{\mathbb{R}^2}\frac{\|f(x+\cdot)-f(\cdot)\|^p_{L^q}}{|x|^{2+sp}}\,dx\\ &\leq C\|f\|^{(m-2)p}_{L^{r(m-2)}}\|f\|^p_{\dot{B}^s_{q,p}}.\end{aligned}$$

The Hölder inequality directly gives

$$\||f|^{m-2}f\|_{L^p}\leq C\|f\|_{L^q}\||f|^{m-2}\|_{L^r}=C\|f\|_{L^q}\|f\|^{m-2}_{L^{r(m-2)}}.$$

This concludes the proof. ∎

We shall also use the following two lemmas (see [11, 19, 48, 49]).

LEMMA 2.3. *Consider the following advection fractional-diffusion equation with* $0<\beta<1$ *in* $\mathbb{R}^n$*:*

$$\begin{cases}\partial_t\theta+(u\cdot\nabla)\theta+\Lambda^\beta\theta=0,\\ \nabla\cdot u=0,\\ \theta(x,0)=\theta_0(x).\end{cases}$$

Let $T>0$ *be given. Suppose that the* θ *is bounded and the drift* u *satisfying* $u(t)\in C^{1-\beta}$ *for any* $t\in[0,T]$*. Assume* $\theta\in L^\infty((0,T],L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n))$ *with initial data* $\theta_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$*. Then the solution* θ *is Hölder continuous for any positive time* $0<t\leq T$*. Moreover, it holds*

$$\|\theta\|_{L^\infty(0,T;\,C^\ell(\mathbb{R}^n))}\leq C\|\theta_0\|_{L^\infty},$$

where the constant C *and* $\ell>0$ *depend on* β *and* $\|u\|_{C^{1-\beta}}$ *only.*

LEMMA 2.4. *Consider the following advection fractional-diffusion equation with* $0<\beta<1$ *in* $\mathbb{R}^n$*:*

$$\begin{cases}\partial_t\theta+(u\cdot\nabla)\theta+\Lambda^\beta\theta=0,\\ \nabla\cdot u=0,\\ \theta(x,0)=\theta_0(x).\end{cases}$$

Let $T>0$ *be given and* u *be a vector field in* $L^\infty((0,T],C^{1-\beta+\zeta}(\mathbb{R}^n))$ *for any* $\zeta\in(0,\beta)$*, then any bounded solution* θ *in* $(0,T]\times\mathbb{R}^n$ *actually belongs to space* $C^{1,\zeta}$*. Moreover, it holds*

$$\|\theta\|_{L^\infty((0,T],C^{1,\zeta}(\mathbb{R}^n))}\leq C\|\theta_0\|_{L^\infty([0,T]\times\mathbb{R}^n)},$$

where the constant C *depends on* β *and* $\|u\|_{C^{1-\beta+\zeta}}$ *only.*

The following lemma is inspired by [31, Proposition 7.1]. We give the detailed proof in the Appendix A.

LEMMA 2.5. *Consider the advection fractional-diffusion equation (3.5), namely*

$$\partial_t G+(u\cdot\nabla)G+\Lambda^\alpha G=[\mathcal{R}_\alpha, u\cdot\nabla]\theta+\Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{2.6}$$

Let $\alpha+\beta\le 1$ with $\frac{1}{2}<\alpha<1$. Suppose G admits the following bound

$$\sup_{0\le t\le T}\|G(t)\|_{L^q}<\infty, \qquad q>\frac{2}{\alpha} \quad (\textit{we may assume } q<\tfrac{2}{1-\alpha}),$$

for any given $T>0$, then for any $0<s\le 3\alpha-2$, it holds

$$\sup_{0\le t\le T}\|G(t)\|_{B^s_{r,\infty}}<\infty,$$

where r is given by

$$\frac{2}{2\alpha-1}<r\le\frac{2q}{2-(1-\alpha)q}.$$

We remark that Lemma 2.5 improves [31, Proposition 7.1] in two ways: on the one hand, we relax the condition $q>\frac{2}{2\alpha-1}$ to $q>\frac{2}{\alpha}$; one the other hand, we can take $r=\frac{2q}{2-(1-\alpha)q}>q$.

3. The proof of Theorem 1.1

This section proves Theorem 1.1. Since the local existence result for (1.1) is standard (see, e.g., [41]), we only provide the global *a priori* estimates.

Basic energy estimates show that (u,θ) of problem (1.1) obeys the global bounds

$$\|\theta(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2\,d\tau\le\|\theta_0\|_{L^2}, \quad \|\theta(t)\|_{L^p}\le\|\theta_0\|_{L^p}, \quad \forall p\in[1,\infty], \tag{3.1}$$

and

$$\|u(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\alpha}{2}}u(\tau)\|_{L^2}^2\,d\tau\le C(t,u_0,\theta_0). \tag{3.2}$$

Now we apply operator curl to the first equation in problem (1.1) to obtain the following vorticity $w=\nabla\times u$ equation

$$\partial_t w+(u\cdot\nabla)w+\Lambda^\alpha w=\partial_{x_1}\theta. \tag{3.3}$$

However, the "vortex stretching" term $\partial_{x_1}\theta$ appears to prevent us from proving any global bound for w. To circumvent this difficulty, a natural idea would be to eliminate $\partial_{x_1}\theta$ from the vorticity equation. To this end, we generalize the idea of Hmidi, Keraani, and Rousset [28, 29] to introduce a new quantity. More precisely, we set $\mathcal{R}_\alpha$ as

$$\mathcal{R}_\alpha:=\partial_{x_1}\Lambda^{-\alpha},$$

apply $\mathcal{R}_\alpha$ to the θ equation in problem (1.1)

$$\partial_t\mathcal{R}_\alpha\theta+(u\cdot\nabla)\mathcal{R}_\alpha\theta=-[\mathcal{R}_\alpha, u\cdot\nabla]\theta-\Lambda^{\beta-\alpha}\partial_{x_1}\theta, \tag{3.4}$$

and combine with Equation (3.3) to obtain that

$$G=\omega-\mathcal{R}_\alpha\theta$$

satisfies

$$\partial_t G+(u\cdot\nabla)G+\Lambda^\alpha G=[\mathcal{R}_\alpha,u\cdot\nabla]\theta+\Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{3.5}$$

Moreover, as in [50], the velocity u can be rewritten as

$$u=\nabla^\perp\Delta^{-1}\omega=\nabla^\perp\Delta^{-1}(G+\mathcal{R}_\alpha\theta)=\nabla^\perp\Delta^{-1}G+\nabla^\perp\Delta^{-1}\mathcal{R}_\alpha\theta:=u_G+u_\theta. \tag{3.6}$$

We need the following global *a priori* bound of L^2 norm for G. When $\alpha>\frac{3}{4}$ and $\beta>4-5\alpha$, the proof is similar to [50, Lemma 2.7].

Lemma 3.1. *Assume $\alpha>0$, $\beta>0$ and $\alpha+\beta\le 1$ and consider the initial-value problem (1.1). If $\alpha>\frac{3}{4}$ and $\beta>4-5\alpha$, then, for any corresponding solution (u,θ) of the problem (1.1), there exist some constant C such that for any $T>0$,*

$$\|G(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\alpha}{2}}G(\tau)\|_{L^2}^2\,d\tau\le C<\infty \tag{3.7}$$

for any $t\in[0,T]$.

Next we establish the following global *a priori* bound for $\|G(t)\|_{L^m}$.

Lemma 3.2. *Assume that (u_0,θ_0) satisfies the assumptions stated in Theorem 1.2. Let $\frac{3}{4}<\alpha<1$ and $\alpha+\beta=1$. For any m satisfying*

$$2<m<\min\left\{\frac{8}{2-\alpha},\frac{1}{1-\alpha}\right\},\quad (10-9\alpha-4\alpha^2)m<8(1-\alpha), \tag{3.8}$$

or equivalently

$$2<m<m_0,\quad m_0=\begin{cases}\dfrac{8(1-\alpha)}{10-9\alpha-4\alpha^2}, & \dfrac{3}{4}<\alpha\le\dfrac{7+\sqrt{145}}{24},\\ \dfrac{1}{1-\alpha}, & \dfrac{7+\sqrt{145}}{24}<\alpha\le\dfrac{6}{7},\\ \dfrac{8}{2-\alpha}, & \dfrac{6}{7}<\alpha<1,\end{cases} \tag{3.9}$$

G admits the following global bound for any $0\le t\le T$,

$$\|G(t)\|_{L^m}^m+\int_0^T\|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m\,d\tau\le C(T,u_0,\theta_0)<\infty. \tag{3.10}$$

Proof. We start with the Gagliardo–Nirenberg inequality (see [26, 50])

$$\|\Lambda^{\gamma\beta}\theta\|_{L_t^{\frac{1}{\gamma}}L_x^{\frac{1}{\gamma}}}\le C\|\Lambda^{\frac{\beta}{2}}\theta\|_{L_t^2L_x^2}^{2\gamma}\|\theta\|_{L_t^\infty L_x^\infty}^{1-2\gamma},\quad 0<\gamma<\frac{1}{2}. \tag{3.11}$$

Multiplying Equation (3.5) by $|G|^{m-2}G$ and integrating over $\mathbb{R}^2$ yields

$$
\begin{aligned}
&\frac{1}{m}\frac{d}{dt}\|G(t)\|_{L^m}^m+\int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx\\
&=\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_{x_1}\theta\,|G|^{m-2}G\,dx+\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_\theta\cdot\nabla]\theta\,|G|^{m-2}G\,dx\\
&\quad+\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx.
\end{aligned}\tag{3.12}
$$

By the maximum principle (see [20]) and the Sobolev embedding, the dissipative term has the lower bound

$$
\int_{\mathbb{R}^2}(\Lambda^\alpha G)|G|^{m-2}G\,dx\ge\widetilde{C}\|\Lambda^{\frac{\alpha}{2}}G^{\frac{m}{2}}\|_{L^2}^2\ge\widetilde{C}\|G\|^m_{L^{\frac{2m}{2-\alpha}}},\tag{3.13}
$$

where $\widetilde{C}>0$ is an absolute constant. By the Hölder inequality and Equation (2.5)

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_{x_1}\theta\,|G|^{m-2}G\,dx\Big|&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|\Lambda^{1-\alpha+(1-\gamma)\beta}(|G|^{m-2}G)\|_{L^{\frac{1}{1-\gamma}}}\\
&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{B^{1-\alpha+(1-\gamma)\beta}_{2,\frac{1}{1-\gamma}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}}.
\end{aligned}
$$

We use the embedding inequality

$$
H^{\frac{\alpha}{2}}\hookrightarrow B^s_{2,\frac{1}{1-\gamma}},\qquad s<\frac{\alpha}{2}.\tag{3.14}
$$

Consequently, for

$$
1-\alpha+(1-\gamma)\beta<\frac{\alpha}{2}\quad\text{or}\quad\gamma>\frac{2\beta+2-3\alpha}{2\beta},\tag{3.15}
$$

we have

$$
\Big|\int_{\mathbb{R}^2}\Lambda^{\beta-\alpha}\partial_x\theta\,|G|^{m-2}G\,dx\Big|\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}}.\tag{3.16}
$$

To deal with the second term of the r.h.s. of the inequality (3.12), we choose $s_1=\gamma\beta$ and s_2 satisfying

$$
2-2\alpha-\gamma\beta<s_2<\frac{\alpha}{2}.
$$

Notice that such s_2 exists as long as

$$
2-2\alpha-\gamma\beta<\frac{\alpha}{2}.\tag{3.17}
$$

Then, by Lemma 2.1, Lemma 2.2, and the embedding inequality (3.14)

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_\theta\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big|\\
&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|\theta\|_{L^\infty}\||G|^{m-2}G\|_{W^{s_2,\frac{1}{1-\gamma}}}\\
&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|\theta_0\|_{L^\infty}\|G\|_{B^{s_2}_{2,\frac{1}{1-\gamma}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}}\\
&\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}}.
\end{aligned}\tag{3.18}
$$

To estimate the third term of the r.h.s. of the inequality (3.12), we take $\delta>0$ to be sufficiently small, say

$$\delta=\frac{4\alpha-3}{100}$$

and

$$s_2=1-\alpha+\delta.$$

For $m-1<q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, we apply Lemma 2.1 and Lemma 2.2 to obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big| \\
&\le C\|G\|_{L^q}\|\theta\|_{L^\infty}\||G|^{m-2}G\|_{W^{s_2-\frac{\delta}{2},p}} \\
&\le C\|\theta_0\|_{L^\infty}\|G\|_{L^q}\|G\|^{m-2}_{L^{(m-2)\times\frac{q}{m-2}}}\|G\|_{B^{s_2-\frac{\delta}{2}}_{\frac{q}{q-(m-1)},p}} \qquad (q>m-1)\\
&\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{B^{s_2-\frac{\delta}{2}}_{\frac{q}{q-(m-1)},p}}.
\end{aligned}$$

For $q\le 2(m-1)$, we use the Besov embedding (2.1), namely

$$H^{s_2-1+\frac{2(m-1)}{q}}\hookrightarrow B^{s_2-\frac{\delta}{2}}_{\frac{q}{q-(m-1)},p}$$

to obtain

$$\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big|\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|_{H^{s_2-1+\frac{2(m-1)}{q}}}. \tag{3.19}$$

We further require $q>\frac{4(m-1)}{3\alpha-2\delta}$ to obtain the following interpolation inequality

$$\|G\|_{H^{-\alpha+\delta+\frac{2(m-1)}{q}}}\le C\|G\|^{1-\mu}_{L^2}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}},$$

where

$$\mu=\frac{-2\alpha+2\delta+\frac{4(m-1)}{q}}{\alpha}\in(0,1).$$

Therefore, for $\frac{4(m-1)}{3\alpha-2\delta}<q<2(m-1)$, one can conclude that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^2}[\mathcal{R}_\alpha,u_G\cdot\nabla]\theta\,|G|^{m-2}G\,dx\Big| &\le C\|\theta_0\|_{L^\infty}\|G\|^{m-1}_{L^q}\|G\|^{1-\mu}_{L^2}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}\\
&\le C\|G\|^{m-1}_{L^q}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}.
\end{aligned}\tag{3.20}$$

Substituting the estimates (3.13), (3.16), (3.18), and (3.20) into the inequality (3.12), we find

$$\frac{d}{dt}\|G(t)\|^m_{L^m}+\|G\|^m_{L^{\frac{2m}{2-\alpha}}}\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|^{m-2}_{L^{\frac{2(m-2)}{1-2\gamma}}}+C\|G\|^{m-1}_{L^q}\|G\|^{\mu}_{H^{\frac{\alpha}{2}}}. \tag{3.21}$$

It follows from the Gagliardo–Nirenberg inequalities that

$$\|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}\le C\|G\|^{1-\lambda_1}_{L^m}\|G\|^{\lambda_1}_{L^{\frac{2m}{2-\alpha}}},\quad \lambda_1=\frac{(1+2\gamma)m-4}{\alpha(m-2)}, \tag{3.22}$$

$$\|G\|_{L^q} \le C\|G\|_{L^m}^{1-\lambda_2}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{\lambda_2}, \quad \lambda_2 = \frac{2-\frac{2m}{q}}{\alpha}. \tag{3.23}$$

In order for $\lambda_1, \lambda_2 \in (0,1)$, we impose the following restrictions

$$\frac{4-m}{2m} \le \gamma \le \frac{m-(2-\alpha)(m-2)}{2m}, \quad m \le q \le \frac{2m}{2-\alpha}. \tag{3.24}$$

In view of the inequalities (3.22) and (3.23), we obtain

$$\begin{aligned} &C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|_{L^{\frac{2(m-2)}{1-2\gamma}}}^{m-2} \\ &\le C\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}}\|G\|_{L^m}^{(m-2)(1-\lambda_1)}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-2)\lambda_1} \\ &\le \frac{\widetilde{C}}{4}\|G\|_{L^{\frac{2m}{2-\alpha}}}^m + C(\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} &C\|G\|_{L^q}^{m-1}\|G\|_{H^{\frac{\alpha}{2}}}^{\mu} \\ &\le C\|G\|_{L^m}^{(m-1)(1-\lambda_2)}\|G\|_{L^{\frac{2m}{2-\alpha}}}^{(m-1)\lambda_2}\|G\|_{H^{\frac{\alpha}{2}}}^{\mu} \\ &\le \frac{\widetilde{C}}{4}\|G\|_{L^{\frac{2m}{2-\alpha}}}^m + C\|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned} \tag{3.26}$$

Inserting the estimates (3.25) and (3.26) into the inequality (3.21), it holds that

$$\begin{aligned} \frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m \le{}& C(\|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}\|G\|_{H^{\frac{\alpha}{2}}})^{\frac{m}{m-(m-2)\lambda_1}}\|G\|_{L^m}^{\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1}} \\ &+ C\|G\|_{H^{\frac{\alpha}{2}}}^{\frac{m\mu}{m-(m-1)\lambda_2}}\|G\|_{L^m}^{\frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2}}. \end{aligned} \tag{3.27}$$

It is easy to check that

$$\frac{m(m-2)(1-\lambda_1)}{m-(m-2)\lambda_1} \le m, \quad \frac{m(m-1)(1-\lambda_2)}{m-(m-1)\lambda_2} \le m,$$

and for m satisfying inequality (3.8)

$$\frac{m\mu}{m-(m-1)\lambda_2} \le 2 \quad \Big(\text{due to } m < \frac{1}{1-\alpha}\Big).$$

We choose $\gamma \in (0, \frac{1}{2})$ such that

$$\gamma \le \frac{8-(2-\alpha)m}{2m(2+\alpha)} \quad \Big(\text{due to } m < \frac{8}{2-\alpha}\Big). \tag{3.28}$$

Later we explain why such γ can be selected. Then, for γ satisfying the condition (3.28), we have

$$\frac{m}{m-(m-2)\lambda_1} \le \frac{2}{2\gamma+1}.$$

In summary, by Lemma 3.1 we have obtained

$$\frac{d}{dt}\|G(t)\|_{L^m}^m + \|G\|_{L^{\frac{2m}{2-\alpha}}}^m \le C\Big(1 + \|\Lambda^{\gamma\beta}\theta\|_{L^{\frac{1}{\gamma}}}^{\frac{1}{\gamma}} + \|G\|_{H^{\frac{\alpha}{2}}}^2\Big)(1+\|G\|_{L^m}^m). \tag{3.29}$$

Gronwall's inequality then implies the desired result.

To complete the proof, we explain that q and γ can be selected to satisfy all the restrictions stated above. The number q should satisfy

$$\max\left\{\frac{4(m-1)}{3\alpha-2\delta},\, m\right\} < q < \min\left\{2(m-1),\, \frac{2m}{2-\alpha}\right\}.$$

Direct computations yield that the number q can be fixed if we select $\delta < \frac{3\alpha-2}{2}$. Putting all the restrictions (3.15), (3.17), (3.24), (3.28), and $0<\gamma<\frac{1}{2}$ on γ, we have

$$\underline{\mathcal{B}}(\alpha) < \gamma < \overline{\mathcal{B}}(\alpha), \tag{3.30}$$

where

$$\underline{\mathcal{B}}(\alpha) = \max\left\{0,\, \frac{2\beta+2-3\alpha}{2\beta},\, \frac{4-5\alpha}{2\beta},\, \frac{4-m}{2m}\right\},$$

$$\overline{\mathcal{B}}(\alpha) = \min\left\{\frac{1}{2},\, \frac{1-\alpha}{\beta},\, \frac{m-(2-\alpha)(m-2)}{2m},\, \frac{8-(2-\alpha)m}{2m(2+\alpha)}\right\}.$$

Inserting $\beta = 1-\alpha$ into $\underline{\mathcal{B}}(\alpha)$ and $\overline{\mathcal{B}}(\alpha)$, the restriction on γ reduces to

$$\max\left\{0,\, \frac{4-5\alpha}{2(1-\alpha)},\, \frac{4-m}{2m}\right\} < \gamma < \min\left\{\frac{1}{2},\, \frac{m-(2-\alpha)(m-2)}{2m},\, \frac{8-(2-\alpha)m}{2m(2+\alpha)}\right\}.$$

Moreover, the number m should obey

$$2 < m < \min\left\{\frac{8}{2-\alpha},\, \frac{2}{2-2\alpha+\delta}\right\}.$$

Due to the arbitrariness of $\delta > 0$, it gives

$$2 < m < \min\left\{\frac{8}{2-\alpha},\, \frac{1}{1-\alpha}\right\}. \tag{3.31}$$

Invoking direct computation yields that for $m > 2$

$$\frac{1}{2} \geq \frac{m-(2-\alpha)(m-2)}{2m} \geq \frac{8-(2-\alpha)m}{2m(2+\alpha)}.$$

As a consequence of the above fact, the condition (3.30) reduces to

$$\max\left\{0,\, \frac{4-5\alpha}{2(1-\alpha)},\, \frac{4-m}{2m}\right\} < \gamma < \frac{8-(2-\alpha)m}{2m(2+\alpha)}.$$

It is easy to check that the γ would work if the following restrictions are met

$$\frac{4-5\alpha}{2(1-\alpha)} < \frac{8-(2-\alpha)m}{2m(2+\alpha)}, \quad \frac{4-m}{2m} < \frac{8-(2-\alpha)m}{2m(2+\alpha)},$$

which leads to

$$(10-9\alpha-4\alpha^2)m < 8(1-\alpha). \tag{3.32}$$

This ends the proof of Lemma 3.2. $\square$

We are ready to prove Theorem 1.1.

Proof. (**Proof of Theorem 1.1.**) The proof follows from the idea in [31]. It consists of two steps. The first step shows that $u_G \in C^\sigma$ with $\sigma > \alpha = 1-\beta$. The second step is to conclude the desired result by invoking the recent progress on the generalized critical surface quasi-geostrophic equation. We then conclude the global regularity of our solution once the regularity of θ is known. Now we present the details.

According to Lemma 2.5, if G satisfies

$$\sup_{0\le t\le T} \|G(t)\|_{L^m} \le C \qquad \text{for some } m > \frac{2}{\alpha}, \tag{3.33}$$

then G can be shown to satisfy

$$\sup_{0\le t\le T} \|G(t)\|_{B^{3\alpha-2}_{\widetilde{r}_0,\infty}} \le C(T,u_0,\theta_0) < \infty, \quad \widetilde{r}_0 = \frac{2m}{2-(1-\alpha)m}. \tag{3.34}$$

By Lemma 3.2, we indeed have obtained

$$\sup_{0\le t\le T} \|G(t)\|_{L^m} \le C \qquad \text{for any } 2 < m < m_0. \tag{3.35}$$

It is worthwhile to notice that for $\alpha > \frac{3}{4}$, we have

$$\frac{8}{2-\alpha} > \frac{2}{\alpha}, \quad \frac{1}{1-\alpha} > \frac{2}{\alpha}.$$

On the other hand, for $\frac{3}{4} < \alpha \le \frac{7+\sqrt{145}}{24}$, we need the restriction

$$\frac{8(1-\alpha)}{10-9\alpha-4\alpha^2} = m_0 > m > \frac{2}{\alpha},$$

which leads to the requirement

$$\alpha > \frac{10}{13} \approx 0.7692.$$

As a consequence, for $m > \frac{2}{\alpha}$ and $\sigma = 2\alpha - \frac{2}{m} > \alpha$, we have

$$\begin{aligned}
\|u_G\|_{C^\sigma} &= \|\nabla^\perp \Delta^{-1} G\|_{C^\sigma} \\
&\approx \|\nabla^\perp \Delta^{-1} G\|_{B^\sigma_{\infty,\infty}} \quad (0<\sigma<1) \\
&\le C\|G\|_{L^2} + C\|G\|_{B^{\sigma-1}_{\infty,\infty}} \\
&\le C\|G\|_{L^2} + C\|G\|_{B^{3\alpha-2}_{\widetilde{r}_0,\infty}} \qquad \left(\widetilde{r}_0 = \frac{2m}{2-(1-\alpha)m}\right) \\
&\le C(T,u_0,\theta_0) < \infty.
\end{aligned} \tag{3.36}$$

This yields the global C^σ bound on u_G. Therefore, according to the bound $\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$ and the properties of Besov spaces, we can conclude

$$\begin{aligned}
\|u_\theta\|_{C^\alpha} &= \|\nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta\|_{C^\alpha} \\
&\approx \|\nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta\|_{B^\alpha_{\infty,\infty}} \\
&\le C\|\theta\|_{L^2} + C\|\theta\|_{B^0_{\infty,\infty}}
\end{aligned}$$

$$\begin{aligned}&\le C\|\theta\|_{L^2}+C\|\theta\|_{L^\infty}\\&\le C(T,u_0,\theta_0)<\infty.\end{aligned}$$

Combining the above two bounds shows

$$\|u\|_{C^\alpha}\le\|u_G\|_{C^\alpha}+\|u_\theta\|_{C^\alpha}\le C(T,u_0,\theta_0)<\infty,\quad \alpha=1-\beta.$$

We now return to the equation of θ and treat it as a generalized critical surface quasi-geostrophic (SQG) type equation

$$\begin{cases}\partial_t\theta+(u\cdot\nabla)\theta+\Lambda^\beta\theta=0,\\ u=u_G+u_\theta,\ u_\theta=-\nabla^\perp\Lambda^{-2-\alpha}\partial_{x_1}\theta.\end{cases}\tag{3.37}$$

By Lemma 2.3, θ is Hölder continuous, namely $\|\theta\|_{C^\eta}<\infty$ for some $\eta>0$, which, in turn, implies

$$\|u_\theta\|_{C^{\alpha+\eta}}=\|\nabla^\perp\Lambda^{-2-\alpha}\partial_{x_1}\theta\|_{C^{\alpha+\eta}}\le C\|\theta\|_{L^2}+C\|\theta\|_{C^\eta}\le C(T,u_0,\theta_0)<\infty.$$

Moreover, it follows from the inequality (3.36) that

$$\|u_G\|_{C^\sigma}\le C(T,u_0,\theta_0)<\infty,\quad \sigma>\alpha=1-\beta.$$

Therefore,

$$\|u\|_{C^\varrho}\le C(T,u_0,\theta_0)<\infty,\quad \varrho=\min\{\alpha+\eta,\sigma\}>\alpha=1-\beta,$$

which, together with Lemma 2.4, implies that θ becomes immediately differentiable. In particular, we have

$$\int_0^T\|\nabla\theta(t)\|_{L^\infty}\,dt\le C(T,u_0,\theta_0)<\infty,\tag{3.38}$$

and

$$\|\omega(t)\|_{L^\infty}\le\|\omega_0\|_{L^\infty}+\int_0^t\|\nabla\theta(\tau)\|_{L^\infty}\,d\tau<\infty.\tag{3.39}$$

Then the inequalities (3.38) and (3.39) imply (u,θ) is the desired classical solution. This completes the proof of Theorem 1.1. ∎

4. *A priori* estimates for Boussinesq Equation (1.1) with $\alpha+\beta<1$

Our next goal is to prove Theorem 1.2, namely the eventual regularity for the supercritical case $\alpha+\beta<1$. This section serves as a preparation. We establish *a priori* bound for u_G. We need to distinguish two cases, namely $\alpha>\frac{4}{5}$ and $\alpha\le\frac{4}{5}$.

We note that Lemma 3.1 works for both the critical case and the supercritical case $\alpha+\beta\le 1$. Next we establish a global *a priori* bound for $\|G\|_{L^5}$ when $\alpha+\beta<1$ with $\alpha>\frac{4}{5}$.

Lemma 4.1. *Consider Equation (1.1) with $\alpha+\beta<1$ and $\alpha>\frac{4}{5}$. Then,*

$$\|G(t)\|_{L^5}^5+\int_0^t\|G(\tau)\|^5_{L^{\frac{10}{2-\alpha}}}\,d\tau\le C(T)<\infty,\tag{4.1}$$

$$\sup_{0\le t\le T}\|G(t)\|_{B^{3\alpha-2}_{5,\infty}}\le C(T)<\infty, \tag{4.2}$$

and

$$\sup_{0\le t\le T}\|u_G\|_{\mathrm{Lip}}\le C(T)<\infty. \tag{4.3}$$

Proof. As in the proof of Lemma 3.2, the inequality (3.29) remains valid as long as

$$2<m<\min\Big\{\frac{8}{2-\alpha},\,\frac{2}{2-2\alpha+\delta}\Big\},$$

$$\underline{\mathcal{B}}(\alpha)<\gamma<\overline{\mathcal{B}}(\alpha), \tag{4.4}$$

where

$$\underline{\mathcal{B}}(\alpha)=\max\Big\{0,\,\frac{2\beta+2-3\alpha}{2\beta},\,\frac{4-5\alpha}{2\beta},\,\frac{4-m}{2m}\Big\}=\max\Big\{0,\,\frac{4-5\alpha}{2\beta},\,\frac{4-m}{2m}\Big\},$$

$$\overline{\mathcal{B}}(\alpha)=\min\Big\{\frac{1}{2},\,\frac{1-\alpha}{\beta},\,\frac{m-(2-\alpha)(m-2)}{2m},\,\frac{8-(2-\alpha)m}{2m(2+\alpha)}\Big\}=\frac{8-(2-\alpha)m}{2m(2+\alpha)},$$

due to $\frac{1-\alpha}{\beta}\ge\frac{1}{2}\ge\frac{m-(2-\alpha)(m-2)}{2m}\ge\frac{8-(2-\alpha)m}{2m(2+\alpha)}$ and $\beta<1-\alpha$. Therefore, γ satisfying condition (4.4) can be selected if

$$\max\Big\{0,\,\frac{m(4-5\alpha)(2+\alpha)}{8-(2-\alpha)m}\Big\}<\beta<1-\alpha. \tag{4.5}$$

Now for the case $\alpha>\frac{4}{5}$ and $\alpha+\beta<1$, we choose $m=5$. As a result, the differential inequality (3.27) becomes

$$\frac{d}{dt}\|G(t)\|^5_{L^5}+\|G\|^5_{L^{\frac{10}{2-\alpha}}}\le C(1+\|\Lambda^{\gamma\beta}\theta\|^{\frac{1}{\gamma}}_{L^{\frac{1}{\gamma}}}+\|G\|^2_{H^{\frac{\alpha}{2}}})(1+\|G\|^5_{L^5}), \tag{4.6}$$

which, by Gronwall's inequality, implies

$$\|G(t)\|^5_{L^5}+\int_0^T\|G(\tau)\|^5_{L^{\frac{10}{2-\alpha}}}\,d\tau\le C<\infty. \tag{4.7}$$

The inequality (4.2) is a direct consequence of Lemma 2.5, and the inequality (4.3) follows from the inequality (4.2). In fact, by the Besov embedding and the inequality (4.2),

$$\|\nabla u_G\|_{L^\infty}\le C\|\nabla\nabla^\perp\Delta^{-1}G\|_{L^\infty}\le C\|G\|_{B^{3\alpha-2}_{5,\infty}}\le C.$$

This yields the global Lipschitz bound on u_G. □

When $\alpha\le\frac{4}{5}$, the following global bound for u_G can be established.

LEMMA 4.2. *Consider Equation (1.1) with the supercritical indices $\alpha+\beta<1$ and $\alpha\le\frac{4}{5}$. Then for any m satisfying*

$$2<m<\widetilde{m}_0,\qquad \widetilde{m}_0:=\min\Big\{\frac{1}{1-\alpha},\,\frac{8}{2-\alpha},\,\frac{8\beta}{(4-5\alpha)(2+\alpha)+(2-\alpha)\beta}\Big\},$$

we have

$$\|G(t)\|_{L^m}^m + \int_0^t \|G(\tau)\|_{L^{\frac{2m}{2-\alpha}}}^m \, d\tau \le C(T) < \infty. \tag{4.8}$$

In addition, if $\frac{2}{\alpha} < m < \widetilde{m}_0$*, then*

$$\sup_{0\le t\le T} \|G(t)\|_{B^{3\alpha-2}_{\widetilde{r}_0,\infty}} \le C(T) < \infty, \qquad \widetilde{r}_0 = \frac{2m}{2-(1-\alpha)m}. \tag{4.9}$$

As a consequence, for β *satisfying*

$$\frac{-3(3\alpha-2)+\sqrt{\alpha^2-204\alpha+164}}{8} < \beta < 1-\alpha, \tag{4.10}$$

(of course, it requires $\alpha > \frac{10}{13} \approx 0.7692$*), it holds*

$$\sup_{0\le t\le T} \|u_G\|_{C^{1-\mu}} \le C(T) < \infty, \qquad 0 < \mu < \beta. \tag{4.11}$$

Proof. The proof of the inequality (4.8) is very similar to that of Lemma 3.2. We remark that the constraint

$$m < \frac{8\beta}{(4-5\alpha)(2+\alpha)+(2-\alpha)\beta}$$

follows from condition (4.5), namely

$$\frac{m(4-5\alpha)(2+\alpha)}{8-(2-\alpha)m} < \beta < 1-\alpha.$$

The inequality (4.9) follows from the inequality (4.8) and Lemma 2.5, and the inequality (4.11) is a consequence of the inequality (4.9) and the constraint imposed on β. More precisely, for β satisfying the inequality (4.10), we have

$$\frac{2}{2\alpha-1+\beta} < \frac{8\beta}{(4-5\alpha)(2+\alpha)+(2-\alpha)\beta},$$

and then we can choose m satisfying

$$\frac{2}{2\alpha-1+\beta} < m < \frac{8\beta}{(4-5\alpha)(2+\alpha)+(2-\alpha)\beta}.$$

Set $\mu = \frac{2}{m} - 2\alpha + 1$, clearly $\mu < \beta$. Then

$$\begin{aligned}
\|u_G\|_{C^{1-\mu}} &\approx \|u_G\|_{B^{1-\mu}_{\infty,\infty}} \\
&= \|\nabla^\perp \Delta^{-1} G\|_{B^{1-\mu}_{\infty,\infty}} \\
&\le C\|G\|_{L^2} + C\|G\|_{B^{-\mu+\frac{2}{\widetilde{r}_0}}_{\widetilde{r}_0,\infty}} \\
&= C\|G\|_{L^2} + C\|G\|_{B^{3\alpha-2}_{\widetilde{r}_0,\infty}}, \qquad \left(\widetilde{r}_0 = \frac{2m}{2-(1-\alpha)m}\right) \\
&\le C(T).
\end{aligned} \tag{4.12}$$

Due to the restrictions

$$\frac{2}{2\alpha-1+\beta}<\min\left\{\frac{1}{1-\alpha},\frac{8}{2-\alpha}\right\},$$

we need

$$\beta>3-4\alpha. \tag{4.13}$$

Therefore, β should satisfy

$$\max\left\{\frac{-3(3\alpha-2)+\sqrt{\alpha^2-204\alpha+164}}{8},3-4\alpha\right\}<\beta<1-\alpha. \tag{4.14}$$

Observing the restriction $\alpha>\frac{3}{4}$, the final restriction on β is

$$\frac{-3(3\alpha-2)+\sqrt{\alpha^2-204\alpha+164}}{8}<\beta<1-\alpha, \tag{4.15}$$

which would work as long as $\alpha>\frac{10}{13}\approx 0.7692$. The proof of Lemma 4.2 is complete. ◻

5. The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. In order to achieving this goal, we first need to establish Proposition 1.1. We consider the difference

$$\delta_h\theta(x,t)=\theta(x+h,t)-\theta(x,t).$$

We will use the pointwise equality (see, e.g., [17, 20])

$$2f(x)\Lambda^\beta f(x)=\Lambda^\beta[f^2(x)]+D_\beta[f](x),\quad 0<\beta<2,$$

where

$$D_\beta[f](x)=C(\beta)\,\text{p.v.}\int_{\mathbb{R}^2}\frac{[f(x)-f(x+y)]^2}{|y|^{2+\beta}}\,dy.$$

Following [16], we have

$$\mathcal{L}(\delta_h\theta)^2+D_\beta[\delta_h\theta]=0, \tag{5.1}$$

where $\mathcal{L}$ denotes the differential operator

$$\mathcal{L}=\partial_t+u\cdot\nabla_x+(\delta_h u)\cdot\nabla_h+\Lambda^\beta.$$

Similarly as in [21], for

$$v(x,t;h)=\frac{\delta_h\theta(x,t)}{(\xi^2(t)+|h|^2)^{\frac{\gamma}{2}}},$$

and for any bounded differentiable function $\xi:[0,\infty)\mapsto[0,\infty)$, we have

$$\begin{aligned}\mathcal{L}v^2+\frac{D_\beta[\delta_h\theta]}{(\xi^2+|h|^2)^\gamma}&=2\gamma|\dot{\xi}|\frac{\xi}{\xi^2+|h|^2}v^2-2\gamma\frac{h\cdot\delta_h u}{\xi^2+|h|^2}v^2\\&\le 2\gamma|\dot{\xi}|\frac{\xi}{\xi^2+|h|^2}v^2+2\gamma\frac{|h|\,|\delta_h u|}{\xi^2+|h|^2}v^2,\end{aligned} \tag{5.2}$$

where $\delta_h u = \nabla^{\perp}\Lambda^{-1-\alpha}\delta_h\theta$. It is worthwhile to state that when $\xi(t)=0$, $\|v\|_{L^\infty_{x,h}}$ is equivalent to the Hölder seminorm $[\theta(t)]_{C^\gamma} := \sup_{x\neq y}\frac{|\theta(x)-\theta(y)|}{|x-y|^\gamma}$.

Parallel to Lemma 3.1 in [21], we have the following lower bound on $D_\beta[\delta_h\theta](x)$.

LEMMA 5.1. *Let $0<\beta<1$ and $0<\gamma<1$. Then there exists a positive constant $C_1 = C_1(\beta,\gamma)\geq 1$ such that*

$$\frac{D_\beta[\delta_h\theta](x)}{(\xi^2+|h|^2)^\gamma} \geq \frac{1}{C_1|h|^\beta}\left[\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\right]^{\frac{\beta}{1-\gamma}}(v(x;h))^2. \tag{5.3}$$

Proof. Let χ be a smooth radially cutoff function that vanishes on $|x|\leq 1$ and is identically 1 for $|x|\geq 2$ and such that $|\chi'|\leq 2$. For $R\geq 4|h|$, we can conclude

$$\begin{aligned}
D_\beta[\delta_h\theta](x) &\geq C(\beta)\int_{\mathbb{R}^2}\frac{[\delta_h\theta(x)-\delta_h\theta(x+y)]^2}{|y|^{2+\beta}}\chi(\frac{|y|}{R})\,dy\\
&\geq C(\beta)[\delta_h\theta(x)]^2\int_{\mathbb{R}^2}\frac{1}{|y|^{2+\beta}}\chi(\frac{|y|}{R})\,dy - 2C(\beta)|\delta_h\theta(x)|\cdot\Big|\int_{\mathbb{R}^2}\frac{\delta_h\theta(x+y)}{|y|^{2+\beta}}\chi(\frac{|y|}{R})\,dy\Big|\\
&\geq C(\beta)\frac{[\delta_h\theta(x)]^2}{R^\beta} - 2C(\beta)|\delta_h\theta(x)|\cdot\Big|\int_{|y|\geq R}[\theta(x+y)-\theta(x)]\delta_{-h}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{2+\beta}}\Big)\,dy\Big|\\
&\geq C(\beta)\frac{[\delta_h\theta(x)]^2}{R^\beta} - 2CC(\beta)|h||\delta_h\theta(x)|\int_{|y|\geq R}\frac{|\delta_y\theta(x)|}{(\xi^2+|y|^2)^{\frac{\gamma}{2}}}\frac{(\xi^2+|y|^2)^{\frac{\gamma}{2}}}{|y|^{3+\beta}}\,dy\\
&\geq C(\beta)\frac{[\delta_h\theta(x)]^2}{R^\beta} - 2CC(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}\int_R^\infty\frac{(\xi^2+\rho^2)^{\frac{\gamma}{2}}}{\rho^{2+\beta}}\,d\rho\\
&\geq C(\beta)\frac{[\delta_h\theta(x)]^2}{R^\beta} - 2CC(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}\Big(\frac{\xi^\gamma}{R^{1+\beta}}+\frac{1}{R^{1+\beta-\gamma}}\Big),
\end{aligned}\tag{5.4}$$

where we have used the following fact, for $|y|\geq\frac{3}{4}R$,

$$\Big|\delta_{-h}\Big(\frac{\chi(\frac{|y|}{R})}{|y|^{2+\beta}}\Big)\Big| \leq C|h|\Big(\frac{1}{|y|^{3+\beta}}+\frac{|\chi'(\frac{|y|}{R})|I_{R\leq|y|\leq 2R}}{R|y|^{2+\beta}}\Big)\leq\frac{C|h|}{|y|^{3+\beta}}.$$

Now we take R as

$$R=\left[\frac{8C\|v\|_{L^\infty_{x,h}}}{|v(x;h)|}\right]^{\frac{1}{1-\gamma}}|h|\geq 4|h|.$$

Then, we easily obtain

$$\begin{aligned}
&2CC(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}\Big(\frac{\xi^\gamma}{R^{1+\beta}}+\frac{1}{R^{1+\beta-\gamma}}\Big)\\
&\leq\frac{2CC(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}}{R^\beta}\Big(\frac{\xi^\gamma}{(8C)^{\frac{1}{1-\gamma}}|h|}\Big[\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big]^{\frac{1}{1-\gamma}}+\frac{1}{8C|h|^{1-\gamma}}\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big)\\
&\leq\frac{2CC(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}}{R^\beta}\Big(\frac{\xi^\gamma}{8C|h|}\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}+\frac{1}{8C|h|^{1-\gamma}}\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big)\\
&\leq\frac{C(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}}{4R^\beta}\Big(\frac{\xi^\gamma}{|h|}+\frac{|h|^\gamma}{|h|}\Big)\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C(\beta)|h||\delta_h\theta(x)|\|v\|_{L^\infty_{x,h}}}{2R^\beta}\frac{(\xi^2+|h|^2)^{\frac{\gamma}{2}}}{|h|}\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\\
&\le C(\beta)\frac{[\delta_h\theta(x)]^2}{2R^\beta}.
\end{aligned}
\tag{5.5}
$$

Combining the inequalities (5.4) and (5.5), we immediately have

$$
D_\beta[\delta_h\theta](x)\ge C(\beta)\frac{[\delta_h\theta(x)]^2}{2R^\beta}=\frac{[\delta_h\theta(x)]^2}{C_1|h|^\beta}\Big[\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big]^{\frac{\beta}{1-\gamma}},
$$

where

$$
C_1=\frac{2(8C)^{\frac{\beta}{1-\gamma}}}{C(\beta)}. \tag{5.6}
$$

This completes the proof of Lemma 5.1. ∎

We select a suitable $\xi=\xi(t)$ to serve our purpose.

LEMMA 5.2. *Let $0<\beta<1$ and $0<\gamma<1$. For $\xi_0>0$ sufficiently large (to be fixed later), we define*

$$
\xi(t)=\begin{cases}\Big[\xi_0^\beta-\dfrac{\beta}{16C_1\gamma}t\Big]^{\frac{1}{\beta}}, & \text{if } 0\le t\le \widehat{T},\\ 0, & \text{if } \widehat{T}\le t<\infty,\end{cases}
$$

where C_1 is as in Equation (5.6) and $\widehat{T}=\frac{16C_1\gamma\xi_0^\beta}{\beta}$. Equivalently, $\xi(t)$ satisfies the following ordinary differential equation

$$
\dot{\xi}(t)=-\frac{1}{16C_1\gamma}\xi^{1-\beta}(t),\quad \xi(0)=\xi_0>0. \tag{5.7}
$$

Moreover, the following inequality holds

$$
2\gamma|\dot{\xi}|\frac{\xi}{\xi^2+|h|^2}v^2\le\frac{1}{8C_1|h|^\beta}v^2. \tag{5.8}
$$

Proof. Simple calculations allow us to show

$$
2\gamma|\dot{\xi}|\frac{\xi}{\xi^2+|h|^2}v^2\le\frac{\xi^{2-\beta}}{8C_1(\xi^2+|h|^2)}v^2\le\frac{(\xi^2+|h|^2)^{1-\frac{\beta}{2}}}{8C_1(\xi^2+|h|^2)}v^2\le\frac{1}{8C_1|h|^\beta}v^2.
$$

This completes the proof. ∎

The following lemma provides an estimate for $\delta_h u$.

LEMMA 5.3. *Let $\alpha+\gamma<1$ and $\rho\ge 4|h|$ be arbitrarily fixed. Then,*

$$
|\delta_h u(x)|\le C\Big(\rho^{\alpha+\frac{\beta}{2}}\big[D_\beta[\delta_h\theta](x)\big]^{\frac12}+|h|\|v\|_{L^\infty_{x,h}}\big[\frac{1}{\rho^{1-\alpha-\gamma}}+\frac{\xi^\gamma}{\rho^{1-\alpha}}\big]\Big)+C(T)|h|^{1-\mu}. \tag{5.9}
$$

Proof. Recall Equation $(A.1)_2$, namely

$$
u=u_G+u_\theta,\quad u_\theta=-\nabla^\perp\Lambda^{-2-\alpha}\partial_1\theta.
$$

As a consequence of the result on [51, p.73],

$$\begin{aligned} u_\theta &= -\mathcal{F}^{-1}\Big(\frac{\xi^\perp}{|\xi|^{2+\alpha}}\Big) * \partial_1\theta \\ &= -\mathcal{F}^{-1}\Big(\frac{\xi^\perp}{|\xi|^{1+2-(1-\alpha)}}\Big) * \partial_1\theta \\ &= C(\alpha)\frac{x^\perp}{|x|^{2-\alpha}} * \partial_1\theta \\ &= C(\alpha)\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^{2-\alpha}}\partial_1\theta(y)\,dy, \end{aligned} \tag{5.10}$$

where $*$ stands for the convolution symbol. By using that the kernel of $x^\perp$ has zero average on the unit sphere, it then gives

$$\begin{aligned} \delta_h u_\theta &= C(\alpha)\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^{2-\alpha}}(\delta_h\partial_{y_1}\theta)(y)\,dy \\ &= C(\alpha)\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^{2-\alpha}}\partial_{y_1}(\delta_h\theta)(y)\,dy \\ &= C(\alpha)\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^{2-\alpha}}\partial_{y_1}[(\delta_h\theta)(y)-(\delta_h\theta)(x)]\,dy \\ &= C(\alpha)\int_{\mathbb{R}^2}\partial_{y_1}(\frac{(x-y)^\perp}{|x-y|^{2-\alpha}})[\delta_h\theta(y)-\delta_h\theta(x)]\,dy \\ &= C(\alpha)\int_{\mathbb{R}^2}\partial_{y_1}(\frac{y^\perp}{|y|^{2-\alpha}})[\delta_h\theta(x+y)-\delta_h\theta(x)]\,dy \\ &= \delta_h u_\theta^{(\mathrm{in})} + \delta_h u_\theta^{(\mathrm{out})}, \end{aligned} \tag{5.11}$$

where

$$\delta_h u_\theta^{(\mathrm{in})} = C(\alpha)\int_{\mathbb{R}^2}\partial_{y_1}\Big(\frac{y^\perp}{|y|^{2-\alpha}}\Big)\Big[1-\chi(\frac{|y|}{\rho})\Big]\Big(\delta_h\theta(x+y)-\delta_h\theta(x)\Big)\,dy,$$

$$\delta_h u_\theta^{(\mathrm{out})} = C(\alpha)\int_{\mathbb{R}^2}\partial_{y_1}\Big(\frac{y^\perp}{|y|^{2-\alpha}}\Big)\chi\Big(\frac{|y|}{\rho}\Big)\Big(\delta_h\theta(x+y)-\delta_h\theta(x)\Big)\,dy.$$

Thanks to Equation (1.3), one has

$$|\delta_h u_G| \le \|u_G\|_{C^{1-\mu}}|h|^{1-\mu} \le C(T)|h|^{1-\mu}.$$

It thus follows from Young's inequality that

$$\begin{aligned} |\delta_h u_\theta^{(\mathrm{in})}| &\le C(\alpha)\int_{|y|\le\rho}\frac{1}{|y|^{2-\alpha}}|\delta_h\theta(x+y)-\delta_h\theta(x)|\,dy \\ &\le C(\alpha)\Big(\int_{|y|\le\rho}\frac{1}{|y|^{2-2\alpha-\beta}}\,dy\Big)^{\frac12}\Big(\int_{|y|\le\rho}\frac{(\delta_h\theta(x+y)-\delta_h\theta(x))^2}{|y|^{2+\beta}}\,dy\Big)^{\frac12} \\ &\le C\rho^{\alpha+\frac{\beta}{2}}\big[D_\beta[\delta_h\theta](x)\big]^{\frac12}. \end{aligned} \tag{5.12}$$

Finally, direct calculation yields, for $\rho\ge 4|h|$,

$$|\delta_h u_\theta^{(\mathrm{out})}| \le C(\alpha)\Big|\int_{|y|\ge\frac{\rho}{2}}\delta_{-h}\Big(\partial_{y_1}(\frac{y^\perp}{|y|^{2-\alpha}})\chi(\frac{|y|}{\rho})\Big)[\theta(x+y)-\theta(x)]\,dy\Big|$$

$$
\begin{aligned}
&\le C|h|\int_{|y|\ge\frac{\rho}{2}}\frac{1}{|y|^{3-\alpha}}|\theta(x+y)-\theta(x)|\,dy\\
&\le C|h|\int_{|y|\ge\frac{\rho}{2}}\frac{(\xi^2+|y|^2)^{\frac{\gamma}{2}}}{|y|^{3-\alpha}}\frac{|\delta_y\theta(x)|}{(\xi^2+|y|^2)^{\frac{\gamma}{2}}}\,dy\\
&\le C|h|\|v\|_{L^\infty_{x,h}}\int_{|y|\ge\frac{\rho}{2}}\frac{(\xi^2+|y|^2)^{\frac{\gamma}{2}}}{|y|^{3-\alpha}}\,dy\\
&\le C|h|\|v\|_{L^\infty_{x,h}}\int_{|y|\ge\frac{\rho}{2}}\Big(\frac{\xi^\gamma}{|y|^{3-\alpha}}+\frac{|y|^\gamma}{|y|^{3-\alpha}}\Big)\,dy\\
&\le C|h|\|v\|_{L^\infty_{x,h}}\Big[\frac{1}{\rho^{1-\alpha-\gamma}}+\frac{\xi^\gamma}{\rho^{1-\alpha}}\Big],
\end{aligned}
\tag{5.13}
$$

where in the last inequality we have applied the fact $\alpha+\gamma<1$. Combining all the above estimates, we obtain Equation(5.9). The proof of Lemma 5.3 is completed. □

The following lemma provides an upper bound for the term involving $\delta_h u$ under the condition $\|v\|_{L^\infty_{x,h}}\le M$.

LEMMA 5.4. *Let $1-\alpha-\beta<\gamma<1-\alpha$ with $0<\alpha,\beta<1$, and assume that*

$$\|v\|_{L^\infty_{x,h}}\le M:=\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma}.$$

Then there exists some constant $C_2=C_2(\beta,\gamma)\ge 1$ such that if

$$\xi_0=(C_2\|\theta_0\|_{L^\infty})^{\frac{1}{1-\alpha-\beta}},$$

then

$$2\gamma\frac{|h||\delta_h u|}{\xi^2+|h|^2}v^2\le\frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}+\frac{1}{8C_1|h|^\beta}v^2 \tag{5.14}$$

for any $|h|\le\min\{\xi_0,\,(16C_1C(T))^{-\frac{1}{\beta-\mu}}\}$.

Proof. By Young's inequality and Equation (5.9),

$$
\begin{aligned}
&2\gamma\frac{|h|}{\xi^2+|h|^2}|\delta_h u|v^2\\
\le&\frac{C2\gamma|h|}{\xi^2+|h|^2}\Big(\rho^{\alpha+\frac{\beta}{2}}\big[D_\beta[\delta_h\theta](x)\big]^{\frac12}+|h|\|v\|_{L^\infty_{x,h}}\Big[\frac{1}{\rho^{1-\alpha-\gamma}}+\frac{\xi^\gamma}{\rho^{1-\alpha}}\Big]\Big)v^2+C(T)\frac{|h|^{2-\mu}}{\xi^2+|h|^2}v^2\\
\le&\frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}+\frac{C\gamma|h|^2}{\xi^2+|h|^2}\Big\{\frac{\gamma v^2}{(\xi^2+|h|^2)^{1-\gamma}}\rho^{2\alpha+\beta}+\frac{\|v\|_{L^\infty_{x,h}}}{\rho^{1-\alpha-\gamma}}+\frac{\|v\|_{L^\infty_{x,h}}\xi^\gamma}{\rho^{1-\alpha}}\Big\}v^2\\
&+C(T)\frac{|h|^{2-\mu}}{\xi^2+|h|^2}v^2.
\end{aligned}
\tag{5.15}
$$

Now we take

$$\rho=4(\xi^2+|h|^2)^{\frac12}.$$

Direct calculations show that for $|h|\le\xi_0$,

$$\frac{\gamma v^2}{(\xi^2+|h|^2)^{1-\gamma}}\rho^{2\alpha+\beta}=\frac{4^{2\alpha+\beta}\gamma v^2}{(\xi^2+|h|^2)^{1-\gamma}}(\xi^2+|h|^2)^{\frac{2\alpha+\beta}{2}}$$

$$
\begin{aligned}
&\le C\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2\gamma}}\frac{1}{(\xi^2+|h|^2)^{\frac{\beta}{2}}}(\xi^2+|h|^2)^{\frac{2\alpha+\beta}{2}+\frac{\beta}{2}+\gamma-1}\\
&\le C\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2\gamma}}\frac{1}{(\xi^2+|h|^2)^{\frac{\beta}{2}}}\xi_0^{2\alpha+2\beta+2\gamma-2}\\
&\le C\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2-2\alpha-2\beta}}\frac{1}{|h|^\beta},
\end{aligned}
\tag{5.16}
$$

where we have used the fact

$$
1-\alpha-\beta<\gamma<1-\alpha\Rightarrow 2\gamma>2\alpha+2\beta+2\gamma-2>0.
$$

Therefore,

$$
\begin{aligned}
&\frac{\|v\|_{L^\infty_{x,h}}}{\rho^{1-\alpha-\gamma}}+\frac{\|v\|_{L^\infty_{x,h}}\xi^\gamma}{\rho^{1-\alpha}}\\
&\le\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma}\rho^{\alpha+\gamma-1}+\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma}\xi^\gamma\rho^{\alpha-1}\\
&\le\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}(\xi^2+|h|^2)^{\frac{\alpha+\gamma-1+\beta}{2}}+\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}\xi^\gamma(\xi^2+|h|^2)^{\frac{\alpha-1+\beta}{2}}\\
&\le\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}(\xi^2+|h|^2)^{\frac{\alpha+\gamma-1+\beta}{2}}+\frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}(\xi^2+|h|^2)^{\frac{\alpha-1+\beta+\gamma}{2}}\\
&\le\frac{C\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}\xi_0^{\alpha+\gamma-1+\beta}+\frac{C\|\theta_0\|_{L^\infty}}{\xi_0^\gamma(\xi^2+|h|^2)^{\frac{\beta}{2}}}\xi_0^{\alpha-1+\beta+\gamma}\\
&\le\frac{C\|\theta_0\|_{L^\infty}}{\xi_0^{1-\alpha-\beta}|h|^\beta},
\end{aligned}
\tag{5.17}
$$

where the following fact has also been applied

$$
1-\alpha-\beta<\gamma<1-\alpha\Rightarrow\gamma>\alpha+\gamma-1+\beta>0,
$$

Substituting the inequalities (5.16) and (5.17) into the inequality (5.15), we obtain the estimate

$$
\begin{aligned}
2\gamma\frac{|h|}{\xi^2+|h|^2}|\delta_h u|v^2&\le\frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}+C\gamma\Big\{\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2-2\alpha-2\beta}}\frac{1}{|h|^\beta}+\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\alpha-\beta}|h|^\beta}\Big\}v^2\\
&\quad+C(T)\frac{|h|^{2-\mu}}{\xi^2+|h|^2}v^2\\
&\le\frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}+\frac{C(T)}{|h|^\mu}v^2+C\gamma\Big\{\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2-2\alpha-2\beta}}+\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\alpha-\beta}}\Big\}\frac{1}{|h|^\beta}v^2\\
&\le\frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}+\frac{1}{8C_1|h|^\beta}v^2.
\end{aligned}
\tag{5.18}
$$

Here we have used the following conditions

$$
C\gamma\Big\{\frac{\|\theta_0\|_{L^\infty}^2}{\xi_0^{2-2\alpha-2\beta}}+\frac{\|\theta_0\|_{L^\infty}}{\xi_0^{1-\alpha-\beta}}\Big\}\le\frac{1}{16C_1},
$$

$$
\frac{C(T)}{|h|^\mu}\le\frac{1}{16C_1|h|^\beta}.
$$

The first condition is fulfilled if we select ξ_0 large enough. In fact, it is sufficient to choose ξ_0 as

$$\xi_0 = (C_2\|\theta_0\|_{L^\infty})^{\frac{1}{1-\alpha-\beta}}, \tag{5.19}$$

where $C_2 = 48CC_1\gamma$. The second condition can be guaranteed by the assumption $|h| \le (16C_1C(T))^{-\frac{1}{\beta-\mu}}$. This completes the proof of Lemma 5.4. ◻

Proof. (**Proof of Proposition 1.1.**) By the definition of v,

$$\|v(0)\|_{L^\infty_{x,h}} \le \frac{2\|\theta_0\|_{L^\infty}}{\xi_0^\gamma} = \frac{M}{2}.$$

Define

$$T_0 = \sup\{t \ge 0 \colon \|v(s)\|_{L^\infty_{x,h}} < M, \forall s \in [0,t]\}.$$

T_0 is the first time for which $\|v(s)\|_{L^\infty_{x,h}}$ reaches the value M. We claim that $T_0 = +\infty$. First, $T_0 > 0$ due to the continuity of $\|v(t)\|_{L^\infty_{x,h}}$ in time variable t. Note that the continuity of v with respect to variable x and h and $|v(x,t)| \to 0$ as $|x| \to \infty$, there exist $(\widetilde{x},\widetilde{h})$ such that

$$|v(\widetilde{x},T_0;\widetilde{h})| = \|v(\widetilde{x},T_0;\widetilde{h})\|_{L^\infty_{x,h}} = M.$$

The value $\widetilde{h}$ satisfies $|\widetilde{h}| \le \xi_0$. Indeed, for any $|h| \ge \xi_0$, we have

$$|v(x,t;h)| \le \frac{2\|\theta_0\|_{L^\infty}}{|h|^\gamma} \le \frac{2\|\theta_0\|_{L^\infty}}{\xi_0^\gamma} = \frac{M}{2}.$$

By Lemmas 5.1 and 5.4, we can conclude that for any $t \in (0, T_0]$

$$\mathcal{L}v^2 + \frac{D_\beta[\delta_h\theta]}{(\xi^2+|h|^2)^\gamma} \le \frac{1}{4C_1|h|^\beta}v^2 + \frac{D_\beta[\delta_h\theta](x)}{2(\xi^2+|h|^2)^\gamma}, \quad |h| \le \xi_0.$$

Making use of the lower bound established in Lemma 5.1, we see that

$$\mathcal{L}v^2 + \frac{1}{4C_1|h|^\beta}\Big(\Big[\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big]^{\frac{\beta}{1-\gamma}} - 1\Big)v^2 + \frac{1}{4C_1|h|^\beta}\Big[\frac{|v(x;h)|}{\|v\|_{L^\infty_{x,h}}}\Big]^{\frac{\beta}{1-\gamma}} v^2 \le 0, \tag{5.20}$$

for any $|h| \le \xi_0$. Let ε be small enough such that

$$\|v(s)\|_{L^\infty_{x,h}} \ge \frac{M}{2}, \ \forall s \in [T_0-\varepsilon, T_0).$$

Let $(\widetilde{x},\widetilde{h}) = (\widetilde{x}(s),\widetilde{h}(s))$ be the point at which v^2 attains its maximum value of M^2. At this point, we have

$$\partial_x v^2 = \partial_h v^2 = 0, \, \Lambda^\beta v^2 \ge 0.$$

Therefore, Equation (5.20) yields

$$(\partial_s v^2)(\widetilde{x},s;\widetilde{h}) + \frac{M^2}{16C_1\xi_0^\beta} \le \mathcal{L}v^2(\widetilde{x},s;\widetilde{h}) + \frac{1}{4C_1|\widetilde{h}|^\beta}v^2(\widetilde{x},s;\widetilde{h}) \le 0,$$

which implies

$$(\partial_s v^2)(\widetilde{x},s;\widetilde{h}) \leq -\frac{M^2}{16C_1\xi_0^\beta}, \quad \forall s\in[T_0-\varepsilon,T_0).$$

Following an argument in [16, Appendix B], one may show that for almost every s in $s\in[T_0-\varepsilon,T_0)$ we have

$$\frac{d}{ds}\|v(s)\|^2_{L^\infty_{x,h}} \leq (\partial_s v^2)(\widetilde{x},s;\widetilde{h}) \leq -\frac{M^2}{16C_1\xi_0^\beta}.$$

Consequently, we get $\|v(T_0)\|_{L^\infty_{x,h}} < M$, which leads to a contradiction. As a result, we thus conclude $T_0=+\infty$. In fact, notice that the selected ξ_0 as in Equation (5.19), the value $\widehat{T}$ can be formulated as

$$\widehat{T} = \frac{16C_1\gamma\xi_0^\beta}{\beta} = \frac{16C_1\gamma}{\beta}(48CC_1\gamma\|\theta_0\|_{L^\infty})^{\frac{\beta}{1-\alpha-\beta}} = C_5\gamma^{\frac{1-\alpha}{1-\alpha-\beta}}\|\theta_0\|_{L^\infty}^{\frac{\beta}{1-\alpha-\beta}},$$

where the constant $C_5=C_5(\alpha,\beta)\geq 1$. Since $\xi(t)\equiv 0$ for all $t\geq\widehat{T}$, the Hölder seminorm of θ can be bounded by

$$[\theta(t)]_{C^\gamma} = \|v(t)\|_{L^\infty_{x,h}} \leq M = \frac{4\|\theta_0\|_{L^\infty}}{\xi_0^\gamma} \leq C_5\gamma^{-\frac{\gamma}{1-\alpha-\beta}}\|\theta_0\|_{L^\infty}^{-\frac{\gamma+\alpha+\beta-1}{1-\alpha-\beta}}.$$

As a special consequence,

$$\begin{aligned}\|u_\theta\|_{C^{\gamma+\alpha}} &= \|\nabla^\perp\Lambda^{-2-\alpha}\partial_x\theta\|_{C^{\gamma+\alpha}}\\ &\leq C\|\theta\|_{L^2} + C\|\theta\|_{C^\gamma}\\ &\leq C<\infty,\end{aligned}$$

where $\gamma>1-\alpha-\beta \Rightarrow \gamma+\alpha>1-\beta$. Recall the condition on u_G,

$$\|u_G\|_{C^{1-\mu}} \leq C(T)<\infty, \quad \mu<\beta \Rightarrow 1-\mu>1-\beta.$$

Therefore,

$$\|u\|_{C^s} \leq C(T)<\infty, \quad \text{for some } s>1-\beta,$$

which together with Lemma 2.4 implies that the solution $\theta\in L^\infty([\widehat{T},\infty),C^{1,\zeta}(\mathbb{R}^2))$ for some $\zeta>0$. This completes the proof of Proposition 1.1. ∎

With Proposition 1.1 at our disposal, we are now ready to prove Theorem 1.2. To begin with, the global existence of a weak solution of the system (1.1) can be easily obtained (see for example [32, 41]). Thus it is sufficient to show that weak solutions of system (3.37) are eventually regular.

Proof. (**Proof of Theorem 1.2.**) By Lemma 4.1, for $\alpha>\frac{4}{5}$ and $\alpha+\beta<1$,

$$\sup_{0\leq t\leq T}\|u_G(t)\|_{\mathrm{Lip}} \leq C<\infty.$$

By Lemma 4.2, for

$$0.7692\approx\frac{10}{13}<\alpha\leq\frac{4}{5}, \quad \frac{-3(3\alpha-2)+\sqrt{\alpha^2-204\alpha+164}}{8}<\beta<1-\alpha,$$

we obtain

$$\sup_{0\le t\le T}\|u_G\|_{C^{1-\mu}}\le C<\infty, \quad 0<\mu<\beta.$$

Proposition 1.1 then implies that there exists $\widehat{T}$ such that

$$\theta\in L^\infty([\widehat{T},\infty)\times C^{1,\zeta}(\mathbb{R}^2))$$

for some $\zeta>0$. In particular,

$$\nabla\theta\in L^\infty([\widehat{T},\infty)\times L^\infty(\mathbb{R}^2)),$$

which especially implies (u,θ) is a classical solution on $[\widehat{T},\infty)$. Thus, we complete the proof of Theorem 1.2. ∎

Appendix A. Proof of eventual regularity for generalized surface quasi-geostrophic type equations and Lemma 2.5. This appendix serves two main purposes. The first is to state an eventual regularity result and a global regularity type result for a generalized surface quasi-geostrophic (SQG) equation following [16,21]. These results for the generalized SQG equation may be useful for future study. The last one is to present the proof of Lemma 2.5.

We first state the eventual regularity result and a global regularity type result for the generalized SQG type equation,

$$\begin{cases}\partial_t\theta+(u\cdot\nabla)\theta+\Lambda^\beta\theta=0,\\ u=\nabla^\perp\Lambda^{-1-\alpha}\theta,\\ \theta(x,0)=\theta_0(x),\end{cases}\tag{A.1}$$

where θ is a scalar real-valued function, $\alpha\in(0,1)$ and $\beta\in(0,1)$ are fixed parameters with $\alpha+\beta<1$. The SQG equation and the generalized SQG type equations have been studied extensively recently due to their geophysical applications and mathematical importance (see, e.g., [15,46]). The global regularity problem concerning these equations have been studied extensively and significant progress has been made (see, e.g., [4,7,8,10,12–18, 20,21,23–25,27,35–38,44,45,52–54,56,58]).

THEOREM A.1. *Consider the system (A.1) with $\alpha>0,\beta>0$ and $\alpha+\beta<1$. Assume that $\theta_0\in H^\sigma(\mathbb{R}^2)$ for some $\sigma>2$ and let θ be a corresponding global Leray–Hopf type weak solution. Then there exists $\widehat{T}>0$, more precisely,*

$$\widehat{T}=C\gamma^{\frac{1-\alpha}{1-\alpha-\beta}}\|\theta_0\|_{L^\infty}^{\frac{\beta}{1-\alpha-\beta}},$$

such that $\theta\in L^\infty([\widehat{T},\infty),C^\gamma(\mathbb{R}^2))$ for some $1-\alpha-\beta<\gamma<1-\alpha$,

$$[\theta(t)]_{C^\gamma}\le C\gamma^{-\frac{\gamma}{1-\alpha-\beta}}\|\theta_0\|_{L^\infty}^{-\frac{\gamma+\alpha+\beta-1}{1-\alpha-\beta}},$$

for some constant $C=C(\alpha,\beta)\ge 1$.

A regularity result for the system (A.1) similar to [21, Theorem 1.3] can also be established.

THEOREM A.2. *Consider the system (A.1) with $\alpha > 0, \beta > 0$ and $\alpha + \beta < 1$. Assume that $\theta_0 \in H^2(\mathbb{R}^2)$ with $\|\theta_0\|_{L^2}^{\frac{\alpha+\beta}{2}} \|\theta_0\|_{\dot{H}^2}^{\frac{2-\alpha-\beta}{2}} = A$. There exist $\alpha_1 = \alpha_1(A) \in (0,1)$, $\beta_1 = \beta_1(A) \in (0,1)$ such that, for every $\alpha \in [\alpha_1, 1)$, $\beta \in [\beta_1, 1)$, system (A.1) admits a unique global smooth solution with*

$$\theta \in L^\infty(0,T;H^2) \cap L^2(0,T;H^{2+\frac{\beta}{2}})$$

for any given $T > 0$.

We remark that the system (A.1) has the scaling property that, if θ is a solution of the system (A.1), then for any $\lambda > 0$ the functions

$$\theta_\lambda(x,t) = \lambda^{\alpha+\beta-1}\theta(\lambda x, \lambda^\beta t),$$

are also solutions of the system (A.1) with the corresponding initial data $\theta_{0,\lambda}(x) = \lambda^{\alpha+\beta-1}\theta_0(\lambda x)$. It is an obvious fact that the quantity $\|\cdot\|_{L^2}^{\frac{\alpha+\beta}{2}} \|\cdot\|_{\dot{H}^2}^{\frac{2-\alpha-\beta}{2}}$ is scaling invariant.

We now turn to the proofs of theorems A.1 and A.2. The proof of Theorem A.1 is a simple consequence of Proposition 1.1 (with $u_G \equiv 0$). To prove Theorem A.2, we first state a local existence result for the SQG equation (A.1).

LEMMA A.1. *Let $\theta_0 \in H^2$ be given, then the supercritical SQG equation (A.1) has a unique local in time solution such that*

$$\theta \in L^\infty(0,T_0;H^2) \cap L^2(0,T_0;H^{2+\frac{\beta}{2}}), \tag{A.2}$$

where T_0 is given by

$$T_0 = \frac{1}{C_4 \|\theta_0\|_{L^2}^{\frac{2\alpha+\beta}{2}} \|\theta_0\|_{\dot{H}^2}^{2-\frac{2\alpha+\beta}{2}}}.$$

Proof. We only provide the proof for the key component of Lemma A.1, namely the local H^2-bound of θ. Multiplying Equation $(\mathrm{A}.1)_2$ by $\Delta^2\theta$, integrating over whole space and using the divergence-free condition, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{\dot{H}^2}^2 + \|\theta\|_{\dot{H}^{2+\frac{\beta}{2}}}^2 \\
&\le -\int \Delta\big((u\cdot\nabla)\theta\big)\Delta\theta\,dx \\
&\le -\int (\Delta u\cdot\nabla)\theta\Delta\theta\,dx - 2\int \nabla u\nabla^2\theta\Delta\theta\,dx \\
&\le C\|\Delta u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\theta\|_{L^{\frac{4}{2\alpha+\beta}}}\|\Delta\theta\|_{L^{\frac{4}{2-\beta}}} + C\|\nabla^2\theta\|_{L^2}\|\Delta\theta\|_{L^{\frac{4}{2-\beta}}}\|\nabla u\|_{L^{\frac{4}{\beta}}} \\
&\le C\|\Delta\theta\|_{L^2}\|\theta\|_{\dot{H}^{2-\frac{2\alpha+\beta}{2}}}\|\theta\|_{\dot{H}^{2+\frac{\beta}{2}}} + C\|\nabla^2\theta\|_{L^2}\|\theta\|_{\dot{H}^{2+\frac{\beta}{2}}}\|\theta\|_{\dot{H}^{2-\frac{2\alpha+\beta}{2}}} \\
&\le \frac{1}{2}\|\theta\|_{\dot{H}^{2+\frac{\beta}{2}}}^2 + C\|\theta\|_{\dot{H}^2}^2\|\theta\|_{\dot{H}^{2-\frac{2\alpha+\beta}{2}}}^2 \\
&\le \frac{1}{2}\|\theta\|_{\dot{H}^{2+\frac{\beta}{2}}}^2 + C\|\theta\|_{\dot{H}^2}^{4-\frac{2\alpha+\beta}{2}}\|\theta\|_{L^2}^{\frac{2\alpha+\beta}{2}},
\end{aligned} \tag{A.3}$$

where the last inequality follows from the following interpolation

$$\|\theta\|_{\dot{H}^{2-\frac{2\alpha+\beta}{2}}} \le C\|\theta\|_{\dot{H}^2}^{1-\frac{2\alpha+\beta}{4}}\|\theta\|_{L^2}^{\frac{2\alpha+\beta}{4}}.$$

The inequality (A.3) immediately implies that

$$\frac{d}{dt}\|\theta(t)\|_{\dot{H}^2}^2 \le C\|\theta\|_{\dot{H}^2}^{4-\frac{2\alpha+\beta}{2}}\|\theta\|_{L^2}^{\frac{2\alpha+\beta}{2}}.$$

It is easy to check that the H^2 norm of θ does not blow up before time

$$T_0 = \frac{1}{C_4\|\theta_0\|_{L^2}^{\frac{2\alpha+\beta}{2}}\|\theta_0\|_{\dot{H}^2}^{2-\frac{2\alpha+\beta}{2}}}.$$

This completes the proof of Lemma A.1. ∎

Next we will prove Theorem A.2.

Proof. (**Proof of Theorem A.2.**) It follows from the eventual regularity theorem (Theorem A.1) that the weak solution becomes smooth after time $\widehat{T}$ given by

$$\widehat{T} = \frac{16C_1\gamma\xi_0^\beta}{\beta} = \frac{16C_1\gamma}{\beta}(24CC_1\gamma\|\theta_0\|_{L^\infty})^{\frac{\beta}{1-\alpha-\beta}} = C_5\gamma^{\frac{1-\alpha}{1-\alpha-\beta}}\|\theta_0\|_{L^\infty}^{\frac{\beta}{1-\alpha-\beta}},$$

where the constant $C_5 \ge 1$. The smooth solution cannot blow up in finite time as long as

$$\widehat{T} \le T_0,$$

which is equivalent to

$$C_5\gamma^{\frac{1-\alpha}{1-\alpha-\beta}}\|\theta_0\|_{L^2}^{\frac{\beta}{2(1-\alpha-\beta)}}\|\theta_0\|_{\dot{H}^2}^{\frac{\beta}{2(1-\alpha-\beta)}} \le \frac{1}{C_4\|\theta_0\|_{L^2}^{\frac{2\alpha+\beta}{2}}\|\theta_0\|_{\dot{H}^2}^{2-\frac{2\alpha+\beta}{2}}},$$

where we have used the interpolation $\|\theta_0\|_{L^\infty} \le C\|\theta_0\|_{L^2}^{\frac{1}{2}}\|\theta_0\|_{\dot{H}^2}^{\frac{1}{2}}$. The above inequality can be rewritten as

$$\|\theta_0\|_{L^2}^{\frac{(\alpha+\beta)(2-2\alpha-\beta)}{2(1-\alpha-\beta)}}\|\theta_0\|_{\dot{H}^2}^{\frac{(2-\alpha-\beta)(2-2\alpha-\beta)}{2(1-\alpha-\beta)}} \le (C_4C_5)^{-1}\gamma^{-\frac{1-\alpha}{1-\alpha-\beta}}. \tag{A.4}$$

Now we assume that

$$\|\theta_0\|_{L^2}^{\frac{\alpha+\beta}{2}}\|\theta_0\|_{\dot{H}^2}^{\frac{2-\alpha-\beta}{2}} = A.$$

Therefore, the inequality (A.4) holds if we select γ such that

$$\gamma \le (C_4C_5)^{-\frac{1-\alpha-\beta}{1-\alpha}}A^{-\frac{2-2\alpha-\beta}{1-\alpha}}. \tag{A.5}$$

Recall that

$$\gamma > 1-\alpha-\beta.$$

Thus, there exist $\alpha_1 = \alpha_1(A) \in (0,1)$, $\beta_1 = \beta_1(A) \in (0,1)$ such that $1-\alpha_1-\beta_1$ is close enough to zero, namely $\alpha_1+\beta_1$ is close enough to one. It is not hard to find that for every $\alpha \in [\alpha_1, 1)$, $\beta \in [\beta_1, 1)$, the solution of the supercritical SQG equation (A.1) does not blow up in finite time. Thus, the proof of Theorem A.2 is completed. ∎

Finally we end this section with the proof of Lemma 2.5.

Proof. (**Proof of Lemma 2.5.**) We apply inhomogeneous blocks Δ_k ($k\in\mathbb{N}$) operator to the combined Equation (2.6) to obtain

$$\partial_t\Delta_k G+\Lambda^\alpha\Delta_k G=\Delta_k[\mathcal{R}_\alpha,u\cdot\nabla]\theta-\Delta_k(u\cdot\nabla G)+\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{A.6}$$

Multiplying both sides of (A.6) by $|\Delta_k G|^{r-2}\Delta_k G$, integrating the result over space $\mathbb{R}^2$, and using the divergence-free condition, we get

$$\frac{1}{r}\frac{d}{dt}\|\Delta_k G\|_{L^r}^r+\int_{\mathbb{R}^2}(\Lambda^\alpha\Delta_k G)|\Delta_k G|^{r-2}\Delta_k G\,dx=I_1^k+I_2^k+I_3^k, \tag{A.7}$$

where

$$I_1^k=\int_{\mathbb{R}^2}\Delta_k[\mathcal{R}_\alpha,u\cdot\nabla]\theta\,|\Delta_k G|^{r-2}\Delta_k G\,dx,$$

$$I_2^k=-\int_{\mathbb{R}^2}\Delta_k(u\cdot\nabla G)\,|\Delta_k G|^{r-2}\Delta_k G\,dx,$$

$$I_3^k=\int_{\mathbb{R}^2}\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta\,|\Delta_k G|^{r-2}\Delta_k G\,dx.$$

Keep in mind the following lower bound

$$\int_{\mathbb{R}^2}(\Lambda^\alpha\Delta_k G)|\Delta_k G|^{r-2}\Delta_k G\,dx\geq c2^{\alpha k}\|\Delta_k G\|_{L^r}^r,\quad k\geq 0,$$

for an absolute constant $c>0$ independent of k. Now we recall the following commutator estimate, whose proof will be given at the end of this section

$$\|\Delta_k[\mathcal{R}_\alpha,u\cdot\nabla]\theta\|_{L^r}\leq C(2^{2(1-\alpha)k}\|u\|_{\dot{B}^\alpha_{r,\infty}}\|\theta\|_{L^\infty}+\|u\|_{L^2}\|\theta\|_{L^2}),\quad \left(\alpha>\frac{1}{2}\right). \tag{A.8}$$

According to the following simple fact

$$\begin{aligned}
\|u\|_{\dot{B}^\alpha_{r,\infty}}&\leq C\|\Lambda^\alpha u\|_{L^r}\\
&\leq C\|\Lambda^{\alpha-1}\Lambda u\|_{L^r}\\
&\leq C\|\Lambda^{\alpha-1}\omega\|_{L^r}\\
&\leq C\|\Lambda^{\alpha-1}G\|_{L^r}+C\|\Lambda^{\alpha-1}\mathcal{R}_\alpha\theta\|_{L^r}\\
&\leq C\|G\|_{L^q}+C\|G\|_{L^2}+C\|\theta\|_{L^r}\qquad\left(r\leq\frac{2q}{2-(1-\alpha)q}\right)\\
&\leq C(T,u_0,\theta_0),
\end{aligned}\tag{A.9}$$

it directly gives

$$\|\Delta_k[\mathcal{R}_\alpha,u\cdot\nabla]\theta\|_{L^r}\leq C2^{2(1-\alpha)k}.$$

As a result, we obtain

$$|I_1^k|\leq C2^{2(1-\alpha)k}\|\Delta_k G\|_{L^r}^{r-1}. \tag{A.10}$$

Employing the above estimate (A.9), it follows from the proof of the estimate (7.17) in [31] that

$$\begin{aligned}|I_2^k| &\le C\|\Lambda^\alpha u\|_{L^r} 2^{(1-\alpha+\frac{2}{r})k}\|\Delta_k G\|_{L^r}^{r-1}\Big(\|\Delta_k G\|_{L^r} + \sum_{m\le k-1} 2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G\|_{L^r}\\ &\quad + \sum_{m\ge k-1} 2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G\|_{L^r}\Big)\\ &\le C2^{(1-\alpha+\frac{2}{r})k}\|\Delta_k G\|_{L^r}^{r-1}\Big(\|\Delta_k G\|_{L^r} + \sum_{m\le k-1} 2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G\|_{L^r}\\ &\quad + \sum_{m\ge k-1} 2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G\|_{L^r}\Big). \end{aligned} \tag{A.11}$$

Finally, using the Bernstein inequality, we are led to

$$\begin{aligned}|I_3^k| &\le C\|\Delta_k\Lambda^{\beta-\alpha}\partial_{x_1}\theta\|_{L^r}\|\Delta_k G\|_{L^r}^{r-1}\\ &\le C2^{(\beta+1-\alpha)k}\|\Delta_k\theta\|_{L^r}\|\Delta_k G\|_{L^r}^{r-1}\\ &\le C2^{(\beta+1-\alpha)k}\|\Delta_k G\|_{L^r}^{r-1}.\end{aligned} \tag{A.12}$$

Inserting the above estimates I_1^k, I_2^k, and I_3^k into the right-hand side of Equation (A.7) yields

$$\begin{aligned}\frac{d}{dt}\|\Delta_k G\|_{L^r} + c2^{\alpha k}\|\Delta_k G\|_{L^r} &\le C2^{2(1-\alpha)k} + C2^{(\beta+1-\alpha)k} + C2^{(1-\alpha+\frac{2}{r})k}L(t)\\ &\le C2^{2(1-\alpha)k} + C2^{(1-\alpha+\frac{2}{r})k}L(t),\end{aligned} \tag{A.13}$$

where

$$L(t) = \|\Delta_k G\|_{L^r} + \sum_{m\le k-1} 2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G\|_{L^r} + \sum_{m\ge k-1} 2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G\|_{L^r},$$

and we have used the fact

$$\alpha+\beta\le 1 \Rightarrow \beta+1-\alpha \le 2(1-\alpha).$$

We get the following by integrating Equation (A.13) w.r.t.

$$\begin{aligned}\|\Delta_k G(t)\|_{L^r} &\le e^{-ct2^{\alpha k}}\|\Delta_k G_0\|_{L^r} + C2^{(2-3\alpha)k}\\ &\quad + C2^{(1-\alpha+\frac{2}{r})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}} L(\tau)\,d\tau.\end{aligned} \tag{A.14}$$

Multiplying both the left- and right-hand sides of Equation (A.14) by 2^{sk} with $s\le 3\alpha-2$ and taking the supremum with respect to k leads to

$$\|G(t)\|_{B^s_{r,\infty}} \le C + \|G_0\|_{B^s_{r,\infty}} + M_1 + M_2 + M_3,$$

where

$$M_1 = C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r})k}2^{sk}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\|\Delta_k G(\tau)\|_{L^r}\,d\tau\Big),$$

$$M_2 = C\sup_{k\ge -1}\Big(2^{(1-\alpha+\frac{2}{r})k}2^{sk}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\le k-1} 2^{(1+\frac{2}{r})(m-k)}\|\Delta_m G(\tau)\|_{L^r}\,d\tau\Big),$$

$$M_3 = C \sup_{k\geq -1} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\geq k-1} 2^{(\alpha-\frac{2}{r})(k-m)} \|\Delta_m G(\tau)\|_{L^r}\, d\tau \Big).$$

Thanks to the condition $r > \frac{2}{2\alpha-1}$, we choose k_0 as

$$C 2^{-(2\alpha-1-\frac{2}{r})k_0} \leq \frac{1}{6}.$$

By the aid of the Bernstein inequality and the convolution Young's inequality, we can conclude (we may assume $r \geq q$, otherwise it is more simple)

$$\begin{aligned}
M_1 &= C \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \|\Delta_k G(\tau)\|_{L^r}\, d\tau \Big) \\
&\quad + C \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \|\Delta_k G(\tau)\|_{L^r}\, d\tau \Big) \\
&\leq C \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+s+\frac{2}{q})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \|\Delta_k G(\tau)\|_{L^q}\, d\tau \Big) \\
&\quad + C \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} 2^{sk} \|\Delta_k G(\tau)\|_{L^r}\, d\tau \Big) \\
&\leq C \|G\|_{L^\infty(0,T;L^q)} \sup_{-1\leq k\leq k_0} 2^{(1-2\alpha+s+\frac{2}{q})k} + C \sup_{k>k_0} 2^{-(2\alpha-1-\frac{2}{r})k} \|G\|_{L^\infty(0,T;B^s_{r,\infty})} \\
&\leq C \|G\|_{L^\infty(0,T;L^q)} + C 2^{-(2\alpha-1-\frac{2}{r})k_0} \|G\|_{L^\infty(0,T;B^s_{r,\infty})} \\
&\leq C \|G\|_{L^\infty(0,T;L^q)} + \frac{1}{6} \|G\|_{L^\infty(0,T;B^s_{r,\infty})},
\end{aligned}$$

$$\begin{aligned}
M_2 &= C \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r})(m-k)} \|\Delta_m G(\tau)\|_{L^r}\, d\tau \Big) \\
&\quad + C \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r}-s)(m-k)} 2^{ms} \|\Delta_m G(\tau)\|_{L^r}\, d\tau \Big) \\
&\leq C \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r})(m-k)} 2^{(\frac{2}{q}-\frac{2}{r})m} \|\Delta_m G(\tau)\|_{L^q}\, d\tau \Big) \\
&\quad + C \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \Big(\sum_{m\leq k-1} 2^{(1+\frac{2}{r}-s)(m-k)} \Big) \|G\|_{L^\infty(0,T;B^s_{r,\infty})}\, d\tau \Big) \\
&\leq C \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r})(m-k)} 2^{(\frac{2}{q}-\frac{2}{r})k} \|G\|_{L^\infty(0,T;L^q)}\, d\tau \Big) \\
&\quad + C \|G\|_{L^\infty(0,T;B^s_{r,\infty})} \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \Big(\sum_{m\leq k-1} 2^{(1+\frac{2}{r}-s)(m-k)} \Big) d\tau \Big) \\
&\leq C \|G\|_{L^\infty(0,T;L^q)} \sup_{-1\leq k\leq k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} 2^{sk} 2^{(\frac{2}{q}-\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}} \sum_{m\leq k-1} 2^{(1+\frac{2}{r})(m-k)}\, d\tau \Big) \\
&\quad + C \|G\|_{L^\infty(0,T;B^s_{r,\infty})} \sup_{k>k_0} \Big(2^{(1-\alpha+\frac{2}{r})k} \int_0^t e^{-c(t-\tau)2^{\alpha k}}\, d\tau \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|G\|_{L^\infty(0,T;L^q)} \sup_{-1\leq k\leq k_0}\Big(2^{(1-\alpha+s+\frac{2}{q})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\,d\tau\Big)\\
&\quad+C\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\sup_{k>k_0}2^{-(2\alpha-1-\frac{2}{r})k}\\
&\leq C\|G\|_{L^\infty(0,T;L^q)} \sup_{-1\leq k\leq k_0}\Big(2^{(1-2\alpha+s+\frac{2}{q})k}\Big)+C2^{-(2\alpha-1-\frac{2}{r})k_0}\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\\
&\leq C\|G\|_{L^\infty(0,T;L^q)}+\frac{1}{6}\|G\|_{L^\infty(0,T;B^s_{r,\infty})}.
\end{aligned}
$$

Similarly, the term M_3 obeys the following estimate

$$
\begin{aligned}
M_3&=C\sup_{-1\leq k\leq k_0}\Big(2^{(1-\alpha+\frac{2}{r})k}2^{sk}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\geq k-1}2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G(\tau)\|_{L^r}\,d\tau\Big)\\
&\quad+C\sup_{k>k_0}\Big(2^{(1-\alpha+\frac{2}{r})k}2^{sk}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\geq k-1}2^{(\alpha-\frac{2}{r})(k-m)}\|\Delta_m G(\tau)\|_{L^r}\,d\tau\Big)\\
&\leq C\sup_{-1\leq k\leq k_0}\Big(2^{(1-\alpha+s+\frac{2}{q})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\geq k-1}2^{(\alpha-\frac{2}{q})(k-m)}\|\Delta_m G(\tau)\|_{L^q}\,d\tau\Big)\\
&\quad+C\sup_{k>k_0}\Big(2^{(1-\alpha+\frac{2}{r})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\sum_{m\geq k-1}2^{(\alpha+s-\frac{2}{r})(k-m)}2^{sm}\|\Delta_m G(\tau)\|_{L^r}\,d\tau\Big)\\
&\leq C\|G\|_{L^\infty(0,T;L^q)}\sup_{-1\leq k\leq k_0}\Big(2^{(1-\alpha+s+\frac{2}{q})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\,d\tau\Big)\\
&\quad+C\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\sup_{k>k_0}\Big(2^{(1-\alpha+\frac{2}{r})k}\int_0^t e^{-c(t-\tau)2^{\alpha k}}\,d\tau\Big)\\
&\leq C\|G\|_{L^\infty(0,T;L^q)}\sup_{-1\leq k\leq k_0}\Big(2^{(1-2\alpha+s+\frac{2}{q})k}\Big)+C\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\sup_{k>k_0}\Big(2^{-(2\alpha-1-\frac{2}{r})k}\Big)\\
&\leq C\|G\|_{L^\infty(0,T;L^q)}+C2^{-(2\alpha-1-\frac{2}{r})k_0}\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\\
&\leq C\|G\|_{L^\infty(0,T;L^q)}+\frac{1}{6}\|G\|_{L^\infty(0,T;B^s_{r,\infty})},
\end{aligned}
$$

where we need the condition $q>\frac{2}{\alpha}$. Therefore, it follows that

$$
\|G\|_{L^\infty(0,T;B^s_{r,\infty})}\leq C+\|G_0\|_{B^s_{r,\infty}}+C\|G\|_{L^\infty(0,T;L^q)}+\frac{1}{2}\|G\|_{L^\infty(0,T;B^s_{r,\infty})}.
$$

Finally, we arrive at

$$
\sup_{0\leq t\leq T}\|G(t)\|_{B^s_{r,\infty}}<\infty,\qquad 0<s\leq 3\alpha-2,
$$

where $\frac{2}{2\alpha-1}<r\leq\frac{2q}{2-(1-\alpha)q}$.

We finally prove the commutator estimate (A.8) to end this section. With suitable modification of the proof of [31, Proposition 4.5], we can obtain the desired estimate. More precisely, we rewrite it as follows:

$$\begin{aligned}\Delta_k[\mathcal{R}_\alpha, u\cdot\nabla]\theta &= \sum_{|j-k|\le 4}\Delta_k\Big([\mathcal{R}_\alpha, S_{j-1}u\cdot\nabla]\Delta_j\theta\Big)+\sum_{|j-k|\le 4}\Delta_k\Big([\mathcal{R}_\alpha,\Delta_j u\cdot\nabla]S_{j-1}\theta\Big)\\ &\quad+\sum_{j-k\ge -4}\Delta_k\Big([\mathcal{R}_\alpha,\Delta_j u\cdot\nabla]\widetilde{\Delta}_j\theta\Big)\\ &:=\widetilde{N}_1^k+\widetilde{N}_2^k+\widetilde{N}_3^k.\end{aligned}$$

According to the estimation of J_1 and J_2 in [31, Proposition 4.5], we immediately get the following estimates by viewing $f=u$ and $g=\nabla\theta$

$$\begin{aligned}\|\widetilde{N}_1^k\|_{L^r}&\le C\sum_{|j-k|\le 4}2^{(1-\alpha)k}\big\|2^{2k}|x|^\alpha h_0(2^k x)\big\|_{L^1}\|S_{j-1}u\|_{\dot B^\alpha_{r,\infty}}\|\Delta_j\nabla\theta\|_{L^\infty}\\ &\le C\sum_{|j-k|\le 4}2^{(1-\alpha)k}2^{-\alpha k}\|u\|_{\dot B^\alpha_{r,\infty}}2^j\|\theta\|_{L^\infty}\\ &\le C2^{2(1-\alpha)k}\|u\|_{\dot B^\alpha_{r,\infty}}\|\theta\|_{L^\infty},\end{aligned}$$

$$\begin{aligned}\|\widetilde{N}_2^k\|_{L^r}&\le C\sum_{|j-k|\le 4}2^{(1-2\alpha)j}\|u\|_{\dot B^\alpha_{r,\infty}}\|S_{j-1}\nabla\theta\|_{L^\infty}\\ &\le C\sum_{|j-k|\le 4}2^{(1-2\alpha)j}\|u\|_{\dot B^\alpha_{r,\infty}}2^j\|\theta\|_{L^\infty}\\ &\le C2^{2(1-\alpha)k}\|u\|_{\dot B^\alpha_{r,\infty}}\|\theta\|_{L^\infty},\end{aligned}$$

where h_0 is a Schwartz function. However, the last term $\widetilde{N}_3$ should be treated differently, without using the commutator structure. The divergence-free condition plays an important role in proving $\widetilde{N}_3$. Indeed, by Bernstein's lemma and the divergence-free condition

$$\begin{aligned}\|\widetilde{N}_3^k\|_{L^r}&\le C\sum_{j-k\ge -4}\Big(\Big\|\Delta_k\big(\mathcal{R}_\alpha(\Delta_j u\cdot\nabla\widetilde{\Delta}_j\theta)\big)\Big\|_{L^r}+\Big\|\Delta_k\big(\Delta_j u\cdot\nabla\mathcal{R}_\alpha\widetilde{\Delta}_j\theta\big)\Big\|_{L^r}\Big)\\ &\le C\sum_{j-k\ge -4}\Big(\Big\|\Delta_k\big(\mathcal{R}_\alpha\nabla\cdot(\Delta_j u\widetilde{\Delta}_j\theta)\big)\Big\|_{L^r}+\Big\|\Delta_k\nabla\cdot\big(\Delta_j u\mathcal{R}_\alpha\widetilde{\Delta}_j\theta\big)\Big\|_{L^r}\Big)\\ &\le C\sum_{j-k\ge -4}\Big(2^{(2-\alpha)k}\|\Delta_j u\widetilde{\Delta}_j\theta\|_{L^r}+2^k\|\Delta_j u\mathcal{R}_\alpha\widetilde{\Delta}_j\theta\|_{L^r}\Big)\\ &\le C\Big(2^{(2-2\alpha)k}\sum_{j-k\ge -4}2^{-\alpha(j-k)}2^{\alpha j}\|\Delta_j u\|_{L^r}\|\theta\|_{L^\infty}\\ &\qquad+2^{(2-2\alpha)k}\sum_{j-k\ge -4, j\ge 0}2^{(1-2\alpha)(j-k)}2^{\alpha j}\|\Delta_j u\|_{L^r}\|\theta\|_{L^\infty}+\|u\|_{L^2}\|\theta\|_{L^2}\Big)\\ &\le C\Big(2^{(2-2\alpha)k}\sum_{j-k\ge -4}2^{-\alpha(j-k)}\|u\|_{\dot B^\alpha_{r,\infty}}\|\theta\|_{L^\infty}\\ &\qquad+2^{(2-2\alpha)k}\sum_{j-k\ge -4, j\ge 0}2^{(1-2\alpha)(j-k)}\|u\|_{\dot B^\alpha_{r,\infty}}\|\theta\|_{L^\infty}+\|u\|_{L^2}\|\theta\|_{L^2}\Big)\qquad\Big(\alpha>\frac{1}{2}\Big)\\ &\le C2^{(2-2\alpha)k}\|u\|_{\dot B^\alpha_{r,\infty}}\|\theta\|_{L^\infty}+C\|u\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

Collecting the above three estimates, we immediately obtain

$$\|\Delta_k[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^r} \le C2^{(2-2\alpha)k}\|u\|_{\dot{B}^\alpha_{r,\infty}}\|\theta\|_{L^\infty} + C\|u\|_{L^2}\|\theta\|_{L^2},$$

which is nothing but the desired commutator estimate (A.8). Consequently, this completes the proof of Lemma 2.5. ∎

REFERENCES

[1] D. Adhikari, C. Cao, H. Shang, J. Wu, X. Xu, and Z. Ye, *Global regularity results for the 2D Boussinesq equations with partial dissipation*, J. Diff. Eqs., 260, 1893–1917, 2016.

[2] D. Adhikari, C. Cao, J. Wu, and X. Xu, *Small global solutions to the damped two-dimensional Boussinesq equations*, J. Diff. Eqs., 256, 3594–3613, 2014.

[3] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Springer, 343, 2011.

[4] L. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math., 171, 1903–1930, 2010.

[5] C. Cao and J. Wu, *Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal., 208, 985–1004, 2013.

[6] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203, 497–513, 2006.

[7] D. Chae, P. Constantin, and J. Wu, *Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations*, Arch. Ration. Mech. Anal., 202, 35–62, 2011.

[8] D. Chae and J. Lee, *Global well-posedness in the super-critical dissipative quasigeostrophic equations*, Commun. Math. Phys., 233, 297–311, 2003.

[9] D. Chae and J. Wu, *The 2D Boussinesq equations with logarithmically supercritical velocities*, Adv. Math., 230, 1618–1645, 2012.

[10] Q. Chen, C. Miao, and Z. Zhang, *A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation*, Commun. Math. Phys., 271, 821–838, 2007.

[11] K. Choi, *Persistence of Hölder continuity for non-local integro-differential equations*, Discrete Contin. Dyn. Syst., 33, 1741–1771, 2013.

[12] P. Constantin, *Energy spectrum of quasigeostrophic turbulence*, Phys. Rev. Lett., 89, 184501, 2002.

[13] P. Constantin, D. Córdoba, and J. Wu, *On the critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J., 50, 97–107, 2001.

[14] P. Constantin, G. Iyer, and J. Wu, *Global regularity for a modified critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J. 57 (Special Issue), 97–107, 2001.

[15] P. Constantin, A. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar*, Nonlinearity 7, 1495–1533, 1994.

[16] P. Constantin, A. Tarfulea, and V. Vicol, *Long time dynamics of forced critical SQG*, Math. Phys., 335, 93–141, 2015.

[17] P. Constantin and V. Vicol, *Nonlinear maximum principles for dissipative linear nonlocal operators and applications*, Geom. Funct. Anal., 22, 1289–1321, 2012.

[18] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal., 30, 937–948, 1999.

[19] P. Constantin and J. Wu, *Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation*, Ann. Inst. H. Poincare Anal. Non Lineaire, 25, 1103–1110, 2008.

[20] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Commun. Math. Phys., 249, 511–528, 2004.

[21] M. Coti Zelati and V. Vicol, *On the global regularity for the supercritical SQG equation*, Indiana Univ. Math. J., to appear.

[22] R. Danchin and M. Paicu, *Global existence results for the anisotropic Boussinesq system in dimension two*, Math. Models Meth. Appl. Sci., 21, 421–457, 2011.

[23] M. Dabkowski, *Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation*, Geom. Funct. Anal., 21, 1–13, 2011.

[24] H. Dong, *Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness*, Discrete Contin. Dyn. Syst., 26, 1197–1211, 2010.

[25] H. Dong and D. Li, *Spatial analyticity of the solutions to the subcritical dissipative quasi-geostrophic equations*, Arch. Ration. Mech. Anal., 189, 131–158, 2008.

[26] H. Hajaiej, L. Molinet, T. Ozawa, and B. Wang, *Sufficient and necessary conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized*

Boson equations, T. Ozawa, M. Sugimoto (Eds.), RIMS Kkyroku Bessatsu B26: Harmonic Analysis and Nonlinear Partial Differential Equations, 5, 159–175, 2011.

[27] T. Hmidi and S. Keraani, *Global solutions of the super-critical 2D quasi-geostrophic equation in Besov spaces*, Adv. Math., 214, 618–638, 2007.

[28] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for a Boussinesq–Navier–Stokes system with critical dissipation*, J. Diff. Eqs., 249, 2147–2174, 2010.

[29] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for Euler-Boussinesq system with critical dissipation*, Commun. Part. Diff. Eqs., 36, 420–445, 2011.

[30] T.Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst., 12, 1–12, 2005.

[31] Q. Jiu, C. Miao, J. Wu, and Z. Zhang, *The 2D incompressible Boussinesq equations with general critical dissipation*, SIAM J. Math. Anal., 46, 3426–3454, 2014.

[32] Q. Jiu, J. Wu, and W. Yang, *Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation*, J. Nonlinear Science, 25, 37–58, 2015.

[33] N. Ju, *The maximum principle and the global attractor for the dissipative 2d quasi-geostrophic equations*, Commun. Math. Phys., 255, 161–181, 2005.

[34] D. KC, D. Regmi, L. Tao, and J. Wu, *The 2D Euler-Boussinesq equations with a singular velocity*, J. Diff. Eqs., 257, 82–108, 2014.

[35] A. Kiselev, *Regularity and blow-up for active scalars*, Math. Model. Nat. Phenom., 5(4), 225–255, 2010.

[36] A. Kiselev, *Nonlocal maximum principles for active scalars*, Adv. Math., 227, 1806–1826, 2011.

[37] A. Kiselev and F. Nazarov, *A variation on a theme of Caffarelli and Vasseur*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 370, 58–72, 2010.

[38] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math., 167, 445-453, 2007.

[39] M. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal., 199, 739–760, 2011.

[40] A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Diff. Eqs., 255, 2636–2654, 2013.

[41] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2001.

[42] C. Miao, J. Wu, and Z. Zhang, *Littlewood–Paley Theory and Applications to Fluid Dynamics Equations*, Monographs on Modern Pure Mathematics, Science Press, Beijing, 142, 2012.

[43] C. Miao and L. Xue, *On the global well-posedness of a class of Boussinesq–Navier–Stokes systems*, NoDEA Nonlinear Diff. Eqs. Appl., 18, 707–735, 2011.

[44] C. Miao and L. Xue, *On the regularity of a class of generalized quasi-geostrophic equations*, J. Diff. Eqs., 251, 2789–2821, 2011.

[45] C. Miao and L. Xue, *Global wellposedness for a modified critical dissipative quasi-geostrophic equation*, J. Diff. Eqs., 252, 792–818, 2012.

[46] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer, New York, 1987.

[47] T. Runst and W. Sickel, *Sobolev spaces of fractional order*, Nemytskij Operators and Nonlinear Partial Differential Equations, Walter de Gruyter, Berlin, 1996.

[48] L. Silvestre, *Hölder estimates for advection fractional-diffusion equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 11(5), 843–855, 2012.

[49] L. Silvestre, *On the differentiability of the solution to an equation with drift and fractional diffusion*, Indiana Univ. Math. J., 61, 557–584, 2012.

[50] A. Stefanov and J. Wu, *A global regularity result for the 2D Boussinesq equations with critical dissipation*, arXiv:1411.1362v3 math.AP.

[51] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Unviersity Press, Princeton, NJ, 1970.

[52] J. Wu, *Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces*, SIAM J. Math. Anal., 36, 1014-1030, 2004/2005.

[53] J. Wu, *The two-dimensional quasi-geostrophic equation with critical or supercritical dissipation*, Nonlinearity, 18, 139-154, 2005.

[54] J. Wu, *Existence and uniqueness results for the 2-D dissipative quasi-geostrophic equation*, Nonlinear Anal., 67, 3013–3036, 2007.

[55] J. Wu and X. Xu, *Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation*, Nonlinearity, 27, 2215–2232, 2014.

[56] J. Wu, X. Xu, and Z. Ye, *Global smooth solutions to the n-dimensional damped models of incompressible fluid mechanics with small initial datum*, J. Nonlinear Science, 25, 157–192, 2015.

[57] X. Xu, *Global regularity of solutions of 2D Boussinesq equations with fractional diffusion*, Nonlinear Anal., 72, 677–681, 2010.

[58] L. Xue and X. Zheng, *Note on the well-posedness of a slightly supercritical surface quasi-geostrophic equation*, J. Diff. Eqs., 253, 795–813, 2012.

[59] W. Yang, Q. Jiu, and J. Wu, *Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation*, J. Diff. Eqs., 257, 4188–4213, 2014.

[60] Z. Ye, *A note on global regularity results for 2D Boussinesq equations with fractional dissipation*, Ann.Polon.Math., to appear.

[61] Z. Ye and X. Xu, *Remarks on global regularity of the 2D Boussinesq equations with fractional dissipation*, Nonlinear Anal., 125, 715–724, 2015.

[62] Z. Ye and X. Xu, *Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation*, J. Diff. Eqs., 260, 6716-6744, 2016.

[63] Z. Ye, X. Xu, and L. Xue, *On the global regularity of the 2D Boussinesq equations with fractional dissipation*, submitted, 2014.