Calc. Var. (2018) 57:153
https://doi.org/10.1007/s00526-018-1438-3

Calculus of Variations

On the Hölder regularity of the weak solution to a drift–diffusion system with pressure

Qianyun Miao[1] · Liutang Xue[2]

Received: 17 July 2017 / Accepted: 7 September 2018

Abstract

In this paper we address the regularity issue of weak solution for the following linear drift–diffusion system with pressure

$$\partial_t u + b \cdot \nabla u - \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad u|_{t=0}(x) = u_0(x),$$

where $x \in \mathbb{R}^n$ and b is a given divergence-free vector field. Under some assumptions of the drift field b in the critical sense, and for the initial data $u_0 \in (L^2(\mathbb{R}^n))^n$, we prove that there exists a weak solution $u(t)$ to this system such that $u(t)$ for any time $t > 0$ is α-Hölder continuous with $\alpha \in (0, 1)$. The proof of the Hölder regularity result utilizes a maximum-principle type method to improve the regularity of weak solution step by step.

Mathematics Subject Classification 76D03 · 35Q35 · 35Q30 · 35K15

1 Introduction

We consider the Cauchy problem of the following linear drift–diffusion system with pressure

$$\begin{cases} \partial_t u + b \cdot \nabla u - \Delta u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n, n \geq 2, u(x, t) = \big(u_1(x, t), u_2(x, t), \ldots, u_n(x, t)\big)$ is the unknown vector field of $\mathbb{R}^n$, and the drift velocity $b(x, t) = (b_1(x, t), b_2(x, t), \ldots, b_n(x, t))$ is a given divergence-free vector field (i.e. $\operatorname{div} b = 0$). The pressure field p can be derived from u and b by the

Communicated by L. Caffarelli.

✉ Liutang Xue
xuelt@bnu.edu.cn

Qianyun Miao
qianyunm@math.pku.edu.cn

1 School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

2 Laboratory of Mathematics and Complex Systems (MOE), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

Published online: 26 September 2018

expression formula

$$p = (-\Delta)^{-1}\mathrm{div}\,\big(b\cdot\nabla u\big). \tag{1.2}$$

The drift–diffusion system (1.1) corresponds to the Stokes system with a drift term, which is a coupled system instead of a scalar equation.

The drift–diffusion system with pressure (1.1) shares a fundamental scaling property: under the following scaling transformations that for each $s \in \mathbb{R}$ and for every $\lambda > 0$,

$$u(x,t) \mapsto u^{(\lambda)}(x,t) := \lambda^s u(\lambda x, \lambda^2 t), \tag{1.3}$$

$$b(x,t) \mapsto b^{(\lambda)}(x,t) := \lambda b(\lambda x, \lambda^2 t), \tag{1.4}$$

$$p(x,t) \mapsto p^{(\lambda)}(x,t) := \lambda^{s+1} p(\lambda x, \lambda^2 t), \tag{1.5}$$

the drift–diffusion system with pressure (1.1) remains invariant, that is,

$$\partial_t u^{(\lambda)} + b^{(\lambda)}\cdot\nabla u^{(\lambda)} - \Delta u^{(\lambda)} + \nabla p^{(\lambda)} = 0, \quad \nabla\cdot u^{(\lambda)} = 0.$$

If the system (1.1) does not have the nonlocal pressure term, it is essentially not a coupled system and each component satisfies the same equation; thus we may assume $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is a scalar field, and it reduces to the classical drift–diffusion equation

$$\partial_t u + b\cdot\nabla u - \Delta u = 0, \quad u|_{t=0} = u_0, \tag{1.6}$$

with $x \in \mathbb{R}^n$, $n \geq 2$, and b a given divergence-free vector field. The Hölder regularity issue of the weak solution for (1.6) with a given vector field b has been a classical problem, which is as follows: *under which conditions of* b, *the weak solution* u *is Hölder continuous for any* $t > 0$? The required regularity on b is usually expressed as $b \in (\mathcal{X})^n = \mathcal{X}\times\cdots\mathcal{X}$, with $\mathcal{X}$ some suitable space–time function space. Noting that the scaling transformations corresponding to the system (1.6) are (1.3), (1.4), we here call that b is under a critical assumption if $b^{(\lambda)}$ defined by (1.4) is invariant under the assumed norm $\|\cdot\|_{\mathcal{X}}$ for all $\lambda > 0$, that is, $\|b^{(\lambda)}\|_{\mathcal{X}} = \|b\|_{\mathcal{X}}$ (e.g. $\mathcal{X} = L^p([0,\infty); L^q(\mathbb{R}^n))$ with $\frac{2}{p} + \frac{n}{q} = 1$, $p \in [2,\infty]$); we call that the assumption of b is subcritical (resp. supercritical) if $b^{(\lambda)}$ has smaller (resp. larger) norm than b for all $\lambda > 0$ small enough.

So far, by using current methods it seems impossible to obtain a Hölder regularity result for (1.6) with a supercritical assumption on b, since the drift part of the equation would be much stronger than the diffusion part at small scales (corresponding to small λ). If b satisfies the subcritical assumption, the drift part would be relatively negligible compared with diffusion at small scales, and one generally can treat the Eq. (1.6) as a perturbation of the linear heat equation, so that some regularity results can be achieved. For instance, if $b \in (L^p_t L^q_x)^n$ with $\frac{2}{p} + \frac{n}{q} < 1$, $p \in (2,\infty]$ ([1]) or b satisfies some Kato's class condition (e.g. [30]), on can get the desired Hölder regularity result. If b satisfies the critical assumption, the regularity problem is more subtle. Since the drift part would not be negligible at any scale, in order to get some regularity results depending on the scaling-invariant norms of b, one has to use the non-perturbative methods. As far as we know, the variations of De Giorgi–Nash–Moser theory ([7,22,23]) seem to be the only workable ways to derive the Hölder regularity result, which states that weak solutions of (1.6) are α-Hölder continuous for all $t > 0$ and for small $\alpha \in (0,1)$. There are some noticeable works in this direction concerning b in different scaling-invariant functional spaces: one can refer to [17, Chapter 3] for the condition that $b \in (L^p_t L^q_x)^n$ with $\frac{2}{p}+\frac{n}{q} = 1$, $p \in [2,\infty]$; and for b belonging to a space–time Morrey space, one can see [24]; and for b belonging to $(L^\infty_t W^{-1,\infty}_x)^n$, or more generally for b belonging to $(L^\infty_t BMO^{-1}_x)^n$, one can respectively see [25] and [11,27]. We also refer to [31] and [26] for the Hölder regularity result with the divergence-free velocity field b satisfying a form of boundedness conditions.

For the drift–diffusion system with pressure (1.1), if the drift velocity field b satisfies the subcritical condition, it seems that the perturbative methods can also be applied to get the Hölder regularity result, and one can see [31] for the regularity result under the condition that b belongs to the Kato's class. But if b satisfies the critical assumption, since it is hard in extending the De Giorgi–Nash–Moser type methods to the coupled systems, there is not much regularity results for the weak solution of (1.1). We here only mention a regularity result of Hölder continuity: Silvestre and Vicol [29] proved that if $u_0 \in (C^\alpha(\mathbb{R}^n))^n$, and $b \in \big(L^p([0,T]; \mathcal{M}^{\frac{2-p}{p}})\big)^n$ with $p \in [1,\infty)$, $T > 0$, the L^1-based Morrey–Campanato space $\mathcal{M}^\beta$ ($\beta \in [-1,1]$) defined by

$$\mathcal{M}^\beta := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \Big| \|f\|_{\mathcal{M}^\beta} = \sup_{x\in\mathbb{R}^n} \sup_{0<r<1} \frac{1}{r^\beta} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(z) - \bar{f}(x,r)| \mathrm{d}z < \infty \Big\}, \tag{1.7}$$

and $\bar{f}(x,t)$ chosen to be 0 if $\beta \in [-1,0)$, the average of f over $B_r(x)$ if $\beta \in [0,1]$, then there exists a weak solution u to the system (1.1) which preserves the C^α-regularity over all $[0,T]$. The proof of [29] relies on a maximum-principle type argument to control the growth of some local average of u (one can see [10] for the same method applied to the kinematic dynamo equations, and see [14,15] for similar methods applied to the surface quasi-geostrophic equation).

In this paper, motivated by [29], we address the regularity problem of weak solution for the coupled system (1.1) with b satisfying some critical assumptions and $u_0 \in (L^2(\mathbb{R}^n))^n$, and we derive the Hölder regularity estimate depending on the scaling-invariant norms of b. Our main result is as follows.

Theorem 1.1 *Let $T > 0$ be any given. Assume that $b : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is a divergence-free vector field satisfying that $b \in \big(L^p([0,T]; \dot{M}^p(\mathbb{R}^n))\big)^n$ with $p \in [1,2]$ and*

$$\dot{M}^p(\mathbb{R}^n) := \begin{cases} \dot{W}^{1,\infty}(\mathbb{R}^n), & \text{if } \ p = 1; \\ \dot{C}^{\frac{2-p}{p}}(\mathbb{R}^n), & \text{if } \ p \in (1,2); \\ L^\infty(\mathbb{R}^n), & \text{if } \ p = 2. \end{cases} \tag{1.8}$$

For $p \in [1,2)$, additionally suppose that

$$P_{\le 1} b \in (L^1([0,T]; L^\infty(\mathbb{R}^n)))^n, \tag{1.9}$$

where $P_{\le 1}$ is the low-frequency operator given by (1.14) below. Let $u_0 \in (L^2(\mathbb{R}^n))^n$, then there is a weak solution (see Definition 2.1 below) $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$ to the drift–diffusion system (1.1) such that $u \in \big(L^\infty((0,T); C^\alpha(\mathbb{R}^n))\big)^n$ for any $\alpha \in (0,1)$. More precisely, for every $t' > 0$, we have

$$\|u\|_{L^\infty([t',T);\dot{C}^\alpha(\mathbb{R}^n))} \le \frac{C\|u_0\|_{L^2(\mathbb{R}^n)}}{t'^{\frac{n+2\alpha}{4}}} \exp\Big\{ C \int_0^T \|b(\tau)\|^p_{\dot{M}^p(\mathbb{R}^n)} \mathrm{d}\tau \Big\}, \tag{1.10}$$

where the constant C depends only on n, α, p.

Besides, if $p = 2$ in the above, the obtained weak solution $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$ is unique.

For Theorem 1.1, we first prove the existence of weak solution to the drift–diffusion system with pressure (1.1). The proof in nature shares much similarity with the existence proof of the

Leray–Hopf weak solution (e.g. see [18,20]) for the incompressible Navier–Stokes system (i.e. $b = u$ in (1.1)). But since the drift field b is a given vector field in the system (1.1) while the drift field b is just the unknown velocity field u in the Navier–Stokes system, there are also many different points, especially concerning the convergence of the terms involving b_ϵ and p_ϵ in the approximate system (2.9). We present the detailed proof of existence part in Sect. 2.1.

For the uniqueness result at $p = 2$ case in Theorem 1.1, the assumption on the divergence-free drift field is $b \in (L^2([0, T); L^\infty(\mathbb{R}^n)))^n$, and it is reminiscent of the Serrin's uniqueness criteria (e.g. see [18,28]) at endpoint case for the Navier–Stokes system. Here, by first mollifying the weak solutions u^i $(i = 1, 2)$ and considering the approximate Eq. (2.24), and then passing to the limit, we manage to prove the crucial equality (2.22) at $p = 2$ case, which can be used to yield the uniqueness result. One can see Sect. 2.2 for the uniqueness proof in detail.

In order to prove the Hölder regularity result, which is the core of Theorem 1.1, we mainly apply a novel idea of [14,21] to the procedure of [29] to improve the regularity step by step. Recalling that the general strategy of [29] is to show that for any $x \in \mathbb{R}^n$ and $\xi > 0$,

$$\int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y < \big(f(t)\omega(\xi)\big)^2 = \big(f(t)\big)^2 \xi^{2\alpha}, \quad \forall t \geq 0, \quad (1.11)$$

with $\bar{u}$ the average given by (2.38) and $f(t)$ some chosen time-dependent function, which according to Campanato [5] yields the α-Hölder regularity of $u(t)$ (see also Lemma 2.4 below). Here, in difference with [29], we introduce a new modulus $\omega(\xi, \xi_0)$ defined by (2.35) to substitute $\omega(\xi) = \xi^\alpha$, which is derived by only replacing the function $\omega(\xi)$ at the range $(0, \xi_0]$ with its tangent line at the point $(\xi_0, \omega(\xi_0))$. By setting $\xi_0 = \xi_0(t)$ given by (2.36) a time-dependent function, we firstly prove that the following strict preservation holds: for every $x \in \mathbb{R}^n$ and $\xi > 0$,

$$I_1(x, \xi, t) = \xi^n \int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y < \big(f_1(t)\omega(\xi, \xi_0(t))\big)^2, \quad \forall t \geq 0, \quad (1.12)$$

with some suitable $f_1(t)$. Since $\omega(\xi, 0+) = \omega(\xi) = \xi^\alpha$ and $\xi_0(t) = 0$ for $t \geq t_1$ with $t_1 > 0$ which can be chosen arbitrarily small, we in fact get an improvement of regularity after a short time

$$I_2(x, \xi, t) = \xi^{n-2\alpha} \int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y \leq f_1(t)^2, \quad \forall t \geq t_1.$$

We then prove that this quantity $I_2(x, \xi, t)$ will further strictly preserve the modulus $f_2(t - t_1)\omega(\xi, \xi_0(t - t_1))$ with some appropriate $f_2(t)$, which implies that

$$I_3(x, \xi, t) = \xi^{n-4\alpha} \int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y \leq f_2(t - t_1)^2, \quad \forall t \geq 2t_1.$$

By repeating the process for a finite time, say n_α-time, we get

$$\xi^{-2\alpha} \int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y \leq f_{n_\alpha}\big(t - (n_\alpha - 1)t_1\big)^2, \quad \forall t \geq n_\alpha t_1,$$

with $f_{n_\alpha}(\cdot)$ some chosen function, which ensures the Hölder regularity of $u(t)$ for $t \geq n_\alpha t_1$. Due to the arbitrariness of t_1, we indeed obtain the desired C^α-regularity of weak solution $u(t)$ for every $t > 0$, which also satisfies the explicit Hölder regularity estimate (1.10), as desired. The main proof of the regularity result is given in Sect. 2.3.

Remark 1.2 The Morrey–Campanato space $\mathcal{M}^{\frac{2-p}{p}}$ considered in [29] has the following properties: if $p = 1$, it corresponds to the space of Lipschitz functions $W^{1,\infty}(\mathbb{R}^n)$; and if $p \in (1, 2)$, it is exactly the space of Hölder continuous functions $C^{\frac{2-p}{p}}(\mathbb{R}^n)$; and if $p = 2$, it is the class of functions having bounded mean oscillation $BMO(\mathbb{R}^n)$. Thus for $p \in [1, 2)$, due to that the assumption on b is essentially the same and the regularity condition on u_0 is removed, Theorem 1.1 generalizes the regularity continuity result of [29] in this case to the Hölder regularity result. But if $p = 2$, since $\dot{M}^2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ is continuously embedded in $\mathcal{M}^0 = BMO(\mathbb{R}^n)$ and the class $BMO(\mathbb{R}^n)$ is strictly larger than the space $L^\infty(\mathbb{R}^n)$ (e.g. see [9, Chapter 6]), Theorem 1.1 needs a stronger assumption $b \in (L^2([0, T); L^\infty(\mathbb{R}^n)))^n$ to ensure the Hölder regularization, rather than the assumption $b \in (L^2([0, T); BMO(\mathbb{R}^n)))^n$ used in [29].

The main technical reason is that the preservation of (1.11) for all time (the symbol "$<$" in (1.11) can be replaced by "$\le$") guarantees that $u(t) \in \dot{C}^\alpha(\mathbb{R}^n)$ for all $t \ge 0$ (see Campanato [5] or (2.34) below), which implies that $|u(x, t) - u(y, t)| \le Cf(t)|x - y|^\alpha$ for all $x, y \in \mathbb{R}^n$ and this property plays an important role in the estimates of terms containing the drift b (see [29]), but for the following preservation used in this paper [or its variants like (1.12)]

$$\int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y \le \big(f(t)\omega(\xi, \xi_0(t))\big)^2, \quad \forall \xi > 0, x \in \mathbb{R}^n, \forall t \ge 0, \tag{1.13}$$

with $\omega(\xi, \xi_0(t))$ defined by (2.35)–(2.37), it is not so clear that such a preservation will imply an analogous pointwise estimates of $u(t)$, more precisely, it is not clear whether or not we can use (1.13) to get the estimate that $|u(x, t) - u(y, t)| \le Cf(t)\omega(|x - y|, \xi_0(t))$ for all $x, y \in \mathbb{R}^n$ and $t \ge 0$.

Remark 1.3 If the spatial dimension $n = 3$ and $b = u$ (noting that the corresponding scaling transformations are (1.3)–(1.5) with $s = 1$), then the system (1.1) reduces to the classical 3D incompressible Navier–Stokes system, and associated with $u_0 \in (L^2(\mathbb{R}^3))^3$, the Navier–Stokes system generates the Leray-Hopf weak solution $u \in (L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^3)))^n$ (e.g. see [18]). Noticing that the condition (1.9) is only used in the existence part of Theorem 1.1, thus Theorem 1.1 directly leads to that under the additional condition that $u \in (L^p([0, T); \dot{M}^p(\mathbb{R}^3)))^3$ $(1 \le p \le 2)$, there is a Leray-Hopf weak solution $u(t)$ (unique at $p = 2$ case) which is Hölder continuous with any index $\alpha \in (0, 1)$ for any $t \in (0, T)$. This result is consistent and compatible with some previous regularity results of 3D Navier–Stokes system which state that under the condition that $u \in (L^p([0, T); \dot{M}^p(\mathbb{R}^3)))^3$ (for $p = 2$ see [13], and for $p \in [1, 2)$ see [2,3,12], and for [6,12,16] etc. for various generalizations), the corresponding Leray-Hopf weak solution u is infinitely smooth (and also unique) on $\mathbb{R}^3 \times (0, T)$.

Remark 1.4 It is not clear for the authors to show the uniqueness of the constructed weak solution stated in Theorem 1.1 at $p \in [1, 2)$ cases. The main reason is that under the assumptions of drift field b and the energy estimate (2.1) of weak solution u^i $(i = 1, 2)$, we do not know how to show that the last line of (2.24) has a limit as $\epsilon \to 0$ and moreover the limit vanishes.

The following notations are used throughout this paper.

- C stands for a constant which may be different from line to line, and $X \lesssim Y$ means that there is a harmless constant C such that $X \le CY$, and $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$ simultaneously.

- We use $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ to denote the ball of $\mathbb{R}^n$, and use $B_r(x_0)^c := \{x \in \mathbb{R}^n : |x-x_0| \geq r\}$ as the complementary set of $B_r(x_0)$; we also abbreviate $B_r(0)$ and $B_r(0)^c$ as B_r and B_r^c respectively. The notation $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ corresponds to the unit ball of $\mathbb{R}^n$.
- For $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ two vectors, $A \otimes B$ is the tensor product of A and B which corresponds to a $n \times n$ matrix with each (i, j)-element equaling $A_i B_j$.
- For a vector field $v = (v_1, \ldots, v_n)$ and a function space X, the notation $(X)^n$ is the abbreviation of the product space $X \times \cdots \times X$, and $v \in (X)^n$ means that $v_i \in X$ for each $i \in \{1, \ldots, n\}$. For a matrix valued function $V = (V_{ij})_{n\times n}$, the notation $V \in (X)^{n\times n}$ means that $V_{ij} \in X$ for every $i, j \in \{1, \ldots, n\}$.
- The notation $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}([0, T))$ or $\mathcal{D}([0, T) \times \mathbb{R}^n)$ denotes the space of C^∞-smooth functions with compact support on $\mathbb{R}^n$, $[0, T)$ or $[0, T) \times \mathbb{R}^n$, respectively. Denote by $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{D}'([0, T))$ or $\mathcal{D}'([0, T) \times \mathbb{R}^n)$ the space of distributions, which is the dual space of $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}([0, T))$ or $\mathcal{D}([0, T) \times \mathbb{R}^n)$ (see [18]).
- The notation $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing C^∞-smooth functions, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions which is the dual space of $\mathcal{S}(\mathbb{R}^n)$.
- For $m \in \mathbb{N}$, $r \in [1, +\infty]$, $s \in \mathbb{R}$, we denote by $W^{m,r}(\mathbb{R}^n)$ ($\dot{W}^{m,r}(\mathbb{R}^n)$) and $H^s(\mathbb{R}^n)$ ($\dot{H}^s(\mathbb{R}^n)$) the usual L^r-based and L^2-based inhomogeneous (homogenous) Sobolev spaces, and by $C^\beta(\mathbb{R}^n)$, $\dot{C}^\beta(\mathbb{R}^n)$ with $\beta \in (0, 1)$ the inhomogeneous and homogeneous Hölder spaces (e.g. see [8]).
- We use $\mathcal{F}(f)$ (or $\hat{f}$) and $\mathcal{F}^{-1}(f)$ to denote the Fourier transform and the inverse Fourier transform of a function f, that is, $\mathcal{F}(f)(\zeta) = \int_{\mathbb{R}^n} e^{ix\cdot\zeta} f(x)\mathrm{d}x$ and $\mathcal{F}^{-1}(g)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} e^{ix\cdot\zeta} g(\zeta)\mathrm{d}\zeta$.
- Denote by $P_{\leq 1}$ the low frequency operator which is defined as a multiplier operator:
$$P_{\leq 1} := \psi(D) = \mathcal{F}^{-1}(\psi)*, \tag{1.14}$$
where $D = \frac{\nabla}{i} = \frac{\nabla}{\sqrt{-1}}$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ is such that $\psi \equiv 1$ on $B_1(0)$, $\mathrm{supp}\,\psi \subset B_2(0)$.

2 Proof of Theorem 1.1

The outline of the proof is as follows. In Sect. 2.1, we give the proof of the existence of a weak solution to the drift–diffusion system (1.1); we show the uniqueness proof at $p = 2$ case in Sect. 2.2; we then prove the core Hölder regularization result in Sect. 2.3; and in Sect. 2.4 we present the proof of some auxiliary results used in Sect. 2.3.

2.1 Existence of weak solution to the drift–diffusion system with pressure (1.1)

First we introduce the definition of weak solution (i.e. distributional solution) for the system (1.1).

Definition 2.1 (*Weak solutions*) Let $T > 0$ be any given. For a divergence-free vector field $b \in (L^1([0, T); L^2_{\mathrm{loc}}(\mathbb{R}^n)))^n$, we call that a vector field $u : [0, T) \times \mathbb{R}^n \to \mathbb{R}^n$ is a weak solution to the drift–diffusion system with pressure (1.1), if it satisfies the following properties.

(1) $u \in \big(L^\infty([0, T); L^2(\mathbb{R}^n)) \cap L^2([0, T); \dot{H}^1(\mathbb{R}^n))\big)^n$ satisfies that
$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\nabla \otimes u(\tau)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}\tau \leq \|u_0\|^2_{L^2(\mathbb{R}^n)}, \quad \forall t \in [0, T). \tag{2.1}$$

(2) $\lim_{t\to 0+}\|u(t)-u_0\|_{L^2(\mathbb{R}^n)}=0$.
(3) There is a distribution $p\in\mathcal{D}'([0,T)\times\mathbb{R}^n)$ such that (u,b) solves the first equation of (1.1) in the distributional sense, that is, for every test function $\chi=(\chi_1,\dots,\chi_n)\in(\mathcal{D}([0,T)\times\mathbb{R}^n))^n$,

$$-\int_0^T\int_{\mathbb{R}^n}\Big(u\cdot(\partial_t\chi+b\cdot\nabla\chi+\Delta\chi)+p\,\mathrm{div}\,\chi\Big)\mathrm{d}x\mathrm{d}t=\int_{\mathbb{R}^n}u_0(x)\chi(x,0)\mathrm{d}x. \quad (2.2)$$

(4) For any test function $\widetilde{\chi}\in\mathcal{D}([0,T)\times\mathbb{R}^n)$,

$$\int_0^T\int_{\mathbb{R}^n}u\cdot\nabla\widetilde{\chi}(x,\tau)\,\mathrm{d}x\mathrm{d}\tau=0. \quad (2.3)$$

Recalling that $e^{t\Delta}$ is the heat semigroup and $\mathbb{P}:=\mathrm{Id}+\nabla(-\Delta)^{-1}\mathrm{div}$ is the Leray projection operator (e.g. see [18, Chapter 11]), we also have the following equivalence results about different formulations of weak solutions to the system (1.1).

Proposition 2.2 *Let $b\in(L^1([0,T);L^\infty(\mathbb{R}^n)))^n$ be a divergence-free vector field. Assume that $u\in(L^\infty([0,T);L^2(\mathbb{R}^n)))^n$ is a vector field of $\mathbb{R}^n$ satisfies (2.3). Then the following statements are equivalent.*

(1) *There exists a distribution $p\in\mathcal{D}'([0,T)\times\mathbb{R}^n)$ such that $\partial_t u-\Delta u+\nabla\cdot(b\otimes u)+\nabla p=0$ in $(\mathcal{D}'([0,T)\times\mathbb{R}^n))^n$ and $\lim_{t\to 0}u=u_0$ in $(\mathcal{S}'(\mathbb{R}^n))^n$.*
(2) *u satisfies that $\partial_t u-\Delta u+\mathbb{P}\nabla\cdot(b\otimes u)=0$ in $(\mathcal{D}'([0,T)\times\mathbb{R}^n))^n$ and $\lim_{t\to 0}u=u_0$ in $(\mathcal{S}'(\mathbb{R}^n))^n$.*
(3) *u satisfies that $u=e^{t\Delta}u_0-\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(b\otimes u)\mathrm{d}\tau$.*

Proof of Proposition 2.2 Recall that if $b=u$, the drift–diffusion system (1.1) reduces to the classical incompressible Navier–Stokes system, and Lemarié-Rieusset in [18, Theorems 11.1, 11.2] proved the above equivalence under very general assumptions that

$$u\in(L^2_{uloc,x}L^2_t([0,T)\times\mathbb{R}^n))^n \quad (2.4)$$

and the following decaying condition

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}^n}\frac{1}{R^n}\int_{t_0}^{t_1}\int_{|x-x_0|\le R}|u|^2\mathrm{d}x\mathrm{d}t=0,\quad 0\le t_0<t_1<T, \quad (2.5)$$

where the notation $L^p_{uloc,x}L^q_t([0,T)\times\mathbb{R}^n)$ for every $1\le p,q<\infty$ is the space of Lebesgue measurable functions f on $[0,T)\times\mathbb{R}^n$ such that the norm $\sup_{x_0\in\mathbb{R}^n}\Big(\int_{|x-x_0|\le 1}(\int_0^T|f(x,t)|^q\mathrm{d}t)^{p/q}\mathrm{d}x\Big)^{1/p}$ is finite. It is clear that our assumption $u\in(L^\infty([0,T);L^2(\mathbb{R}^n)))^n$ guarantees the conditions (2.4), (2.5). Also note that in the case of Navier–Stokes system, the *a priori* information of $u\otimes u$ is that $u\otimes u\in(L^1_{uloc,x}L^1_t([0,T)\times\mathbb{R}^n)^{n\times n}$, while in our case of drift–diffusion system (1.1), we get $b\otimes u\in(L^1([0,T);L^2(\mathbb{R}^n)))^{n\times n}$, which directly implies $b\otimes u\in(L^1_{uloc,x}L^1_t([0,T)\times\mathbb{R}^n))^{n\times n}$. Hence we can follow the same arguments as [18, Theorems 11.1, 11.2] to prove Proposition 2.2, and we here omit the details. □

We now sketch the proof that the drift–diffusion system (1.1) generates a weak solution $u:\mathbb{R}^n\times[0,T)\to\mathbb{R}^n$.

Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$ and $\eta\in\mathcal{D}(\mathbb{R})$ be two smooth cut-off functions such that $\int_{\mathbb{R}^n}\varphi\mathrm{d}x=1$ and $\mathrm{supp}\,\eta\subset(-1,1)$, $\int_{\mathbb{R}}\eta\mathrm{d}t=1$. Set $\varphi_\epsilon(x):=\frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon})$, $\eta_\epsilon(t):=\frac{1}{\epsilon}\eta(\frac{t}{\epsilon})$, $\epsilon>0$. Denote by $u_{0,\epsilon}(x) = \varphi_\epsilon * u_0(x)$, by $b_\epsilon(x,t) = (\eta_\epsilon \varphi_\epsilon) * b(x,t)$ if $t \in [\epsilon, T-\epsilon]$, while $b_\epsilon(x,t) = 0$ if $t \in [0,\epsilon) \cup (T-\epsilon, T)$. We consider the following approximate system that for $\epsilon > 0$,

$$\begin{cases} \partial_t u + b_\epsilon \cdot \nabla u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u_{0,\epsilon}. \end{cases} \tag{2.6}$$

Due to that $u_0 \in (L^2(\mathbb{R}^n))^n$, $b \in \big(L^p([0,T]; \dot{M}^p(\mathbb{R}^n))\big)^n$ and $P_{\leq 1} b \in \big(L^1([0,T]; L^\infty(\mathbb{R}^n))\big)^n$, then for every $\epsilon > 0$, we have $u_{0,\epsilon} \in (H^m(\mathbb{R}^n))^n$, and $b_\epsilon \in \big(L^\infty([0,T]; W^{m,\infty}(\mathbb{R}^n))\big)^n$, $m \in \mathbb{N} \cap (\frac{n}{2}+2, \infty)$, which can be seen by the follows: $\|u_{0,\epsilon}\|_{H^m(\mathbb{R}^n)} \leq C\epsilon^{-m}\|u_0\|_{L^2(\mathbb{R}^n)}$, and

$$\begin{aligned} \|b_\epsilon\|_{L^\infty([0,T];W^{m,\infty}(\mathbb{R}^n))} &= \|b_\epsilon\|_{L^\infty([\epsilon,T-\epsilon];W^{m,\infty}(\mathbb{R}^n))} \\ &\leq C\epsilon^{-1}\|\varphi_\epsilon * b\|_{L^1([0,T];W^{m,\infty})} \leq C\epsilon^{-1}\|\varphi_\epsilon\|_{W^{m,1}}\|b\|_{L^1([0,T];L^\infty)} \\ &\leq C\epsilon^{-m-1}\big(\|b\|_{L^p([0,T];\dot{M}^p(\mathbb{R}^n))} + \|P_{\leq 1} b\|_{L^1([0,T];L^\infty(\mathbb{R}^n))}\big). \end{aligned}$$

By means of the mild formulation of u

$$u(x,t) = e^{t\Delta}u_{0,\epsilon} - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (b_\epsilon \otimes u)(x,\tau)\mathrm{d}\tau, \tag{2.7}$$

and by using the following estimate (from Plancherel's therem and Hölder's inequality) that for every $0 < T_1 < T$,

$$\begin{aligned} &\left\| \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot (b_\epsilon \otimes u)\mathrm{d}\tau \right\|_{L^\infty([0,T_1];H^m(\mathbb{R}^n))} \\ &\leq C \left\| \int_0^t e^{-(t-\tau)|\zeta|^2}|\zeta|(1+|\zeta|^2)^{\frac{m}{2}}|\mathcal{F}(b_\epsilon \otimes u)(\zeta,\tau)|\mathrm{d}\tau \right\|_{L^\infty([0,T_1];L^2)} \\ &\leq C \left\| \left(\int_0^t e^{-2(t-\tau)|\zeta|^2}|\zeta|^2\mathrm{d}\tau\right)^{1/2} \left(\int_0^t (1+|\zeta|^2)^m|\mathcal{F}(b_\epsilon \otimes u)(\zeta,\tau)|^2\mathrm{d}\tau\right)^{1/2} \right\|_{L^\infty([0,T_1];L^2)} \\ &\leq C\left(\int_0^{T_1} \|b_\epsilon \otimes u(\tau)\|_{H^m}^2\mathrm{d}\tau\right)^{1/2} \leq CT_1^{1/2}\|b_\epsilon\|_{L^\infty([0,T];W^{m,\infty})}\|u\|_{L^\infty([0,T_1];H^m)}, \end{aligned}$$

we can apply Picard's iteration to show that there is a time $T_1 > 0$ depending only on $\|b_\epsilon\|_{L^\infty([0,T);W^{m,\infty})}$ and dimension n so that the Eq. (2.7) admit a unique solution $u_\epsilon \in (C([0,T_1]; H^m(\mathbb{R}^n)))^n$. From the following estimate

$$\begin{aligned} &\|\mathbb{P}(b_\epsilon \cdot \nabla u_\epsilon)\|_{L^\infty([0,T_1];H^{m-2}(\mathbb{R}^n))} + \|b_\epsilon \cdot \nabla u_\epsilon\|_{L^\infty([0,T_1];H^{m-2}(\mathbb{R}^n))} \\ &\quad \leq C\|b_\epsilon\, u_\epsilon\|_{L^\infty([0,T_1];H^{m-1}(\mathbb{R}^n))} \leq C\|b_\epsilon\|_{L^\infty([0,T_1];W^{m,\infty}(\mathbb{R}^n))}\|u_\epsilon\|_{L^\infty([0,T);H^m(\mathbb{R}^n))}, \end{aligned}$$

and using (2.7), we have (e.g. see [18, Chapter 11])

$$\partial_t u_\epsilon = \Delta u_\epsilon - \mathbb{P}(b_\epsilon \cdot \nabla u_\epsilon),$$

and thanks to that $u \in (C([0,T_1]; H^m(\mathbb{R}^n)))^n$ and $b_\epsilon \in (C([0,T_1]; W^{m,\infty}(\mathbb{R}^n)))^n$, we can obtain $u_\epsilon \in (C^1([0,T_1]; H^{m-2}(\mathbb{R}^n)))^n$. We define the function

$$p_\epsilon = (-\Delta)^{-1}\operatorname{div}\big(b_\epsilon \cdot \nabla u_\epsilon\big) = (-\Delta)^{-1}\operatorname{div}\nabla \cdot (b_\epsilon \otimes u_\epsilon) \tag{2.8}$$

which belongs to $L^\infty([0,T); H^m(\mathbb{R}^n))$, and thus (from the definition of $\mathbb{P}$)

$$\partial_t u_\epsilon + b_\epsilon \cdot \nabla u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = 0, \quad \operatorname{div} u_\epsilon = 0. \tag{2.9}$$

Moreover, noting that the time increment $T_1 > 0$ is a uniform constant and is independent of the starting time, we can consider the time interval $[T_1, 2T_1]$, $[2T_1, 3T_1]$, ..., and finally $[kT_1, (k+1)T_1] \cap [0, T)$ for some $k \in \mathbb{N}$, so that by using the time connectivity, we obtain a unique solution $u_\epsilon \in (C([0, T); H^m(\mathbb{R}^n)) \cap C^1([0, T); H^{m-2}(\mathbb{R}^n)))^n$ to the system (2.6). Thanks to the continuous embedding $H^m(\mathbb{R}^n) \hookrightarrow W^{2,\infty}(\mathbb{R}^n)$ and $H^{m-2}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $m > \frac{n}{2} + 2$, we see that u_ϵ and p_ϵ satisfy the system (2.9) in the classical pointwise sense.

Since $\|u_{0,\epsilon}\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}$, in view of the divergence-free property of b_ϵ and the classical energy estimate, we get the following L^2-estimate of u_ϵ:

$$\|u_\epsilon(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\nabla u_\epsilon(\tau)\|^2_{L^2(\mathbb{R}^n)} \mathrm{d}\tau \leq \|u_0\|^2_{L^2(\mathbb{R}^n)}, \quad \forall t \in [0, T), \tag{2.10}$$

which corresponds to that $u_\epsilon \in \big(L^\infty([0, T); L^2(\mathbb{R}^n)) \cap L^2([0, T); \dot{H}^1(\mathbb{R}^n))\big)^n$ uniformly in ϵ. Owing to the weak (weak-$*$) compactness lemmas (e.g. see [4, Theorems 3.16, 3.18]), this implies that there exists a vector field $u = (u_1, \ldots, u_n)$ such that u_ϵ, up to a subsequence, denoting by u_{ϵ_k}, weakly converges to u in the space $\big(L^\infty([0, T); L^2(\mathbb{R}^n)) \cap L^2([0, T); \dot{H}^1(\mathbb{R}^n))\big)^n$ (weakly-* converges in L^∞-topology) as $\epsilon_k \to 0$. Since for any $T > 0$, $u_{\epsilon_k} \in L^2([0, T) \times \mathbb{R}^n)$ uniformly in ϵ, u_{ϵ_k} (up a subsequence if necessary) is weakly convergent to u in $L^2([0, T) \times \mathbb{R}^n)$, which also implies that $u_{\epsilon_k} \to u$ in $\mathcal{D}'([0, T) \times \mathbb{R}^n)$. By setting $\beta(t) \in \mathcal{D}([0, T))$, and using the weak convergence of βu_{ϵ_k} in $(L^2([0, T) \times \mathbb{R}^n))^n$ and the weak convergence of $\nabla \otimes u_{\epsilon_k}$ in $(L^2([0, T) \times \mathbb{R}^n))^{n\times n}$ [from the convergence in $(\mathcal{D}')^{n\times n}$ and uniform control (2.10)], we obtain that

$$\begin{aligned}
&\int\int_{\mathbb{R}^n} |\beta(t)|^2 |u(x,t)|^2 \mathrm{d}x\mathrm{d}t + 2\int |\beta(t)|^2 \left(\int_0^t \int_{\mathbb{R}^n} |\nabla \otimes u(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau\right) \mathrm{d}t \\
&\leq \liminf_{\epsilon_k\to 0} \int\int_{\mathbb{R}^n} |\beta(t)|^2 |u_{\epsilon_k}(x,t)|^2 \mathrm{d}x\mathrm{d}t + 2\int |\beta(t)|^2 \left(\int_0^t \int_{\mathbb{R}^n} |\nabla \otimes u_{\epsilon_k}(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau\right) \mathrm{d}t \\
&\leq \int |\beta(t)|^2 \mathrm{d}t \, \|u_0\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

For any $t_0 > 0$, we choose $\beta(t) = \frac{1}{\sqrt{\delta}}\theta(\frac{t-t_0}{\delta})$, $\delta > 0$ with $\theta \in \mathcal{D}(\mathbb{R})$ satisfying $\int |\theta(t)|^2 \mathrm{d}t = 1$, then we get

$$\limsup_{\delta\to 0} \int\int_{\mathbb{R}^n} \frac{1}{\delta}\left|\theta\left(\frac{t-t_0}{\delta}\right)\right|^2 |u(x,t)|^2 \mathrm{d}x\mathrm{d}t + 2\int_0^{t_0}\int_{\mathbb{R}^n} |\nabla u(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau \leq \|u_0\|^2_{L^2(\mathbb{R}^n)}.$$

If $t_0 > 0$ is a Lebesgue point of the measurable function $t \mapsto \|u(t)\|_{L^2(\mathbb{R}^n)}$, the limit in the left-hand side of the above inequality equals $\|u(t_0)\|^2_{L^2(\mathbb{R}^n)}$, thus we prove the inequality (2.1) for almost every $t \in [0, T)$ (later we shall show that (2.1) indeed holds for any $t \in [0, T)$), which also implies that

$$\|u\|_{L^\infty([0,T);L^2(\mathbb{R}^n))} \leq \|u_0\|_{L^2(\mathbb{R}^n)}, \quad \text{and} \quad \|u\|_{L^2([0,T);L^2(\mathbb{R}^n))} \leq \|u_0\|_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

By using $b \in (L^p([0, T); \dot{M}^p(\mathbb{R}^n)))^n$, $p \in [1, 2]$, and $P_{\leq 1} b \in (L^1([0, T); L^\infty(\mathbb{R}^n)))^n$, we have that (from the high-low frequency decomposition)

$$b \in (L^1([0, T); L^\infty(\mathbb{R}^n)))^n, \tag{2.12}$$

and also $b \in (L^1([0, T); L^4_{\mathrm{loc}}(\mathbb{R}^n)))^n$, thus b_ϵ strongly converges to b in $(L^1([0, T); L^4_{\mathrm{loc}}(\mathbb{R}^n)))^n$ as $\epsilon \to 0$ (e.g. see [8, Appendix C.4]).

It is clear to see that u_{ϵ_k} and b_{ϵ_k} satisfy (2.2), (2.3) with (u, b) replaced by $(u_{\epsilon_k}, b_{\epsilon_k})$. We also find that for every test function $\chi \in (\mathcal{D}([0,T)\times\mathbb{R}^n))^n$,

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^n}(u_{\epsilon_k}\otimes b_{\epsilon_k})\cdot(\nabla\otimes\chi)\mathrm{d}x\mathrm{d}\tau-\int_0^T\int_{\mathbb{R}^n}(u\otimes b)\cdot(\nabla\otimes\chi)\mathrm{d}x\mathrm{d}\tau\right|\\
&\leq\left|\int_0^T\int_{\mathbb{R}^n}(u_{\epsilon_k}\otimes(b_{\epsilon_k}-b))\cdot(\nabla\otimes\chi)\mathrm{d}x\mathrm{d}\tau\right|+\left|\int_0^T\int_{\mathbb{R}^n}((u_{\epsilon_k}-u)\otimes b)\cdot(\nabla\otimes\chi)\mathrm{d}x\mathrm{d}\tau\right|\\
&\leq\|u_{\epsilon_k}\|_{L^\infty([0,T);L^2)}\|b_{\epsilon_k}-b\|_{L^1([0,T);L^4_{\mathrm{loc}})}\|\nabla\otimes\chi\|_{L^\infty([0,T);L^4)}\\
&+\left|\int_0^T\int_{\mathbb{R}^n}((u_{\epsilon_k}-u)\otimes b)\cdot(\nabla\otimes\chi)\mathrm{d}x\mathrm{d}\tau\right|\\
&\longrightarrow 0,\qquad \text{as } \epsilon_k\to 0,
\end{aligned}\tag{2.13}
$$

where the last convergence (2.13) is deduced from (2.10), the strong convergence result of b_{ϵ_k}, the weak (weak-*) convergence of u_{ϵ_k} in $(L^\infty([0,T);L^2(\mathbb{R}^n)))^n$ and the fact $b\cdot(\nabla\otimes\chi)\in\big(L^1([0,T);L^2(\mathbb{R}^n))\big)^n$. From (2.8), we next intend to prove that

$$
p_{\epsilon_k}\to p:=(-\Delta)^{-1}\mathrm{div}\,\mathrm{div}\,(b\otimes u),\quad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}^n).\tag{2.14}
$$

Recall that $\mathcal{R}_i\mathcal{R}_j:=\partial_{x_i}\partial_{x_j}(-\Delta)^{-1}$ $(i,j=1,\ldots,n)$ is a Fourier multiplier operator with multiplier $m(\zeta)=-\frac{\zeta_i\zeta_j}{|\zeta|^2}$ which has the following expression formula (e.g. see [9, Theorem 4.13])

$$
\mathcal{R}_i\mathcal{R}_jf(x)=a_{ij}f(x)+\mathcal{T}_{ij}f(x),\tag{2.15}
$$

where $a_{ij}=-\frac{1}{n}$ for $i=j$ and $a_{ij}=0$ for $i\neq j$, and $\mathcal{T}_{ij}$ is a singular integral operator

$$
\mathcal{T}_{ij}f(x):=\text{p.v.}\int_{\mathbb{R}^n}K_{ij}(x-y)f(y)\mathrm{d}y=\lim_{\epsilon\to0}\int_{|x-y|>0}K_{ij}(x-y)f(y)\mathrm{d}y,\tag{2.16}
$$

with the kernel $K_{ij}(x)=\frac{\Omega_{ij}(\hat{x})}{|x|^n}$, $\forall x\neq 0$, $\hat{x}=\frac{x}{|x|}$, and $\Omega_{ij}(\hat{x})\in C^\infty(\mathbb{S}^{n-1})$ satisfying the zero-average property. We thus have that for every $\chi\in(\mathcal{D}([0,T)\times\mathbb{R}^n))^n$,

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}(p_{\epsilon_k}-p)(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau&=\int_0^T\int_{\mathbb{R}^n}\mathcal{R}_i\mathcal{R}_j(b_{\epsilon_k,i}\,u_{\epsilon_k,j}-b_iu_j)(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau\\
&=\int_0^T\int_{\mathbb{R}^n}(b_{\epsilon_k,i}\,u_{\epsilon_k,j}-b_iu_j)\,\mathcal{R}_i\mathcal{R}_j(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau\\
&=a_{ij}\int_0^T\int_{\mathbb{R}^n}(b_{\epsilon_k,i}\,u_{\epsilon_k,j}-b_iu_j)\,(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_0^T\int_{\mathbb{R}^n}(b_{\epsilon_k,i}\,u_{\epsilon_k,j}-b_iu_j)\,\mathcal{T}_{ij}(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau,
\end{aligned}\tag{2.17}
$$

where the Einstein summation convention on repeated indices is also used. Similarly as (2.13), we obtain

$$
a_{ij}\left|\int_0^T\int_{\mathbb{R}^n}(b_{\epsilon_k,i}\,u_{\epsilon_k,j}-b_iu_j)\,(\mathrm{div}\,\chi)\mathrm{d}x\mathrm{d}\tau\right|\to 0,\qquad \text{as } \epsilon_k\to 0.\tag{2.18}
$$

Let χ be supported in the space–time domain $[\tau_0,\tau_1]\times B_{R_0}$ with $0\leq\tau_0<\tau_1<T$ and some $R_0>0$. By letting $R\geq R_0$ be some constant chosen later and using the support property, we see that for every $x \in B_{2R}^c$,

$$
\begin{aligned}
|\mathcal{T}_{ij}(\operatorname{div}\chi)(x,\tau)| &= \left|\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(\widehat{x-y})}{|x-y|^n}(\operatorname{div}\chi)(y,\tau)\mathrm{d}y\right| \\
&\le C\int_{B_{R_0}}\frac{1}{|x-y|^n}|(\operatorname{div}\chi)(y,\tau)|\mathrm{d}y \le C\frac{1}{|x|^n}\|\operatorname{div}\chi\|_{L^\infty([0,T);L^1)}.
\end{aligned} \tag{2.19}
$$

For any $\varepsilon > 0$, by using (2.10), (2.11), (2.12), (2.19) and Hölder's inequality, we deduce that

$$
\begin{aligned}
&\left|\int_0^T\int_{B_{2R}^c}(b_{\epsilon_k,i}\,u_{\epsilon_k,j} - b_i u_j)\,\mathcal{T}_{ij}(\operatorname{div}\chi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\le C\|\operatorname{div}\chi\|_{L_t^\infty L_x^1}\int_0^T\int_{B_{2R}^c}\frac{|b_{\epsilon_k,i}\,u_{\epsilon_k,j} - b_i u_j|}{|x|^n}\mathrm{d}x\mathrm{d}\tau \\
&\le C\|\operatorname{div}\chi\|_{L_t^\infty L_x^1}\left(\|b_{\epsilon_k}\|_{L_t^1L_x^\infty}\|u_{\epsilon_k}\|_{L_t^\infty L_x^2} + \|b\|_{L_t^1L_x^\infty}\|u\|_{L_t^\infty L_x^2}\right)\left(\int_{B_{2R}^c}\frac{1}{|x|^{2n}}\mathrm{d}x\right)^{1/2} \\
&\le C\|\operatorname{div}\chi\|_{L_t^\infty L_x^1}\|b\|_{L_t^1L_x^\infty}\|u_0\|_{L^2(\mathbb{R}^n)}\frac{1}{R^{n/2}} < \varepsilon,
\end{aligned} \tag{2.20}
$$

where the last inequality can be ensured by choosing some fixed R such that

$$
R \ge \max\left\{2R_0, \frac{2C}{\varepsilon}\|\operatorname{div}\chi\|_{L_t^\infty L_x^1}\|b\|_{L_t^1L_x^\infty}\|u_0\|_{L^2}\right\}.
$$

By arguing as (2.13), and using the facts that $\|\mathcal{T}_{ij}(\operatorname{div}\chi)\|_{L_t^\infty L_x^4} \le C\|\nabla\chi\|_{L_t^\infty L_x^4}$ and $\|\mathcal{T}_{ij}(\operatorname{div}\chi)\|_{L_t^\infty L_x^2} \le C\|\chi\|_{L_t^\infty H_x^1}$, we also infer that

$$
\begin{aligned}
&\left|\int_0^T\int_{B_{2R}}(b_{\epsilon_k,i}\,u_{\epsilon_k,j} - b_i u_j)\,\mathcal{T}_{ij}(\operatorname{div}\chi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\le \left|\int_0^T\int_{B_{2R}}(u_{\epsilon_k,j}\,(b_{\epsilon_k,i} - b_i))\,\mathcal{T}_{ij}(\operatorname{div}\chi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\quad + \left|\int_0^T\int_{B_{2R}}((u_{\epsilon_k,j} - u_j)\,b_i)\mathcal{T}_{ij}(\operatorname{div}\chi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\le \|u_{\epsilon_k}\|_{L^\infty([0,T);L^2)}\|b_{\epsilon_k} - b\|_{L^1([0,T);L^4_{\mathrm{loc}})}\|\nabla\chi\|_{L^\infty([0,T);L^4)} \\
&\quad + \left|\int_0^T\int_{\mathbb{R}^n}(u_{\epsilon_k,j} - u_j)\,(b_i\,\mathcal{T}_{ij}(\operatorname{div}\chi))\mathrm{d}x\mathrm{d}\tau\right| \\
&\longrightarrow 0, \qquad \text{as } \epsilon_k \to 0.
\end{aligned} \tag{2.21}
$$

Hence, gathering (2.17) and the estimates (2.18), (2.20), (2.21) leads to

$$
\lim_{\epsilon_k\to 0}\left|\int_0^T\int_{\mathbb{R}^n}(p_{\epsilon_k} - p)(\operatorname{div}\chi)\mathrm{d}x\mathrm{d}\tau\right| < \varepsilon,
$$

thus from the arbitrariness of $\varepsilon > 0$, we conclude the desired convergence (2.14). Therefore we can pass the limit $\epsilon_k \to 0$ to show that u indeed satisfies (2.2) and (2.3), that is, u solves the drift–diffusion system (1.1) in the distributional sense.

Next we show that the solution u is weakly continuous from $[0, T)$ to $(L^2(\mathbb{R}^n))^n$ after a redefinition on a null set of $[0, T)$. Indeed, from $\partial_t u = \Delta u + \mathbb{P}\nabla\cdot(b\otimes u)$ in the distributional sense and the following estimate

$$\|\mathbb{P}\nabla\cdot(b\otimes u)\|_{L^1([0,T);H^{-2})}\le C\|b\otimes u\|_{L^1([0,T);L^2)}\le\|b\|_{L^1([0,T);L^\infty)}\|u\|_{L^\infty([0,T);L^2)},$$

we have $\partial_t u\in(L^1([0,T);H^{-2}(\mathbb{R}^n)))^n$, thus $u(t)$ for every $t\in[0,T)$ is continuous in $(\mathcal{S}'(\mathbb{R}^n))^n$. Due to that the inequality (2.1) is satisfied for almost every $t\in[0,T)$, there is a null set $N\subset[0,T)$ such that (2.1) holds on $[0,T)\setminus N$, and we can redefine the values of u on N so that $\|u(t)\|_{L^2(\mathbb{R}^n)}\le C$ for every $t\in N$. Hence, by using the facts that $t\mapsto u$ is bounded in L^2-norm for every $t\in[0,T)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can prove that u is weakly continuous in $(L^2(\mathbb{R}^n))^n$ for every $t\in[0,T)$, as desired.

We now prove the energy estimate (2.1) for every $t\in[0,T)$ and also $\lim_{t\to0+}\|u(t)-u_0\|_{L^2(\mathbb{R}^n)}=0$. Indeed, for any $t\in[0,T)$, recalling that (2.1) is valid for every $t\in[0,T)\setminus N$ with null set N, there exists a sequence of times $\{t_j\}_{j=1}^\infty\subset[0,T)\setminus N$ such that $t_j\to t$ as $j\to\infty$, thus we deduce that $\|u(t)\|^2_{L^2(\mathbb{R}^n)}\le\liminf_{t_j\to t}\|u(t_j)\|^2_{L^2(\mathbb{R}^n)}$ and $\int_0^t\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}\tau=\lim_{t_j\to t}\int_0^{t_j}\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x\mathrm{d}\tau$; hence (2.1) holds for every $t\in[0,T)$. For the strong continuity property of u at time $t=0$, due to that $u(t)$ is weakly L^2-continuous at $t=0$, we only need to show that $\lim_{t\to0+}\|u(t)\|^2_{L^2(\mathbb{R}^n)}=\|u_0\|^2_{L^2(\mathbb{R}^n)}$. But this equality can be seen from $\|u_0\|^2_{L^2(\mathbb{R}^n)}\le\liminf_{t\to0+}\|u(t)\|^2_{L^2(\mathbb{R}^n)}$ which is from the weak convergence, and also $\limsup_{t\to0+}\|u(t)\|^2_{L^2(\mathbb{R}^n)}\le\|u_0\|^2_{L^2(\mathbb{R}^n)}$ which is ensured by (2.1).

Therefore, based on the above analysis, we construct a weak solution $u:[0,T)\times\mathbb{R}^n\to\mathbb{R}^n$ to the drift–diffusion system with pressure (1.1) in the sense of Definition 2.1.

2.2 Uniqueness of weak solutions to the drift–diffusion system with pressure (1.1) at $p=2$ case

Assume that u^1 and u^2 are two weak solutions to the system (1.1) associated with the same initial data $u_0\in(L^2(\mathbb{R}^n))^n$, that is, u^i $(i=1,2)$ belongs to $\big(L^\infty([0,T);L^2(\mathbb{R}^n))\cap L^2([0,T);\dot H^1(\mathbb{R}^n))\big)^n$ and satisfies (2.2), (2.3).

We first have the following result which plays an important role in the uniqueness issue.

Proposition 2.3 *Let $T>0$ be any given. Assume that u^1 and u^2 defined on $[0,T)\times\mathbb{R}^n$ are two weak solutions to the system (1.1) with the same data $u_0\in(L^2(\mathbb{R}^n))^n$. Additionally suppose that b is a divergence-free (in distributional sense) vector field satisfying $b\in(L^2([0,T);L^\infty(\mathbb{R}^n)))^n$. Then the map $t\mapsto\int_{\mathbb{R}^n}u^1(x,t)\cdot u^2(x,t)\mathrm{d}x$ for every $t\in[0,T)$ is continuous, and we have the equality that for every $0\le s<t<T$,*

$$\int_{\mathbb{R}^n}u^1(x,t)\cdot u^2(x,t)\mathrm{d}x+2\int_s^t\int_{\mathbb{R}^n}(\nabla\otimes u^1)\cdot(\nabla\otimes u^2)\mathrm{d}x\mathrm{d}\tau=\int_{\mathbb{R}^n}u^1(x,s)\cdot u^2(x,s)\mathrm{d}x. \tag{2.22}$$

Proof of Proposition 2.3 Let $\varphi_\epsilon(x):=\frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon})$, $\eta_\epsilon(t):=\frac{1}{\epsilon}\eta(\frac{t}{\epsilon})$, $\epsilon>0$ with $\varphi\in\mathcal{D}(\mathbb{R}^n)$ and $\eta\in\mathcal{D}(\mathbb{R})$ satisfying $\int_{\mathbb{R}^n}\varphi\mathrm{d}x=1$ and $\operatorname{supp}\eta\subset(-1,1)$, $\int_{\mathbb{R}}\eta\mathrm{d}t=1$. Set $\omega_\epsilon(x,t):=\varphi_\epsilon(x)\eta_\epsilon(t)$. Then $\omega_\epsilon*u^i$ $(i=1,2)$ is a smooth function on $[\epsilon,T-\epsilon]\times\mathbb{R}^n$, and we have

$$\begin{aligned}\partial_t\big((\omega_\epsilon*u^1)\cdot(\omega_\epsilon*u^2)\big)&=\big(\partial_t(\omega_\epsilon*u^1)\big)\cdot(\omega_\epsilon*u^2)+(\omega_\epsilon*u^1)\cdot\big(\partial_t(\omega_\epsilon*u^2)\big)\\&=(\omega_\epsilon*\partial_tu^1)\cdot(\omega_\epsilon*u^2)+(\omega_\epsilon*u^1)\cdot(\omega_\epsilon*\partial_tu^2),\end{aligned} \tag{2.23}$$

where the notation $*$ means the space–time convolution. Since u^i $(i=1,2)$ solves the first equation of (1.1) in the distributional sense, we infer that for $i,j\in\{1,2\}$,

$$
\begin{aligned}
(\omega_\epsilon * \partial_t u^i)\cdot(\omega_\epsilon * u^j) &= (\omega_\epsilon * \Delta u^i)\cdot(\omega_\epsilon * u^j)\\
&\quad - (\omega_\epsilon * \nabla p^i)\cdot(\omega_\epsilon * u^j)\\
&\quad - (\omega_\epsilon * \nabla\cdot(b\otimes u^i))\cdot(\omega_\epsilon * u^j)\\
&= \nabla\cdot\big((\omega_\epsilon * \nabla\otimes u^i)\cdot(\omega_\epsilon * u^j)\big) - (\omega_\epsilon * \nabla\otimes u^i)\cdot(\omega_\epsilon * \nabla\otimes u^j)\\
&\quad - \nabla\cdot\big((\omega_\epsilon * p^i)\cdot(\omega_\epsilon * u^j)\big)\\
&\quad - \nabla\cdot\big((\omega_\epsilon * (b\otimes u^i))\cdot(\omega_\epsilon * u^j)\big) + (\omega_\epsilon * (b\otimes u^i))\cdot(\omega_\epsilon * \nabla\otimes u^j).
\end{aligned}
$$

Let $\gamma(t)\in\mathcal{D}([\epsilon, T-\epsilon])$ and $\psi(x)\in\mathcal{D}(\mathbb{R}^n)$ with $\psi\equiv 1$ on B_1. Observe that for a vector field $F_\epsilon\in(L^1([\epsilon, T-\epsilon]\times\mathbb{R}^n))^n$ uniformly in ϵ and $R>0$, we get

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}\nabla\cdot F_\epsilon(x,t)\gamma(t)\psi\big(\frac{x}{R}\big)\mathrm{d}x\mathrm{d}t &= -\frac{1}{R}\int_0^T\int_{|x|\ge R}\gamma(t)F_\epsilon(x,t)\cdot(\nabla\psi)\big(\frac{x}{R}\big)\mathrm{d}x\mathrm{d}t\\
&\longrightarrow 0, \qquad \text{as } R\to\infty.
\end{aligned}
$$

Also noticing that

$$
\begin{aligned}
&\|(\omega_\epsilon * (\nabla\otimes u^i))\cdot(\omega_\epsilon * u^j)\|_{L^1([\epsilon,T-\epsilon]\times\mathbb{R}^n)}\\
&\quad\le \|\omega_\epsilon * (\nabla\otimes u^i)\|_{L^2([\epsilon,T-\epsilon]\times\mathbb{R}^n)}\|\omega_\epsilon * u^j\|_{L^2([\epsilon,T-\epsilon]\times\mathbb{R}^n)}\\
&\quad\le \|\nabla\otimes u^i\|_{L^2([0,T)\times\mathbb{R}^n)}\|u^j\|_{L^2([0,T)\times\mathbb{R}^n)} < T^{\frac{1}{2}}\|u_0\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\omega_\epsilon * (b\otimes u^i))\cdot(\omega_\epsilon * u^j)\|_{L^1([\epsilon,T-\epsilon]\times\mathbb{R}^n)}\\
&\quad\le \|\omega_\epsilon * (b\otimes u^i)\|_{L^1([\epsilon,T-\epsilon];L^2(\mathbb{R}^n))}\|\omega_\epsilon * u^j\|_{L^\infty([\epsilon,T-\epsilon];L^2(\mathbb{R}^n))}\\
&\quad\le \|b\|_{L^1([0,T);L^\infty)}\|u^i\|_{L^\infty([0,T);L^2)}\|u^j\|_{L^\infty([0,T);L^2)} \le \|b\|_{L^1([0,T);L^\infty(\mathbb{R}^n))}\|u_0\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

and (from $p^i = (-\Delta)^{-1}\mathrm{div}\,\mathrm{div}\,(b\otimes u^i)$, see Proposition 2.2)

$$
\begin{aligned}
&\|(\omega_\epsilon * p^i)\cdot(\omega_\epsilon * u^j)\|_{L^1([\epsilon,T-\epsilon]\times\mathbb{R}^n)} \le \|p^i\|_{L^1([0,T);L^2(\mathbb{R}^n))}\|u^j\|_{L^\infty([0,T);L^2(\mathbb{R}^n))}\\
&\quad\le \|b\|_{L^1([0,T);L^\infty)}\|u^i\|_{L^\infty([0,T);L^2)}\|u^j\|_{L^\infty([0,T);L^2)} \le \|b\|_{L^1([0,T);L^\infty(\mathbb{R}^n))}\|u_0\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

we take an inner product of the equality (2.23) with $\gamma(t)\psi(\frac{x}{R})$ $(R>0)$ and then let $R\to\infty$, we find that in $\mathcal{D}'([\epsilon, T-\epsilon])$,

$$
\begin{aligned}
&\partial_t\int_{\mathbb{R}^n}(\omega_\epsilon * u^1)\cdot(\omega_\epsilon * u^2)\mathrm{d}x + 2\int_{\mathbb{R}^n}(\omega_\epsilon * (\nabla\otimes u^1))\cdot(\omega_\epsilon * (\nabla\otimes u^2))\mathrm{d}x\\
&= \int_{\mathbb{R}^n}(\omega_\epsilon * (b\otimes u^1))\cdot(\omega_\epsilon * (\nabla\otimes u^2))\mathrm{d}x + \int_{\mathbb{R}^n}(\omega_\epsilon * (\nabla\otimes u^1))\cdot(\omega_\epsilon * (b\otimes u^2))\mathrm{d}x\\
&= \int_{\mathbb{R}^n}(\omega_\epsilon * (b\otimes u^1))\cdot(\omega_\epsilon * (\nabla\otimes u^2))\mathrm{d}x - \int_{\mathbb{R}^n}(\omega_\epsilon * u^1)\cdot(\omega_\epsilon * \nabla\cdot(b\otimes u^2))\mathrm{d}x.
\end{aligned}
\tag{2.24}
$$

Now we pass ϵ to 0 in the above quantities. Notice that as $\epsilon\to 0$, for some $g\in L^2([0,T)\times\mathbb{R}^n)$, $\omega_\epsilon * g$[1] strongly converges to g in $L^2([0,T)\times\mathbb{R}^n)$. We thus obtain that in $\mathcal{D}'([0,T))$,

$$
\partial_t\int_{\mathbb{R}^n}(\omega_\epsilon * u^1)\cdot(\omega_\epsilon * u^2)\mathrm{d}x \to \partial_t\int_{\mathbb{R}^n}u^1\cdot u^2\mathrm{d}x, \tag{2.25}
$$

[1] This function can be defined on $[0,T)\times\mathbb{R}^n$ by first defining $\omega_\epsilon * g$ as space–time convolution on $[\epsilon, T-\epsilon]\times\mathbb{R}^n$ and then extending it by 0 on $([0,\epsilon)\cup(T-\epsilon,T))\times\mathbb{R}^n$.

and

$$2\int_{\mathbb{R}^n}(\omega_\epsilon*(\nabla\otimes u^1))\cdot(\omega_\epsilon*(\nabla\otimes u^2))\mathrm{d}x\to 2\int_{\mathbb{R}^n}(\nabla\otimes u^1)\cdot(\nabla\otimes u^2)\mathrm{d}x, \tag{2.26}$$

and

$$\int_{\mathbb{R}^n}(\omega_\epsilon*(b\otimes u^1))\cdot(\omega_\epsilon*(\nabla\otimes u^2))\mathrm{d}x\to\int_{\mathbb{R}^n}(b\otimes u^1)\cdot(\nabla\otimes u^2)\mathrm{d}x=\int_{\mathbb{R}^n}b\cdot(\nabla\otimes u^2)\cdot u^1\mathrm{d}x. \tag{2.27}$$

Since $b\in(L^2([0,T);L^\infty(\mathbb{R}^n)))^n$ with $\operatorname{div} b=0$, and $u^2\in(L^2([0,T);H^1(\mathbb{R}^n)))^n$, we have $\nabla\cdot(b\otimes u^2)=b\cdot\nabla u^2$ in $(\mathcal{D}'([0,T)\times\mathbb{R}^n))^n$. Indeed, we may first get the equality for the smooth vector field $\omega_\epsilon*b$ and $\omega_\epsilon*u^2$, and then pass to the limit $\epsilon\to 0$. Thus we infer that in $\mathcal{D}'([0,T))$,

$$\int_{\mathbb{R}^n}(\omega_\epsilon*u^1)\cdot(\omega_\epsilon*\nabla\cdot(b\otimes u^2))\mathrm{d}x=\int_{\mathbb{R}^n}(\omega_\epsilon*u^1)\cdot(\omega_\epsilon*(b\cdot(\nabla\otimes u^2)))\mathrm{d}x.$$

Due to that $\|\omega_\epsilon*u^1\|_{L^\infty([0,T);L^2(\mathbb{R}^n))}\le\|u^1\|_{L^\infty([0,T);L^2(\mathbb{R}^n))}\le\|u_0\|_{L^2(\mathbb{R}^n)}$, as $\epsilon\to 0$, we have $\omega_\epsilon*u^1$ (up to a subsequence, still denoting by $\omega_\epsilon*u^1$) weakly-$*$ converges to u^1 in $(L^\infty([0,T);L^2(\mathbb{R}^n)))^n$ (e.g. see [4, Theorem 13.6]). From

$$\begin{aligned}\|b\cdot(\nabla\otimes u^2)\|_{L^1([0,T);L^2(\mathbb{R}^n))}&\le\|b\|_{L^2([0,T);L^\infty)}\|\nabla\otimes u^2\|_{L^2([0,T)\times\mathbb{R}^n)}\\&\le\|b\|_{L^2([0,T);L^\infty(\mathbb{R}^n))}\|u_0\|_{L^2},\end{aligned} \tag{2.28}$$

we also deduce that $\omega_\epsilon*(b\cdot(\nabla\otimes u^2))$ strongly converges to $b\cdot(\nabla\otimes u^2)$ in $L^1([0,T);L^2(\mathbb{R}^n))$. Hence for every $\gamma(t)\in\mathcal{D}([0,T))$,

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^n}\gamma(t)(\omega_\epsilon*u^1)\cdot(\omega_\epsilon*(b\cdot(\nabla\otimes u^2)))\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^n}\gamma(t)u^1\cdot(b\cdot(\nabla\otimes u^2))\mathrm{d}x\mathrm{d}t\right|\\&=\left|\int_0^T\int_{\mathbb{R}^n}\gamma(t)\left(\omega_\epsilon*u^1-u^1\right)\cdot(b\cdot(\nabla\otimes u^2))\mathrm{d}x\mathrm{d}t\right|\\&\quad+\left|\int_0^T\int_{\mathbb{R}^n}\gamma(t)(\omega_\epsilon*u^1)\cdot\left(\omega_\epsilon*(b\cdot(\nabla\otimes u^2))-b\cdot(\nabla\otimes u^2)\right)\mathrm{d}x\mathrm{d}t\right|\\&\le\left|\int_0^T\int_{\mathbb{R}^n}\gamma(t)\left(\omega_\epsilon*u^1-u^1\right)\cdot(b\cdot(\nabla\otimes u^2))\mathrm{d}x\mathrm{d}t\right|\\&\quad+\|\gamma\|_{L^\infty}\|\omega_\epsilon*(b\cdot(\nabla\otimes u^2))-b\cdot(\nabla\otimes u^2)\|_{L^1([0,T);L^2)}\|u_0\|_{L^2}\\&\longrightarrow 0,\qquad\text{as }\epsilon\to 0,\end{aligned}$$

which directly leads to that in $\mathcal{D}'([0,T))$,

$$\int_{\mathbb{R}^n}(\omega_\epsilon*u^1)\cdot(\omega_\epsilon*\nabla\cdot(b\otimes u^2))\mathrm{d}x\to\int_{\mathbb{R}^n}u^1\cdot(b\cdot(\nabla\otimes u^2))\mathrm{d}x=\int_{\mathbb{R}^n}(b\cdot(\nabla\otimes u^2))\cdot u^1\mathrm{d}x. \tag{2.29}$$

Gathering (2.24) and the convergence results (2.25)–(2.29), we conclude that in $\mathcal{D}'([0,T))$,

$$\partial_t\int_{\mathbb{R}^n}(u^1\cdot u^2)\mathrm{d}x+2\int_{\mathbb{R}^n}\left((\nabla\otimes u^1)\cdot(\nabla\otimes u^2)\right)\mathrm{d}x=0. \tag{2.30}$$

Since u^i $(i=1,2)$ is a weak solution to the drift–diffusion system (1.1), it also satisfies that following integral equation

$$u^i(x,t)=e^{t\Delta}u_0-\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b\cdot\nabla u^i)(x,\tau)\mathrm{d}\tau. \tag{2.31}$$

We claim that

$$u^i \ (i=1,2) \text{ is strongly } L^2-\text{continuous for every } t\in[0,T). \tag{2.32}$$

Indeed, from $u_0\in(L^2(\mathbb{R}^n))^n$, $e^{t\Delta}u_0$ for every t is continuous in $(L^2(\mathbb{R}^n))^n$; while recalling that for $g\in L^{\frac{2}{2-\beta}}([0,T);\dot H^{-\beta}(\mathbb{R}^n))$ with some $\beta\in[0,1]$, the function $t\mapsto \int_0^t e^{(t-\tau)\Delta}\mathbb{P}g(\tau)\mathrm{d}\tau$ is continuous in t with values in $L^2(\mathbb{R}^n)$ (see [12, Pg. 392]), thus thanks to (2.28), the function $\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b\cdot\nabla u^i)\mathrm{d}\tau$ is also L^2-continuous about the time variable; hence together with (2.31), the assertion (2.32) is followed.

Therefore, by virtue of (2.32) the map $t\mapsto\int_{\mathbb{R}^n}u^1\cdot u^2\mathrm{d}x$ for every $t\in[0,T)$ is continuous. We can integrate the equality (2.30) to get the desired equality (2.22). □

Based on Proposition 2.3 and the energy estimate (2.1), we now prove the uniqueness result. We have that for every $t\in[0,T)$,

$$\begin{aligned}
\|u^1(t)-u^2(t)\|^2_{L^2(\mathbb{R}^2)} &= \|u^1(t)\|^2_{L^2(\mathbb{R}^n)}+\|u^2(t)\|^2_{L^2(\mathbb{R}^n)}-2\int_{\mathbb{R}^n}u^1(x,t)\cdot u^2(x,t)\,\mathrm{d}x\\
&= \|u^1(t)\|^2_{L^2(\mathbb{R}^n)}+\|u^2(t)\|^2_{L^2(\mathbb{R}^n)}+4\int_0^t\int_{\mathbb{R}^n}(\nabla\otimes u^1)\cdot(\nabla\otimes u^2)\mathrm{d}x\mathrm{d}\tau-2\|u_0\|^2_{L^2(\mathbb{R}^n)}\\
&\le -2\int_0^t\int_{\mathbb{R}^n}|\nabla\otimes u^1|^2\mathrm{d}x\mathrm{d}\tau-2\int_0^t\int_{\mathbb{R}^n}|\nabla\otimes u^2|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad+4\int_0^t\int_{\mathbb{R}^n}(\nabla\otimes u^1)\cdot(\nabla\otimes u^2)\mathrm{d}x\mathrm{d}\tau\\
&\le -2\int_0^t\int_{\mathbb{R}^n}|\nabla\otimes u^1(x,\tau)-\nabla\otimes u^2(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Hence $u^1\equiv u^2$ on $[0,T)\times\mathbb{R}^n$, and we conclude the uniqueness of weak solutions for system (1.1) at the $p=2$ case.

2.3 Proof of Hölder regularity result

Throughout this subsection, let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be a radially symmetric test function such that

$$\phi\equiv1,\ \text{ on } B_{1/2};\qquad \operatorname{supp}\phi\subset B_1;\qquad \int_{\mathbb{R}^n}\phi\,\mathrm{d}x=1.$$

We first recall the definition of the L^2-based Morrey–Campanato space $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$, which is very useful in the proof: the Morrey–Campanato space $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$ with $\lambda\in(0,n+2)$ is the set of $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned}
\|f\|_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} &:= \sup_{x_0\in\mathbb{R}^n,r>0}\left(\frac{1}{r^\lambda}\int_{B_r(x_0)}|f(x)-\bar f(x_0,r)|^2\phi\left(\frac{x-x_0}{r}\right)\mathrm{d}x\right)^{1/2}\\
&= \sup_{x_0\in\mathbb{R}^n,r>0}\left(\frac{1}{r^{\lambda-n}}\int_{B_1}|f(x_0+ry)-\bar f(x_0,r)|^2\phi(y)\,\mathrm{d}y\right)^{1/2}<\infty,
\end{aligned}\tag{2.33}$$

with $\bar f(x_0,r):=\int_{B_1}f(x_0+ry)\phi(y)\mathrm{d}y$. The L^2-based Morrey–Campanato space $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$ has the following important equivalence property (e.g. see [19, Pg. 361]).

Lemma 2.4 *We have*

$$\mathcal{L}^{2,\lambda}(\mathbb{R}^n) = \begin{cases} BMO(\mathbb{R}^n), & \text{if } \lambda = n, \\ \dot{C}^{\frac{\lambda-n}{2}}(\mathbb{R}^n), & \text{if } \lambda \in (n, n+2), \\ \dot{W}^{1,\infty}(\mathbb{R}^n), & \text{if } \lambda = n+2, \end{cases} \tag{2.34}$$

where $BMO(\mathbb{R}^n)$ is the space of functions with bounded mean oscillation (e.g. see [9, Chapter 6]*).*

In this subsection, the following modulus of continuity is also of frequent use

$$\omega(\xi, \xi_0) = \begin{cases} (1-\alpha)\xi_0^\alpha + \alpha\xi_0^{\alpha-1}\xi, & \text{if } 0 < \xi \le \xi_0, \\ \xi^\alpha, & \text{if } \xi > \xi_0, \end{cases} \tag{2.35}$$

where $\alpha \in (0, 1)$, $\xi_0 = \xi_0(t)$ satisfies that

$$\dot{\xi}_0 = -\rho\xi_0^{-1}, \quad \xi_0(0) = A_0, \tag{2.36}$$

with some constants $\rho, A_0 > 0$, that is,

$$\xi_0(t) = \sqrt{A_0^2 - 2\rho t}. \tag{2.37}$$

Noticing that as $\xi_0 \to 0$, $\omega(\xi, \xi_0)$ reduces to $\omega(\xi, 0+) = \omega(\xi) = \xi^\alpha$ which is the modulus of continuity of the α-Hölder continuity.

Since we can first consider the approximate solution u_ϵ ($\epsilon > 0$) solving the system (2.9) with $p_\epsilon = (-\Delta)^{-1}\mathrm{div}\,(b_\epsilon \cdot \nabla u_\epsilon)$ and then pass to the limit $\epsilon \to 0$, we here only focus on the *a priori* estimates and always assume that $u(x, t) \in (C([0, T); H^m(\mathbb{R}^n)) \cap C^1([0, T); H^{m-2}(\mathbb{R}^n)))^n$ ($m \in (\frac{n}{2} + 2, \infty) \cap \mathbb{N}$) is a classical solution to the Eqs. (1.1), (1.2). In order to show the *a priori* estimate concerning the improvement of the solution from $u(t) \in L^2(\mathbb{R}^n)$ to $u(t) \in \dot{C}^\alpha(\mathbb{R}^n)$, we divide the proof into three steps.

Step 1 A *priori* estimate of $u(t)$ on the improvement from $L^2(\mathbb{R}^n)$ to $\mathcal{L}^{2,2\alpha}(\mathbb{R}^n)$.

By setting the weighted mean of u on $B_\xi(x)$ as

$$\bar{u}(x, \xi, t) = \int_{B_1} u(x + \xi y, t)\phi(y)\mathrm{d}y, \tag{2.38}$$

with $\phi \in \mathcal{D}(\mathbb{R}^n)$ the same test function in (2.33), we firstly intend to control the following quantity

$$I_1(x, \xi, t) = \xi^n \int_{B_1} |u(x + \xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y)\mathrm{d}y \tag{2.39}$$

through the suitable modulus of continuity $\omega_1(\xi, t)^2$, which is precisely given by

$$\omega_1(\xi, t) = f_1(t)\omega(\xi, \xi_0) = \begin{cases} f_1(t)\big((1-\alpha)\xi_0^\alpha + \alpha\xi_0^{\alpha-1}\xi\big), & \text{if } 0 < \xi \le \xi_0, \\ f_1(t)\xi^\alpha, & \text{if } \xi > \xi_0, \end{cases} \tag{2.40}$$

with $f_1(t) > 0$ a non-decreasing differentiable function chosen later [see (2.70) below]. That is, for such defined I_1 and ω_1, we shall prove that for every $x \in \mathbb{R}^n$ and $\xi > 0$,

$$I_1(x, \xi, t) < \omega_1(\xi, t)^2, \quad \text{for all } t \in [0, T). \tag{2.41}$$

Before proving (2.41), we first show the direct consequence of this uniform inequality (2.41). From the expression formula of (2.37), we see that $\xi_0(t_1) = 0$ at the time $t_1 := \frac{A_0^2}{2\rho}$

with ρ given by (2.66) below, thus for $t \geq t_1$, $\omega_1(\xi, t)$ reduces to $f_1(t)\xi^\alpha$, and (2.41) implies that for every $x \in \mathbb{R}^n$ and $\xi > 0$,

$$\xi^{n-2\alpha} \int_{B_1} |u(x+\xi y, t) - \bar{u}(x, \xi, t)|^2 \phi(y) \mathrm{d}y \leq f_1(t)^2, \quad \forall t \geq t_1, \tag{2.42}$$

which according to (2.33) means that $u(t)$ for every $t \geq t_1$ belongs to the Morrey–Campanato space $\mathcal{L}^{2,2\alpha}(\mathbb{R}^n)$.

Next we proceed to prove (2.41) for all $x \in \mathbb{R}^n$, $\xi > 0$ and $t \in [0, T)$. Observing that

$$\begin{aligned} I_1(x, \xi, 0) &= \xi^n \int_{B_1} |u_0(x+\xi y) - \bar{u}(x, \xi, 0)|^2 \phi(y) \mathrm{d}y \\ &= \xi^n \int_{B_1} |u_0(x+\xi y)|^2 \phi(y) \mathrm{d}y - \xi^n |\bar{u}(x, \xi, 0)|^2 \leq \int_{B_\xi(x)} |u_0(y)|^2 \mathrm{d}y \leq \|u_0\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.43}$$

and $\omega_1(\xi, 0) \geq f_1(0)\omega(0+, \xi_0(0)) = (1-\alpha) f_1(0) A_0^\alpha$, we infer that by choosing $f_1(0)$ to be large enough so that

$$f_1(0) \geq \frac{\sqrt{2}\|u_0\|_{L^2(\mathbb{R}^n)}}{(1-\alpha)A_0^\alpha}, \tag{2.44}$$

we have $I_1(x, \xi, 0) < \omega_1(\xi, 0)^2$ for all $x \in \mathbb{R}^n$ and $\xi > 0$.

Now we assume that the strict inequality $I_1(x, \xi, t) < \omega_1(\xi, t)^2$ is firstly lost at some time $t_* \in (0, T)$ (without loss of generality). Since u is a smooth function that has spatial decay at infinity, from the time continuity, we get

$$I_1(x, \xi, t_*) \leq \omega_1(\xi, t_*), \quad \text{for all } x \in \mathbb{R}^n, \xi > 0. \tag{2.45}$$

A direct consequence of (2.45) is the following result.

Lemma 2.5 *Assume that the assumption (2.45) is satisfied, then there exists a positive constant $C = C(n, \alpha)$ so that for every $(x, \xi) \in \mathbb{R}^n \times (0, \infty)$ and for every $j \in \mathbb{N}$,*

$$\xi^n \int_{B_{2^j}} |u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)|^2 \mathrm{d}y \leq C(j+1)^2 2^{j(n+2\alpha)} \omega_1(\xi, t_*)^2. \tag{2.46}$$

Proof of Lemma 2.5 Thanks to the change of variables, the support property of ϕ and Hölder's inequality, we deduce that

$$\begin{aligned} &\xi^n \int_{B_{2^j}} |u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)|^2 \mathrm{d}y \\ &= \xi^n 2^{(j+1)n} \int_{B_{1/2}} |u(x+2^{j+1}\xi y, t_*) - \bar{u}(x, \xi, t_*)|^2 \mathrm{d}y \\ &\leq \xi^n 2^{(j+1)n} \\ &\quad \times \int_{B_1} |u(x+2^{j+1}\xi y, t_*) - \bar{u}(x, \xi, t_*)|^2 \phi(y) \mathrm{d}y \\ &\leq \xi^n 2^{(j+1)n+1} \int_{B_1} |u(x+2^{j+1}\xi y, t_*) - \bar{u}(x, 2^{j+1}\xi, t_*)|^2 \phi(y) \mathrm{d}y \\ &\quad + \xi^n 2^{(j+1)n+1} \\ &\quad \times \left(\sum_{k=0}^{j} |\bar{u}(x, 2^{k+1}\xi, t_*) - \bar{u}(x, 2^k\xi, t_*)| \right)^2 \end{aligned}$$

$$
\begin{aligned}
&\le 2\omega_1(2^{j+1}\xi, t_*)^2 + \xi^n 2^{(j+1)n+1}(j+1) \\
&\quad \times \sum_{k=0}^{j} \int_{B_1} |u(x+2^k\xi y, t_*) - \bar{u}(x, 2^{k+1}\xi, t_*)|^2 \phi(y)\mathrm{d}y \\
&\le 2\omega_1(2^{j+1}\xi, t_*)^2 + \xi^n 2^{(j+2)n+1}(j+1) \\
&\quad \times \sum_{k=0}^{j} \int_{B_{1/2}} |u(x+2^{k+1}\xi y, t_*) - \bar{u}(x, 2^{k+1}\xi, t_*)|^2 \mathrm{d}y \\
&\le 2\omega_1(2^{j+1}\xi, t_*)^2 + 2^{(j+2)n+1}(j+1) \sum_{k=0}^{j} 2^{-(k+1)n}\omega_1(2^{k+1}\xi, t_*)^2 \\
&\le C_0 2^{(j+2)n}(j+1)^2\omega_1(2^{j+1}\xi, t_*)^2. \qquad (2.47)
\end{aligned}
$$

Then we control $\omega_1(2^{j+1}\xi, t_*)$ from $\omega_1(\xi, t_*)$. If $0 < \xi \le \xi_0$, we get $\omega_1(\xi, t_*) \ge (1-\alpha) f_1(t_*)\xi_0^\alpha$ and $\omega_1(2^{j+1}\xi, t_*) \le 2^{(j+1)\alpha} f_1(t_*)\xi_0^\alpha$, thus $\omega_1(2^{j+1}\xi, t_*) \le \frac{1}{1-\alpha} 2^{(j+1)\alpha}\omega_1(\xi, t_*)$. Whereas if $\xi > \xi_0$, we directly see that $\omega_1(2^{j+1}\xi, t_*) \le 2^{(j+1)\alpha}\omega_1(\xi, t_*)$. Hence $\omega_1(2^{j+1}\xi, t_*) \le \frac{1}{1-\alpha} 2^{(j+1)\alpha}\omega_1(\xi, t_*)$ for all $\xi > 0$. Inserting this estimate into (2.47) yields (2.46). □

Besides, we have the following breakdown criterion.

Lemma 2.6 *Let $t_* \in (0, T)$ be the first time that the strict preservation (2.41) is lost, then there exists some $x \in \mathbb{R}^n$ and $\xi > 0$ such that*

$$
I_1(x, \xi, t_*) = \omega_1(\xi, t_*)^2. \qquad (2.48)
$$

The proof of this lemma is postponed in Sect. 2.4.

Since $I_1(x, \xi, t_*)$ attains its maximum at x for fixed (x, ξ) in (2.48), we get

$$
\begin{aligned}
\nabla_x I_1 = 2\xi^n \int_{B_1} &(u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)) \cdot (\nabla_x \otimes u(x+\xi y, t_*) \\
&-\nabla_x \otimes \bar{u}(x, \xi, t_*))\, \phi(y)\mathrm{d}y = 0. \qquad (2.49)
\end{aligned}
$$

From the definition of $\bar{u}$ (2.38), we also see that

$$
\int_{B_1} (u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)) \cdot \nabla_x \otimes u(x+\xi y, t_*)\phi(y)\mathrm{d}y = 0. \qquad (2.50)
$$

By virtue of the fact $I_1(x, \xi, t) < \omega_1(\xi, t)^2$ for all $t \in [0, t_*)$, we have

$$
\partial_t \omega_1(\xi, t_*)^2 \le \partial_t I_1(x, \xi, t_*),
$$

which leads to that

$$
\begin{aligned}
\omega_1(\xi, t_*)\partial_t\omega_1(\xi, t_*) &\le \xi^n \int_{B_1} (u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)) \cdot (\partial_t u(x+\xi y, t_*) \\
&\qquad -\partial_t \bar{u}(x, \xi, t_*))\, \phi(y)\mathrm{d}y \\
&= \xi^n \int_{B_1} (u(x+\xi y, t_*) - \bar{u}(x, \xi, t_*)) \cdot \partial_t u(x+\xi y, t_*)\phi(y)\mathrm{d}y. \qquad (2.51)
\end{aligned}
$$

Plugging the Eq. (1.1) into the right-hand-side of (2.51) yields

$$\begin{aligned}\omega_1(\xi,t_*)\partial_t\omega_1(\xi,t_*) \le &-\xi^n\int_{B_1}(u(x+\xi y,t_*)-\bar u(x,\xi,t_*))\cdot(b(x+\xi y,t_*)\\&\cdot\nabla_x\otimes u(x+\xi y,t_*))\,\phi(y)\mathrm{d}y\\&+\xi^n\int_{B_1}(u(x+\xi y,t_*)-\bar u(x,\xi,t_*))\cdot\Delta_x u(x+\xi y,t_*)\phi(y)\mathrm{d}y\\&-\xi^n\int_{B_1}(u(x+\xi y,t_*)-\bar u(x,\xi,t_*))\cdot\nabla_x p(x+\xi y,t_*)\phi(y)\mathrm{d}y\\:=&\ \mathcal{C}(x,\xi,t_*)+\mathcal{D}(x,\xi,t_*)+\mathcal{P}(x,\xi,t_*).\end{aligned}\tag{2.52}$$

Recalling that $\omega_1(\xi,t)$ and $\xi_0(t)$ are defined by (2.40) and (2.36) respectively, we obtain that for $0<\xi<\xi_0(t_*)$,

$$\begin{aligned}\partial_t\omega_1(\xi,t_*)=&f_1'(t_*)\left((1-\alpha)\xi_0^\alpha+\alpha\xi_0^{\alpha-1}\xi\right)\\&+f_1(t_*)\left(\alpha(1-\alpha)\xi_0^{\alpha-1}\dot\xi_0-\alpha(1-\alpha)\xi_0^{\alpha-2}\dot\xi_0\xi\right)\\=&f_1'(t_*)\left((1-\alpha)\xi_0^\alpha+\alpha\xi_0^{\alpha-1}\xi\right)-\rho\alpha(1-\alpha)f_1(t_*)\left(\xi_0^{\alpha-2}-\xi_0^{\alpha-3}\xi\right)\\\ge&(1-\alpha)f_1'(t_*)\xi_0^\alpha-\rho\alpha(1-\alpha)f_1(t_*)\xi_0^{\alpha-2},\end{aligned}\tag{2.53}$$

and for $\xi\ge\xi_0(t_*)$,

$$\partial_t\omega_1(\xi,t_*)=f_1'(t_*)\xi^\alpha.\tag{2.54}$$

For the contribution from the convection term $\mathcal{C}$, taking advantage of (2.50), $\nabla_x\otimes u(x+\xi y,t_*)=\frac{1}{\xi}\nabla_y\otimes u(x+\xi y,t_*)=\frac{1}{\xi}\nabla_y\otimes\big(u(x+\xi y,t_*)-\bar u(x,\xi,t_*)\big)$, the divergence-free property of b and the integration by parts, we get

$$\begin{aligned}\mathcal{C}=&-\xi^{n-1}\int_{B_1}(u(x+\xi y,t_*)-\bar u(x,\xi,t_*))\\&\cdot\left(\left(b(x+\xi y,t_*)-\bar b(x,\xi,t_*)\right)\cdot\nabla_y\otimes u(x+\xi y,t_*)\right)\phi(y)\mathrm{d}y\\=&-\frac{1}{2}\xi^{n-1}\int_{B_1}\left(b(x+\xi y,t_*)-\bar b(x,\xi,t_*)\right)\cdot\nabla_y\left(|u(x+\xi y,t_*)-\bar u(x,\xi,t_*)|^2\right)\phi(y)\mathrm{d}y\\=&\frac{1}{2}\xi^{n-1}\int_{B_1}|u(x+\xi y,t_*)-\bar u(x,\xi,t_*)|^2\left(b(x+\xi y,t_*)-\bar b(x,\xi,t_*)\right)\cdot\nabla\phi(y)\mathrm{d}y.\end{aligned}\tag{2.55}$$

Thanks to the following estimate [deduced from (1.8)] that for every $y\in B_1$:

$$\begin{aligned}|b(x+\xi y,t_*)-\bar b(x,\xi,t_*)|\le&\int_{B_1}|b(x+\xi y,t_*)-b(x+\xi z,t_*)|\phi(z)\mathrm{d}z\\\le&\|b(t_*)\|_{\dot M^p}\xi^{\frac{2-p}{p}}\int_{B_1}|y-z|^{\frac{2-p}{p}}\phi(z)\mathrm{d}z\le2\|b(t_*)\|_{\dot M^p}\xi^{\frac{2-p}{p}},\end{aligned}\tag{2.56}$$

and using Lemma 2.5, we infer that

$$\begin{aligned}\mathcal{C}(x,\xi,t_*) &\le \frac{1}{\xi}C_0\|b(t_*)\|_{\dot{M}^p}\xi^{\frac{2-p}{p}}\xi^n\int_{B_1}|u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*)|^2\mathrm{d}y\\ &\le C\|b(t_*)\|_{\dot{M}^p}\xi^{\frac{2}{p}-2}\omega_1(\xi,t_*)^2\\ &\le \begin{cases} C\|b(t_*)\|_{\dot{M}^p}\omega_1(\xi,t_*)f(t_*)\xi^{\frac{2}{p}-2}\xi_0^\alpha, & \text{if } 0<\xi\le\xi_0(t_*);\\ C\|b(t_*)\|_{\dot{M}^p}\omega_1(\xi,t_*)f(t_*)\xi^{\frac{2}{p}+\alpha-2}, & \text{if } \xi>\xi_0(t_*).\end{cases}\end{aligned} \tag{2.57}$$

For the contribution of the dissipation term $\mathcal{D}$, we shall use the following crucial lemma which is a modification of [29, Lemma 3.4]:

Lemma 2.7 *For fixed x, ξ and t_* appearing in (2.48), there exists a small constant $c_*>0$ depending only on n and ϕ so that*

$$\begin{aligned}-\mathcal{D}(x,\xi,t_*) &\ge c_*\frac{\big((n+2)I_1(x,\xi,t_*)-\xi\partial_\xi I_1(x,\xi,t_*)\big)^2}{\xi^2 I_1(x,\xi,t_*)}\\ &= c_*\frac{\big((n+2)\omega_1(\xi,t_*)-2\xi\partial_\xi\omega_1(\xi,t_*)\big)^2}{\xi^2}.\end{aligned} \tag{2.58}$$

For the proof of Lemma 2.7, one can see Sect. 2.4 below.

According to Lemma 2.7, if $0<\xi\le\xi_0(t_*)$, we see that $\omega_1(\xi,t_*)=f_1(t_*)\Big((1-\alpha)\xi_0^\alpha+\alpha\xi_0^{\alpha-1}\xi\Big)\le f_1(t_*)\xi_0^\alpha$, $\partial_\xi\omega_1(\xi,t_*)=\alpha f_1(t_*)\xi_0^{\alpha-1}$, and

$$\begin{aligned}(n+2)\omega_1(\xi,t_*)-2\xi\partial_\xi\omega_1(\xi,t_*) &= f_1(t_*)\Big((n+2)(1-\alpha)\xi_0^\alpha+n\alpha\xi_0^{\alpha-1}\xi\Big)\\ &\ge 2f_1(t_*)(1-\alpha)\xi_0^\alpha,\end{aligned}$$

thus

$$\mathcal{D}(x,\xi,t_*)\le -c_*\frac{4(1-\alpha)^2f_1(t_*)^2\xi_0^{2\alpha}}{\xi^2}\le -4c_*(1-\alpha)^2\omega_1(\xi,t_*)f_1(t_*)\xi_0^\alpha\xi^{-2}; \tag{2.59}$$

whereas if $\xi>\xi_0(t_*)$, we see that $\omega_1(\xi,t_*)=f_1(t_*)\xi^\alpha$, $\partial_\xi\omega_1(\xi,t_*)=\alpha f_1(t_*)\xi^{\alpha-1}$ and $(n+2)\omega_1(\xi,t_*)-2\xi\partial_\xi\omega_1(\xi,t_*)=f_1(t_*)(n+2-2\alpha)\xi^\alpha$, thus

$$\mathcal{D}(x,\xi,t_*)\le -c_*\frac{4(1-\alpha)^2f_1(t_*)^2\xi^{2\alpha}}{\xi^2}\le -4c_*(1-\alpha)^2\omega_1(\xi,t_*)f_1(t_*)\xi^{\alpha-2}. \tag{2.60}$$

Next we consider the contribution of the pressure term $\mathcal{P}$. Thanks to the following expression [deduced from (1.2) and divergence-free property of u]

$$\begin{aligned}\nabla_x p(x+\xi y,t_*) &= \nabla_x(-\Delta_x)^{-1}\mathrm{div}_x\,(b(x+\xi y,t_*)\cdot\nabla_x\otimes u(x+\xi y,t_*))\\ &= \frac{1}{\xi}\nabla_y(-\Delta_y)^{-1}\mathrm{div}_y\,\big(b(x+\xi y,t_*)\cdot\nabla_y\otimes(u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*))\big)\\ &= \frac{1}{\xi}\nabla_y(-\Delta_y)^{-1}\mathrm{div}_y\,\Big(\big(b(x+\xi y,t_*)-\bar{b}(x,\xi,t_*)\big)\\ &\quad\cdot\nabla_y\otimes(u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*))\Big),\end{aligned}$$

we have

$\mathcal{P}(x,\xi)$

$$= -\frac{1}{\xi}\int_{B_1}\big(u(x+y\xi)-\bar u(x,\xi)\big)\cdot\nabla_y(-\Delta_y)^{-1}\,\mathrm{div}_y\,\big((b(x+\xi y)-\bar b(x,\xi))\cdot\nabla_y$$
$$\otimes\,(u(x+\xi y)-\bar u(x,\xi))\big)\phi(y)\mathrm{d}y$$
$$= -\frac{1}{\xi}\int_{B_1}\big(u_k(x+y\xi)-\bar u_k(x,\xi)\big)\,\partial_{y_k}(-\Delta_y)^{-1}\partial_{y_i}\big((b_j(x+\xi y)$$
$$-\,\bar b_j(x,\xi))\,\partial_{y_j}(u_i(x+\xi y)-\bar u_i(x,\xi))\big)\phi(y)\mathrm{d}y$$
$$= -\frac{1}{\xi}\int_{B_1}\big(u_k(x+y\xi)-\bar u_k(x,\xi)\big)$$
$$\times\;\partial_{y_k}\partial_{y_i}\partial_{y_j}(-\Delta_y)^{-1}\big((b_j(x+\xi y)-\bar b_j(x,\xi))\,(u_i(x+\xi y)-\bar u_i(x,\xi))\big)\phi(y)\mathrm{d}y,$$

where we suppressed the t_*-dependence in the above formula and we also used the Einstein convention on repeated indices. Through the integration by parts and using the divergence-free property of u and b, we find

$$\mathcal{P}(x,\xi)$$
$$=\xi^{n-1}\int_{\mathbb{R}^n}\partial_{y_i}\partial_{y_j}(-\Delta_y)^{-1}\left(\big(b_i(x+\xi y)-\bar b_i(x,\xi)\big)\big(u_j(x+\xi y)-\bar u_j(x,\xi)\big)\right)(u(x+y\xi)$$
$$-\bar u(x,\xi))\cdot\nabla\phi(y)\mathrm{d}y$$
$$=\xi^{n-1}\int_{\mathbb{R}^n}\big(b_i(x+\xi y)-\bar b_i(x,\xi)\big)\big(u_j(x+\xi y)-\bar u_j(x,\xi)\big)\,\mathcal{R}_i\mathcal{R}_j\,((u(x+y\xi)$$
$$-\bar u(x,\xi))\cdot\nabla\phi(y))\,\mathrm{d}y,$$

where $\mathcal{R}_i=\partial_{y_i}(-\Delta)^{-\frac12}$ $(i=1,\dots,n)$ is the classical Riesz transform (e.g. see [9, Eq. (4.7)]). Note that the operator $\mathcal{R}_i\mathcal{R}_j:=\mathcal{F}^{-1}(-\frac{\zeta_i\zeta_j}{|\zeta|^2})$ $(i,j=1,\dots,n)$ has the expression formula [see (2.15)]

$$\mathcal{R}_i\mathcal{R}_j g(y)=a_{ij}g(y)+\mathcal{T}_{ij}g(y), \tag{2.61}$$

where $a_{ij}=-\frac1n$ for $i=j$ and $a_{ij}=0$ for $i\neq j$, and $\mathcal{T}_{ij}$ is a singular integral operator

$$\mathcal{T}_{ij}g(y):=\mathrm{p.v.}\int_{\mathbb{R}^n}K_{ij}(y-z)g(z)\mathrm{d}z=\lim_{\epsilon\to0}\int_{|y-z|>0}K_{ij}(y-z)g(z)\mathrm{d}z, \tag{2.62}$$

with the kernel $K_{ij}(y)=\frac{\Omega_{ij}(\hat y)}{|y|^n}$, $\forall y\neq0$, $\hat y=\frac{y}{|y|}$, and $\Omega_{ij}(\hat y)\in C^\infty(\mathbb{S}^{n-1})$ satisfying the zero-average property. Thus taking advantage of (2.56), we get

$$\mathcal{P}\le C\|b\|_{\dot M^p}\xi^{n+\frac2p-2}\int_{B_1}|u(x+\xi y)-\bar u(x,\xi)|^2\mathrm{d}y$$
$$+\,C\|b\|_{\dot M^p}\xi^{n+\frac2p-2}\int_{|y|\le2}|u(x+\xi y)-\bar u(x,\xi)|\Big|\mathcal{T}_{ij}\Big(\big(u(x+\xi y)-\bar u(x,\xi)\big)\cdot\nabla\phi(y)\Big)\Big|\mathrm{d}y$$
$$+\,C\|b\|_{\dot M^p}\xi^{n+\frac2p-2}\int_{|y|\ge2}|y|^{\frac{2-p}{p}}|u(x+\xi y)-\bar u(x,\xi)|\Big|\mathcal{T}_{ij}\Big(\big(u(x+\xi y)-\bar u(x,\xi)\big)\cdot\nabla\phi(y)\Big)\Big|\mathrm{d}y$$
$$:=\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3.$$

The first term $\mathcal{P}_1$ can be exactly estimated as (2.57). For the second term $\mathcal{P}_2$, by virtue of the Hölder inequality, the Calderón–Zygmund theorem (e.g. see [9, Chapter 4]) and Lemma 2.5, we obtain

$$\mathcal{P}_2\le C\|b\|_{\dot M^p}\xi^{n+\frac2p-2}\left(\int_{|y|\le2}|u(x+\xi y)-\bar u(x,\xi)|^2\mathrm{d}y\right)^{1/2}$$

$$\times\left(\int_{\mathbb{R}^n}\left|\big(u(x+\xi y)-\bar u(x,\xi)\big)\cdot\nabla\phi(y)\right|^2\mathrm{d}y\right)^{1/2}$$
$$\le C\|b\|_{\dot M^p}\xi^{n+\frac{2}{p}-2}\int_{|y|\le 2}|u(x+\xi y)-\bar u(x,\xi)|^2\mathrm{d}y\le C\|b\|_{\dot M^p}\xi^{\frac{2}{p}-2}\omega_1(\xi,t_*)^2. \tag{2.63}$$

For $\mathcal{P}_3$, from the integration by parts and the mean-free property of $(u-\bar u)\phi$, we find

$$\mathcal{T}_{ij}\Big(\big(u(x+\xi y)-\bar u(x,\xi)\big)\cdot\nabla\phi(y)\Big)=\text{p.v.}\int_{\mathbb{R}^n}K_{ij}(y-z)\Big(\big(u(x+\xi z)-\bar u(x,\xi)\big)\cdot\nabla_z\phi(z)\Big)\,\mathrm{d}z$$
$$=\text{p.v.}\int_{\mathbb{R}^n}\phi(z)\big(u(x+\xi z)-\bar u(x,\xi)\big)\cdot\nabla K_{ij}(y-z)\mathrm{d}z$$
$$=\text{p.v.}\int_{\mathbb{R}^n}\phi(z)\big(u(x+\xi z)-\bar u(x,\xi)\big)\cdot\big(\nabla K_{ij}(y-z)-\nabla K_{ij}(y)\big)\mathrm{d}z,$$

thus thanks to the following estimate (deduced from the mean value theorem and support property)

$$|K_{ij}(y-z)-K_{ij}(y)|\le C_0|\nabla^2K_{ij}(y)||z|\le C_0\frac{1}{|y|^{n+2}},\quad\forall y\in B_2^c,z\in B_1,$$

and using Hölder's inequality and (2.46), we infer that

$$\mathcal{P}_3\le C\|b\|_{\dot M^p}\xi^{n+\frac{2}{p}-2}\int_{|y|\ge 2}|y|^{\frac{2-p}{p}}\frac{1}{|y|^{n+2}}|u(x+\xi y)-\bar u(x,\xi)|\mathrm{d}y$$
$$\times\left(\int_{B_1}|u(x+\xi z)-\bar u(x,\xi)|^2\phi(z)\mathrm{d}z\right)^{1/2}$$
$$\le C\|b\|_{\dot M^p}\omega_1(\xi,t_*)\xi^{\frac{n}{2}+\frac{2}{p}-2}\sum_{j=1}^{\infty}\int_{2^j\le|y|\le 2^{j+1}}\frac{1}{|y|^{n+3-2/p}}|u(x+\xi y)-\bar u(x,\xi)|\mathrm{d}y$$
$$\le C\|b\|_{\dot M^p}\omega_1(\xi,t_*)\xi^{\frac{n}{2}+\frac{2}{p}-2}\sum_{j=1}^{\infty}\frac{1}{2^{j(n/2+3-2/p)}}\left(\int_{|y|\le 2^{j+1}}|u(x+\xi y)-\bar u(x,\xi)|^2\mathrm{d}y\right)^{1/2}$$
$$\le C\|b\|_{\dot M^p}\omega_1(\xi,t_*)^2\xi^{\frac{2}{p}-2}\sum_{j=1}^{\infty}2^{-j(3-2/p-\alpha)}(j+2)$$
$$\le C\|b\|_{\dot M^p}\xi^{\frac{2}{p}-2}\omega_1(\xi,t_*)^2. \tag{2.64}$$

Collecting the estimates on $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ yields

$$\mathcal{P}(x,\xi,t_*)\le C\|b\|_{\dot M^p}\xi^{\frac{2}{p}-2}\omega_1(\xi,t_*)^2. \tag{2.65}$$

Hence we insert the above estimates (2.53), (2.54), (2.57), (2.59)–(2.60) and (2.65) into (2.52) to get that for $0<\xi\le\xi_0(t_*)$,

$$f_1'(t_*)\xi_0^\alpha-\rho\alpha(1-\alpha)f_1(t_*)\xi_0^{\alpha-2}\le C\|b(t_*)\|_{\dot M^p}f_1(t_*)\xi^{\frac{2}{p}-2}\xi_0^\alpha-4c_*(1-\alpha)^2f_1(t_*)\xi_0^\alpha\xi^{-2},$$

and for $\xi>\xi_0(t_*)$,

$$f_1'(t_*)\xi^\alpha\le C\|b(t_*)\|_{\dot M^p}f_1(t_*)\xi^{\frac{2}{p}+\alpha-2}-4c_*(1-\alpha)^2f_1(t_*)\xi^{\alpha-2}.$$

If $0 < \xi \leq \xi_0(t_*)$, by letting

$$\rho := \frac{2c_*(1-\alpha)}{\alpha}, \tag{2.66}$$

we have

$$f_1'(t_*) \leq f_1(t_*) \left(C\|b(t_*)\|_{\dot{M}^p} \xi^{\frac{2}{p}-2} - 2c_*(1-\alpha)^2 \xi^{-2} \right); \tag{2.67}$$

whereas if $\xi > \xi_0(t_*)$,

$$f_1'(t_*) \leq f_1(t_*) \left(C\|b(t_*)\|_{\dot{M}^p} \xi^{\frac{2}{p}-2} - 4c_*(1-\alpha)^2 \xi^{-2} \right). \tag{2.68}$$

By maximizing values of the right-hand-side terms of the above two formulas, we find that

$$f_1'(t_*) \leq C_1 \|b(t_*)\|_{\dot{M}^p}^p f_1(t_*), \tag{2.69}$$

with $C_1 > 0$ a constant depending only on n, α, p.

Hence, if we set $f_1(t)$ to be defined by $f_1'(t) = 2C_1\|b(t)\|_{\dot{M}^p} f_1(t)$, that is,

$$f_1(t) = f_1(0) \exp\left\{ 2C_1 \int_0^t \|b(\tau)\|_{\dot{M}^p}^p \mathrm{d}\tau \right\} \tag{2.70}$$

where $f_1(0)$ can be chosen as $\frac{2\|u_0\|_{L^2(\mathbb{R}^n)}}{(1-\alpha)A_0^\alpha}$ (satisfying (2.44)), we have that (2.69) can not hold true and thus $I_1(x,\xi,t) < \omega_1(\xi,t)^2$ for all $(x,\xi,t) \in \mathbb{R}^n \times (0,\infty) \times [0,T)$ with $\omega_1(\xi,t) = f_1(t)\omega(\xi,\xi_0)$.

Step 2 A priori estimate of $u(t)$ on the improvement from $\mathcal{L}^{2,2\alpha}(\mathbb{R}^n)$ to $\mathcal{L}^{2,n}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

We can further repeat the above process to show a similar improvement. By setting

$$I_i(x,\xi,t) := \xi^{n-2\alpha(i-1)} \int_{B_1} |u(x+\xi y,t) - \bar{u}(x,\xi,t)|^2 \phi(y)\mathrm{d}y, \quad \text{for every } i = 1,2\ldots, \tag{2.71}$$

the conclusion of the above step (2.42) reads as

$$I_2(x,\xi,t) \leq f_1(t)^2, \quad \forall t \in [t_1,T), \tag{2.72}$$

with $t_1 = \frac{A_0^2}{2\rho}$ and $f_1(t)$ defined by (2.70). Based on (2.72), we shall similarly prove that

$$I_2(x,\xi,t) < \omega_2(\xi,t-t_1)^2, \quad \text{for all } t \in [t_1,T), \tag{2.73}$$

with $\omega_2(\xi,\cdot) = f_2(\cdot)\omega(\xi,\xi_0(\cdot))$ and $f_2(\cdot) > 0$ a non-decreasing function chosen later [see (2.81)].

Since $I_2(x,\xi,t_1) \leq f_1(t_1)^2$ and $\omega_2(\xi,0) \geq \omega_2(0,0) = f_2(0)(1-\alpha)A_0^\alpha$, we see that (2.73) holds true for $t = t_1$ by choosing $f_2(0) = \frac{2f_1(t_1)}{A_0^\alpha(1-\alpha)}$. If we suppose that $t_* \in (t_1,T)$ is the first time that the strict inequality (2.73) is lost, then there exists some $x \in \mathbb{R}^n$ and $\xi > 0$ such that

$$I_2(x,\xi,t_*) = \omega_2(\xi,t_*-t_1)^2, \tag{2.74}$$

and by denoting $t_{*,1} := t_* - t_1$, similarly as above we get

$$\begin{aligned} &\omega_2(\xi,t_{*,1})\partial_t\omega_2(\xi,t_{*,1}) \\ &\leq -\xi^{n-2\alpha} \int_{B_1} \big(u(x+\xi y,t_*) - \bar{u}(x,\xi,t_*)\big) \cdot \big(b(x+\xi y,t_*) \cdot \nabla_x \otimes u(x+\xi y,t_*)\big)\phi(y)\mathrm{d}y \end{aligned} \tag{2.75}$$

$$
\begin{aligned}
&+\xi^{n-2\alpha}\int_{B_1}\big(u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*)\big)\cdot\Delta_x u(x+\xi y,t_*)\phi(y)\mathrm{d}y\\
&-\xi^{n-2\alpha}\int_{B_1}\big(u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*)\big)\cdot\nabla_x p(x+\xi y,t_*)\phi(y)\mathrm{d}y\\
:=\ &\overline{\mathcal{C}}(x,\xi,t_*)+\overline{\mathcal{D}}(x,\xi,t_*)+\overline{\mathcal{P}}(x,\xi,t_*).
\end{aligned}\tag{2.76}
$$

Noting that [analogous with (2.46)]

$$
\begin{aligned}
&\xi^{n-2\alpha}\int_{B_{2^j}}|u(x+\xi y,t_*)-\bar{u}(x,\xi,t_*)|^2\mathrm{d}y\le C(j+1)^2 2^{j(n+2\alpha)}\omega_2(\xi,t_{*,1})^2,\\
&\qquad\forall(x,\xi)\in\mathbb{R}^n\times(0,\infty),
\end{aligned}\tag{2.77}
$$

and by arguing as (2.53), (2.54), (2.57), (2.63)–(2.65) and (2.58)–(2.60), we infer that

$$
\partial_t\omega_2(\xi,t_{*,1})\ge\begin{cases}(1-\alpha)f_2'(t_{*,1})\xi_0^\alpha-\rho\alpha(1-\alpha)f_2(t_{*,1})\xi_0^{\alpha-2}, & \text{if } 0<\xi\le\xi_0(t_{*,1}),\\ f_2'(t_{*,1})\xi^\alpha, & \text{if } \xi>\xi_0(t_{*,1}),\end{cases}
$$

and

$$
\overline{\mathcal{C}}(x,\xi,t_*)+\overline{\mathcal{P}}(x,\xi,t_*)\le\begin{cases}C\|b(t_*)\|_{\dot{M}^p}\omega_2(\xi,t_{*,1})f(t_{*,1})\xi^{\frac{2}{p}-2}\xi_0^\alpha, & \text{if } 0<\xi\le\xi_0(t_{*,1}),\\ C\|b(t_*)\|_{\dot{M}^p}\omega_2(\xi,t_{*,1})f(t_{*,1})\xi^{\frac{2}{p}+\alpha-2}, & \text{if } \xi>\xi_0(t_{*,1}),\end{cases}
$$

and (with $c_*>0$ the same constant appearing in Lemma 2.7)

$$
\begin{aligned}
\overline{\mathcal{D}}(x,\xi,t_*)&\le -c_*\frac{\big((n+2-2\alpha)\omega_2(\xi,t_{*,1})-2\xi\partial_\xi\omega_2(\xi,t_{*,1})\big)^2}{\xi^2}\\
&\le\begin{cases}-4c_*(1-\alpha)^2\omega_2(\xi,t_{*,1})f_2(t_{*,1})\xi_0^\alpha\xi^{-2}, & \text{if } 0<\xi\le\xi_0(t_{*,1}),\\ -4c_*(1-\alpha)^2\omega_2(\xi,t_{*,1})f_2(t_{*,1})\xi^{\alpha-2}, & \text{if } \xi>\xi_0(t_{*,1}).\end{cases}
\end{aligned}
$$

Hence, we obtain that for every $0<\xi\le\xi_0(t_{*,1})$,

$$
\begin{aligned}
&f_2'(t_{*,1})\xi_0^\alpha-\rho\alpha(1-\alpha)f_2(t_{*,1})\xi_0^{\alpha-2}\le C\|b(t_*)\|_{\dot{M}^p}f_2(t_{*,1})\xi^{\frac{2}{p}-2}\xi_0^\alpha\\
&\quad-4c_*(1-\alpha)^2 f_2(t_{*,1})\xi_0^\alpha\xi^{-2},
\end{aligned}\tag{2.78}
$$

and for every $\xi>\xi_0(t_{*,1})$,

$$
f_2'(t_{*,1})\xi^\alpha\le C\|b(t_*)\|_{\dot{M}^p}f_2(t_{*,1})\xi^{\frac{2}{p}+\alpha-2}-4c_*(1-\alpha)^2 f_2(t_{*,1})\xi^{\alpha-2}.\tag{2.79}
$$

By choosing $\rho>0$ as (2.66), we see that for some constant $C_2>0$ depending only on n,α,p,

$$
f_2'(t_{*,1})\le C_2\|b(t_*)\|_{\dot{M}^p}^p f_2(t_{*,1}).\tag{2.80}
$$

But if we set $f_2(\cdot)>0$ as

$$
f_2(t-t_1)=\frac{2f_1(t_1)}{A_0^\alpha(1-\alpha)}\exp\left\{2C_2\int_{t_1}^t\|b(\tau)\|_{\dot{M}^p}^p\mathrm{d}\tau\right\},\quad\forall t\ge t_1,\tag{2.81}
$$

we see that (2.80) does not hold true, which in turn concludes the uniform inequality (2.73). Due to that $\xi_0(t-t_1)=0$ for all $t\ge 2t_1$, the preservation (2.73) and the definition of $f(t)$

(2.70) imply that

$$\begin{aligned} I_3(x,\xi,t) &= \xi^{n-4\alpha}\int_{B_1}|u(x+\xi y,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y \le f_2(t-t_1)^2 \\ &\le \frac{16\|u_0\|^2_{L^2(\mathbb{R}^n)}}{A_0^{4\alpha}(1-\alpha)^4}\exp\left\{4C\int_0^t\|b(\tau)\|^p_{\dot{M}^p}\mathrm{d}\tau\right\}, \quad \forall t\in[2t_1,T), \end{aligned} \tag{2.82}$$

which also guarantees that $u(t)$ $(t \ge 2t_1)$ belongs to the Morrey–Campanato space $\mathcal{L}^{2,4\alpha}(\mathbb{R}^n)$.

If $\alpha \in (\frac{1}{2}, 1)$ and $n = 2$, we see that $n - 2\alpha > 0$ and $n - 4\alpha < 0$, thus by letting $\bar{\theta} = \frac{4\alpha-n}{2\alpha} \in (0, 1)$, and using (2.72) and (2.82), we have that for every $t \in [2t_1, T)$,

$$\begin{aligned} \int_{B_1}|u(x+\xi y,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y &\le I_2(x,\xi,t)^{\bar{\theta}}I_3(x,\xi,t)^{1-\bar{\theta}} \\ &\le f_1(t)^{2\bar{\theta}}f_2(t-t_1)^{2-2\bar{\theta}} \\ &\le \frac{16\|u_0\|^2_{L^2(\mathbb{R}^n)}}{A_0^{n}(1-\alpha)^{n/\alpha}}\exp\left\{4C\int_0^t\|b(\tau)\|^p_{\dot{M}^p}\mathrm{d}\tau\right\}, \end{aligned}$$

which ensures that $u(t)$ $(t \ge 2t_1)$ belongs to the Morrey–Campanato space $\mathcal{L}^{2,n}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

For other scope of α and n, we can iterate the above process for a finite times to show the desired estimate. Under the condition that for $i = 1, 2, \dots, [\frac{n}{2\alpha}]$ (with $[\frac{n}{2\alpha}]$ the integer part of the number $\frac{n}{2\alpha}$)

$$I_{i+1}(x,\xi,t) \le f_i(t-(i-1)t_1)^2, \quad \forall t\in[it_1,T), \tag{2.83}$$

we intend to show that

$$I_{i+1}(x,\xi,t) < \omega_{i+1}(\xi,t-it_1)^2, \quad \forall t\in[it_1,T), \tag{2.84}$$

where $\omega_{i+1}(\xi,t) = f_{i+1}(t-it_1)\omega(\xi,\xi_0(t-it_1))$ and $f_{i+1}(t-it_1)$ is a suitable non-decreasing function (see (2.89)). Indeed, firstly by choosing $f_{i+1}(0) = \frac{2f_i(t_1)}{A_0^{\alpha}(1-\alpha)}$ we see that (2.84) is satisfied for $t = it_1$; then if we assume that $t_* \in (t_1, T)$ is the first time that (2.84) is lost, then there exists some $x \in \mathbb{R}^n$ and $\xi > 0$ such that

$$I_{i+1}(x,\xi,t_*) = \omega_{i+1}(\xi,t_*-it_1)^2, \tag{2.85}$$

and by denoting $t_{*,i} := t_* - it_1$, we can deduce (as above) that for every $0 < \xi \le \xi_0(t_{*,i})$,

$$\begin{aligned} & f'_{i+1}(t_{*,i})\xi_0^{\alpha} - \rho\alpha(1-\alpha)f_{i+1}(t_{*,i})\xi_0^{\alpha-2} \le C\|b(t_*)\|_{\dot{M}^p}f_{i+1}(t_{*,i})\xi^{\frac{2}{p}-2}\xi_0^{\alpha} \\ & -4c_*(1-\alpha)^2f_{i+1}(t_{*,i})\xi_0^{\alpha}\xi^{-2}, \end{aligned} \tag{2.86}$$

and for every $\xi > \xi_0(t_{*,i})$,

$$f'_{i+1}(t_{*,i})\xi^{\alpha} \le C\|b(t_*)\|_{\dot{M}^p}f_{i+1}(t_{*,i})\xi^{\frac{2}{p}+\alpha-2} - 4c_*(1-\alpha)^2f_{i+1}(t_{*,i})\xi^{\alpha-2}, \tag{2.87}$$

where $c_* > 0$ is just the same number appraring in Lemma 2.7. By choosing $\rho > 0$ as (2.66), we see that for some constant $C_{i+1} > 0$ depending only on n, α, p,

$$f'_{i+1}(t_{*,i}) \le C_{i+1}\|b(t_*)\|^p_{\dot{M}^p}f_{i+1}(t_{*,i}). \tag{2.88}$$

But if we set $f_{i+1}(t-it_1)>0$ as

$$f_{i+1}(t-it_1)=\frac{2f_i(t_1)}{A_0^{\alpha}(1-\alpha)}\exp\left\{2C_{i+1}\int_{it_1}^{t}\|b(\tau)\|_{\dot{M}^p}^{p}\mathrm{d}\tau\right\},\quad \forall t\geq it_1 \tag{2.89}$$

we conclude that (2.88) does not hold, which in turn proves the inequality (2.84). Furthermore, thanks to $\xi_0(t-it_1)=0$ for all $t\geq(i+1)t_1=\frac{(i+1)A_0}{2\rho}$, we have

$$\begin{aligned}I_{i+2}(x,\xi,t)&=\xi^{n-(i+1)2\alpha}\int_{B_1}|u(x+\xi y,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y\leq f_{i+1}(t-it_1)^2\\&\leq\frac{4^{i+1}\|u_0\|_{L^2(\mathbb{R}^n)}^2}{(1-\alpha)^{2(i+1)}A_0^{2(i+1)\alpha}}\exp\left\{4C\int_0^t\|b(\tau)\|_{\dot{M}^p}^{p}\mathrm{d}\tau\right\},\quad\forall t\in[(i+1)t_1,T).\end{aligned} \tag{2.90}$$

Notice that we finally can obtain the estimates of $I_i(x,\xi,t)$ and $I_{i+1}(x,\xi,t)$ for $i=i_\alpha:=[\frac{n}{2\alpha}]$. Due to that $n-2\alpha(i_\alpha+1)<0$ and $n-2\alpha i_\alpha\geq 0$, by setting $\theta_\alpha:=\frac{2\alpha(i_\alpha+1)-n}{2\alpha}\in(0,1]$, and from (2.90), we conclude that for all $t\geq(i_\alpha+1)t_1$,

$$\begin{aligned}\int_{B_1}|u(x+\xi y,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y&\leq I_{i_\alpha+1}(x,\xi,t)^{\theta_\alpha}I_{i_\alpha+2}(x,\xi,t)^{1-\theta_\alpha}\\&\leq f_{i_\alpha}(t-(i_\alpha-1)t_1)^{2\theta_\alpha}f_{i_\alpha+1}(t-i_\alpha t_1)^{2(1-\theta_\alpha)}\\&\leq\frac{4^{i_\alpha+1}\|u_0\|_{L^2(\mathbb{R}^n)}^2}{A_0^n(1-\alpha)^{n/\alpha}}\exp\left\{4C\int_0^t\|b(\tau)\|_{\dot{M}^p}^{p}\mathrm{d}\tau\right\},\end{aligned} \tag{2.91}$$

which implies that $u(t)$ $(t\geq(i_\alpha+1)t_1)$ belongs to the Morrey–Campanato space $\mathcal{L}^{2,n}(\mathbb{R}^n)=BMO(\mathbb{R}^n)$.

Step 3 A priori estimate of $u(t)$ on the improvement from $\mathcal{L}^{2,n}(\mathbb{R}^n)$ to $\mathcal{L}^{2,n+2\alpha}(\mathbb{R}^n)=\dot{C}^{\alpha}(\mathbb{R}^n)$.

Based on (2.91), we can further intend to show that for every $t\in[(i_\alpha+1)t_1,T)$,

$$\int_{B_1}|u(x+y\xi,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y\leq\omega_{i_\alpha+2}(\xi,t-(i_\alpha+1)t_1)^2, \tag{2.92}$$

with $\omega_{i_\alpha+2}(\xi,\tilde{t})=f_{i_\alpha+2}(\tilde{t})\omega(\xi,\xi_0(\tilde{t}))$ $(\tilde{t}:=t-(i_\alpha+1)t_1)$ and $f_{i_\alpha+2}(\cdot)$ an appropriate non-decreasing function. Indeed, the proof is almost the same as the deduction at the above two steps, and by letting $f_{i_\alpha+2}$ be defined as:

$$f_{i_\alpha+2}(t-(i_\alpha+1)t_1)=f_{i_\alpha+2}(0)\exp\left\{2C_{i_\alpha+2}\int_{(i_\alpha+1)t_1}^{t}\|b(\tau)\|_{\dot{M}^p}^{p}\mathrm{d}\tau\right\}, \tag{2.93}$$

with $f_{i_\alpha+2}(0)=\frac{2}{A_0^{\alpha}(1-\alpha)}f_{i_\alpha}(2t_1)^{\theta_\alpha}f_{i_\alpha+1}(t_1)^{1-\theta_\alpha}$ and $C_{i_\alpha+2}$ some constant depending only on n,α,p, we can show that (2.92) holds true. (2.92) also guarantees that for all $t\in[(i_\alpha+2)t_1,T)$,

$$\begin{aligned}&\xi^{-2\alpha}\int_{B_1}|u(x+\xi y,t)-\bar{u}(x,\xi,t)|^2\phi(y)\mathrm{d}y\\&\leq f_{i_\alpha+2}(t-(i_\alpha+1)t_1)^2\\&\leq\frac{4^{i_\alpha+2}\|u_0\|_{L^2(\mathbb{R}^n)}^2}{A_0^{n+2\alpha}(1-\alpha)^{(n+2\alpha)/\alpha}}\exp\left\{4C\int_0^t\|b(\tau)\|_{\dot{M}^p}^{p}\mathrm{d}\tau\right\},\end{aligned} \tag{2.94}$$

which corresponds to that $u(t)$ $(t\in[(i_\alpha+2)t_1,T))$ belongs to the Morrey–Campanato space $\mathcal{L}^{2,n+2\alpha}(\mathbb{R}^n)$, or equivalently, Hölder space $\dot{C}^{\alpha}(\mathbb{R}^n)$.

Therefore, for $t' > 0$ any small number fixed, we can choose t_1 and A_0 small enough so that $t' \in [(i_\alpha + 2)t_1, T)$. Recalling that $t_1 = \frac{A_0^2}{2\rho}$, $\rho = \frac{2c_*(1-\alpha)}{\alpha}$, $i_\alpha = [\frac{n}{2\alpha}]$, we can let

$$A_0 = \left(\frac{2c_*(1-\alpha)}{\left([\frac{n}{2\alpha}] + 2\right)\alpha} t' \right)^{1/2}, \tag{2.95}$$

that is, $t' = 2(i_\alpha + 2)t_1$, thus we conclude that $u(t)$ for every $t \in [t', T)$ belongs to the Hölder space $\dot{C}^\alpha(\mathbb{R}^n)$, and (2.94) with such an A_0 immediately leads to (1.10). In combination with the preservation of L^2-norm of $u(t)$, this moreover yields that $u(t)$ for every $t \in [t', T)$ belongs to the inhomogeneous Hölder space $C^\alpha(\mathbb{R}^n)$, and due to the arbitrariness of t', we complete the proof of Theorem 1.1.

2.4 Proofs of auxiliary results

We first show the proof of the crucial breakdown scenario (2.48).

Proof of Lemma 2.6 We prove by contradiction. We suppose that there is no point $(x, \xi) \in \mathbb{R}^n \times (0, \infty)$ so that the equality $I_1(x, \xi, t_*) = \omega_1(\xi, t_*)^2$ holds. Denoting by

$$F_1(x, \xi, t) = \frac{I_1(x, \xi, t)}{\omega_1(\xi, t)^2}, \tag{2.96}$$

and from $F_1(x, \xi, t) < 1$ for all $t \in (0, t_*)$, we see that $F_1(x, \xi, t_*) \leq 1$ for all $x \in \mathbb{R}^n$ and $\xi > 0$. Since we suppose that (2.48) is not true, we necessarily get $F_1(x, \xi, t_*) < 1$ for all $x \in \mathbb{R}^n$ and $\xi > 0$.

The following deduction is divided into several cases according to the values of (x, ξ). Recalling that $\omega_1(\xi, t)$ is defined by (2.40), if ξ is large enough so that $\omega_1(\xi, t_*)^2 \geq 2\|u_0\|_{L^2}^2$ and $\xi \geq A_0 = \xi_0(0)$, that is, for $\xi \geq C_1 := \max\{(\frac{\sqrt{2}\|u_0\|_{L^2}}{f_1(t_*)})^{\frac{1}{\alpha}}, A_0\}$, then by using the energy estimate $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$ for all $t \in (0, T)$ and the nondecreasing property $\omega_1(\xi, t) \geq \omega_1(\xi, t_*)$ for $\xi \geq A_0$ and $t \in [t_*, T)$, we argue as (2.43) to get that for all $t \in [t_*, T)$,

$$I_1(x, \xi, t) \leq \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \leq \frac{1}{2}\omega_1(\xi, t_*)^2 \leq \frac{1}{2}\omega_1(\xi, t)^2. \tag{2.97}$$

If $\xi > 0$ is small enough and $\xi_0(t_*) > 0$, there exists a small constant $h_1 = h_1(t_*) > 0$ such that $\xi_0(t_* + h_1) > 0$ and $t_* + h_1 < T$, then due to that $\omega_1(\xi, t) \geq \omega_1(0, t) = (1-\alpha) f_1(t)\, \xi_0(t)^\alpha > 0$ for $t \in [t_*, t_* + h_1]$ and

$$\lim_{\xi \to 0+} I_1(x, \xi, t) \leq \lim_{\xi \to 0+} C\|\nabla u(t)\|_{L^\infty}^2 \xi^{n+2} = 0, \quad \text{for every } t \in [t_*, t_* + h_1],$$

there exists a small constant $c_1 = c_1(t_*, h_1) > 0$ so that for all $0 < \xi \leq c_1$ and $t \in [t_*, t_* + h_1]$ we have

$$I_1(x, \xi, t) \leq \frac{1}{2}\omega_1(\xi, t)^2. \tag{2.98}$$

If $\xi > 0$ is small enough and $\xi_0(t_*) = 0$, then $\omega_1(\xi, t) = f_1(t)\xi^\alpha$ for all $\xi > 0$ and $t \in [t_*, T)$, and by virtue of the estimate $I_1(x, \xi, t) \leq C\|\nabla u(t)\|_{L^\infty}^2 \xi^{n+2}$ for $t \in [t_*, T)$, there also exists small constant $h_2 > 0$ and $c_1' = c_1'(t_*, h_2) > 0$ so that (2.98) also holds for $0 < \xi \leq c_1'$ and $t \in [t_*, t_* + h_2]$. Thus by denoting $\tilde{c}_1 := \min\{c_1, c_1'\}$, we only need to consider the case $\tilde{c}_1 \leq \xi \leq C_1$. Since u has the spatial decay near infinity, there exists a small constant $h_3 = h_3(t_*) > 0$ and a large number $M = M(t_*, h_3) > 0$ such that for all $t \in [t_*, t_* + h_3]$ and $|z| \geq M$,

$$|u(z,t)| \leq \frac{1}{2} C_1^{-n/2} f_1(t_*)\omega(\tilde{c}_1, \xi_0(t_* + h_2)) \leq \frac{1}{2} C_1^{-n/2}\omega_1(\tilde{c}_1, t),$$

then for $x \in \{|x| \geq M + C_1\}$ and $t \in [t_*, t_* + h_3]$, we have

$$I_1(x,\xi,t) = \xi^n \int_{B_1} |u(x+\xi y,t)|^2 \phi(y)\mathrm{d}y - \xi^n |\bar{u}(x,\xi,t)|^2 \leq \frac{1}{2}\omega_1(\tilde{c}_1,t)^2 \leq \frac{1}{2}\omega_1(\xi,t)^2. \tag{2.99}$$

Then it suffices to consider the continuous function $F_1(x,\xi,t) = \frac{I_1(x,\xi,t)}{\omega_1(\xi,t)^2}$ on the compact set

$$\mathcal{K} = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^+ \middle| |x| \leq M + C_1,\ \tilde{c}_1 \leq \xi \leq C_1\right\};$$

and due to $F_1(x,\xi,t_*) < 1$, there exists a small constant $h_4 = h_4(t_*) > 0$ such that $F_1(x,\xi,t) < 1$ for all $t \in [t_*, t_* + h_4]$ and $(x,\xi) \in \mathcal{K}$.

Let $h = \min\{h_1, h_2, h_3, h_4\}$, then $h > 0$ and $F_1(x,\xi,t) < 1$ on $t \in [t_*, t_* + h]$ for all $(x,\xi) \in \mathbb{R}^n \times (0,\infty)$, which clearly contradicts with the definition of t_*. Hence there is indeed some $(x,\xi) \in \mathcal{K}$ so that $F_1(x,\xi,t_*) = 1$, that is, the scenario (2.48) holds as claimed. □

Next we turn to proving the key Lemma 2.7.

Proof of Lemma 2.7 Note that the quantity $I_1(x,\xi,t_*)$ defined by (2.39) can be expressed as

$$I_1(x,\xi,t_*) = \frac{\xi^n}{2}\int_{B_1}\int_{B_1} |u(x+\xi y,t_*) - u(x+\xi z,t_*)|^2 \phi(y)\phi(z)\mathrm{d}y\mathrm{d}z. \tag{2.100}$$

Since at the point $x \in \mathbb{R}^n$ the quantity $I_1(\cdot,\xi,t_*)$ attains its maximum, we have

$$\begin{aligned}
0 \geq\ & \Delta_x I_1(x,\xi,t_*)\\
=\ & \xi^n \int_{B_1}\int_{B_1} \big(u(x+\xi y,t_*) - u(x+\xi z,t_*)\big)\\
& \cdot \big(\Delta_x u(x+\xi y,t_*) - \Delta_x u(x+\xi z,t_*)\big)\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\\
& + \xi^n \int_{B_1}\int_{B_1} |\nabla\otimes u(x+\xi y,t_*) - \nabla\otimes u(x+\xi z,t_*)|^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\\
=\ & 2\mathcal{D}(x,\xi,t_*) + \xi^n \int_{B_1}\int_{B_1} |\nabla\otimes u(x+\xi y,t_*) - \nabla\otimes u(x+\xi z,t_*)|^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z,
\end{aligned}$$

which implies that

$$-\mathcal{D}(x,\xi,t_*) \geq \frac{\xi^n}{2}\int_{B_1}\int_{B_1} |\nabla\otimes u(x+\xi y,t_*) - \nabla\otimes u(x+\xi z,t_*)|^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z. \tag{2.101}$$

Notice also that

$$\begin{aligned}
& \partial_\xi I_1(x,\xi,t_*)\\
& = \frac{n}{2}\xi^{n-1}\int_{B_1}\int_{B_1} |u(x+\xi y,t_*) - u(x+\xi z,t_*)|^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\\
& + \xi^n \int_{B_1}\int_{B_1} \big(u(x+\xi y,t_*) - u(x+\xi z,t_*)\big)\cdot\big(y\cdot\nabla\otimes u(x+\xi y,t_*)\\
& - z\cdot\nabla\otimes u(x+\xi z,t_*)\big)\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

□

We shall adapt the following key lemma (see [29, Lemma 3.5]) to analyse the relations among the quantities $I_1(x,\xi,t_*)$, $\partial_\xi I_1(x,\xi,t_*)$ and $\mathcal{D}(x,\xi,t_*)$.

Lemma 2.8 *Let $f=(f_1,\ldots,f_n):B_1\to\mathbb{R}^n$ be a vector field satisfying $f\in(H^1(\mathbb{R}^n))^n$, and $\phi\in\mathcal{D}(\mathbb{R}^n)$ be a radially-symmetric non-increasing (in radius) smooth function supported on B_1. Then there is a constant C depending only on n and ϕ such that*

$$\begin{aligned}&\iint_{B_1\times B_1}(f(y)-f(z))^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z-\iint_{B_1\times B_1}\big(y\cdot\nabla\otimes f(y)\\&\quad-z\cdot\nabla\otimes f(z)\big)\cdot(f(y)-f(z))\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\\&\le C\left(\iint_{B_1\times B_1}(\nabla\otimes f(y)-\nabla\otimes f(z))^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\right)^{1/2}\\&\quad\times\left(\iint_{B_1\times B_1}(f(y)-f(z))^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z\right)^{1/2}.\end{aligned}$$

Due to that the proof of this lemma presented in [29] seems a little bit obscure, we here give an explicit and elementary proof of Lemma 2.8, which is placed in the end of this subsection.

The estimate on the dissipation term (2.58) is now a direct consequence of Lemma 2.8. By letting $f(y)=u(x+\xi y,t_*)$ and using the scaling transform, we get

$$2I_1(x,\xi,t_*)-\big(\xi\partial_\xi I_1(x,\xi,t_*)-nI_1(x,\xi,t_*)\big)\le C\left(-2\xi^2\mathcal{D}(x,\xi,t_*)\right)^{1/2}(2I_1(x,\xi,t_*))^{1/2},$$

which leads to

$$-\mathcal{D}(x,\xi,t_*)\ge c_*\frac{\big((n+2)I_1(x,\xi,t_*)-\xi\partial_\xi I_1(x,\xi,t_*)\big)^2}{\xi^2I_1(x,\xi,t_*)}.\tag{2.102}$$

The second equality of (2.58) is just followed from the equality (2.48).

At last we show the proof of the crucial Lemma 2.8.

Proof of Lemma 2.8 Observe that from [29, Lemma 3.5] it suffices to control the quantity $\mathcal{E}$ defined as

$$\begin{aligned}\mathcal{E}:=&\iint_{B_1\times B_1}\int_0^1\Big(\big(\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(y)\big)\cdot y\\&-\big(\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(z)\big)\cdot z\Big)^2\phi(y)\phi(z)\mathrm{d}s\mathrm{d}y\mathrm{d}z,\end{aligned}$$

by the following quantity

$$\Pi:=\iint_{B_1\times B_1}|\nabla\otimes f(y)-\nabla\otimes f(z)|^2\phi(y)\phi(z)\mathrm{d}y\mathrm{d}z,\tag{2.103}$$

more precisely, we only need to prove that

$$\mathcal{E}\le C\Pi.$$

Denoting by

$$\mathcal{E}_1:=\iint_{B_1\times B_1}\int_0^1\big(\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(y)\big)^2\phi(y)\phi(z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}z,\tag{2.104}$$

and from $\mathcal{E}\le C_0\mathcal{E}_1+C_0\Pi$, it moreover reduces to show the following inequality

$$\mathcal{E}_1\le C\Pi.\tag{2.105}$$

For simplicity, we here also suppose that $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies that[2]

$$2^{-1-\frac{1}{1-|y|}} \leq \phi(y) \leq 2^{-\frac{1}{1-|y|}} \quad \text{for } \; 3/4 \leq |y| < 1. \tag{2.106}$$

If $y \in B_{3/4}$, $z \in B_{3/4}$, we see that $\phi(y) \approx \phi(z) \approx \phi(sy + (1-s)z)$ for $s \in [0, 1]$, thus by the changing of variables we have

$$\begin{aligned}
&\iint_{B_{3/4}\times B_{3/4}} \int_0^1 |\nabla \otimes f(sy + (1-s)z) - \nabla \otimes f(y)|^2 \phi(y)\phi(z)\, \mathrm{d}s\mathrm{d}y\mathrm{d}z \\
&\leq C \iint_{B_{3/4}\times B_{3/4}} \int_0^{1/2} |\nabla \otimes f(sy + (1-s)z) - \nabla \otimes f(y)|^2 \phi(y)\phi(sy + (1-s)z)\, \mathrm{d}s\mathrm{d}y\mathrm{d}z \\
&\quad + C \iint_{B_{3/4}\times B_{3/4}} \int_{1/2}^1 |\nabla \otimes f(sy + (1-s)z) - \nabla \otimes f(z)|^2 \phi(sy + (1-s)z)\phi(z)\, \mathrm{d}s\mathrm{d}y\mathrm{d}z \\
&\quad + C \iint_{B_{3/4}\times B_{3/4}} |\nabla \otimes f(y) - \nabla \otimes f(z)|^2 \phi(y)\phi(z)\, \mathrm{d}y\mathrm{d}z \\
&\leq C \iint \int_0^{1/2} |\nabla \otimes f(\tilde{z}) - \nabla \otimes f(y)|^2 (1-s)^{-n} \phi(y)\phi(\tilde{z})\, \mathrm{d}s\mathrm{d}y\mathrm{d}\tilde{z} \\
&\quad + C \iint \int_{1/2}^1 |\nabla \otimes f(\tilde{y}) - \nabla \otimes f(z)|^2 s^{-n} \phi(\tilde{y})\phi(z)\, \mathrm{d}s\mathrm{d}\tilde{y}\mathrm{d}z + C\Pi \\
&\leq C\Pi.
\end{aligned} \tag{2.107}$$

Without loss of generality, we assume $|y| \leq |z|$ in the sequel, since otherwise it can be similarly treated. For the case $y \in B_{3/4}$ and $z \in B_{3/4}^c$, we infer that there is an absolute constant $s_0 \in (0, 1)$ such that $\phi(y) \leq C_0\phi(sy + (1-s)z)$ for $s \in [s_0, 1]$ (the dangerous case is that y and z is of the same direction), and also from $|sy + (1-s)z| \leq |z|$ we get $\phi(z) \leq \phi(sy + (1-s)z)$, thus we can follow the above argument by dividing the interval $s \in [0, 1]$ into $s \in [0, s_0]$ and $s \in [s_0, 1]$ to show that

$$\int_{B_{3/4}^c} \int_{B_{3/4}} \int_0^1 |\nabla \otimes f(sy + (1-s)z) - \nabla \otimes f(y)|^2 \phi(y)\phi(z)\, \mathrm{d}s\mathrm{d}y\mathrm{d}z \leq C\Pi. \tag{2.108}$$

By denoting $\mathcal{C}_j := \{y \in B_1 : 2^{-j-1} \leq \phi(y) \leq 2^{-j}\}$ for $j \in \mathbb{N}$, we next consider $y \in \mathcal{C}_j$ and $z \in \mathcal{C}_{j+k}$ for $j \in \mathbb{N} \cap [4, \infty)$ and $k \in \mathbb{N}$, and for such y and z, from (2.106), it leads to that

$$y \in \left\{\mathbb{R}^n : 1 - \frac{1}{j} \leq |y| \leq 1 - \frac{1}{j+1}\right\}, \quad \text{and} \quad z \in \left\{\mathbb{R}^n : 1 - \frac{1}{j+k} \leq |z| \leq 1 - \frac{1}{j+k+1}\right\}.$$

Let $\theta \in [0, 2\pi)$ be the angle between $y \in \mathcal{C}_j$ and $z \in \mathcal{C}_{j+k}$ so that $\hat{z} = \hat{y}e^{i\theta}$, where $\hat{y} := \frac{y}{|y|}$ and $\hat{z} := \frac{z}{|z|}$. If $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, then the considered case is similar to the case of $y \in B_{3/4}$ and $z \in B_{3/4}^c$, and we omit the details. If $\theta \in [0, \frac{\pi}{2})$ (the case of $\theta \in (\frac{3\pi}{2}, 2\pi)$ is the same), we deduce that

$$\begin{aligned}
|sy + (1-s)z| &= \left|s|y|\hat{y} + (1-s)|z|e^{i\theta}\hat{y}\right| \\
&= \sqrt{|y|^2s^2 + 2|y||z|s(1-s)\cos\theta + (1-s)^2|z|^2} \leq s|y| + (1-s)|z|
\end{aligned}$$

[2] One can similarly consider the test function ϕ with other faster spatial decay, like $\phi(y) \approx 2^{-\frac{1}{1-|y|^2}}$ for $3/4 \leq |y| < 1$.

and

$$|sy+(1-s)z| \geq \sqrt{|y|^2s^2+|z|^2(1-s)^2} \geq |y|\sqrt{s^2+(1-s)^2} \geq \frac{\sqrt{2}}{2}|y| \geq \frac{1}{2},$$

which implies that $\phi(\frac{1}{2}) \geq \phi(sy+(1-s)z) \geq \phi(s|y|+(1-s)|z|)$. According to (2.106) and the following facts that

$$s|y|+(1-s)|z| \leq s\left(1-\frac{1}{j+1}\right)+(1-s)\left(1-\frac{1}{j+k+1}\right)=1-\left(\frac{s}{j+1}+\frac{1-s}{j+k+1}\right),$$

we obtain that

$$\phi(s|y|+(1-s)|z|) \geq 2^{-\frac{(j+1)(j+k+1)}{s(j+k+1)+(1-s)(j+1)}} = 2^{-(j+1)}2^{-\frac{(1-s)(j+1)k}{j+1+sk}} \geq \frac{1}{2}\phi(y)2^{-\frac{(1-s)(j+1)k}{j+1+sk}}, \tag{2.109}$$

thus by choosing σ to be $\frac{1}{4}$ if $k \leq 4$, and to be $\frac{1}{k^2}$ if $k \geq 4$, we find that $\phi(sy+(1-s)z) \approx \phi(s|y|+(1-s)|z|) \approx \phi(y)$ for every $s \in [1-\sigma, 1]$ and $y \in \mathcal{C}_j \cap \{|sy+(1-s)z| \geq 1-\frac{1}{j}\}$, and furthermore

$$\begin{aligned}
&\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\int_{y\in\mathcal{C}_j}\int_{z\in\mathcal{C}_{j+k},\theta\in[0,\frac{\pi}{2})}\int_{1-\sigma}^{1}|\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(z)|^2\phi(y)\phi(z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}z\\
&\quad\leq C\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\int_{\tilde{y}\in\{1-\frac{1}{j}\leq|\tilde{y}|\leq1-\frac{1}{j+k+1}\}}\int_{z\in\mathcal{C}_{j+k}}\int_{1-\sigma}^{1}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\phi(\tilde{y})\phi(z)s^{-n}\mathrm{d}s\mathrm{d}\tilde{y}\mathrm{d}z\\
&\qquad+C\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\int_{\tilde{y}\in\{\frac{1}{2}\leq|\tilde{y}|\leq1-\frac{1}{j}\}}\int_{z\in\mathcal{C}_{j+k}}\int_{1-\sigma}^{1}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\frac{\phi(y)}{\phi(\tilde{y})}\phi(\tilde{y})\phi(z)s^{-n}\mathrm{d}s\mathrm{d}\tilde{y}\mathrm{d}z\\
&\quad\leq C\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\sum_{l=0}^{k}\int_{\tilde{y}\in\mathcal{C}_{j+l}}\int_{z\in\mathcal{C}_{j+k}}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\sigma\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z\\
&\qquad+C\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\sum_{l=1}^{j-2}\int_{\tilde{y}\in\mathcal{C}_{j-l}}\int_{z\in\mathcal{C}_{j+k}}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\sigma\,2^{-l}\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z\\
&\quad\leq C\sum_{j=4}^{\infty}\sum_{k=4}^{\infty}\sum_{l=0}^{k}\frac{1}{k^2}\int_{\tilde{y}\in\mathcal{C}_{j+l}}\int_{z\in\mathcal{C}_{j+k}}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z\\
&\qquad+C\left(\sum_{l=1}^{\infty}2^{-l}\right)\sum_{j=4}^{\infty}\sum_{k=4}^{\infty}\frac{1}{k^2}\int_{\tilde{y}\in B_1}\int_{z\in\mathcal{C}_{j+k}}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z+C\Pi\\
&\quad\leq C\sum_{j=4}^{\infty}\sum_{l=0}^{\infty}\sum_{k\geq\max\{4,l\}}\frac{1}{k^2}\int_{\tilde{y}\in\mathcal{C}_{j+l}}\int_{z\in\mathcal{C}_{j+k}}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z\\
&\qquad+C\sum_{k=4}^{\infty}\frac{1}{k^2}\int_{\tilde{y}\in B_1}\int_{z\in B_1}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z+C\Pi\\
&\quad\leq C\sum_{j=4}^{\infty}\sum_{l=0}^{\infty}\frac{1}{\max\{4^2,l^2\}}\int_{\tilde{y}\in\mathcal{C}_{j+l}}\int_{z\in B_{3/4}^c}|\nabla\otimes f(\tilde{y})-\nabla\otimes f(z)|^2\,\phi(\tilde{y})\phi(z)\,\mathrm{d}y\mathrm{d}z+C\Pi\\
&\quad\leq C\Pi.
\end{aligned}\tag{2.110}$$

Now for the case $s \in [0, 1-\sigma]$, from $\phi(sy+(1-s)z) \geq \phi(s|y|+(1-s)|z|) \geq \phi(z)$ and (2.109), we infer that

$$\frac{\phi(z)}{\phi(sy+(1-s)z)} \leq \frac{\phi(z)}{\phi(s|y|+(1-s)|z|)} \leq \frac{2^{-(j+k)}}{2^{-(j+1)}2^{-\frac{(1-s)(j+1)k}{j+1+sk}}} \leq 2^{-k+1}2^{(1-s)k} = 2^{-sk+1},$$

thus we have

$$\begin{aligned}
&\sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\int_{y\in\mathcal{C}_j}\int_{z\in\mathcal{C}_{j+k},\theta\in[0,\frac{\pi}{2})}\int_0^{1-\sigma}|\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(y)|^2\phi(y)\phi(z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}z\\
&\leq \sum_{j=4}^{\infty}\sum_{k\in\mathbb{N}}\int_{y\in\mathcal{C}_j}\int_{z\in\mathcal{C}_{j+k},\theta\in[0,\frac{\pi}{2})}\int_{\frac{1}{2}}^{1-\sigma}|\nabla\otimes f(sy+(1-s)z)\\
&\quad-\nabla\otimes f(y)|^2 2^{-sk+1}\phi(y)\phi(sy+(1-s)z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}z\\
&\quad+\int_{y\in B_{3/4}^c}\int_{B_{3/4}^c\cap\{|z|\geq|y|\}}\int_0^{\frac{1}{2}}|\nabla\otimes f(sy+(1-s)z)-\nabla\otimes f(y)|^2\phi(y)\phi(sy+(1-s)z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}z\\
&\leq \sum_{j=4}^{\infty}\sum_{k\in\mathbb{N},k\geq4}\int_{y\in\mathcal{C}_j}\int_{\tilde{z}\in B_1}\int_{\frac{1}{2}}^{1-\frac{1}{k^2}}|\nabla\otimes f(\tilde{z})-\nabla\otimes f(y)|^2(1-s)^{-n}2^{-sk+1}\phi(y)\phi(\tilde{z})\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tilde{z}\\
&\quad+\sum_{j=4}^{\infty}\sum_{k=0}^{4}\int_{y\in\mathcal{C}_j}\int_{\tilde{z}\in B_1}\int_{\frac{1}{2}}^{\frac{3}{4}}|\nabla\otimes f(\tilde{z})-\nabla\otimes f(y)|^2(1-s)^{-n}2^{-sk+1}\phi(y)\phi(\tilde{z})\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tilde{z}\\
&\quad+\int_{y\in B_{3/4}^c}\int_{\tilde{z}\in B_1}\int_0^{\frac{1}{2}}|\nabla\otimes f(\tilde{z})-\nabla\otimes f(y)|^2(1-s)^{-n}\phi(y)\phi(\tilde{z})\,\mathrm{d}s\mathrm{d}y\mathrm{d}\tilde{z}\\
&\leq C\Big(\sum_{k\in\mathbb{N},k\geq4}k^{2n}e^{-\frac{1}{2}k}\Big)\int_{B_{3/4}^c}\int_{\tilde{z}\in B_1}|\nabla\otimes f(\tilde{z})-\nabla\otimes f(y)|^2\phi(y)\phi(\tilde{z})\,\mathrm{d}y\mathrm{d}\tilde{z}\\
&\quad+C\int_{y\in B_1}\int_{\tilde{z}\in B_1}|\nabla\otimes f(\tilde{z})-\nabla\otimes f(y)|^2\phi(y)\phi(\tilde{z})\,\mathrm{d}y\mathrm{d}\tilde{z}\\
&\leq C\Pi.
\end{aligned}\tag{2.111}$$

Gathering the above estimates (2.107)–(2.111) leads to the desired inequality (2.105), and furthermore concludes Lemma 2.8. □

Acknowledgements The authors would like to express their deep gratitude to the anonymous referee for the helpful comments to improve this paper. The authors would also like to express their gratitude to Prof. Songying Li, Prof. Qi S. Zhang and Prof. Luis Silvestre for insightful discussion and comments. Q. Miao was supported by China Postdoctoral Science Foundation (No. 197520). L. Xue was partially supported by National Natural Science Foundation of China (Grants Nos. 11401027, 11671039 and 11771043).

References

1. Aronson, D.G.: Non-negative solutions of linear parabolic equations. Ann. Sc. Norm. Super. Pisa **22**(3), 607–694 (1968)
2. Beirao da Veiga, H.: Concerning the regularity problem for the solutions of the Navier–Stokes equations. C. R. Acad. Sci. Paris Sér. I **321**, 405–408 (1995)
3. Beirao da Veiga, H.: A new regularity class for the Navier-Stokes equations in R^n. Chin. Ann. Math. **16B**, 407–412 (1995)
4. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, Berlin (2011)

5. Campanato, S.: Propietá di hölderianitá di alcune classi di funzioni. Ann. Sc. Norm. Super. Pisa **17**, 175–188 (1963)
6. Chen, Q., Zhang, Z.: Space-time estimates in the Besov spaces and the Navier–Stokes equations. Methods Appl. Anal. **13**(1), 107–122 (2006)
7. De Giorgi, E.: Sulla dierenziabilita e lánaliticita delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **3**(3), 25–43 (1957)
8. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19, 2nd edn. American Mathematical Society, Providence (2010)
9. Duoandikoetxea, J.: Fourier Analysis. Graduate Studies in Mathematics, vol. 29. American Mathematical Society, Providence (2000)
10. Friedlander, S., Suen, A.: Hölder continuity of solutions to kinematic dynamo equations. J. Math. Fluid Mech. **16**, 691–700 (2014)
11. Friedlander, S., Vicol, V.: Global well-posedness for an advection–diffusion equation arising in magneto-geostrophic dynamics. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**(2), 283–301 (2011)
12. Germain, P.: Multipliers, paramultipliers, and weak-strong uniqueness for the Navier–Stokes equations. J. Differ. Equ. **226**, 373–428 (2006)
13. Giga, Y.: Solutions for semilinear Parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system. J. Differ. Equ. **62**, 186–212 (1986)
14. Kiselev, A.: Nonlocal maximum principle for active scalars. Adv. Math. **227**(5), 1806–1826 (2011)
15. Kiselev, A., Nazarov, F., Volberg, A.: Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. Invent. Math. **167**, 445–453 (2007)
16. Kozono, H., Ogawa, T., Taniuchi, Y.: The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations. Math. Z. **242**, 251–278 (2002)
17. Ladyženskaya, O.A., Solonikov, V.A., Ural'tseva, N.N.: Linear and Quasilinear Equations of Parabolic Type. Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI (1967)
18. Lemarié-Rieusset, P.G.: Recent Developments in the Navier–Stokes Problem. Chapman and Hall, London (2002)
19. Lemarié-Rieusset, P.G.: The Navier–Stokes Problem in the 21st Century. Chapman and Hall, London (2016)
20. Leray, J.: Essai sur le mouvement d'un fluide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
21. Miao, C., Xue, L.: On the regularity issues of a class of drift–diffusion equations with nonlocal diffusion. Preprint. ArXiv:1507.04806 [math.AP]
22. Moser, J.: On Harnack's theorem for elliptic differential equations. Commun. Pure Appl. Math. **14**, 577–591 (1961)
23. Nash, J.: Continuity of solutions of parabolic and elliptic equations. Am. J. Math. **80**, 931–954 (1958)
24. Nazarov, A.I., Ural'tseva, N.N.: The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients. Algebra Analiz **23**(1), 136–168 (2011)
25. Osada, H.: Diffusion processes with generators of generalized divergence form. J. Math. Kyoto Univ. **27**(4), 597–619 (1987)
26. Semenov, Y.A.: Regularity theorems for parabolic equations. J. Funct. Anal. **231**(2), 375–417 (2006)
27. Seregin, G., Silvestre, L., Šverak, V., Zlatoš, A.: On divergence-free drifts. J. Differ. Equ. **252**(1), 505–540 (2012)
28. Serrin, J.: The initial value problem for the Navier–Stokes equations. In: 1963 Nonlinear Problems (Proceedings of Symposium, Madison, Wis.), pp. 69–98 (1962)
29. Silvestre, L., Vicol, V.: Hölder continuity for a drift–diffusion equation with pressure. Ann. Inst. H. Poincaré-AN **29**, 637–652 (2012)
30. Zhang, Q.S.: Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$. Manuscripta Math. **93**(3), 381–390 (1997)
31. Zhang, Q.S.: A strong regularity result for parabolic equations. Commun. Math. Phys. **244**(2), 245–260 (2004)