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Global well-posedness for a modified critical dissipative quasi-geostrophic equation

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ABSTRACT

In this paper we consider the following modified quasi-geostrophic equation

$$\partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \qquad u = |D|^{\alpha-1}\mathcal{R}^\perp\theta, \quad x\in\mathbb{R}^2$$

with $\nu > 0$ and $\alpha \in]0,1[\cup]1,2[$. When $\alpha \in]0,1[$, the equation was firstly introduced by Constantin, Iyer and Wu (2008) in [11]. Here, by using the modulus of continuity method, we prove the global well-posedness of the system. As a byproduct, we also show that for every $\alpha \in]0,2[$, the Lipschitz norm of the solution has a uniform exponential upper bound.

1. Introduction

In this paper we focus on the following modified 2D dissipative quasi-geostrophic equation

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \\ u = |D|^{\alpha-1}\mathcal{R}^\perp\theta, \qquad \theta|_{t=0} = \theta_0(x) \end{cases} \tag{1.1}$$

with $\nu > 0$, $\alpha \in]0,1[\cup]1,2[$, $|D|^\beta = (-\Delta)^{\frac{\beta}{2}}$ is defined via the Fourier transform

$$\widehat{|D|^\beta f}(\zeta) = |\zeta|^\beta \hat{f}(\zeta)$$

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and

$$\mathcal{R}^{\perp}\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) := |D|^{-1}(\partial_2\theta, -\partial_1\theta)$$

where $\mathcal{R}_i$ $(i = 1, 2)$ are the usual Riesz transforms (cf. [16]).

When $\alpha = 0$, this model describes the evolution of the vorticity of a two-dimensional damped inviscid incompressible fluid. The case of $\alpha = 1$ just is the critical dissipative quasi-geostrophic equation which arises in the geostrophic study of rotating fluids (cf. [8]). Although when $\alpha = 2$ the flow term in (1.1) vanishes, we can still view the model introduced in [17] as a meaningful generalization of this endpoint case, where the model is derived from the study of the full magnetohydrodynamic equations and the divergence-free three-dimensional velocity u satisfies $u = M[\theta]$ with M a nonlocal differential operator of order 1. We also refer to [3–5] for other related generalized quasi-geostrophic models.

For convenience, we here recall the well-known 2D quasi-geostrophic equation

$$(QG)_\alpha \quad \begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta = 0, \\ u = \mathcal{R}^{\perp}\theta, \qquad \theta(0,x) = \theta_0(x), \end{cases}$$

where $\nu \geqslant 0$ and $0 \leqslant \alpha \leqslant 2$. When $\nu > 0$, $\alpha \in]0,1[\cup]1,2[$, we observe that the system (1.1) is almost the same with the quasi-geostrophic equation, and its only difference lies on introducing an extra $|D|^{\alpha-1}$ in the definition of u. When $\alpha \in]0,1[$, $|D|^{\alpha-1}$ is a negative derivative operator and always plays a good role; while when $\alpha \in]1,2[$, $|D|^{\alpha-1}$ is a positive derivative operator and always takes a bad part. Moreover, corresponding to the dissipation operator $|D|^\alpha$ in the equation $(QG)_\alpha$, this additional operator makes it be a new balanced state: the flow term $u\cdot\nabla\theta$ scales the same way as the dissipative term $|D|^\alpha\theta$, i.e., Eq. (1.1) is scaling invariant under the transformation

$$\theta(t,x) \mapsto \theta_\lambda(t,x) := \theta\big(\lambda^\alpha t, \lambda x\big), \quad \text{with } \lambda > 0.$$

We note that in the sense of scaling invariance, the system (1.1) is similar to the critical quasi-geostrophic equation $(QG)_1$.

Recently, when $\alpha \in]0,1[$, Constantin, Iyer and Wu in [11] introduced this modified quasi-geostrophic equation and proved the global regularity of Leray–Hopf weak solutions to the system with L^2 initial data. Basically, they use the method from Caffarelli–Vasseur [2] which deals with the same issue of 2D critical quasi-geostrophic equation $(QG)_1$. We also remark that partially because of its simple form and its internal analogy with the 3D Euler/Navier–Stokes equations, the quasi-geostrophic equation $(QG)_\alpha$, especially the critical one $(QG)_1$, has been extensively considered (see e.g. [1–3,7–10,12,14,18,23] and references therein). While global regularity of Navier–Stokes equations remains an outstanding challenge in mathematical physics, the global issue of the 2D critical dissipative quasi-geostrophic equation has been in a satisfactory state. In [10], Constantin, Cordoba and Wu showed the global well-posedness of the classical solution under the condition that the zero-dimensional L^∞ norm of the data is small. This smallness assumption was firstly removed by Kiselev, Nazarov and Volberg in [18], where they obtained the global well-posedness for the arbitrary periodic smooth initial data by using a modulus of continuity method. Almost at the same time, Caffarelli and Vasseur in [2] resolved the problem to establish the global regularity of weak solutions associated with L^2 initial data by exploiting the De Giorgi method. We also cite the work of Abidi and Hmidi [1] and Dong and Du [14], as extended work of [18], in which the authors proved the global well-posedness with the initial data belonging to the (critical) space $\dot{B}^0_{\infty,1}$ and H^1 respectively without the additional periodic assumption.

The main goal in this paper is to prove the global well-posedness of the system (1.1) with $\alpha \in]0,1[\cup]1,2[$. In contrast with the work of [11], we here basically follow the pathway of [18] to obtain the global results by constructing suitable moduli of continuity. Precisely, we have

Theorem 1.1. *Let $\nu > 0$, $\alpha \in]0,2[$ and $\theta_0 \in H^m(\mathbb{R}^2)$, $m > 1$, then there exists a unique global solution*

$$\theta \in \mathcal{C}\big([0,\infty[; H^m\big) \cap L^2_{\mathrm{loc}}\big([0,\infty[; H^{m+\frac{\alpha}{2}}\big) \cap \mathcal{C}^\infty\big(]0,\infty[\times \mathbb{R}^2\big)$$

to the modified quasi-geostrophic equation (1.1). *Moreover, if θ_0 also satisfies $\nabla\theta_0 \in L^\infty(\mathbb{R}^2)$, we get the uniform exponential bound of the Lipschitz norm*

$$\sup_{t\geqslant 0}\big\|\nabla\theta(t)\big\|_{L^\infty} \leqslant C\|\nabla\theta_0\|_{L^\infty}\exp\big\{C\|\theta_0\|_{L^\infty}\big\}, \tag{1.2}$$

where C is an absolute constant depending only on α, ν.

The proof is divided into two parts. First through applying the energy method and para-differential techniques, we obtain the local existence results (Proposition 4.1) and further build the blowup criterion (Proposition 4.2). Then we adopt the nonlocal maximum principle method of Kiselev–Nazarov–Volberg and finally manage to remove all the possible breakdown scenarios by constructing suitable moduli of continuity.

Remark 1.1. The main new ingredient in the global existence part is a suitable modulus of continuity with the explicit formula (5.12). For every $\alpha \in]0, 2[$, such MOC has a logarithmic growth near infinity, and this further yields the uniform exponential bound of the Lipschitz norm of the solution. In particular, when $\alpha = 1$, (1.2) is an improvement of the corresponding bound in [20], where it is a double exponential type.

The paper is organized as follows. In Section 2, we present some preparatory results. In Section 3, some facts about modulus of continuity are discussed. In Section 4, we obtain the local results and establish blowup criterion. Finally, we prove the global existence in Section 5.

2. Preliminaries

In this preparatory section, we present some notations, give the definitions and some related results of the Sobolev spaces and Besov spaces, and we also provide some necessary classical estimates.

We begin by introducing some notations.

- ◇ Throughout this paper C stands for a constant which may be different from line to line. We sometimes use $A \lesssim B$ instead of $A \leqslant CB$, and use $A \lesssim_{\beta,\gamma,\ldots} B$ instead of $A \leqslant C(\beta,\gamma,\ldots)B$ with $C(\beta,\gamma,\ldots)$ a constant depending on $\beta,\gamma,\ldots$. For $A \approx B$ we mean $A \lesssim B \lesssim A$.
- ◇ Denote by $\mathcal{D}(\mathbb{R}^n)$ the space of test functions, by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing smooth functions, by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ the quotient space of tempered distributions which modulo polynomials.
- ◇ $\mathcal{F}f$ or $\hat{f}$ denotes the Fourier transform, that is $\mathcal{F}f(\zeta) = \hat{f}(\zeta) = \int_{\mathbb{R}^n} e^{-ix\cdot\zeta} f(x)\,\mathrm{d}x$, while $\mathcal{F}^{-1}f$ the inverse Fourier transform, namely, $\mathcal{F}^{-1}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\zeta} f(\zeta)\,\mathrm{d}\zeta$.
- ◇ Denote by $[\gamma]$ the integer part of the real number γ, and by $N!$ the factorial of the positive integer N.

Now we give the definition of L^2-based Sobolev space. For $s \in \mathbb{R}$, the inhomogeneous Sobolev space

$$H^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^n);\ \|f\|_{H^s}^2 := \int_{\mathbb{R}^n} \big(1+|\zeta|^2\big)^s \big|\hat{f}(\zeta)\big|^2\,\mathrm{d}\zeta < \infty \right\}.$$

Also one can define the corresponding homogeneous space:

$$\dot{H}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n);\ \|f\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^n} |\zeta|^{2s}\big|\hat{f}(\zeta)\big|^2\,\mathrm{d}\zeta < \infty \right\}.$$

To define Besov space we need the following dyadic partition of unity (see e.g. [6]). Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{D}(\mathbb{R}^n)$ be supported respectively in the ball $\{\zeta \in \mathbb{R}^n\colon\ |\zeta| \leqslant \frac{4}{3}\}$ and the shell $\{\zeta \in \mathbb{R}^n\colon\ \frac{3}{4} \leqslant |\zeta| \leqslant \frac{8}{3}\}$ such that

$$\chi(\zeta) + \sum_{j\geqslant 0} \varphi\big(2^{-j}\zeta\big) = 1, \quad \forall \zeta \in \mathbb{R}^n; \qquad \sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\zeta\big) = 1, \quad \forall \zeta \neq 0.$$

For all $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the nonhomogeneous Littlewood–Paley operators

$$\Delta_{-1} f := \chi(D) f; \qquad \Delta_j f := \varphi\big(2^{-j}D\big) f, \qquad S_j f := \sum_{-1\leqslant k\leqslant j-1} \Delta_k f, \quad \forall j \in \mathbb{N}.$$

And the homogeneous Littlewood–Paley operators can be defined as follows

$$\dot{\Delta}_j f := \varphi\big(2^{-j}D\big) f; \qquad \dot{S}_j f := \sum_{k\in\mathbb{Z},\, k\leqslant j-1} \dot{\Delta}_k f, \quad \forall j \in \mathbb{Z}.$$

Now we introduce the definition of Besov spaces. Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the nonhomogeneous Besov space

$$B^s_{p,r} := \big\{ f \in \mathcal{S}'\big(\mathbb{R}^n\big);\ \|f\|_{B^s_{p,r}} := \big\| \big\{2^{js} \|\Delta_j f\|_{L^p}\big\}_{j\geqslant -1} \big\|_{\ell^r} < \infty \big\},$$

and the homogeneous space

$$\dot{B}^s_{p,r} := \big\{ f \in \mathcal{S}'\big(\mathbb{R}^n\big)/\mathcal{P}\big(\mathbb{R}^n\big);\ \|f\|_{\dot{B}^s_{p,r}} := \big\| \big\{2^{js} \|\dot{\Delta}_j f\|_{L^p}\big\}_{j\in\mathbb{Z}} \big\|_{\ell^r(\mathbb{Z})} < \infty \big\}.$$

We point out that for all $s \in \mathbb{R}$, $B^s_{2,2} = H^s$ and $\dot{B}^s_{2,2} = \dot{H}^s$.

Next we introduce two kinds of space–time Besov spaces. The first one is the classical space–time Besov space $L^\rho([0, T], B^s_{p,r})$, abbreviated by $L^\rho_T B^s_{p,r}$, which is the set of tempered distribution f such that

$$\|f\|_{L^\rho_T B^s_{p,r}} := \big\| \big\| \big\{2^{js} \|\Delta_j f\|_{L^p}\big\}_{j\geqslant -1} \big\|_{\ell^r} \big\|_{L^\rho([0,T])} < \infty.$$

The second one is the Chemin–Lerner's mixed space–time Besov space $\widetilde{L}^\rho([0, T], B^s_{p,r})$, abbreviated by $\widetilde{L}^\rho_T B^s_{p,r}$, which is the set of tempered distribution f satisfying

$$\|f\|_{\widetilde{L}^\rho_T B^s_{p,r}} := \big\| \big\{2^{qs} \|\Delta_q f\|_{L^\rho_T L^p}\big\}_{q\geqslant -1} \big\|_{\ell^r} < \infty.$$

Due to Minkowiski's inequality, we immediately obtain

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r}, \quad \text{if } r \geqslant \rho \quad \text{and} \quad \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \quad \text{if } \rho \geqslant r.$$

These can similarly extend to the homogeneous one $L^\rho_T \dot{B}^s_{p,r}$ and $\widetilde{L}^\rho_T \dot{B}^s_{p,r}$.

Bernstein's inequality is fundamental in the analysis involving Besov spaces (see [6]).

Lemma 2.1. *Let $f \in L^a$, $1 \leqslant a \leqslant b \leqslant \infty$. Then for every $(k,q) \in \mathbb{N}^2$ there exists a constant $C > 0$ such that*

$$\sup_{|\alpha|=k} \left\| \partial^\alpha S_q f \right\|_{L^b} \leqslant C 2^{q(k+n(\frac{1}{a}-\frac{1}{b}))} \| S_q f \|_{L^a},$$

$$C^{-1} 2^{qk} \| \Delta_q f \|_{L^a} \leqslant \sup_{|\alpha|=k} \left\| \partial^\alpha \Delta_q f \right\|_{L^a} \leqslant C 2^{qk} \| \Delta_q f \|_{L^a}.$$

Finally we state the maximum principle for the transport-diffusion equation (cf. [12]).

Proposition 2.2. *Let u be a smooth divergence-free vector field and f be a smooth function. Assume that θ is the smooth solution of the equation*

$$\partial_t \theta + u \cdot \nabla \theta + \nu |D|^\alpha \theta = f, \qquad \operatorname{div} u = 0,$$

with initial datum θ_0 and $\nu \geqslant 0$, $0 \leqslant \alpha \leqslant 2$, then for every $p \in [1,\infty]$ we have

$$\left\| \theta(t) \right\|_{L^p} \leqslant \| \theta_0 \|_{L^p} + \int_0^t \left\| f(\tau) \right\|_{L^p} \mathrm{d}\tau. \tag{2.1}$$

3. Moduli of continuity

In this section, we discuss the moduli of continuity which will play a key role in our global existence part.

We suppose that ω is a modulus of continuity (MOC), i.e., a continuous, increasing, concave function on $[0,\infty)$ such that $\omega(0) = 0$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has the modulus of continuity ω if $|f(x) - f(y)| \leqslant \omega(|x-y|)$ for all $x, y \in \mathbb{R}^n$, and that f has the strict modulus of continuity if the inequality is strict for $x \neq y$.

Next we introduce the pseudo-differential operators $\mathcal{R}_{\alpha,j}$ which may be termed as the modified Riesz transforms.

Proposition 3.1. *Let $\alpha \in]0,2[$, $1 \leqslant j \leqslant n$, $n \geqslant 2$, then for every $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathcal{R}_{\alpha,j} f(x) = |D|^{\alpha-1} \mathcal{R}_j f(x) = c_{\alpha,n} \text{p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+\alpha}} f(x-y)\,\mathrm{d}y, \tag{3.1}$$

where $c_{\alpha,n}$ is the normalization constant such that

$$\widehat{\mathcal{R}_{\alpha,j} f}(\zeta) = -i \frac{\zeta_j}{|\zeta|^{2-\alpha}} \hat{f}(\zeta).$$

The proof is placed in Appendix A. Also note that when $\alpha \in]0,1[$, we do not need to introduce the principle value of integral expression in the formula (3.1).

The pseudo-differential operators like the modified Riesz transforms do not preserve the moduli of continuity generally, but they also do not destroy them too much either. Precisely, similarly as the lemma in [18], we have

Lemma 3.2. *If the function θ has the modulus of continuity ω, then $u = (-\mathcal{R}_{\alpha,2}\theta, \mathcal{R}_{\alpha,1}\theta)$ $(\alpha \in]0,2[)$ has the modulus of continuity*

$$\Omega(\xi) = A_\alpha \left(\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha} \, \mathrm{d}\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}} \, \mathrm{d}\eta \right) \tag{3.2}$$

with some absolute constant $A_\alpha > 0$ that may depend on α.

Proof. The modified Riesz transforms are pseudo-differential operators with kernels $K(x) = \frac{S(x')}{|x|^{n-1+\alpha}}$ (in our special case, $n = 2$ and $S(x') = \frac{x_j}{|x|}$, $j = 1, 2$), where $x' = \frac{x}{|x|} \in \mathbb{S}^{n-1}$. The function $S \in C^1(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}} S(x') \, \mathrm{d}\sigma(x') = 0$. Assume that the function $f : \mathbb{R}^n \to \mathbb{R}^m$ has some modulus of continuity ω, that is $|f(x) - f(y)| \leqslant \omega(|x - y|)$ for all $x, y \in \mathbb{R}^n$. Then take any x, y with $|x - y| = \xi$, and consider the difference

$$\int K(x-t) f(t) \, \mathrm{d}t - \int K(y-t) f(t) \, \mathrm{d}t. \tag{3.3}$$

First due to the cancelation property of S we have

$$\left| \int_{|x-t| \leqslant 2\xi} K(x-t) f(t) \, \mathrm{d}t \right| = \left| \int_{|x-t| \leqslant 2\xi} K(x-t) \big(f(t) - f(x) \big) \, \mathrm{d}t \right| \leqslant C \int_0^{2\xi} \frac{\omega(r)}{r^\alpha} \, \mathrm{d}r.$$

Since ω is concave, we obtain

$$\int_0^{2\xi} \frac{\omega(r)}{r^\alpha} \, \mathrm{d}r \leqslant 2^{2-\alpha} \int_0^\xi \frac{\omega(r)}{r^\alpha} \, \mathrm{d}r. \tag{3.4}$$

A similar estimate holds for the second integral in (3.3). Next, set $z = \frac{x+y}{2}$, then

$$\begin{aligned}
&\left| \int_{|x-t| \geqslant 2\xi} K(x-t) f(t) \, \mathrm{d}t - \int_{|y-t| \geqslant 2\xi} K(y-t) f(t) \, \mathrm{d}t \right| \\
&\quad = \left| \int_{|x-t| \geqslant 2\xi} K(x-t) \big(f(t) - f(z) \big) \, \mathrm{d}t - \int_{|y-t| \geqslant 2\xi} K(y-t) \big(f(t) - f(z) \big) \, \mathrm{d}t \right| \\
&\quad \leqslant \int_{|z-t| \geqslant 3\xi} \big| K(x-t) - K(y-t) \big| \big| f(t) - f(z) \big| \, \mathrm{d}t \\
&\qquad + \int_{\frac{3\xi}{2} \leqslant |z-t| \leqslant 3\xi} \big(\big| K(x-t) \big| + \big| K(y-t) \big| \big) \big| f(t) - f(z) \big| \, \mathrm{d}t \\
&\quad = I_1 + I_2.
\end{aligned}$$

To estimate the first integral, we use the smoothness condition of S to get

$$\big| K(x-t) - K(y-t) \big| \leqslant C \frac{|x-y|}{|z-t|^{n+\alpha}} \quad \text{when } |z-t| \geqslant 3\xi,$$

thus

$$I_1 \leqslant C\xi \int_{3\xi}^{\infty} \frac{\omega(r)}{r^{1+\alpha}}\,\mathrm{d}r \leqslant C3^{-\alpha}\xi \int_{\xi}^{\infty} \frac{\omega(3r)}{r^{1+\alpha}}\,\mathrm{d}r \leqslant C\xi \int_{\xi}^{\infty} \frac{\omega(r)}{r^{1+\alpha}}\,\mathrm{d}r.$$

For the second integral, using the concavity of ω and (3.4), we have

$$\begin{aligned} I_2 &\leqslant 2C\omega(3\xi)\xi^{1-\alpha} \int_{\xi\leqslant|x-t|\leqslant\frac{7}{2}\xi} \frac{1}{|x-t|^n}\,\mathrm{d}t \\ &\leqslant C\omega(\xi)\xi^{1-\alpha} \leqslant C2^{\alpha} \int_{\xi}^{2\xi} \frac{\omega(r)}{r^{\alpha}}\,\mathrm{d}r \leqslant C \int_0^{\xi} \frac{\omega(r)}{r^{\alpha}}\,\mathrm{d}r. \qquad \square \end{aligned}$$

Now we consider a special action of the fractional differential operators $|D|^{\alpha}$ ($\alpha \in]0, 2[$) on the function having modulus of continuity. Precisely,

Lemma 3.3. *If the function $\theta : \mathbb{R}^2 \to \mathbb{R}$ has modulus of continuity ω, and especially satisfies $\theta(T_*, x) - \theta(T_*, y) = \omega(\xi)$ at some $T_* > 0$ and $x, y \in \mathbb{R}^2$ with $|x - y| = \xi > 0$, then we have*

$$\begin{aligned} \big[\big(-|D|^{\alpha}\big)\theta\big](x) - \big[\big(-|D|^{\alpha}\big)\theta\big](y) &\leqslant B_{\alpha} \int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta \\ &\quad + B_{\alpha} \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta \end{aligned} \tag{3.5}$$

where $B_{\alpha} > 0$ is an absolute constant.

Remark 3.1. In fact this result has occurred in [24,19], as a generalization of the one in [18], thus we here omit the proof. Also note that due to concavity of ω, both terms on the right-hand side of (3.5) are strictly negative.

4. Local existence and blowup criterion

Our purpose in this section is to prove the following local result:

Proposition 4.1. *Let $\nu > 0$, $0 < \alpha < 2$ and the initial data $\theta_0 \in H^m$, $m > 1$. Then there exists a positive T depending only on α, ν and $\|\theta_0\|_{H^m}$ such that the modified quasi-geostrophic equation (1.1) generates a unique solution $\theta \in \mathcal{C}([0,T], H^m) \cap L^2([0,T], H^{m+\frac{\alpha}{2}})$. Moreover we have $t^{\gamma}\theta \in L^{\infty}([0,T], H^{m+\gamma\alpha})$ for all $\gamma \geqslant 0$, which implies $\theta \in \mathcal{C}^{\infty}(]0,T] \times \mathbb{R}^2)$.*

We further obtain the following criterion for the breakdown of smooth solutions:

Proposition 4.2. *Let T^* be the maximal existence time of θ in $\mathcal{C}([0,T^*[, H^m) \cap L^2([0,T^*[, H^{m+\frac{\alpha}{2}})$, $m > 1$. If $T^* < \infty$ then we necessarily have*

$$\int_0^{T^*} \big\|\theta(t)\big\|^2_{\dot{B}^{\frac{\alpha}{2}}_{\infty,\infty}}\,\mathrm{d}t = \infty,$$

and

$$\int_0^{T^*} \left\| \nabla\theta(t) \right\|_{L^\infty}^{\alpha} \, \mathrm{d}t = \infty. \tag{4.1}$$

The method of proof for Proposition 4.1 is to regularize Eq. (1.1) by the standard Friedrich method, and then pass to the limit for the regularization parameter.

Denote the frequency cutoff operator $\mathcal{J}_\epsilon : L^2(\mathbb{R}^2) \to H^m(\mathbb{R}^2)$, $\epsilon > 0$, $m \geqslant 0$ by

$$(\mathcal{J}_\epsilon f)(x) = \mathcal{F}^{-1}\big(\hat{f}(\cdot) 1_{B_{1/\epsilon}}(\cdot)\big)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\zeta} \hat{f}(\zeta) 1_{\{|\cdot| \leqslant \frac{1}{\epsilon}\}}(\zeta) \, \mathrm{d}\zeta.$$

The following properties of $\mathcal{J}_\epsilon$ are obvious.

Lemma 4.3. *Let $\mathcal{J}_\epsilon$ be the projection operator defined as above, $m \in \mathbb{R}^+$, $k \in \mathbb{R}^+$, $\delta \in [0, m[$. Then*

(i) *for all $f \in H^m$, $\lim_{\epsilon\to 0} \|\mathcal{J}_\epsilon f - f\|_{H^m} = 0$,*
(ii) *for all $f \in H^m$, $|D|^m(\mathcal{J}_\epsilon f) = \mathcal{J}_\epsilon(|D|^m f)$ and $\Delta_j(\mathcal{J}_\epsilon f) = \mathcal{J}_\epsilon(\Delta_j f)$,*
(iii) *for all $f \in H^m$, $\|\mathcal{J}_\epsilon f - f\|_{H^{m-\delta}} \lesssim \epsilon^\delta \|f\|_{H^m}$ and $\|\mathcal{J}_\epsilon f\|_{H^{m+k}} \lesssim \frac{1}{\epsilon^k} \|f\|_{H^m}$.*

Then we regularize the modified quasi-geostrophic equation (1.1) as follows

$$\begin{cases} \theta_t^\epsilon + \mathcal{J}_\epsilon\big(\big(\mathcal{J}_\epsilon u^\epsilon\big) \cdot \nabla\big(\mathcal{J}_\epsilon \theta^\epsilon\big)\big) + \nu \mathcal{J}_\epsilon |D|^\alpha \theta^\epsilon = 0, \\ u^\epsilon = |D|^{\alpha-1} \mathcal{R}^\perp \theta^\epsilon, \qquad \theta^\epsilon\big|_{t=0} = \mathcal{J}_\epsilon \theta_0. \end{cases} \tag{4.2}$$

For this approximate system, we have

Proposition 4.4. *Let the initial data $\theta_0 \in L^2$. Then for any $\epsilon > 0$ there exists a unique global solution $\theta^\epsilon \in \mathcal{C}^\infty([0,\infty[, \mathcal{J}_\epsilon L^2)$ to the regularized equation* (4.2)*.*

Proof. We can write (4.2) as follows

$$\frac{d}{dt}\theta^\epsilon = F_\epsilon\big(\theta^\epsilon\big), \qquad \theta^\epsilon\big|_{t=0} = \mathcal{J}_\epsilon \theta_0, \tag{4.3}$$

with

$$F_\epsilon\big(\theta^\epsilon\big) = -\mathcal{J}_\epsilon\big(\big(\mathcal{J}_\epsilon u^\epsilon\big) \cdot \nabla\big(\mathcal{J}_\epsilon \theta^\epsilon\big)\big) - \nu \mathcal{J}_\epsilon |D|^\alpha \theta^\epsilon.$$

For every $\epsilon > 0$, we can show that (cf. [21])

$$\big\| F_\epsilon(f) \big\|_{L^2} \lesssim_{\epsilon,\nu} \|f\|_{L^2} + \|f\|_{L^2}^2,$$

and

$$\big\| F_\epsilon(f_1, f_2) \big\|_{L^2} \lesssim_{\epsilon,\nu,\|f_i\|_{L^2}} \|f_1 - f_2\|_{L^2},$$

where f, f_1, f_2 are all in L^2. This means that F_ϵ maps L^2 into L^2 and F_ϵ is locally Lipschitz continuous on L^2. Hence the Cauchy–Lipschitz theorem ensures that for every $\theta_0 \in L^2$, there exists a unique solution $\theta^\epsilon \in \mathcal{C}^1([0, T_\epsilon[, L^2)$ with $T_\epsilon > 0$ is the maximal existence time.

Moreover, using the L^2 energy method, form $\operatorname{div} u^\epsilon = 0$ and $\mathcal{J}_\epsilon \theta^\epsilon \in \mathcal{C}^1([0, T_\epsilon[, \mathcal{J}_\epsilon L^2)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\theta^\epsilon\|_{L^2}^2 + \nu\||D|^{\alpha/2}\mathcal{J}_\epsilon\theta^\epsilon\|_{L^2}^2 = 0.$$

Thus

$$\sup_{t\in[0,T_\epsilon[}\|\theta^\epsilon(t)\|_{L^2} \leqslant \|\mathcal{J}_\epsilon\theta_0\|_{L^2} \leqslant \|\theta_0\|_{L^2}. \tag{4.4}$$

Then the classical continuation criterion guarantees $T_\epsilon = \infty$.

Further, since $\mathcal{J}_\epsilon\theta^\epsilon$ is also a solution of (4.2), from the uniqueness we find $\theta^\epsilon = \mathcal{J}_\epsilon\theta^\epsilon$. □

Remark 4.1. From the proof we know $\theta^\epsilon = \mathcal{J}_\epsilon\theta^\epsilon$, thus (4.2) will be written as follows

$$\begin{cases} \theta_t^\epsilon + \mathcal{J}_\epsilon(u^\epsilon\cdot\nabla\theta^\epsilon) + \nu|D|^\alpha\theta^\epsilon = 0, \\ u^\epsilon = |D|^{\alpha-1}\mathcal{R}^\perp\theta^\epsilon, \quad \theta^\epsilon|_{t=0} = \mathcal{J}_\epsilon\theta_0. \end{cases} \tag{4.5}$$

In the sequel we shall instead base on this form.

Next, we prove the main result in this section.

Proof of Proposition 4.1. *Step* 1: *Uniform bounds.*

We claim that: the regularized solution $\theta^\epsilon \in \mathcal{C}^\infty(\mathbb{R}^+\times\mathbb{R}^2)$ to Eq. (4.2) satisfies

$$\begin{aligned} &\|\theta^\epsilon(t)\|_{B^m_{2,2}}^2 + \nu\||D|^{\frac{\alpha}{2}}\theta^\epsilon\|_{L^2_T B^m_{2,2}}^2 \\ &\quad\leqslant \|\theta_0\|_{B^m_{2,2}}^2 + \frac{C_\alpha}{\nu}\int_0^t \big(\|\theta^\epsilon(\tau)\|_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}^2 + \|\theta^\epsilon(\tau)\|_{\dot H^{\frac{\alpha}{2}}}^2\big)\|\theta^\epsilon(\tau)\|_{B^m_{2,2}}^2\,\mathrm{d}\tau. \end{aligned} \tag{4.6}$$

Indeed, for every $q\in\mathbb{N}$, applying dyadic operator Δ_q to both sides of the regularized equation (4.5) yields

$$\partial_t\Delta_q\theta^\epsilon + \mathcal{J}_\epsilon\big((S_{q+1}u^\epsilon)\cdot\nabla\Delta_q\theta^\epsilon\big) + \nu|D|^\alpha\Delta_q\theta^\epsilon = \mathcal{J}_\epsilon\big(F_q(u^\epsilon,\theta^\epsilon)\big),$$

where

$$F_q(u^\epsilon,\theta^\epsilon) = (S_{q+1}u^\epsilon)\cdot\nabla\Delta_q\theta^\epsilon - \Delta_q(u^\epsilon\cdot\nabla\theta^\epsilon).$$

Taking the L^2 inner product in the above equality with $\Delta_q\theta^\epsilon$ and using the divergence-free property, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta_q\theta^\epsilon\|_{L^2}^2 + \nu\||D|^{\frac{\alpha}{2}}\Delta_q\theta^\epsilon\|_{L^2}^2 &\leqslant \left|\int_{\mathbb{R}^2}\big(F_q(u^\epsilon,\theta^\epsilon)\big)(x)\mathcal{J}_\epsilon\Delta_q\theta^\epsilon(x)\,\mathrm{d}x\right| \\ &\leqslant 2^{-q\frac{\alpha}{2}}\|F_q(u^\epsilon,\theta^\epsilon)\|_{L^2}2^{q\frac{\alpha}{2}}\|\mathcal{J}_\epsilon\Delta_q\theta^\epsilon\|_{L^2} \\ &\leqslant C_0 2^{-q\frac{\alpha}{2}}\|F_q(u^\epsilon,\theta^\epsilon)\|_{L^2}\||D|^{\frac{\alpha}{2}}\Delta_q\theta^\epsilon\|_{L^2}. \end{aligned}$$

Then by virtue of the Young inequality, we deduce

$$\frac{1}{2}\frac{d}{dt}\left\|\Delta_q\theta^\epsilon\right\|_{L^2}^2+\frac{\nu}{2}\left\||D|^{\frac{\alpha}{2}}\Delta_q\theta^\epsilon\right\|_{L^2}^2\leqslant\frac{C_0}{\nu}\left(2^{-q\frac{\alpha}{2}}\left\|F_q\left(u^\epsilon,\theta^\epsilon\right)\right\|_{L^2}\right)^2.$$

Integrating in time leads to

$$\left\|\Delta_q\theta^\epsilon(t)\right\|_{L^2}^2+\nu\left\||D|^{\frac{\alpha}{2}}\Delta_q\theta^\epsilon\right\|_{L_t^2L^2}^2\leqslant\|\Delta_q\mathcal{J}_\epsilon\theta_0\|_{L^2}^2+\frac{C_0}{\nu}\int\left(2^{-q\frac{\alpha}{2}}\left\|F_q\left(u^\epsilon,\theta^\epsilon\right)(\tau)\right\|_{L^2}\right)^2\mathrm{d}\tau. \tag{4.7}$$

From the inequality (A.2) in Appendix A, we know that

$$2^{-q\frac{\alpha}{2}}\left\|F_q\left(u^\epsilon,\theta^\epsilon\right)\right\|_{L^2}\lesssim_\alpha\left\|\theta^\epsilon\right\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,\infty}}\left(\sum_{q'\geqslant q-4}2^{(q-q')(1-\frac{\alpha}{2})}\left\|\Delta_{q'}\theta^\epsilon\right\|_{L^2}+\sum_{|q'-q|\leqslant 4}\left\|\Delta_{q'}\theta^\epsilon\right\|_{L^2}\right). \tag{4.8}$$

Plunging the estimate (4.8) into inequality (4.7), then multiplying both sides by 2^{2qm} ($m>1$) and summing up over $q\in\mathbb{N}$, we obtain

$$\begin{aligned}&\sum_{q\in\mathbb{N}}2^{2qm}\left\|\Delta_q\theta^\epsilon\right\|_{L^2}^2+\nu\sum_{q\in\mathbb{N}}2^{2qm}\left\||D|^{\frac{\alpha}{2}}\Delta_q\theta^\epsilon\right\|_{L_t^2L^2}^2\\&\quad\leqslant\sum_{q\in\mathbb{N}}2^{2qm}\|\Delta_q\theta_0\|_{L^2}^2+\frac{C_\alpha}{\nu}\int_0^t\left\|\theta^\epsilon(\tau)\right\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,\infty}}^2\left\|\theta^\epsilon(\tau)\right\|_{B^m_{2,2}}^2\mathrm{d}\tau.\end{aligned} \tag{4.9}$$

On the other hand, we apply the low frequency operator Δ_{-1} to the regularized system (4.2) to get

$$\partial_t\Delta_{-1}\theta^\epsilon+\nu|D|^\alpha\Delta_{-1}\theta^\epsilon=-\mathcal{J}_\epsilon\Delta_{-1}\left(u^\epsilon\cdot\nabla\theta^\epsilon\right).$$

Multiplying both sides by $\Delta_{-1}\theta^\epsilon$ and integrating in the spatial variable, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Delta_{-1}\theta^\epsilon\right\|_{L^2}^2+\nu\left\||D|^{\frac{\alpha}{2}}\Delta_{-1}\theta^\epsilon\right\|_{L^2}^2&\leqslant\left|\int_{\mathbb{R}^2}\operatorname{div}\Delta_{-1}\left(u^\epsilon\theta^\epsilon\right)(x)\Delta_{-1}\mathcal{J}_\epsilon\theta^\epsilon(x)\,\mathrm{d}x\right|\\&\leqslant\left\|\Delta_{-1}\left(u^\epsilon\theta^\epsilon\right)\right\|_{\dot{H}^{1-\frac{\alpha}{2}}}\left\|\mathcal{J}_\epsilon|D|^{\frac{\alpha}{2}}\Delta_{-1}\theta^\epsilon\right\|_{L^2}\\&\leqslant\frac{C_0}{\nu}\left\|\Delta_{-1}\left(u^\epsilon\theta^\epsilon\right)\right\|_{L^2}^2+\frac{\nu}{2}\left\||D|^{\frac{\alpha}{2}}\Delta_{-1}\theta^\epsilon\right\|_{L^2}^2.\end{aligned}$$

Using the Bernstein inequality, the Sobolev embedding $\dot{H}^{1-\frac{\alpha}{2}}\hookrightarrow L^{\frac{4}{\alpha}}$ ($\alpha\in\,]0,2[$) and the Hölder inequality, we see that

$$\left\|\Delta_{-1}\left(u^\epsilon\theta^\epsilon\right)\right\|_{L^2}\lesssim\left\|\Delta_{-1}\left(u^\epsilon\theta^\epsilon\right)\right\|_{L^{\frac{4}{\alpha+2}}}\lesssim\left\|u^\epsilon\right\|_{L^{\frac{4}{\alpha}}}\left\|\theta^\epsilon\right\|_{L^2}\lesssim\left\|u^\epsilon\right\|_{\dot{H}^{1-\frac{\alpha}{2}}}\left\|\theta^\epsilon\right\|_{L^2}\lesssim\left\|\theta^\epsilon\right\|_{\dot{H}^{\frac{\alpha}{2}}}\left\|\theta^\epsilon\right\|_{L^2},$$

thus from the energy estimate (4.4) we have

$$\left\|\Delta_{-1}\theta^\epsilon(t)\right\|_{L^2}^2+\nu\left\||D|^{\frac{\alpha}{2}}\Delta_{-1}\theta^\epsilon\right\|_{L_t^2L^2}^2\leqslant\|\Delta_{-1}\theta_0\|_{L^2}^2+\frac{C_\alpha}{\nu}\int_0^t\left\|\theta^\epsilon(\tau)\right\|_{\dot{H}^{\frac{\alpha}{2}}}^2\left\|\theta^\epsilon(\tau)\right\|_{B^m_{2,2}}^2\mathrm{d}\tau. \tag{4.10}$$

Multiplying (4.10) by 2^{-2m} and combining it with (4.9) leads to (4.6).

Next, we prove that the solution family (θ^ϵ) is uniformly bounded in H^m. First denote by

$$Z(t) := \|\theta^\epsilon(\tau)\|^2_{L^\infty_t B^m_{2,2}} + \nu \int_0^t \||D|^{\frac{\alpha}{2}}\theta^\epsilon(\tau)\|^2_{B^m_{2,2}}\,\mathrm{d}\tau,$$

then from $B^m_{2,2} \hookrightarrow \dot B^m_{2,2}$, $m > 1$, and by interpolation and Besov embedding, we get

$$\|\theta^\epsilon\|_{L^{p_m}_t \dot B^{\frac{\alpha}{2}}_{\infty,\infty}} \lesssim \|\theta^\epsilon\|_{L^\infty_t B^{m-1}_{\infty,\infty} \cap L^2_t \dot B^{m-1+\frac{\alpha}{2}}_{\infty,\infty}} \lesssim Z(t)^{\frac{1}{2}},$$

with $p_m \in]2, \infty]$ defined by

$$p_m := \begin{cases} \frac{\alpha}{1+\alpha/2-m}, & m \in]1, 1+\frac{\alpha}{2}[, \\ \infty, & m \in [1+\frac{\alpha}{2}, \infty[. \end{cases}$$

Furthermore, from (4.6) and the upper estimate, we find

$$\begin{aligned} Z(t) &\leqslant \|\theta_0\|^2_{B^m_{2,2}} + \frac{C_\alpha}{\nu}\int_0^t \left(\|\theta^\epsilon(\tau)\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}} + \|\theta^\epsilon(\tau)\|^2_{\dot H^{\frac{\alpha}{2}}}\right) Z(\tau)\,\mathrm{d}\tau \\ &\leqslant \|\theta_0\|^2_{B^m_{2,2}} + C_{\alpha,\nu} t^{1-\frac{2}{p_m}} Z(t)^2 + C_{\alpha,\nu}\int_0^t \|\theta^\epsilon(\tau)\|^2_{\dot H^{\frac{\alpha}{2}}} Z(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.11}$$

By the continuity method, we infer that for some $T > 0$ satisfying

$$C_{\alpha,\nu} T^{1-\frac{2}{p_m}} 2\|\theta_0\|^2_{B^m_{2,2}} \exp\{2C_{\alpha,\nu}\|\theta_0\|^2_{L^2}\} < \frac{1}{2} \quad \Longleftrightarrow \quad T < \left(\frac{\exp\{-2C_{\alpha,\nu}\|\theta_0\|^2_{L^2}\}}{4C_{\alpha,\nu}\|\theta_0\|^2_{B^m_{2,2}}}\right)^{\frac{p_m}{p_m-2}},$$

we have

$$Z(t) \leqslant 2\|\theta_0\|^2_{B^m_{2,2}} \exp\{2C_{\alpha,\nu}\|\theta_0\|^2_{L^2}\}, \quad \forall t \in [0, T].$$

From the fact that $\|\cdot\|^2_{B^m_{2,2}}/C_0 \leqslant \|\cdot\|^2_{H^m} \leqslant C_0\|\cdot\|^2_{B^m_{2,2}}$ with C_0 a universal number, we get that for every $t \in [0, T]$,

$$\|\theta^\epsilon\|^2_{L^\infty_T H^m} + \nu\||D|^{\frac{\alpha}{2}}\theta^\epsilon\|^2_{L^2_T H^m} \leqslant 2C_0^2\|\theta_0\|^2_{H^m} \exp\{2C_{\alpha,\nu}\|\theta_0\|^2_{L^2}\}. \tag{4.12}$$

This clearly implies that the family (θ^ϵ) is uniformly bounded in $\mathcal{C}([0, T], H^m) \cap L^2([0, T]; H^{m+\frac{\alpha}{2}})$, $m > 1$, with respect to ϵ.

Step 2: *Strong convergence.*

We firstly claim that the solutions (θ^ϵ) to Eq. (4.5) strongly converge in $\mathcal{C}([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T]; \dot H^{\frac{\alpha}{2}}(\mathbb{R}^2))$. Indeed, for every $0 < \tilde\epsilon < \epsilon$, we assume that θ^ϵ and $\theta^{\tilde\epsilon}$ are two approximate solutions, then from a direct calculation

$$\big(\theta_t^\epsilon - \theta_t^{\tilde\epsilon}, \theta^\epsilon - \theta^{\tilde\epsilon}\big) = -\nu\big(|D|^\alpha\theta^\epsilon - |D|^\alpha\theta^{\tilde\epsilon}, \theta^\epsilon - \theta^{\tilde\epsilon}\big) - \big(\big(\mathcal{J}_\epsilon\big(u^\epsilon\cdot\nabla\theta^\epsilon\big) - \mathcal{J}_{\tilde\epsilon}\big(u^{\tilde\epsilon}\cdot\nabla\theta^{\tilde\epsilon}\big)\big), \theta^\epsilon - \theta^{\tilde\epsilon}\big),$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\theta^\epsilon(t) - \theta^{\tilde\epsilon}(t)\big\|_{L^2}^2 + \nu\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)(t)\big\|_{L^2}^2 \\
&\quad= \big((\mathcal{J}_\epsilon - \mathcal{J}_{\tilde\epsilon})\big(u^\epsilon\cdot\nabla\theta^\epsilon\big), \theta^\epsilon - \theta^{\tilde\epsilon}\big) + \big(\mathcal{J}_{\tilde\epsilon}\big(\big(u^\epsilon - u^{\tilde\epsilon}\big)\cdot\nabla\theta^\epsilon\big), \theta^\epsilon - \theta^{\tilde\epsilon}\big) \\
&\qquad+ \big(\mathcal{J}_{\tilde\epsilon}\big(u^{\tilde\epsilon}\cdot\nabla\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big), \theta^\epsilon - \theta^{\tilde\epsilon}\big) \\
&\quad:= II_1 + II_2 + II_3.
\end{aligned}$$

For II_1, by means of Lemma 4.3, divergence-free condition and the following simple inequality

$$\big\|u^\epsilon\big\|_{\dot H^{1-\alpha/2}} = \big\||D|^{\alpha-1}\mathcal{R}^\perp\theta^\epsilon\big\|_{\dot H^{2-\alpha/2}} \lesssim \big\|\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}} \lesssim \big\|\theta^\epsilon\big\|_{H^m} \lesssim M^{\frac{1}{2}},$$

with $M > 0$ the uniform upper bound from (4.12), we have

$$\begin{aligned}
|II_1| &\lesssim \big\|(\mathcal{J}_\epsilon - \mathcal{J}_{\tilde\epsilon})\big(u^\epsilon\cdot\nabla\theta^\epsilon\big)\big\|_{\dot H^{-\frac{\alpha}{2}}}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2} \\
&\leqslant \epsilon^\alpha C_\nu\big\|u^\epsilon\cdot\nabla\theta^\epsilon\big\|_{L^2}^2 + \frac{\nu}{4}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2}^2 \\
&\leqslant \epsilon^\alpha C_{\alpha,\nu}\big\|u^\epsilon\big\|_{\dot H^{1-\frac{\alpha}{2}}}^2\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2 + \frac{\nu}{4}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2}^2 \\
&\leqslant \epsilon^\alpha C_{\alpha,\nu}M\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2 + \frac{\nu}{4}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2}^2,
\end{aligned}$$

where in the second line we have used the classical product estimate (cf. [15]) that for every $s, t < 1$ and $s + t > 0$,

$$\|fg\|_{\dot H^{s+t-1}} \lesssim_{s,t} \|f\|_{\dot H^s}\|g\|_{\dot H^t}. \tag{4.13}$$

For II_2, using the Young inequality and (4.13) again, we directly obtain

$$\begin{aligned}
|II_2| &\leqslant \big\|\big(u^\epsilon - u^{\tilde\epsilon}\big)\cdot\nabla\theta^\epsilon\big\|_{\dot H^{-\frac{\alpha}{2}}}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2} \\
&\leqslant C_{\alpha,\nu}\big\||D|^{\alpha-1}\mathcal{R}^\perp\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{\dot H^{1-\alpha}}^2\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2 + \frac{\nu}{4}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2}^2 \\
&\leqslant C_{\alpha,\nu}\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2\big\|\theta^\epsilon - \theta^{\tilde\epsilon}\big\|_{L^2}^2 + \frac{\nu}{4}\big\||D|^{\frac{\alpha}{2}}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big\|_{L^2}^2.
\end{aligned}$$

For the last term, II_3, from the divergence-free fact of $u^{\tilde\epsilon}$ and $\mathcal{J}_{\tilde\epsilon}\theta^\epsilon = \mathcal{J}_\epsilon\theta^\epsilon = \theta^\epsilon$ we get

$$II_3 = \big(\big(u^{\tilde\epsilon}\cdot\nabla\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big), \mathcal{J}_{\tilde\epsilon}\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)\big) = \frac{1}{2}\big(u^{\tilde\epsilon}, \nabla\big(\theta^\epsilon - \theta^{\tilde\epsilon}\big)^2\big) = 0.$$

Putting all these estimates together yields that

$$\frac{1}{2}\frac{d}{dt}\big\|\theta^\epsilon - \theta^{\tilde\epsilon}\big\|_{L^2}^2 + \frac{\nu}{2}\big\|\theta^\epsilon - \theta^{\tilde\epsilon}\big\|_{\dot H^{\frac{\alpha}{2}}}^2 \leqslant \epsilon^\alpha C_{\alpha,\nu}M\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2 + C_{\alpha,\nu}\big\|\nabla\theta^\epsilon\big\|_{\dot H^{\frac{\alpha}{2}}}^2\big\|\theta^\epsilon - \theta^{\tilde\epsilon}\big\|_{L^2}^2.$$

Thus the Grönwall inequality leads to the desired result:

$$\begin{aligned}\|\theta^\epsilon-\theta^{\tilde\epsilon}\|^2_{L^\infty_T L^2}+\nu\|\theta^\epsilon-\theta^{\tilde\epsilon}\|^2_{L^2_T\dot H^{\frac{\alpha}{2}}}&\leqslant e^{C_{\alpha,\nu}\||D|^{\frac{\alpha}{2}}\theta^\epsilon\|^2_{L^2_T H^m}}\left(\epsilon^\alpha C_{\alpha,\nu}M\||D|^{\frac{\alpha}{2}}\theta^\epsilon\|^2_{L^2_T H^m}+\|\theta^\epsilon_0-\theta^{\tilde\epsilon}_0\|^2_{L^2}\right)\\&\leqslant e^{C_{\alpha,\nu}M}\left(\epsilon^\alpha+\|\theta^\epsilon_0-\theta^{\tilde\epsilon}_0\|^2_{L^2}\right)\\&\lesssim_{M,\alpha,\nu}a(\epsilon),\end{aligned}\tag{4.14}$$

where $a(\epsilon):=\epsilon^\alpha+\|(Id-\mathcal{J}_\epsilon)\theta_0\|^2_{L^2}$ satisfies that $a(\epsilon)\to 0$ as $\epsilon\to 0$.

From (4.14), we deduce that the solution family (θ^ϵ) is a Cauchy sequence in $\mathcal{C}([0,T];L^2(\mathbb{R}^2))\cap L^2([0,T];\dot H^{\frac{\alpha}{2}}(\mathbb{R}^2))$, so that it converges strongly to a function θ belonging to $\mathcal{C}([0,T];L^2(\mathbb{R}^2))\cap L^2([0,T];\dot H^{\frac{\alpha}{2}}(\mathbb{R}^2))$. This result combined with uniform bound of $Z(t)$ and the interpolation inequality in Sobolev spaces gives that for all $0\leqslant s<m$

$$\begin{aligned}\|\theta^\epsilon-\theta\|_{L^\infty_T H^s}&\leqslant C_s\|\theta^\epsilon-\theta\|^{1-s/m}_{L^\infty_T L^2}\|\theta^\epsilon-\theta\|^{s/m}_{L^\infty_T H^m}\\&\lesssim_{s,M,\alpha,\nu}a(\epsilon)^{\frac12(1-\frac{s}{m})}\end{aligned}$$

and

$$\begin{aligned}\||D|^{\frac{\alpha}{2}}(\theta^\epsilon-\theta)\|_{L^2_T H^s}&\leqslant C_s\left(\||D|^{\frac{\alpha}{2}}(\theta^\epsilon-\theta)\|^{1-s/m}_{L^2_T L^2}\||D|^{\frac{\alpha}{2}}(\theta^\epsilon-\theta)\|^{s/m}_{L^2_T H^m}\right)\\&\lesssim_{s,M,\alpha,\nu}a(\epsilon)^{\frac12(1-\frac{s}{m})}.\end{aligned}$$

Hence we obtain the strong convergence in $\mathcal{C}([0,T];H^s(\mathbb{R}^2))\cap L^2([0,T];H^{s+\frac{\alpha}{2}})$ for all $s<m$. With $1<s<m$, this specially implies strong convergence in $\mathcal{C}([0,T],\mathcal{C}(\mathbb{R}^2))$. Also from the equation

$$\theta^\epsilon_t=-\nu|D|^\alpha\theta^\epsilon-\mathcal{J}_\epsilon\left(u^\epsilon\cdot\nabla\theta^\epsilon\right),$$

we find that θ^ϵ_t strongly converges to $-\nu|D|^\alpha\theta-u\cdot\nabla\theta$ in $L^2([0,T],L^2(\mathbb{R}^2))$. Since $\theta^\epsilon\to\theta$, the distribution limit of θ^ϵ_t has to be θ_t. Thus $\theta\in H^1([0,T],L^2(\mathbb{R}^2))\cap\mathcal{C}([0,T],\mathcal{C}(\mathbb{R}^2))$ is a solution to the original equation (1.1). Using Fatou's lemma, from (4.12), we also have $\theta\in L^\infty([0,T],H^m(\mathbb{R}^2))\cap L^2([0,T],H^{m+\frac{\alpha}{2}}(\mathbb{R}^2))$.

Next, we show that $\theta\in\mathcal{C}([0,T],H^m(\mathbb{R}^2))$. Indeed, from the deduction in *Step 1*, we can find that the formula (4.11) remains true by replacing $\|\theta^\epsilon\|_{L^\infty_t B^m_{2,2}}$ with $\|\theta^\epsilon\|_{\widetilde L^\infty_t B^m_{2,2}}$ in the definition of $Z(t)$. Hence, we in fact obtain $\theta^\epsilon\in\widetilde L^\infty([0,T];B^m_{2,2}(\mathbb{R}^2))$ uniformly in ϵ. Based on this fact and by a classical process (cf. [7]), we can prove the continuity-in-time issue.

Step 3: *Uniqueness.*

Let $\theta^1,\theta^2\in L^\infty([0,T],H^m(\mathbb{R}^2))\cap L^2([0,T];\dot H^{1+\frac{\alpha}{2}}(\mathbb{R}^2))$ be two smooth solutions to the modified quasi-geostrophic equation (1.1) with the same initial data. Denote by $u^i=|D|^{\alpha-1}R^\perp\theta^i$, $i=1,2$, $\delta\theta:=\theta^1-\theta^2$, $\delta u:=u^1-u^2$, then we write the difference equation as

$$\partial_t\delta\theta+u^1\cdot\nabla\delta\theta+\nu|D|^\alpha\delta\theta=-\delta u\cdot\nabla\theta^2,\qquad\delta\theta|_{t=0}\delta\theta_0\equiv 0.$$

We also use the L^2 energy method, and in a similar way as treating the term II_3, we obtain

$$\frac12\frac{d}{dt}\|\delta\theta(t)\|^2_{L^2}\leqslant C_{\alpha,\nu}\|\nabla\theta^2(t)\|^2_{\dot H^{\frac{\alpha}{2}}}\|\delta\theta(t)\|^2_{L^2}.$$

Thus the Grönwall inequality ensures that

$$\left\|\delta\theta(t)\right\|_{L^2} \leqslant \|\delta\theta_0\|_{L^2}\exp\left\{C_{\alpha,\nu}\left\|\nabla\theta^2\right\|^2_{L^2_T\dot{H}^{\frac{\alpha}{2}}}\right\} \equiv 0, \quad \forall t\in[0,T],$$

that is, $\theta^1 \equiv \theta^2$.

Step 4: *Smoothing effect.*

Precisely, we have that for all $\gamma\in\mathbb{R}^+$ and $t\in[0,T]$

$$\begin{aligned}&\left\|t^\gamma\theta(t)\right\|^2_{L^\infty_T H^{m+\gamma\alpha}} + \left\||D|^{\frac{\alpha}{2}}\left(t^\gamma\theta(t)\right)\right\|^2_{L^2_T H^{m+\gamma\alpha}}\\ &\quad\leqslant C\left([\gamma]+1\right)^{2(\gamma-[\gamma])}\left([\gamma]!\right)^2\left(1+T^{2\gamma}\right)Me^{C\gamma M},\end{aligned} \tag{4.15}$$

where M denotes an upper bound of $\|\theta\|^2_{L^\infty_T H^m} + \nu\||D|^{\frac{\alpha}{2}}\theta\|^2_{L^2_T H^m}$ and C is an absolute constant depending only on α, ν, m. Notice that $t^\gamma\theta$ $(\gamma>0)$ satisfies

$$\partial_t\left(t^\gamma\theta\right) + u\cdot\nabla\left(t^\gamma\theta\right) + \nu|D|^\alpha\left(t^\gamma\theta\right) = \gamma t^{\gamma-1}\theta, \qquad \left(t^\gamma\theta\right)\big|_{t=0} = 0, \tag{4.16}$$

which is a linear transport-diffusion equation with the velocity $u=|D|^{\alpha-1}R^\perp\theta$, $\alpha\in\,]0,2[$. We first treat the case $\gamma\in\mathbb{Z}^+$. For $\gamma=1$, in a similar way as obtaining (4.6) and using the maximum principle (2.1), we infer that

$$\begin{aligned}&\left\|t\theta(t)\right\|^2_{B^{m+\alpha}_{2,2}} + \nu\left\||D|^{\frac{\alpha}{2}}\left(t\theta(t)\right)\right\|^2_{L^2_T B^{m+\alpha}_{2,2}}\\ &\quad\lesssim \int_0^T \left(\left\|\theta(t)\right\|^2_{\dot{B}^{\frac{\alpha}{2}}_{\infty,\infty}} + \left\|\theta(t)\right\|^2_{\dot{H}^{\frac{\alpha}{2}}}\right)\left\|t\theta(t)\right\|^2_{B^{m+\alpha}_{2,2}}\,\mathrm{d}t + \left\||D|^{\frac{\alpha}{2}}\theta\right\|^2_{L^2_T B^m_{2,2}} + \left\|t\theta(t)\right\|_{L^2_T L^2}\|\theta\|_{L^2_T L^2}\\ &\quad\lesssim \int_0^T \left(\left\|\theta(t)\right\|^2_{\dot{B}^{\frac{\alpha}{2}}_{\infty,\infty}} + \left\|\theta(t)\right\|^2_{\dot{H}^{\frac{\alpha}{2}}}\right)\left\|t\theta(t)\right\|^2_{B^{m+\alpha}_{2,2}}\,\mathrm{d}t + M + \|\theta_0\|^2_{L^2}T^2.\end{aligned}$$

Grönwall's inequality yields that

$$\left\|t\theta(t)\right\|^2_{B^{m+\alpha}_{2,2}} + \nu\left\||D|^{\frac{\alpha}{2}}\left(t\theta(t)\right)\right\|^2_{L^2_T B^{m+\alpha}_{2,2}} \lesssim M\left(1+T^2\right)e^{CM}, \tag{4.17}$$

where we have used the following estimates

$$\left\||D|^{\frac{\alpha}{2}}\theta\right\|^2_{L^2_T\dot{B}^0_{\infty,\infty}} \leqslant M, \quad\text{and}\quad \|\theta\|^2_{L^2_T\dot{H}^{\frac{\alpha}{2}}} \leqslant \|\theta_0\|^2_{L^2} \leqslant M.$$

Thus (4.15) with $\gamma=1$ follows. Now suppose estimate (4.15) holds for $\gamma=N$, we shall consider the case $N+1$. We use Eq. (4.16) with $\gamma=N+1$, and similarly as obtaining (4.17), only by replacing $\theta(t)$ with $t^N\theta(t)$ and m with $m+N\alpha$, we have

$$\begin{aligned}&\left\|t^{N+1}\theta(t)\right\|^2_{H^{m+(N+1)\alpha}} + \left\||D|^{\frac{\alpha}{2}}\left(t^{N+1}\theta(t)\right)\right\|^2_{L^2_T H^{m+(N+1)\alpha}}\\ &\quad\lesssim e^{CM}\left((N+1)^2\left\||D|^{\frac{\alpha}{2}}\left(t^N\theta(t)\right)\right\|^2_{L^2_T H^{m+N\alpha}} + (N+1)\left\|t^{N+1}\theta(t)\right\|_{L^2_T L^2}\left\|t^N\theta(t)\right\|_{L^2_T L^2}\right)\\ &\quad\lesssim e^{CM}\left((N+1)^2(N!)^2\left(1+T^{2N}\right)Me^{CNM} + \left((N+1)!\right)^2T^{2N+2}\|\theta_0\|^2_{L^2}\right)\\ &\quad\lesssim \left((N+1)!\right)^2\left(1+T^{2N+2}\right)Me^{C(N+1)M},\end{aligned}$$

where in the second line we have used the following estimation (by repeatedly using the maximum principle (2.1) N times)

$$\left\|t^N\theta(t)\right\|_{L^\infty_T L^2} \leqslant NT\left\|t^{N-1}\theta(t)\right\|_{L^\infty_T L^2} \leqslant (N!)T^N\|\theta_0\|_{L^2}.$$

Thus the induction method ensures the estimate (4.15) for all $\gamma\in\mathbb{Z}^+$. Also notice that for $\gamma=0$ the inequality (4.15) is also satisfied. Hence we obtain estimate (4.15) for all $\gamma\in\mathbb{N}$. For the general $\gamma\geqslant 0$, we set $[\gamma]\leqslant\gamma<[\gamma]+1$, and use the interpolation inequality in Sobolev spaces to get

$$\begin{aligned}\left\|t^\gamma\theta\right\|^2_{L^\infty_T H^{m+\gamma\alpha}} &\leqslant \left\|t^{[\gamma]}\theta\right\|^{2([\gamma]+1-\gamma)}_{L^\infty_T H^{m+[\gamma]\alpha}}\left\|t^{[\gamma]+1}\theta\right\|^{2(\gamma-[\gamma])}_{L^\infty_T H^{m+([\gamma]+1)\alpha}}\\ &\lesssim C\big([\gamma]+1\big)^{2(\gamma-[\gamma])}\big([\gamma]!\big)^2\big(1+T^{2\gamma}\big)Me^{C\gamma M}.\end{aligned}$$

Similar estimate holds for $\||D|^{\frac{\alpha}{2}}(t^\gamma\theta(t))\|^2_{L^2_T H^{m+\gamma\alpha}}$.

Therefore, we conclude Proposition 4.1. □

Now, we are devoted to building the blowup criterion.

Proof of Proposition 4.2. We first note that the equation has a natural blowup criterion: if $T^*<\infty$ then necessarily

$$\|\theta\|_{L^\infty([0,T^*[,H^m)}=\infty.$$

Otherwise from the local result, the solution will continue over T^*.

In the same way as obtaining the estimate (4.6), we get the similar result for the original equation

$$\left\|\theta(t)\right\|^2_{H^m}+\nu\left\||D|^{\frac{\alpha}{2}}\theta(\tau)\right\|^2_{L^2_t H^m}\leqslant C_0\|\theta_0\|^2_{H^m}+C_{\alpha,\nu}\int_0^t\Big(\left\|\theta(\tau)\right\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}+\left\|\theta(\tau)\right\|^2_{\dot H^{\frac{\alpha}{2}}}\Big)\left\|\theta(\tau)\right\|^2_{H^m}\,\mathrm{d}\tau.$$

Thus the Grönwall inequality and the energy estimate leads to

$$\begin{aligned}&\|\theta\|^2_{L^\infty_T H^m}+\nu\left\||D|^{\frac{\alpha}{2}}\theta\right\|^2_{L^2([0,T],H^m)}\\ &\quad\leqslant C_0\|\theta_0\|^2_{H^m}\exp\left\{C_{\alpha,\nu}\int_0^T\left\|\theta(t)\right\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}\mathrm{d}t+C_{\alpha,\nu}\int_0^T\left\|\theta(t)\right\|^2_{\dot H^{\frac{\alpha}{2}}}\mathrm{d}t\right\}\\ &\quad\leqslant C_0\|\theta_0\|^2_{H^m}\exp\left\{C_{\alpha,\nu}\int_0^T\left\|\theta(t)\right\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}\mathrm{d}t+C_{\alpha,\nu}\|\theta_0\|^2_{L^2}\right\}.\end{aligned}$$

Furthermore, if $T^*<\infty$ and the integral $\int_0^{T^*}\|\theta(t)\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}\,\mathrm{d}t<\infty$, then we directly have $\sup_{0\leqslant t<T^*}\|\theta(t)\|_{H^m}<\infty$, and this contradicts the upper natural blowup criterion. Thus, if $T^*<\infty$, we necessarily have the equality $\int_0^{T^*}\|\theta(t)\|^2_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}}\,\mathrm{d}t=\infty$.

On the other hand, from interpolation and the maximum principle (2.1), we obtain that for every $t\in[0,T^*[$

$$\begin{aligned}\|\theta(t)\|_{\dot B^{\frac{\alpha}{2}}_{\infty,\infty}} &\lesssim \|\theta(t)\|^{1-\frac{\alpha}{2}}_{\dot B^0_{\infty,\infty}} \|\theta(t)\|^{\frac{\alpha}{2}}_{\dot B^1_{\infty,\infty}} \\ &\lesssim \|\theta(t)\|^{1-\frac{\alpha}{2}}_{L^\infty} \|\nabla\theta(t)\|^{\frac{\alpha}{2}}_{L^\infty} \lesssim \|\theta_0\|^{1-\frac{\alpha}{2}}_{L^\infty} \|\nabla\theta(t)\|^{\frac{\alpha}{2}}_{L^\infty}.\end{aligned}$$

Hence if $T^* < \infty$, we also necessarily need $\int_0^{T^*} \|\nabla\theta(t)\|^{\alpha}_{L^\infty}\,dt = \infty$. □

5. Global existence

In this section, we use the modulus of continuity argument developed by Kiselev, Nazarov and Volberg [18] to prove the global result. Throughout this section, we assume T^* be the maximal existence time of the solution in $\mathcal{C}([0,T^*[,H^m)\cap L^2([0,T^*[,H^{m+\frac{\alpha}{2}})$.

From Proposition 4.1, we know that there exists $T_0 > 0$ such that for all $t \in [0,T_0]$,

$$t^{\frac{1}{\alpha}} \|\theta(t)\|_{H^{m+1}} \leqslant C_{\alpha,T_0,\|\theta_0\|_{H^m}}.$$

Let $\lambda > 0$ be a real number which will be chosen later and $T_1 \in\,]0,T_0[$ (note that if $\theta_0 \in \mathrm{Lip}(\mathbb{R}^2)$, we can choose $T_1 = 0$), then we define the set

$$\mathcal{I} := \big\{T \in [T_1, T^*[\ \big|\ \forall t \in [T_1,T],\ \forall x,y \in \mathbb{R}^2,\ x \neq y,\ |\theta(t,x)-\theta(t,y)| < \omega_\lambda(|x-y|)\big\},$$

where ω is a strict modulus of continuity also satisfying that $\omega'(0) < \infty$, $\lim_{\eta\searrow 0}\omega''(\eta) = -\infty$ and

$$\omega_\lambda(|x-y|) = \omega(\lambda|x-y|).$$

The explicit expression of ω will be shown later (i.e. (5.11) in the below).

We first show that the set $\mathcal{I}$ is nonempty, that is, at least $T_1 \in \mathcal{I}$. The proof is almost the same with the one in [1]. We omit it here and only note that to fit our purpose λ can be taken

$$\lambda = \frac{\omega^{-1}(3\|\theta_0\|_{L^\infty})}{2\|\theta_0\|_{L^\infty}} \|\nabla\theta(T_1)\|_{L^\infty}. \tag{5.1}$$

Thus $\mathcal{I}$ is an interval of the form $[T_1, T_*[$, where $T_* := \sup_{T\in\mathcal{I}} T$. We have three possibilities:

(a) $T_* = T^*$,
(b) $T_* < T^*$ and $T_* \in \mathcal{I}$,
(c) $T_* < T^*$ and $T_* \notin \mathcal{I}$.

For case (a), we necessarily have $T^* = \infty$, since the Lipschitz norm of θ does not blow up from the definition of $\mathcal{I}$ which contradicts with (4.1). This is our goal.

For case (b), we observe that this is just the case treated in [1] or [14] showing that it is impossible. The proof only needs small modification, so we omit it either. We just point out in this case the smoothing effects will also be used, since we need the fact that $\|\nabla^2\theta(T_*)\|_{L^\infty}$ is finite.

Then our task is reduced to get rid of the case (c). We prove by contradiction. If the case (c) is satisfied, then by the time continuity of θ, we necessarily get

$$\sup_{x,y\in\mathbb{R}^2,\,x\neq y} \frac{|\theta(T_*,x)-\theta(T_*,y)|}{\omega_\lambda(|x-y|)} = 1.$$

We further have the following assertion (with its proof in Appendix A).

Lemma 5.1. *If $T_* < T^*$ is the first time that the strict modulus of continuity ω_λ is lost (i.e. case (c)), then there exists $x, y \in \mathbb{R}^2$, $x \neq y$ such that*

$$\theta(T_*, x) - \theta(T_*, y) = \omega_\lambda(\xi), \quad \text{with } \xi := |x - y|. \tag{5.2}$$

Moreover, let $\ell = \frac{x-y}{|x-y|}$ and $v \in \mathbb{S}^1$ be the unit vector perpendicular to ℓ, we have

$$\partial_\ell \theta(T_*, x) = \partial_\ell \theta(T_*, y) = \omega'_\lambda(\xi), \qquad \partial_v \theta(T_*, x) = \partial_v \theta(T_*, y) = 0, \tag{5.3}$$

where $\partial_\ell = \ell \cdot \nabla$ and $\partial_v = v \cdot \nabla$ are the directional derivatives along ℓ and v respectively.

We shall show that this scenario (5.2) cannot happen, more precisely, we shall prove

$$f'(T_*) < 0, \quad \text{with } f(t) := \theta(t, x) - \theta(t, y).$$

This is impossible since we necessarily have $f(t) \leqslant f(T_*)$, for all $0 \leqslant t \leqslant T_*$ from the definition of $\mathcal{I}$.

We see that the modified quasi-geostrophic equation (1.1) can be defined in the classical sense (from the smoothing effect), and thus

$$\begin{aligned} f'(T_*) &= -\big[(u \cdot \nabla\theta)(T_*, x) - (u \cdot \nabla\theta)(T_*, y)\big] + \nu\big[\big(-|D|^\alpha\theta\big)(T_*, x) - \big(-|D|^\alpha\theta\big)(T_*, y)\big] \\ &:= \mathcal{A}_1 + \mathcal{A}_2 \end{aligned}$$

with

$$u = |D|^{\alpha-1}\mathcal{R}^\perp\theta = \mathcal{R}^\perp_\alpha\theta := (-\mathcal{R}_{\alpha,2}\theta, \mathcal{R}_{\alpha,1}\theta)$$

where $\mathcal{R}_{\alpha,j}$ are the modified Riesz transforms introduced in Section 3.

For the first term, $\mathcal{A}_1$, from (5.2), we find that

$$\begin{aligned} \mathcal{A}_1 &= \big[\big(u(T_*, x) - u(T_*, y)\big) \cdot \ell\big]\omega'_\lambda(\xi) \\ &= \big[\big(u(T_*, x) - u(T_*, y)\big) \cdot \ell\big]\lambda\omega'(\lambda\xi). \end{aligned}$$

Lemma 3.2 gives us a rough estimate as follows

$$|\mathcal{A}_1| \leqslant \Omega_\lambda(\xi)\lambda\omega'(\lambda\xi) = \lambda^\alpha\big(\Omega\omega'\big)(\lambda\xi),$$

where $\Omega_\lambda(\xi)$ is defined from (3.2), i.e.,

$$\Omega_\lambda(\xi) = A\left(\int_0^\xi \frac{\omega_\lambda(\eta)}{\eta^\alpha}\,\mathrm{d}\eta + \xi\int_\xi^\infty \frac{\omega_\lambda(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta\right) = \lambda^{\alpha-1}\Omega(\lambda\xi). \tag{5.4}$$

For the second term, $\mathcal{A}_2$, from Lemma 3.3 we get

$$\mathcal{A}_2 \leqslant \nu\lambda^\alpha\Upsilon(\lambda\xi),$$

where

$$\Upsilon(\xi) := B\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta + B\int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta.$$

Thus we obtain

$$f'(T_*) \leqslant \lambda^{\alpha}\big(\Omega\omega' + \nu\Upsilon\big)(\lambda\xi). \tag{5.5}$$

Observe that when $\alpha \in]1,2[$ and ξ is a large number, the integral from 0 to ξ in the expression of Ω will produce much difficulty, roughly speaking, to ensure the right-hand side of (5.5) is negative, we need that the contribution from this part $\omega(\xi)\omega'(\xi) = c\frac{\omega(\xi)}{\xi^{\alpha}}$ with $c > 0$ a fixed small number, thus it seems impossible to construct an appropriate unbounded MOC. However, basically following an idea from [20], we can further develop the contribution of the dissipative term and use the additional dissipation to control this "bad" part of the nonlinearity, so that we can construct an appropriate unbounded MOC to guarantee $f'(T_*) < 0$. Meanwhile, when $\alpha = 1$ this method can also slightly improve the MOC constructed in [18]. Precisely,

Lemma 5.2. *Under the condition of Lemma* 5.1 *and for* $\alpha \in [1,2[$, *we have*

$$\mathcal{A}_2 \leqslant \nu\lambda^{\alpha}\Upsilon(\lambda\xi) + \nu\lambda^{\alpha}\Upsilon^{\perp}(\lambda\xi), \tag{5.6}$$

where $\Upsilon^{\perp} \leqslant 0$ *is a meaningful integral defined from* θ *and* ω. *Correspondingly, we can treat the drift term as follows*

$$\big|\big(u(T_*,x) - u(T_*,y)\big)\cdot\ell\big| \leqslant \lambda^{\alpha-1}\widetilde{\Omega}(\lambda\xi) \tag{5.7}$$

with

$$\widetilde{\Omega}(\xi) = A\left(-\xi\Upsilon^{\perp}(\xi) + \xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta + \xi^{-\alpha+1}\omega(\xi)\right), \tag{5.8}$$

where A *is an absolute constant that may depend on* α.

Remark 5.1. For the reader's convenience, we give the explicit formula of $\Upsilon^{\perp}$. Set $x_0 := (\frac{\xi}{2},0)$, $y_0 := (-\frac{\xi}{2},0)$, thus there exist a unique rotating transform ρ and a unique vector $a \in \mathbb{R}^2$ such that $x = \rho x_0 - a$ and $y = \rho y_0 - a$, where $x, y \in \mathbb{R}^2$ are two points stated in Lemma 5.1. Then we have

$$\Upsilon^{\perp}(\xi) \leqslant -C\int_{B^+_{r_0\xi}(x_0)} \frac{f_{\rho,a}(\eta,\mu)}{|x_0-(\eta,\mu)|^{2+\alpha}}\,\mathrm{d}\eta\,\mathrm{d}\mu, \tag{5.9}$$

where $B^+_{r_0\xi}(x_0) := \{(\eta,\mu)\in\mathbb{R}^2\colon\ |(\eta,\mu)-x_0| \leqslant r_0\xi,\ \mu > 0\}$,

$$f_{\rho,a}(\eta,\mu) := 2\omega(2\eta) - \tilde{\theta}(T_*,\eta,\mu) + \tilde{\theta}(T_*,-\eta,\mu) - \tilde{\theta}(T_*,\eta,-\mu) + \tilde{\theta}(T_*,-\eta,-\mu) \geqslant 0,$$

and $\tilde{\theta}(t,\eta,\mu) := \theta(t,\rho\cdot(\eta,\mu)-a)$. Notice that in the scenario described in Lemma 5.1, the expression of $\Upsilon^{\perp}$ is meaningful for all $\alpha \in [1,2[$.

Proof of Lemma 5.2. This is a direct consequence of Lemmas 5.5 and 5.6 in [22] when $\alpha \in]1,2[$, and they can simply extend to the case $\alpha = 1$. □

Hence when $\alpha \in [1,2[$, based on Lemma 5.2, we also get

$$f'(T_*) \leqslant \lambda^\alpha \big(\widetilde{\Omega}\omega' + \nu\Upsilon + \nu\Upsilon^\perp\big)(\lambda\xi). \tag{5.10}$$

Next we shall construct our special modulus of continuity in the spirit of [18]. Let $0 < \gamma < \delta < 1$ be two small positive numbers chosen later, and define the continuous function ω as follows[1]

$$\text{MOC} \begin{cases} \omega(\xi) = \xi - \xi^{1+\frac{\alpha}{2}} & \text{if } 0 \leqslant \xi \leqslant \delta, \\ \omega'(\xi) = \frac{\gamma}{4\xi} & \text{if } \xi > \delta, \end{cases} \tag{5.11}$$

equivalently,

$$\omega(\xi) = \begin{cases} \xi - \xi^{1+\frac{\alpha}{2}} & \text{if } 0 \leqslant \xi \leqslant \delta, \\ \delta - \delta^{1+\frac{\alpha}{2}} + \frac{\gamma}{4}\log\frac{\xi}{\delta} & \text{if } \xi > \delta. \end{cases} \tag{5.12}$$

Note that, for small δ, the left derivative of ω at δ is about 1, while the right derivative equals $\frac{\gamma}{4\delta} < \frac{1}{4}$. So ω is concave if δ is small enough. Clearly, $\omega(0) = 0$, $\omega'(0) = 1$ and $\lim_{\eta\to 0+}\omega''(\eta) = -\infty$, and ω is unbounded (it has the logarithmic growth at infinity).

Then our target is to show that, for this MOC ω, when $\alpha \in]0,1[$

$$\Omega(\xi)\omega'(\xi) + \nu\Upsilon(\xi) < 0 \quad \text{for all } \xi > 0, \tag{5.13}$$

and when $\alpha \in [1,2[$

$$\widetilde{\Omega}(\xi)\omega'(\xi) + \nu\Upsilon(\xi) + \nu\Upsilon^\perp(\xi) < 0 \quad \text{for all } \xi > 0. \tag{5.14}$$

In the following we shall carefully check these two formulae.

Case I: When $\alpha \in]0,1[$.

Precisely, we shall check the following inequality

$$A\left[\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha}\,\mathrm{d}\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta\right]\omega'(\xi) + \nu B\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta$$
$$+ \nu B\int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta < 0 \quad \text{for all } \xi > 0.$$

We further divide it into two cases.

[1] Note that when $\alpha \in]0,1[$, we once adopted an equivalent expression that $\omega'(\xi) = \frac{\gamma}{4(\xi+\xi^\alpha)}$ if $\xi > \delta$ in the first version of this paper, and the formula like (5.11) (only $\alpha \in]0,1[$) also occurs in [19].

Case I.1: $\alpha \in]0,1[$ and $0 < \xi \leqslant \delta$.
Since $\frac{\omega(\eta)}{\eta} \leqslant \omega'(0) = 1$ for all $\eta > 0$ and $\eta \leqslant \eta^\alpha$ for $\eta \leqslant \delta < 1$, we have

$$\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha}\,d\eta \leqslant \int_0^\xi \frac{\omega(\eta)}{\eta}\,d\eta \leqslant \xi,$$

and

$$\int_\xi^\delta \frac{\omega(\eta)}{\eta^{1+\alpha}}\,d\eta \leqslant \int_\xi^\delta \frac{1}{\eta^\alpha}\,d\eta = \frac{1}{1-\alpha}\left(\delta^{1-\alpha} - \xi^{1-\alpha}\right) \leqslant \frac{1}{1-\alpha}.$$

Furthermore,

$$\int_\delta^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}}\,d\eta = \frac{1}{\alpha}\frac{\omega(\delta)}{\delta^\alpha} + \frac{1}{\alpha}\int_\delta^\infty \frac{\gamma}{4\eta^{1+\alpha}}\,d\eta \leqslant \frac{1}{\alpha} + \frac{1}{\alpha^2}\frac{\gamma}{\delta^\alpha} \leqslant \frac{2}{\alpha},$$

if $\gamma < \alpha\delta$. Clearly $\omega'(\xi) \leqslant \omega'(0) = 1$, so we get that the positive part is bounded by $A\xi\frac{2}{\alpha(1-\alpha)}$.
For the negative part, we have

$$\begin{aligned}\nu B\int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\,d\eta &\leqslant \nu B\int_0^{\frac{\xi}{2}} \frac{\omega''(\xi)2\eta^2}{\eta^{1+\alpha}}\,d\eta \\ &= -\nu B\frac{\alpha(2+\alpha)}{2^{1-\alpha}(2-\alpha)}\xi^{1-\frac{\alpha}{2}} \leqslant -\frac{\alpha}{2}\nu B\xi^{1-\frac{\alpha}{2}}.\end{aligned}$$

But, clearly $\xi(A\frac{2}{\alpha(1-\alpha)} - \frac{\alpha}{2}\nu B\xi^{-\frac{\alpha}{2}}) < 0$ on $]0,\delta]$ when δ is small enough.

Case I.2: $\alpha \in]0,1[$ and $\xi \geqslant \delta$.
For $\eta \leqslant \delta < 1$ we still use $\eta^\alpha \geqslant \eta$ and for $\delta \leqslant \eta \leqslant \xi$ we use $\omega(\eta) \leqslant \omega(\xi)$, then

$$\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha}\,d\eta \leqslant \delta + \frac{\omega(\xi)}{1-\alpha}\left(\xi^{1-\alpha} - \delta^{1-\alpha}\right) \leqslant \omega(\xi)\left(\frac{2}{\alpha} + \frac{\xi^{1-\alpha}}{1-\alpha}\right),$$

where the last inequality is due to $\frac{\alpha}{2}\delta < \omega(\delta) \leqslant \omega(\xi)$ if δ is small enough (i.e. $\delta < (1-\frac{\alpha}{2})^{2/\alpha}$). Also

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}}\,d\eta = \frac{1}{\alpha}\frac{\omega(\xi)}{\xi^\alpha} + \frac{1}{\alpha}\int_\xi^\infty \frac{\gamma}{4\eta^{1+\alpha}}\,d\eta \leqslant \frac{1}{\alpha}\frac{\omega(\xi)}{\xi^\alpha} + \frac{1}{\alpha^2}\frac{\gamma}{2}\frac{1}{\xi^\alpha} \leqslant \frac{2}{\alpha}\frac{\omega(\xi)}{\xi^\alpha}$$

if $\gamma < \alpha^2\delta$ (thus $\gamma/2 \leqslant \alpha\omega(\xi)$) and δ is small enough. Thus the positive term is bounded from above by

$$A\omega(\xi)\left(\frac{2}{\alpha} + \left(\frac{1}{1-\alpha} + \frac{2}{\alpha}\right)\xi^{1-\alpha}\right)\omega'(\xi) \leqslant A\frac{\omega(\xi)}{\xi^\alpha}\frac{2}{\alpha(1-\alpha)}\left(\xi + \xi^\alpha\right)\frac{\gamma}{4\xi} \leqslant \frac{A\delta^{\alpha-1}\gamma}{\alpha(1-\alpha)}\frac{\omega(\xi)}{\xi^\alpha}.$$

For the negative part, we first observe that for $\xi \geqslant \delta$,

$$\omega(2\xi) = \omega(\xi) + \int_{\xi}^{2\xi} \omega'(\eta)\,\mathrm{d}\eta = \omega(\xi) + \frac{\log 2}{2}\gamma \leqslant \frac{3}{2}\omega(\xi),$$

under the same assumptions on δ and γ as above. Also, taking advantage of the concavity we obtain $\omega(2\eta+\xi) - \omega(2\eta-\xi) \leqslant \omega(2\xi)$ for all $\eta \geqslant \frac{\xi}{2}$. Therefore

$$\nu B \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta \leqslant -\nu B \frac{\omega(\xi)}{2} \int_{\frac{\xi}{2}}^{\infty} \frac{1}{\eta^{1+\alpha}}\,\mathrm{d}\eta = -\nu B \frac{2^{\alpha}}{2\alpha} \frac{\omega(\xi)}{\xi^{\alpha}}.$$

But $\frac{\omega(\xi)}{\xi^{\alpha}}(\frac{A\delta^{\alpha-1}\gamma}{\alpha(1-\alpha)} - \nu B \frac{2^{\alpha}}{2\alpha}) < 0$ if γ is small enough (i.e. $\gamma < \min\{\alpha^2\delta, \frac{\nu(1-\alpha)B2^{\alpha}}{2A}\delta^{1-\alpha}\}$).

Case II: When $\alpha \in [1, 2[$.

Precisely, we shall check the following inequality

$$A\left[-\xi \Upsilon^{\perp}(\xi) + \xi^{-\alpha+1}\omega(\xi) + \xi \int_{\xi}^{\infty} \frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta\right]\omega'(\xi) + \nu B \int_{0}^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta$$

$$+ \nu B \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(2\eta+\xi) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta + \nu \Upsilon^{\perp}(\xi) < 0 \quad \text{for all } \xi > 0.$$

We also further divide it into two cases.

Case II.1: $\alpha \in [1, 2[$ and $0 < \xi \leqslant \delta$.

Since $\frac{\omega(\eta)}{\eta} \leqslant \omega'(0) = 1$ for all $\eta > 0$ and $-\Upsilon^{\perp}(\xi) \geqslant 0$, we have $-\xi\Upsilon^{\perp}(\xi) \leqslant -\delta\Upsilon^{\perp}(\xi)$ and $\xi^{-\alpha+1}\omega(\xi) \leqslant \xi^{2-\alpha}$ and

$$\int_{\xi}^{\delta} \frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta \leqslant \int_{\xi}^{\delta} \frac{1}{\eta^{\alpha}}\,\mathrm{d}\eta \leqslant \begin{cases} \frac{1}{\alpha-1}\xi^{1-\alpha}, & \alpha \in]1, 2[, \\ \log(\delta/\xi), & \alpha = 1. \end{cases}$$

Further, integration by parts leads to

$$\begin{aligned}\int_{\delta}^{\infty} \frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta &= \frac{1}{\alpha}\frac{\omega(\delta)}{\delta^{\alpha}} + \frac{1}{\alpha}\int_{\delta}^{\infty} \frac{\gamma}{4\eta^{\alpha+1}}\,\mathrm{d}\eta \\ &\leqslant \frac{1}{\alpha}\frac{1}{\delta^{\alpha-1}} + \frac{\gamma}{4\alpha^2}\frac{1}{\delta^{\alpha}} \leqslant 2\frac{1}{\delta^{\alpha-1}} \leqslant 2\xi^{1-\alpha}.\end{aligned}$$

Clearly $\omega'(\xi) \leqslant \omega'(0) = 1$, so we get that the positive part is bounded by

$$\begin{cases} A(-\delta\Upsilon^{\perp}(\xi) + \xi^{2-\alpha}\frac{4}{\alpha-1}), & \alpha \in]1, 2[, \\ A(-\delta\Upsilon^{\perp}(\xi) + \xi(3 + \log\frac{\delta}{\xi})), & \alpha = 1. \end{cases}$$

For the negative part, we have

$$\nu B\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta\leqslant\nu B\int_0^{\frac{\xi}{2}}\frac{\omega''(\xi)2\eta^2}{\eta^{1+\alpha}}\,\mathrm{d}\eta$$
$$=-\nu B\frac{\alpha(2+\alpha)}{2^{1-\alpha}(2-\alpha)}\xi^{1-\frac{\alpha}{2}}\leqslant-3\nu B\xi^{1-\frac{\alpha}{2}}.$$

But, clearly if δ is chosen small enough, we find that for every $\xi\in\,]0,\delta]$

$$\begin{cases}(-A\delta+\nu)\Upsilon^{\perp}(\xi)+\xi^{2-\alpha}\left(A\dfrac{4}{\alpha-1}-3\nu B\xi^{\frac{\alpha}{2}-1}\right)<0, & \alpha\in\,]1,2[,\\ (-A\delta+\nu)\Upsilon^{\perp}(\xi)+\xi\left(3A+A\log\dfrac{\delta}{\xi}-3\nu B\xi^{-\frac{1}{2}}\right)<0, & \alpha=1.\end{cases}$$

Case II.2: $\alpha\in[1,2[$ and $\xi\geqslant\delta$.
For the positive part we have

$$\int_\xi^\infty\frac{\omega(\eta)}{\eta^{1+\alpha}}\,\mathrm{d}\eta=\frac{1}{\alpha}\frac{\omega(\xi)}{\xi^\alpha}+\frac{1}{\alpha}\int_\xi^\infty\frac{\gamma}{4\eta^{\alpha+1}}\,\mathrm{d}\eta$$
$$\leqslant\frac{1}{\alpha}\frac{\omega(\xi)}{\xi^\alpha}+\frac{\gamma}{4\alpha^2}\frac{1}{\xi^\alpha}\leqslant2\frac{\omega(\xi)}{\xi^\alpha},$$

where we have used the simple fact that $\gamma\leqslant\frac{\delta}{2}\leqslant\omega(\delta)\leqslant\omega(\xi)$. Thus the positive term is bounded from above by

$$A\big(-\xi\Upsilon^{\perp}(\xi)+3\omega(\xi)\xi^{1-\alpha}\big)\omega'(\xi)=A\big(-\xi\Upsilon^{\perp}(\xi)+3\omega(\xi)\xi^{1-\alpha}\big)\frac{\gamma}{4\xi}\leqslant-A\gamma\Upsilon^{\perp}(\xi)+A\gamma\frac{\omega(\xi)}{\xi^\alpha}.$$

For the negative part, in a similar way as treating the corresponding part in Case I.2, we have

$$\nu B\int_{\frac{\xi}{2}}^\infty\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+\alpha}}\,\mathrm{d}\eta\leqslant-\nu B\frac{\omega(\xi)}{2}\int_{\frac{\xi}{2}}^\infty\frac{1}{\eta^{1+\alpha}}\,\mathrm{d}\eta\leqslant-\frac{\nu B}{2}\frac{\omega(\xi)}{\xi^\alpha}.$$

But, clearly $(-A\gamma+\nu)\Upsilon^{\perp}(\xi)+\frac{\omega(\xi)}{\xi^\alpha}(A\gamma-\frac{\nu B}{2})<0$ if γ is small enough.
Therefore both Cases I and II yield $f'(T_*)<0$.
Finally, only case (a) occurs and we obtain $T^*=\infty$. Moreover

$$\big\|\nabla\theta(t)\big\|_{L^\infty}<\lambda,\quad\forall t\in[0,\infty[,$$

where the value of $\lambda\sim C\|\nabla\theta_0\|_{L^\infty}e^{C\|\theta(T_1)\|_{L^\infty}}$ is given by (5.1).

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Appendix A

A.1. The formula for $\mathcal{R}_{\alpha,j} f$

Proof of Proposition 3.1. By applying the Fourier transformation to the operator $\mathcal{R}_{\alpha,j}$ ($\alpha \in]0,2[$), we know that the symbol of $\mathcal{R}_{\alpha,j}$ is $-i\zeta_j/|\zeta|^{2-\alpha}$. Now we want to know the explicit formula of $\mathcal{F}^{-1}(-i\zeta_j/|\zeta|^{2-\alpha})$. From the equality in the distributional sense

$$\frac{\partial}{\partial x_j}|x|^{-(n+\alpha-2)} = -(n+\alpha-2)\text{p.v.}\frac{x_j}{|x|^{n+\alpha}},$$

and the known formula that for every $0 < a < n$ (cf. [16])

$$\big(|x|^{-a}\big)^{\wedge}(\zeta) = \frac{2^{n-a}\pi^{n/2}\Gamma(\frac{n-a}{2})}{\Gamma(\frac{a}{2})}|\zeta|^{-n+a},$$

we directly have

$$\begin{aligned}\left(\text{p.v.}\frac{x_j}{|x|^{n+\alpha}}\right)^{\wedge}(\zeta) &= -\frac{1}{n+\alpha-2}\big(\partial_{x_j}|x|^{-n-\alpha+2}\big)^{\wedge}(\zeta)\\ &= -\frac{i\zeta_j}{n+\alpha-2}\big(|x|^{-n-\alpha+2}\big)^{\wedge}(\zeta)\\ &= -\frac{i\zeta_j}{n+\alpha-2}\frac{2^{2-\alpha}\pi^{n/2}\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{n+\alpha-2}{2})}|\zeta|^{\alpha-2}\\ &= -i\frac{2^{1-\alpha}\pi^{n/2}\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}\cdot\frac{\zeta_j}{|\zeta|^{2-\alpha}}. \qquad \square\end{aligned}$$

A.2. A commutator estimate

The key to the proof of the uniform estimate is the following commutator estimate:

Lemma A.1. *Let v be a divergence-free vector field over $\mathbb{R}^n$. For every $q \in \mathbb{N}$, denote*

$$F_q(v,f) := S_{q+1}v \cdot \nabla\Delta_q f - \Delta_q(v \cdot \nabla f).$$

Then for every $\beta \in]0,1[$, there exists a positive constant C such that

$$2^{-q\beta}\big\|F_q(v,f)\big\|_{L^2} \leqslant C\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\bigg(\sum_{q'\leqslant q+4}2^{q'-q}\|\Delta_{q'}f\|_{L^2} + \sum_{q'\geqslant q-4}2^{(q-q')(1-\beta)}\|\Delta_{q'}f\|_{L^2}\bigg), \tag{A.1}$$

especially, in the case $n = 2$ and $v = |D|^{\alpha-1}\mathcal{R}^{\perp}f$ ($\alpha \in]0,2[$), we further have for every $\beta \in]\max\{0,\alpha-1\},1[$ and every $q \in \mathbb{N}$

$$2^{-q\beta}\big\|F_q(v,f)\big\|_{L^2} \leqslant C\|f\|_{\dot B^{\alpha-\beta}_{\infty,\infty}}\bigg(\sum_{q'\geqslant q-4}2^{(q-q')(1-\beta)}\|\Delta_{q'}f\|_{L^2} + \sum_{|q'-q|\leqslant 4}\|\Delta_{q'}f\|_{L^2}\bigg). \tag{A.2}$$

Moreover, when $\beta = 0$, $\alpha \in]0,1[$, (A.1) and (A.2) hold if we replace $\|v\|_{\dot B^1_{\infty,\infty}}$ by $\|\nabla v\|_{L^\infty}$.

Proof. Using Bony decomposition, we decompose $F_q(v,f)$ into $\sum_{i=1}^6 F_q^i(v,f)$ (cf. [13]), where

$$F_q^1(v,f)=(S_{q+1}v-v)\cdot\nabla\Delta_q f,\qquad F_q^2(v,f)=[\Delta_{-1}v,\Delta_q]\cdot\nabla f,$$

$$F_q^3(v,f)=\sum_{q'\in\mathbb{N}}[S_{q'-1}\widetilde{v},\Delta_q]\cdot\nabla\Delta_{q'}f,\qquad F_q^4(v,f)=\sum_{q'\geqslant -1}\Delta_{q'}\widetilde{v}\cdot\nabla\Delta_q S_{q'+2}f,$$

$$F_q^5(v,f)=-\sum_{q'\in\mathbb{N}}\Delta_q(\Delta_{q'}\widetilde{v}\cdot\nabla S_{q'-1}f),\qquad F_q^6(v,f)=-\sum_{q'\geqslant -1}\operatorname{div}\Delta_q\Big(\Delta_{q'}\widetilde{v}\sum_{i\in\{\pm1,0\}}\Delta_{q'+i}f\Big),$$

where $[A,B]:=AB-BA$ denotes the commutator operator and $\widetilde{v}:=v-\Delta_{-1}v$ denotes the high frequency part of v.

For F_q^1, from the divergence-free property of v we directly obtain that when $1-\beta>0$

$$\begin{aligned}2^{-q\beta}\big\|F_q^1(v,f)\big\|_{L^2}&\lesssim\sum_{q'\geqslant q+1}2^{(1-\beta)(q-q')}2^{q'(1-\beta)}\|\Delta_{q'}v\|_{L^\infty}\|\Delta_q f\|_{L^2}\\&\lesssim\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\|\Delta_q f\|_{L^2}.\end{aligned}$$

For F_q^2, since $F_2^q(v,f)=\sum_{|q'-q|\leqslant1}[\Delta_{-1}v,\Delta_q]\cdot\nabla\Delta_{q'}f$, then from the expression formula of Δ_q and mean value theorem, we get that when $\beta>0$

$$\begin{aligned}2^{-q\beta}\big\|F_q^2(v,f)\big\|_{L^2}&\lesssim 2^{-q\beta}2^{-q}\|\nabla\Delta_{-1}v\|_{L^\infty}\sum_{|q'-q|\leqslant1}2^{q'}\|\Delta_{q'}f\|_{L^2}\\&\lesssim\sum_{-\infty\leqslant j\leqslant -1}2^{j\beta}\big\||D|^{1-\beta}\dot\Delta_j v\big\|_{L^\infty}\sum_{|q'-q|\leqslant1}\|\Delta_{q'}f\|_{L^2}\\&\lesssim\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\sum_{|q'-q|\leqslant1}\|\Delta_{q'}f\|_{L^2}.\end{aligned}$$

For F_q^3, similarly as estimating F_q^2, we infer

$$\begin{aligned}2^{-q\beta}\big\|F_q^3(v,f)\big\|_{L^2}&\lesssim 2^{-q\beta}\sum_{|q'-q|\leqslant4}2^{-q}\|\nabla S_{q'-1}\widetilde{v}\|_{L^\infty}2^{q'}\|\Delta_{q'}f\|_{L^2}\\&\lesssim\sum_{|q'-q|\leqslant4}\sum_{q''\leqslant q'-2}2^{(q''-q')\beta}\big\||D|^{1-\beta}\Delta_{q''}\widetilde{v}\big\|_{L^\infty}\|\Delta_{q'}f\|_{L^2}\\&\lesssim\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\sum_{|q'-q|\leqslant4}\|\Delta_{q'}f\|_{L^2}.\end{aligned}$$

For F_q^4 and F_q^5, from the spectral property and the fact $2^{q'(1-\beta)}\|\Delta_{q'}\widetilde{v}\|_{L^\infty}\approx\|\Delta_{q'}|D|^{1-\beta}\widetilde{v}\|_{L^\infty}$, we have

$$2^{-q\beta}\big\|F_q^4(v,f)\big\|_{L^2}\lesssim\sum_{q'\geqslant q-2}2^{(q-q')(1-\beta)}2^{q'(1-\beta)}\|\Delta_{q'}\widetilde{v}\|_{L^\infty}\|\Delta_q f\|_{L^2}\lesssim\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\|\Delta_q f\|_{L^2},$$

$$\begin{aligned}2^{-q\beta}\big\|F_q^5(v,f)\big\|_{L^2}&\lesssim 2^{-q\beta}\sum_{|q'-q|\leqslant4}2^{q'}\|\Delta_{q'}\widetilde{v}\|_{L^\infty}\sum_{q''\leqslant q'-2}2^{q''-q'}\|\Delta_{q''}f\|_{L^2}\\&\lesssim\|v\|_{\dot B^{1-\beta}_{\infty,\infty}}\sum_{q''\leqslant q+2}2^{q''-q}\|\Delta_{q''}f\|_{L^2}.\end{aligned}$$

Besides, for F_q^5 when $v = |D|^{\alpha-1}\mathcal{R}^\perp f$, we alternatively have the following improvement that when $\beta > \alpha - 1$

$$\begin{aligned} 2^{-q\beta} \big\| F_q^5(v,f) \big\|_{L^2} &\leqslant 2^{-q\beta} \sum_{|q'-q|\leqslant 4} \big\| \Delta_{q'}(Id - \Delta_{-1})|D|^{\alpha-1}\mathcal{R}^\perp f \big\|_{L^2} \|\nabla S_{q'-1} f\|_{L^\infty} \\ &\lesssim \sum_{|q'-q|\leqslant 4} \|\Delta_{q'} f\|_{L^2} \sum_{-\infty \leqslant q'' \leqslant q'-2} 2^{(\alpha-1-\beta)(q'-q'')} \big\| |D|^{\alpha-\beta} \dot{\Delta}_{q''} f \big\|_{L^\infty} \\ &\lesssim \|f\|_{\dot{B}^{\alpha-\beta}_{\infty,\infty}} \sum_{|q'-q|\leqslant 4} \|\Delta_{q'} f\|_{L^2}. \end{aligned}$$

Finally, for F_q^6 we easily have

$$\begin{aligned} 2^{-q\beta} \big\| F_q^6(v,f) \big\|_{L^2} &\lesssim \sum_{q'\geqslant q-3} 2^{(q-q')(1-\beta)} 2^{q'(1-\beta)} \|\Delta_{q'} \widetilde{v}\|_{L^\infty} \sum_{i\in\{\pm 1,0\}} \|\Delta_{q'+i} f\|_{L^2} \\ &\lesssim \|v\|_{\dot{B}^{1-\beta}_{\infty,\infty}} \sum_{q'\geqslant q-4} 2^{(q-q')(1-\beta)} \|\Delta_{q'} f\|_{L^2}. \end{aligned}$$

Combining the above estimates appropriately yields the inequalities (A.1) and (A.2). □

A.3. Proof of Lemma 5.1

Proof of Lemma 5.1. Set $C' := \omega^{-1}(3\|\theta_0\|_{L^\infty})$, then from the maximum principle (2.1), we get

$$\lambda|x-y| \geqslant C' \quad \Rightarrow \quad \big|\theta(T_*,x) - \theta(T_*,y)\big| < \frac{2}{3}\omega_\lambda\big(|x-y|\big). \tag{A.3}$$

Since $\nabla\theta(t) \in \mathcal{C}([T_1,T^*[, H^m(\mathbb{R}^2))$, then for every $\epsilon > 0$, there exists $R > 0$ such that

$$\big\|\nabla\theta(T_*)\big\|_{L^\infty(\mathbb{R}^2\setminus B_R)} \leqslant C_0 \big\|\nabla\theta(T_*)\big\|_{H^m(\mathbb{R}^2\setminus B_R)} \leqslant \epsilon,$$

where B_R is a ball centered at the origin with the radius R and $\mathbb{R}^2 \setminus B_R$ is its complement. Thus for every x, y $(x \neq y)$ satisfying that $\lambda|x-y| \leqslant C'$ and x or y belongs to $\mathbb{R}^2 \setminus B_{R+C'/\lambda}$, we get

$$\big|\theta(T_*,x) - \theta(T_*,y)\big| \leqslant \big\|\nabla\theta(T_*)\big\|_{L^\infty(\mathbb{R}^2\setminus B_R)} |x-y| \leqslant \epsilon|x-y|.$$

Taking advantage of the following inequality from the concavity of ω

$$\frac{\omega(C')}{C'}\lambda|x-y| \leqslant \omega_\lambda\big(|x-y|\big),$$

we can take ϵ small enough such that $\epsilon < \frac{1}{2}\frac{\omega(C')}{C'}\lambda$ to obtain

$$\lambda|x-y| \leqslant C', \qquad x \text{ or } y \in \mathbb{R}^2 \setminus B_{R+\frac{C'}{\lambda}} \quad \Rightarrow \quad \big|\theta(T_*,x) - \theta(T_*,y)\big| < \frac{1}{2}\omega_\lambda\big(|x-y|\big). \tag{A.4}$$

Now it remains to consider the case when $x, y \in B_{R+\frac{C'}{\lambda}}$. From the smoothing effect, we know $\|\nabla^2\theta(T_*)\|_{L^\infty} < \infty$, thus we have (cf. [18])

$$\left\|\nabla\theta(T_*)\right\|_{L^\infty(B_{R+\frac{C'}{\lambda}})} < \lambda\omega'(0).$$

Let $\delta' \ll 1$ small enough, then we see

$$\left\|\theta(T_*)\right\|_{L^\infty(B_{R+\frac{C'}{\lambda}})} < \lambda\left(1-\delta'\right)\frac{\omega(\delta')}{\delta'}.$$

Thus for every x, y $(x \neq y)$ satisfying that $\lambda|x-y| \leqslant \delta'$ and both x, y belongs to $B_{R+C'/\lambda}$, we have

$$\begin{aligned}\left|\theta(T_*,x)-\theta(T_*,y)\right| &\leqslant \left\|\nabla\theta(T_*)\right\|_{L^\infty(B_{R+\frac{C'}{\lambda}})}|x-y| \\ &< \left(1-\delta'\right)\frac{\omega(\delta')}{\delta'}\lambda|x-y| \leqslant \left(1-\delta'\right)\omega_\lambda\left(|x-y|\right).\end{aligned} \tag{A.5}$$

We set

$$\Omega := \left\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2\colon \max\left\{|x|,|y|\right\} \leqslant R+\frac{C'}{\lambda},\ |x-y| \geqslant \frac{\delta'}{\lambda}\right\},$$

then from the above results we necessarily have

$$1 = \sup_{x\neq y}\frac{|\theta(T_*,x)-\theta(T_*,y)|}{\omega_\lambda(|x-y|)} = \sup_{(x,y)\in\Omega}\frac{|\theta(T_*,x)-\theta(T_*,y)|}{\omega_\lambda(|x-y|)}.$$

Thus the conclusion follows from the compactness of Ω.

For (5.3), it is from a direct computation under the scenario (5.2) (cf. Proposition 2.4 in [20]). □

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