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REGULARITY AND SINGULARITY RESULTS FOR THE DISSIPATIVE WHITHAM EQUATION AND RELATED SURFACE WAVE EQUATIONS*

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Abstract. We consider the Cauchy problem for the Whitham equation and related surface wave equations with (fractional) dissipation. We prove global regularity results at the subcritical and critical dissipative cases by applying the method of modulus of continuity, and we show a finite-time singularity result at the supercritical dissipative case.

Keywords. Whitham equation; surface wave equation; global regularity; modulus of continuity; singularity.

AMS subject classifications. 35A01; 35Q35; 35Q86.

1. Introduction

In this paper we address the following Whitham equation with dissipation

$$\partial_t u + u\partial_x u + \mu L\partial_x u + \nu\Lambda^\alpha u = 0, \quad u|_{t=0}(x) = u_0(x), \tag{1.1}$$

where $x\in\mathbb{R}$ (or $\mathbb{T}$), $\nu>0$, $\mu\neq 0$, $\alpha\in]0,2]$, u is a 1D scalar field, the (fractional) differential operator $\Lambda^\alpha := (-\partial_{xx})^{\frac{\alpha}{2}}$, and the Fourier multiplier operator L is defined via

$$\widehat{Lf}(\zeta) = m(\zeta)\widehat{f}(\zeta) = \sqrt{\frac{\tanh\zeta}{\zeta}}\widehat{f}(\zeta). \tag{1.2}$$

Let $K(x) = \mathcal{F}^{-1}\big(m(\zeta)\big)(x) = \mathcal{F}^{-1}\Big(\sqrt{\frac{\tanh\zeta}{\zeta}}\Big)(x)$ be the kernel function of L, we also get

$$Lf(x) = \int_{\mathbb{R}} K(x-y)f(y)\mathrm{d}y. \tag{1.3}$$

When $\nu=0$, Equation (1.1) is the classical Whitham equation, which was introduced by Whitham [47] as an alterative to the Korteweg-de Vries (abbr. KdV) equation

$$\partial_t u + u\partial_x u + \mu\Big(1+\frac{1}{6}\partial_x^2\Big)\partial_x u = 0, \quad u|_{t=0}(x) = u_0(x). \tag{1.4}$$

The symbol $m(\zeta) = \sqrt{\frac{\tanh\zeta}{\zeta}}$ arises from the full frequency dispersion for linear gravity water waves on finite depth, and the first two terms in the Maclaurin series of $m(\zeta)$ are $1-\frac{1}{6}\zeta^2$, just corresponding to the symbol of KdV equation. Since the KdV Equation (1.4), as a long-wave approximation model of water wave equations (see [36]), does not admit breaking solutions due to the strong dispersion effect, Whitham proposed Equation (1.1) (with $\nu=0$) as a simplified mathematical equation of water wave equations to

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study the breaking phenomenon, which is an important and intriguing problem in the water wave theory. Recently, compared with the KdV equation, both numerical simulation and wave-channel experiments [7,46] show the modeling advantages of the Whitham equation, when either short or large waves are concerned. One can also see [34] for more physical relevance and discussion of Whitham Equation (1.1) with $\nu=0$. When $\nu>0$, the dissipation effect is introduced in Equation (1.1), which naturally occurs in many real situations, and one can analogously see [43] for the KdV equation with (fractional) dissipation and see [6,17,20] (and references therein) for various 1D dispersive equations with dissipation.

We also consider the following surface wave equation with fractional dispersion and dissipation

$$\partial_t u + u\partial_x u + \mu\Lambda^\beta \mathcal{H}u + \nu\Lambda^\alpha u = 0, \quad u|_{t=0}(x) = u_0(x), \tag{1.5}$$

where $x\in\mathbb{R}$, $\nu>0$, $\mu\neq 0$, $\alpha\in]0,2]$, $\beta\in[0,1[$, $\mathcal{H}=-\partial_x\Lambda^{-1}$ is the usual Hilbert transform (e.g. see [44]). When $\nu=0$, different values of β lead to various surface wave models: if $\beta=3$, Equation (1.5) is the KdV equation; if $\beta=2$, Equation (1.5) is the well-known Benjamin-Ono equation; if $\beta=1$, Equation (1.5) corresponds to the invisid Burgers equation (after a suitable transformation); if $\beta=1/2$, it is observed by Hur [25] that (1.5) shares the dispersion relation and scaling symmetry analogous to the 2D water wave system in the infinite depth; while if $\beta=0$, Equation (1.5) is proposed by Biello and Hunter [4] as a model for water waves with nonzero constant-valued linearized frequency. When $\nu>0$, we also include the dissipative effect in Equation (1.5).

The Whitham equation (i.e. Equation (1.1) with $\nu=0$) has attracted much attention in recent years, and there have been several noticeable works on the breaking mechanism. Ehrnström, Groves and Wahlén [21] proved the existence of solitary waves, i.e. solutions of the form $u=u(x-ct)$ with $u(x-ct)\to 0$ as $x-ct\to\pm\infty$. Ehrnström and Wahlén [23] constructed the highest, cusped, periodic travelling wave solution to the Whitham equation (similar to the Stokes wave although with a different angle), which solved a long-standing conjecture proposed in Whitham [48]. For the existence and properties of the periodic travelling waves, as well as the corresponding stability versus instability issue for the Whitham equation, one can refer to [8, 9, 22, 27, 28] and references therein. As another breaking scenario proposed by Whitham [48], the so-called "wave breaking", which means that the solution itself is uniformly bounded but its slope becomes unbounded at finite time, was recently proved by Hur [26] for the Whitham equation associated with some smooth data (see also the past work [13, 42] on some Whitham-type equations).

For the surface wave Equation (1.5) with $\nu=0$ and $\beta\in]0,1[$, Castro, Córdoba and Gancedo [11] (see also Hur [25] for the case $\beta=1/2$) proved that by applying the weighted integral method inspired by [15], the smooth solution associated with some data $u_0\in L^2\cap C^{1+\delta}(\mathbb{R})$, $\delta>0$ blows up at finite time (they also prove a similar blowup result for the $\beta=0$ case by using a different method). Hur and Tao [29] considered (1.5) with $\nu=0$ and $0<\beta<1/2$ to show that the wave breaking phenomenon occurs for the equation with some smooth initial data. Later, Hur [26] further extended the same result to the Equation (1.5) with $\nu=0$ and $0<\beta<2/3$. It should be noted that for the KdV-like Equation (1.5) with $\nu=0$, the cases $\beta\in]1,2[$ are more subtle, and based on the numerical stimulations, Klein, Saut et al. [34, 35] conjectured the global well-posedness for the case $\beta>3/2$ as well as the finite-time blowup for the case $1<\beta\leq 3/2$ (one can see [41] for the recent progress on the case $13/7<\beta<2$).

If $\mu=0$ and $\nu>0$, Equations (1.1) and (1.5) reduce to the dissipative Burgers equation: the classical viscous case $\alpha=2$ and the fractional dissipation case $\alpha\in]0,2[$ (cf. [5]).

Fractional dissipation related to Lévy flights also appears in many physical models (e.g. see [12, 38, 43]). Kiselev, Nazarov and Shterenberg [32] proved that for $\alpha\in[1,2]$ and $u_0\in H^s$, $s\geq 3/2-\alpha$, there is a unique global smooth solution to the dissipative Burgers equation (the method used in the case $\alpha=1$ is the original method of modulus of continuity); while for the case $\alpha\in]0,1[$, they showed that the shock singularity similar to the inviscid case occurs (see [18] for another proof using the weighted integral method, and also [2]). Besides, the authors in [32] proved that at the $\alpha=1$ case and associated with rough initial data $u_0\in L^p$ $(1<p<\infty)$, there is a solution which is C^∞-smooth for any $t>0$ (if $u_0\in L^2$, see also [10] for a different method using De Giorgi's iteration), but so far the uniqueness issue remains an interesting open problem. Alibaud and Andreianov [1] showed that starting from some initial data $u_0\in L^\infty$, the uniqueness of weak solution (in the distributional sense) fails for the dissipative Burgers equation for the $0<\alpha<1$ case (while the uniqueness of weak solution is ensured for the $\alpha>1$ case, see [19]). By these results, *the cases $1<\alpha\leq 2$, $\alpha=1$ and $0<\alpha<1$ are called the subcritical, critical and supercritical cases, respectively.*

We also mention a model relevant to the above equations, the dissipative dispersive surface quasi-geostrophic (abbr. SQG) equation

$$\partial_t\theta+v\cdot\nabla\theta+\mu v_2+\nu\Lambda^\alpha\theta=0,\quad v=(v_1,v_2)=\mathcal{R}^\perp\theta=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta),\quad \theta|_{t=0}(x)=\theta_0(x), \tag{1.6}$$

where $x\in\mathbb{R}^2$ (or $\mathbb{T}^2$), $\nu>0$, $\mu\neq 0$, $\mathcal{R}_i=\partial_{x_i}\Lambda^{-1}$ $(i=1,2)$ is the usual Riesz transform (e.g. see [44]). Here θ is a real-valued scalar function that can be interpreted as a buoyancy field, v is the velocity field, μ is the amplitude parameter. Equation (1.6) is a simplified model from the geostrophic fluid dynamics and describes the evolution of a surface buoyancy in the presence of an environmental horizontal buoyancy gradient ([24]). Physically, the background buoyancy gradient generates dispersive waves, thus Equation (1.6) provides a 2D model for the interactions among turbulent motion, dispersive waves and dissipation. By using the modulus of continuity method, Kiselev and Nazarov [31] considered this dissipative dispersive Equation (1.6) for the case $\alpha=1$ and proved the global existence and uniqueness of smooth solution associated with smooth data.

In this paper we are mainly concerned with the following dissipative dispersive Burgers equation which includes Equations (1.1) and (1.5) as examples (see Lemma 2.1 below)

$$\partial_t u+u\partial_x u+\mu L_\beta u+\nu\Lambda^\alpha u=0,\quad u|_{t=0}(x)=u_0(x), \tag{1.7}$$

where $x\in\mathbb{R}$ (or $\mathbb{T}$), $\nu>0$, $\mu\neq 0$, $\alpha\in]0,2]$, $\beta\in[0,1[$, u is a scalar function, and L_β is a Fourier multiplier operator

$$\widehat{L_\beta f}(\zeta)=im_\beta(\zeta)\hat{f}(\zeta), \tag{1.8}$$

with $i^2=-1$, $m_\beta\in C^\infty(\mathbb{R}\setminus\{0\})$ a real-valued odd function which satisfies the following assumptions

(A1) $|m_\beta(\zeta)|\leq C|\zeta|^\beta$ for every $\zeta\in\mathbb{R}$;

(A2) $m_\beta(\zeta)$ is of the Mikhlin-Hörmander type, that is, $m_\beta(\zeta)$ satisfies that for every $\zeta\neq 0$,

$$|\partial^k m_\beta(\zeta)|\leq C|\zeta|^{-k}|m_\beta(\zeta)|,\qquad \text{for } k\in\{1,2,3\}. \tag{1.9}$$

We intend to address Equation (1.7) to show some global regularity results at the subcritical and critical cases $\alpha\in[1,2]$ and a singularity result for the supercritical case $0<\alpha<1$. Our main regularity results are as follows.

THEOREM 1.1 (Global well-posedness in the subcritical and critical dissipative cases). *Assume that $\nu>0$, $\mu\neq 0$, $\alpha\in[1,2]$, $\beta\in[0,1[$ and $u_0\in H^s(\mathbb{R})$, $s>3/2$. Then the dissipative dispersive Burgers Equation (1.7) admits a unique global solution $u\in C([0,\infty[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,\infty[)$.*

In the critical case $\alpha=1$, we moreover have the following global regularity result associated with rough initial data.

THEOREM 1.2 (Global regularity result in the critical dissipative case). *Assume that $\nu>0$, $\mu\neq 0$, $\alpha=1$, $\beta\in[0,1[$, and $u_0\in L^2\cap L^\infty(\mathbb{R})$. Then for the dissipative dispersive Burgers Equation (1.7), there is a global weak solution (see Definition 4.1 below) $u\in L^\infty([0,\infty[;L^2\cap L^\infty(\mathbb{R}))\cap L^2([0,\infty[;\dot{H}^{1/2}(\mathbb{R}))$ such that for any $t'>0$, $u(x,t)$ is C^∞-regular on $\mathbb{R}\times[t',+\infty[$.*

Our singularity result for the case $\alpha\in]0,1[$ and $\beta\in]0,1[$ is stated in the following.

THEOREM 1.3 (Finite time blowup in the supercritical dissipative case). *Let $\nu>0$, $\mu\neq 0$, $\alpha\in]0,1[$, $\beta\in]0,1[$. There exists initial data $u_0\in H^s(\mathbb{R})$, $s>3/2$ (satisfying (5.18) below) and a finite time $T>0$ depending only on u_0, such that for the solution $u(x,t)$ to the dissipative dispersive Burgers Equation (1.7), we have*

$$\limsup_{t\to T}\|\partial_x u(\cdot,t)\|_{L^\infty(\mathbb{R})}=\infty. \tag{1.10}$$

In the proof of Theorems 1.1-1.2, we mainly use the method of modulus of continuity (see Definition 2.1) that originated in [30, 32, 33]. The general idea is to prove that the evolution of considered equation obeys a suitable (stationary or time-dependent) modulus of continuity, and by a contradiction analysis, it reduces to justify the pointwise inequality (2.8) under the scenario (2.7), then by using the equation and Lemma 2.4, and noting that the contribution from the dissipation term is negative, the strategy is to let the negative contribution play a dominant role so that one can prove (2.8).

For the proof of Theorem 1.1, we first show the local well-posedness result for the considered equation, the blowup criterion in terms of the Lipschitz norm of solution, and also the uniform L^∞-estimate of the solution (see Lemma 2.6); then for the subcritical case $\alpha\in]1,2]$, we manage to prove the maximal lifespan solution obeys some stationary (bounded) modulus of continuity (3.3), which implies the desired uniform-in-time Lipschitz regularity. For the critical case $\alpha=1$, we moreover present a refined blowup criterion (see Lemma 3.1) in terms of the σ-Hölder regularity of solution, then in order to show the needed uniform Hölder estimate on the maximal lifespan, we prove the preservation of a suitable stationary unbounded modulus of continuity (3.20) by the evolution, which is pursued in Lemma 3.2; note that the modulus of continuity (3.20) is in a simple form, and is different from the ones used in [31, 32] (where they instead lead to the preservation of Lipschitz regularity).

For the proof of Theorem 1.2, by virtue of the global existence result of weak solutions established in Proposition 4.1 and the regularity criterion in Lemma 4.1, the main point is to show that the weak solution starting from the rough initial data $u_0\in L^2\cap L^\infty$ instantly obtains the required Hölder regularity. To this goal, we construct a family of time-dependent moduli of continuity (4.12)-(4.13), which reduces to the stationary modulus of continuity (3.20) after a time period that can be arbitrarily small, then we manage to prove that the weak solution obeys such moduli of continuity for all time by a careful analysis (see Lemma 4.2), and it yields the needed Hölder regularity after an arbitrarily small time interval, as desired.

The proof of Theorem 1.3 applies the method of weighted integral as in [11,15,18,25], and the general strategy is to prove that some weighted integral of the solution $E(t)$

(see (5.4) below) satisfies $E'(t) \geq \frac{E(t)^2}{C} - C$, which means $E(t)$ blows up at a finite time, and it will in turn yield that the solution can not be globally regular. Here the contributions from both the dispersive and dissipative terms are considered simultaneously, and also the dissipation may have some more advantage than previous works on the purely dispersive equations (e.g. see Lemma 2.6), thus we include the detailed proof for completeness and convenience.

REMARK 1.1. For the dissipative dispersive Burgers Equation (1.7) with $\alpha \in]\beta, 1[$, by Lemma 2.6 below, we have $\sup_{t\in[0,T[} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} < \infty$, which combined with equality (1.10) yields that the singularity in Theorem 1.3 has the "wave breaking" phenomenon.

REMARK 1.2. It seems that the weighted integral method used in Theorem 1.3 fails for the case $\beta = 0$, and thus we do not address this case in Theorem 1.3. Note that by applying a different method, Castro et al. in [11] manage to show a blowup result for the surface wave Equation (1.5) with $\nu = 0$, $\mu < 0$ and $\beta = 0$. If one intends to extend such a blowup result to Equation (1.7) with $\nu = 0$ and $\beta = 0$, some deeper properties of the operator L_0 (like the kernel K_0 satisfying $(-1)^n \partial_y^n K_0(y) \geq 0$ for $y > 0$ and $n = 0, 1$) need to be proved, which are not so clear from our current viewpoint.

REMARK 1.3. By applying the same procedure as that for the surface wave Equation (1.5) with $\beta = 0$ and $\alpha = 1$, one can show the analogous Theorem 1.2 for the dissipative dispersive SQG Equation (1.6).

The paper is organized as follows. In Section 2, we collect some auxiliary results used in the main proof. We show the proof of Theorem 1.1 in Section 3, and we prove Theorem 1.2 in Section 4. Then, Section 5 is devoted to the proof of Theorem 1.3. At last, in the Appendix, we give the details of the proof for Proposition 3.1 concerning the local well-posedness result, and also present an L^∞-estimate of the viscous Burgers equation with forcing.

The following notations are used throughout this paper.

- C stands for a constant which may be different from line to line, and $C(\lambda_1, \lambda_2 \cdots, \lambda_n)$ denotes a constant C depending on the coefficients $\lambda_1, \lambda_2, \cdots, \lambda_n$. $X \lesssim Y$ means that there is a harmless constant C such that $X \leq CY$, and $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$ simultaneously.

- The notation $C_c^\infty(\mathbb{R})$ or $C_c^\infty(\mathbb{R} \times [0, T[)$ denotes the space of C^∞-smooth functions with compact support on $\mathbb{R}$ or $\mathbb{R} \times [0, T[$, respectively. The notation $\mathcal{S}(\mathbb{R})$ is the Schwartz class of rapidly decreasing C^∞-smooth functions, and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions which is the dual space of $\mathcal{S}(\mathbb{R})$.

- For $m \in \mathbb{N}$, $r \in [1, +\infty]$, $s \in \mathbb{R}$, we denote by $W^{m,r}(\mathbb{R})$ ($\dot{W}^{m,r}(\mathbb{R})$) and $H^s(\mathbb{R})$ ($\dot{H}^s(\mathbb{R})$) the usual L^r-based and L^2-based inhomogeneous (homogenous) Sobolev spaces, and by $C^{m,\gamma}(\mathbb{R})$, $\dot{C}^{m,\gamma}(\mathbb{R})$ with $\gamma \in]0, 1[$ the inhomogeneous and homogeneous Hölder spaces (if $m = 0$, we also write $C^{0,\gamma}(\mathbb{R})$ and $\dot{C}^{0,\gamma}(\mathbb{R})$ as $C^\gamma(\mathbb{R})$ and $\dot{C}^\gamma(\mathbb{R})$ for brevity).

- We use $\mathcal{F}(f)$ (or $\widehat{f}$) and $\mathcal{F}^{-1}(f)$ to denote the Fourier transform and the inverse Fourier transform of a function f, that is, $\mathcal{F}(f)(\zeta) = \int_{\mathbb{R}} e^{ix\cdot\zeta} f(x) \mathrm{d}x$ and $\mathcal{F}^{-1}(g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\zeta} g(\zeta) \mathrm{d}\zeta$.

2. Preliminary and auxiliary results

We compile some useful and auxiliary results in this section.

2.1. Some properties related to the dispersive operator L_β.

LEMMA 2.1. *The operator $L\partial_x$ in Equation (1.1) and the operator $\Lambda^\beta\mathcal{H}$ in Equation (1.5) are in the realm of the Fourier multiplier operator L_β introduced in Equation (1.7).*

Proof. The multiplier of the operator $\Lambda^\beta\mathcal{H}$ ($\beta\in[0,1[$) is $i\zeta|\zeta|^{\beta-1}$, and the function $\zeta|\zeta|^{\beta-1}$ is clearly a real-valued odd function satisfying assumptions (A1) and (A2).

The multiplier of the operator $L\partial_x$ is $i\widetilde{m}(\zeta)$ with $\widetilde{m}(\zeta)=\zeta\sqrt{\frac{\tanh\zeta}{\zeta}}$. It is easy to see that $\widetilde{m}(\zeta)$ is a real-valued odd function satisfying $|\widetilde{m}(\zeta)|\le|\zeta|^{1/2}$ for all $\zeta\in\mathbb{R}$ (i.e. assumption (A1) with $\beta=1/2$). Now we verify assumption (A2). Since $\partial_\zeta^k\widetilde{m}(\zeta)$ is either odd or even for $k=1,2,3$, we only need to consider the case $\zeta>0$. By a direct computation, we see that for every $\zeta>0$,

$$\left(\zeta\sqrt{\frac{\tanh\zeta}{\zeta}}\right)'=\frac{1}{2}\sqrt{\frac{\tanh\zeta}{\zeta}}+\frac{1}{2}\sqrt{\frac{\zeta}{\tanh\zeta}}\operatorname{sech}^2\zeta=\frac{1}{2}\sqrt{\frac{\tanh\zeta}{\zeta}}\left(1+\frac{\zeta}{\tanh\zeta}\operatorname{sech}^2\zeta\right),$$

which combined with the properties of $\tanh\zeta$ and $\operatorname{sech}\zeta$ ($|\tanh\zeta|\le 1$ for all ζ, $|\tanh\zeta|\approx|\zeta|$ for $|\zeta|$ small, $|\operatorname{sech}\zeta|\approx e^{-|\zeta|}$ for $|\zeta|$ large and $|\operatorname{sech}\zeta|\approx 1$ for $|\zeta|$ small) leads to

$$|\widetilde{m}'(\zeta)|\le C_0|\zeta|^{-1}|\widetilde{m}(\zeta)|,\quad \forall|\zeta|>0. \tag{2.1}$$

Similar computation also yields that for every $\zeta>0$

$$\begin{aligned}\left(\zeta\sqrt{\frac{\tanh\zeta}{\zeta}}\right)''&=-\frac{\sqrt{\tanh\zeta}}{4\zeta^{3/2}}+\frac{\operatorname{sech}^2\zeta}{4}\left(\frac{2}{\zeta}\sqrt{\frac{\zeta}{\tanh\zeta}}-\frac{\zeta^{1/2}}{\sinh^2\zeta}\sqrt{\tanh\zeta}-4\sqrt{\zeta\tanh\zeta}\right)\\&=\frac{1}{\zeta}\sqrt{\frac{\tanh\zeta}{\zeta}}\left(-\frac{1}{4}+\operatorname{sech}^2\zeta\left(\frac{1}{2}\frac{\zeta}{\tanh\zeta}-\frac{1}{4}\frac{\zeta^2}{\sinh^2\zeta}-\zeta^2\right)\right),\end{aligned}$$

and

$$\begin{aligned}\left(\zeta\sqrt{\frac{\tanh\zeta}{\zeta}}\right)'''&=\frac{1}{\zeta^2}\sqrt{\frac{\tanh\zeta}{\zeta}}\left(\frac{3}{8}+\operatorname{sech}^2\zeta\left(-\frac{3}{8}\frac{\zeta}{\tanh\zeta}-\frac{3}{2}\zeta^2+2\zeta^2\tanh\zeta\right)\right)\\&\quad+\frac{1}{\zeta^2}\sqrt{\frac{\tanh\zeta}{\zeta}}\operatorname{sech}^2\zeta\left(-\frac{3}{8}\frac{\zeta^2}{\sinh^2\zeta}+\frac{1}{2}\frac{\zeta^3}{\sinh^2\zeta\tanh\zeta}-\frac{1}{8}\frac{\zeta^3}{\sinh^3\zeta\cosh\zeta}\right),\end{aligned}$$

we find that

$$|\widetilde{m}''(\zeta)|\le C_0|\zeta|^{-2}|\widetilde{m}(\zeta)|,\ \text{and}\ |\widetilde{m}'''(\zeta)|\le C_0|\zeta|^{-3}|\widetilde{m}(\zeta)|,\qquad \forall|\zeta|>0. \tag{2.2}$$

Estimates (2.1)-(2.2) immediately ensure assumption (A2), as desired. ∎

Now we recall the definition of the dyadic blocks (see e.g. [3]). Let $\chi\in C_c^\infty(\mathbb{R})$ be a non-negative function such that $\chi(\zeta)=1$ if $|\zeta|\le 1/2$ and 0 if $|\zeta|\ge 1$. Let us define another function $\varphi\in C_c^\infty(\mathbb{R})$ by $\varphi(\zeta)=\chi(\zeta/2)-\chi(\zeta)$ which is therefore supported on a corona. Then, we define the Fourier multiplier Δ_j ($j\in\mathbb{N}$) and Δ_{-1} by

$$\widehat{\Delta_j f}(\zeta)=\varphi(2^{-j}\zeta)\widehat{f}(\zeta)\quad\text{and}\quad\widehat{\Delta_{-1}f}(\zeta)=\chi(\zeta)\widehat{f}(\zeta). \tag{2.3}$$

By these operators we have the following Littlewood-Paley decomposition of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$:

$$f = \Delta_{-1} f + \sum_{j\in\mathbb{N}} \Delta_j f \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

For $s \in \mathbb{R}$ and $(p,q) \in [1,\infty]^2$, we thus define the inhomogeneous Besov spaces as the set of $f \in \mathcal{S}'(\mathbb{R})$ so that the following quantity is finite

$$\|f\|_{B^s_{p,q}(\mathbb{R})} := \big\| \{2^{js} \|\Delta_j f\|_{L^p}\}_{j\geq -1} \big\|_{\ell^q}.$$

In particular, we have the equivalence of L^2-based Sobolev space $H^s(\mathbb{R}) = B^s_{2,2}(\mathbb{R})$ and L^∞-based Hölder space $C^\gamma(\mathbb{R}) = B^\gamma_{\infty,\infty}(\mathbb{R})$ with $\gamma \in]0,1[$ (see [3]).

The following lemma deals with the action of the Fourier multiplier $L\partial_x$ into the dyadic blocks.

Lemma 2.2. *Let L_β ($\beta \in [0,1[$) be the Fourier multiplier operator defined by (1.8) with $m_\beta \in C^\infty(\mathbb{R}\setminus\{0\})$ a real-valued odd function satisfying the assumptions (A1)-(A2). Then there exists a constant $C = C(\beta) > 0$ such that for every $p \in [1,\infty]$ and $j \in \mathbb{N}$,*

$$\|\Delta_j L_\beta f\|_{L^p(\mathbb{R})} \leq C 2^{j\beta} \|\Delta_j f\|_{L^p(\mathbb{R})}. \tag{2.4}$$

Proof. From assumptions (A1)-(A2), we see that

$$|\partial^k m_\beta(\zeta)| \leq C|\zeta|^{\beta-k}, \ \forall \zeta \neq 0, \quad \text{for } k \in \{1,2,3\},$$

thus we directly apply [3, Lemma 2.2] to obtain estimate (2.4). □

Next we derive the expression formula of the operator L_β and show the key kernel estimates.

Lemma 2.3. *Let L_β ($\beta \in [0,1[$) be the Fourier multiplier operator defined by formula (1.8), where $m_\beta(\zeta)$ is a real-valued odd function satisfying the assumptions (A1)-(A2). Let $f \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, then we have*

$$\begin{aligned} L_\beta f(x) &= \mathrm{p.v.}\int_{\mathbb{R}} K_\beta(x-y)(f(y)-f(x))\mathrm{d}y \\ &= \mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|x-y|)\,\mathrm{sgn}(x-y)(f(y)-f(x))\mathrm{d}y, \end{aligned} \tag{2.5}$$

where the kernel $K_\beta(x) = \mathcal{F}^{-1}(i m_\beta(\zeta))(x)$ is a real-valued odd function which satisfies that for every $x \neq 0$,

$$|K_\beta(x)| \leq \frac{C}{|x|^{1+\beta}}, \quad \text{and} \quad |K'_\beta(x)| \leq \frac{C}{|x|^{2+\beta}}, \tag{2.6}$$

with $C = C(\beta)$ some positive constant.

Proof. Since $m_\beta(\zeta)$ is a real-valued odd function, it is easy to see that $K_\beta(x) = C_0 \int_{\mathbb{R}} e^{ix\zeta} i m_\beta(\zeta)\mathrm{d}\zeta$ is also a real-valued odd function.

Now we prove estimate (2.6) (one can see [16, Lemma 5.1] for a similar treatment). Let $\chi, \varphi \in C^\infty_c(\mathbb{R})$ be the cutoff functions introduced around (2.3), and it directly yields that for every $N \in \mathbb{Z}$,

$$1 = \chi(2^{-N}\zeta) + \sum_{j=N}^{\infty} \varphi(2^{-j}\zeta), \quad \forall \zeta \in \mathbb{R}.$$

For some $N\in\mathbb{Z}$ chosen later, we have

$$\begin{aligned} K_\beta(x) &= C_0\int_{\mathbb{R}} e^{ix\zeta} i m_\beta(\zeta)\chi(2^{-N}\zeta)\mathrm{d}\zeta + C_0\sum_{j=N}^{\infty}\int_{\mathbb{R}} e^{ix\zeta} i m_\beta(\zeta)\varphi(2^{-j}\zeta)\mathrm{d}\zeta \\ &:= K_{\beta,1}(x)+K_{\beta,2}(x). \end{aligned}$$

For $K_{\beta,1}(x)$, we use assumption (A1) to derive that

$$|K_{\beta,1}(x)|\leq C\int_{|\zeta|\leq 2^N}|\zeta|^\beta\mathrm{d}\zeta\leq C2^{N(1+\beta)}.$$

For $K_{\beta,2}(x)$, by virtue of the integration by parts and the assumptions (A1)-(A2), we find that for every $x\neq 0$ and $\beta\in[0,1[$,

$$\begin{aligned} |K_{\beta,2}(x)| &\leq C\frac{1}{|x|^3}\left|\sum_{j=N}^{\infty}\int_{\mathbb{R}} e^{ix\zeta}\,\partial_x^3\big(m_\beta(\zeta)\varphi(2^{-j}\zeta)\big)\mathrm{d}\zeta\right| \\ &\leq \frac{C}{|x|^3}\sum_{j=N}^{\infty}\int_{2^{j-1}\leq|\zeta|\leq 2^{j+1}}\big(|\partial^3 m_\beta(\zeta)|+|m_\beta(\zeta)|2^{-3j}\big)\mathrm{d}\zeta \\ &\leq \frac{C}{|x|^3}\sum_{j=N}^{\infty}2^{-j(2-\beta)}\leq\frac{C}{|x|^3}2^{-N(2-\beta)}. \end{aligned}$$

For every $x\neq 0$, we choose $N\in\mathbb{Z}$ to be $N=[\log_2\frac{1}{|x|}]+1$ (which implies that $\frac{1}{|x|}\leq 2^N\leq \frac{2}{|x|}$), and gathering the above estimates leads to $|K_\beta(x)|\leq C|x|^{-1-\beta}$. Noting that

$$K_\beta'(x)=-C_0\int_{\mathbb{R}} e^{ix\zeta}\zeta m_\beta(\zeta)\chi(2^{-N}\zeta)\mathrm{d}\zeta - C_0\sum_{j=N}^{\infty}\int_{\mathbb{R}} e^{ix\zeta}\zeta m_\beta(\zeta)\varphi(2^{-j}\zeta)\mathrm{d}\zeta,$$

by using the same argument as above, we obtain that for every $\beta\in[0,1[$ and $x\neq 0$, $|K_\beta'(x)|\leq C|x|^{-2-\beta}$, as desired. □

2.2. Modulus of continuity. First is the definition of the modulus of continuity.

Definition 2.1. *A function $\omega:]0,\infty[\to]0,\infty[$ is called a modulus of continuity (abbr. MOC) if ω is continuous on $]0,\infty[$, nondecreasing, concave, and piecewise C^2 with one-sided derivatives defined at every point in $]0,\infty[$. We say a function $f:\mathbb{R}^d\to\mathbb{R}^l$ obeys the modulus of continuity ω if $|f(x)-f(y)|<\omega(|x-y|)$ for every $x\neq y\in\mathbb{R}^d$.*

We have the following general criterion on the preservation of the modulus of continuity $\omega(\xi,t)$ by some function $u(x,t)$.

Proposition 2.1. *Assume that*

(1) for every $t\geq 0$, $\omega(\xi,t)$ is a MOC and satisfies that its inverse function $\omega^{-1}((2+\epsilon_0)\|\theta(\cdot,t)\|_{L^\infty_x},t)<\infty$ with some $\epsilon_0>0$;

(2) for every fixed point ξ, $\omega(\xi,t)$ is piecewise C^1 in the time variable with one-sided derivatives defined at each point, and that for all ξ near infinity, $\omega(\xi,t)$ is continuous in t uniformly in ξ;

(3) $\omega(0+,t)$ and $\partial_\xi\omega(0+,t)$ are continuous in t with values in $\mathbb{R}\cup\{\pm\infty\}$, and satisfy that for every $t\geq 0$, either $\omega(0+,t)>0$ or $\partial_\xi\omega(0+,t)=\infty$ or $\partial_{\xi\xi}\omega(0+,t)=-\infty$.

Let $u\in C([0,T^*[;H^s(\mathbb{R}^d))\cap C^\infty(\mathbb{R}^d\times]0,T^*[)$, $s>\frac{d}{2}+1$ *be a smooth function and initially* $u_0(x)$ *obey* $\omega(\xi,0)$. *Then for every* $T\in]0,T^*[$, *if for all* $t\in]0,T]$ *and* $x\neq y\in\mathbb{R}^d$ *satisfying the following scenario*

$$\begin{aligned} &u(x,t)-u(y,t)=\omega(\xi,t), \quad \textit{with } \xi=|x-y|, \quad \textit{and} \\ &|u(x',t)-u(y',t)|\leq \omega(|x'-y'|,t), \quad \forall x',y'\in\mathbb{R}^d. \end{aligned} \tag{2.7}$$

one can show that for every $\xi\in\{\xi>0:\omega(\xi,t)\leq 2\|u(\cdot,t)\|_{L^\infty}\}$,

$$-\partial_t\omega(\xi,t)+\big(\partial_t u(x,t)-\partial_t u(y,t)\big)<0, \tag{2.8}$$

then the function $u(x,T)$ *obeys the modulus of continuity* $\omega(\xi,T)$.

Proof. Assume that $t_1\in]0,T^*[$ is the first time that the modulus of continuity $\omega(\xi,t)$ is lost by $u(x,t)$, then there exist two points $x\neq y\in\mathbb{R}^d$ such that the scenario (2.7) holds with $t=t_1$, and for the proof one can see [39, Proposition 3.2] or [30, Theorem 2.2].

If one has inequality (2.8) with $t=t_1$, then it directly yields

$$\partial_t\Big(\frac{u(x,t)-u(y,t)}{\omega(\xi,t)}\Big)\Big|_{t=t_1}=\frac{-\partial_t\omega(\xi,t_1)+\partial_t u(x,t_1)-\partial_t u(y,t_1)}{\omega(\xi,t_1)}<0,$$

which is a clear contradiction to scenario (2.7) and the fact that $u(x,t)$ obeys the MOC $\omega(\xi,t)$ for every $0\leq t<t_1$. Hence, inequality (2.8) in the considered scope (for ξ satisfying $\omega(\xi,t)>2\|u(t)\|_{L^\infty}$ the preservation naturally holds) under scenario (2.7) guarantees that the MOC $\omega(\xi,t)$ is preserved by the function $u(x,t)$, as desired. ∎

The following lemma is concerned with some actions of functions having the modulus of continuity.

LEMMA 2.4. *Assume that* $\omega(\xi,t)$ *for every* $t\geq 0$ *is a modulus of continuity, and scenario* (2.7) *is satisfied. Then the following statements hold.*

(1) We have

$$|(u\partial_x u)(x,t)-(u\partial_x u)(y,t)|\leq\omega(\xi,t)\partial_\xi\omega(\xi,t), \tag{2.9}$$

and

$$\partial_{xx}u(x,t)-\partial_{xx}u(y,t)\leq 2\partial_{\xi\xi}\omega(\xi,t). \tag{2.10}$$

(2) Define $D_\alpha(x,y,t):=-\Lambda^\alpha u(x,t)+\Lambda^\alpha u(y,t)$, $\alpha\in]0,2[$. *Then* $D_\alpha(x,y,t)$ *can be expressed as*

$$D_\alpha(x,y,t)=C_\alpha\,\mathrm{p.v.}\int_{\mathbb{R}}\frac{1}{|z|^{1+\alpha}}\big(u(x+z,t)-u(y+z,t)-\omega(\xi,t)\big)\,\mathrm{d}z, \tag{2.11}$$

and it satisfies that for any $\xi=|x-y|>0$,

$$\begin{aligned} D_\alpha(x,y,t)\leq{}& C_1\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta,t)+\omega(\xi-2\eta,t)-2\omega(\xi,t)}{\eta^{1+\alpha}}\,d\eta \\ &+C_1\int_{\frac{\xi}{2}}^\infty\frac{\omega(2\eta+\xi,t)-\omega(2\eta-\xi,t)-2\omega(\xi,t)}{\eta^{1+\alpha}}\,d\eta, \end{aligned} \tag{2.12}$$

with $C_1>0$ *a constant depending only on* α.

(3) Define $\Phi_\beta(x,y,t):=-\big(L_\beta u(x,t)-L_\beta u(y,t)\big)$, $\beta\in[0,1[$, *where* L_β *is the Fourier multiplier operator introduced in Equation* (1.7). *Then for* $\alpha\in[\beta,2[$ *and for any* $\xi=|x-y|>0$,

$$\Phi_\beta(x,y,t)\le -C_2\xi^{\alpha-\beta}D_\alpha(x,y,t)+C_2\xi\int_\xi^\infty \frac{\omega(\eta,t)}{\eta^{2+\beta}}\mathrm{d}\eta+C_2\xi^{-\beta}\omega(\xi,t), \tag{2.13}$$

with the constant $C_2>0$ *depending only on* α *and* β. *Besides, there also exists a constant* $C_2'=C_2'(\beta)>0$ *such that for any* $\xi=|x-y|>0$,

$$\Phi_\beta(x,y,t)\le C_2'\int_0^\xi \frac{\omega(\eta,t)}{\eta^{1+\beta}}\mathrm{d}\eta+C_2'\xi\int_\xi^\infty \frac{\omega(\eta,t)}{\eta^{2+\beta}}\mathrm{d}\eta. \tag{2.14}$$

Proof. Since the time variable t does not play an essential role in the proof, we suppress it in the functions u, ω, D_α, Φ and Φ_β for simplicity.

(1) The proof of inequalities (2.9) and (2.10) is classical, e.g. see [30, 33], and we omit the details.

(2) Equality (2.11) directly follows from scenario (2.7) and the following expression (see [14])

$$\Lambda^\alpha u(x)=C_\alpha\mathrm{p.v.}\int_{\mathbb{R}}\frac{u(x)-u(x+z)}{|z|^{1+\alpha}}\mathrm{d}z,\quad \forall\alpha\in]0,2[. \tag{2.15}$$

The proof of inequality (2.12) is by now classical, e.g. see [30,33], and we here omit the details.

(3) By using expression (2.5), we see that

$$\begin{aligned}
&|L_\beta u(x)-L_\beta u(y)|\\
=&\left|\mathrm{p.v.}\int_{\mathbb{R}}K_\beta(x-z)\big(u(z)-u(x)\big)\mathrm{d}z-\mathrm{p.v.}\int_{\mathbb{R}}K_\beta(y-z)\big(u(z)-u(y)\big)\mathrm{d}z\right|\\
=&\left|\mathrm{p.v.}\int_{\mathbb{R}}K_\beta(z)\big(u(x-z)-u(x)\big)\mathrm{d}z-\mathrm{p.v.}\int_{\mathbb{R}}K_\beta(z)\big(u(y-z)-u(y)\big)\mathrm{d}z\right|\\
\le&|I(x,y)|+|II(x,y)|,
\end{aligned}$$

with

$$I(x,y):=\mathrm{p.v.}\int_{|z|\le 2\xi}K_\beta(z)\big(u(x-z)-u(x)\big)\mathrm{d}z-\mathrm{p.v.}\int_{|z|\le 2\xi}K_\beta(z)\big(u(y-z)-u(y)\big)\mathrm{d}z, \tag{2.16}$$

and

$$II(x,y):=\int_{|z|\ge 2\xi}K_\beta(z)\big(u(x-z)-u(x)\big)\mathrm{d}z-\int_{|z|\ge 2\xi}K_\beta(z)\big(u(y-z)-u(y)\big)\mathrm{d}z. \tag{2.17}$$

Scenario (2.7) implies that

$$I(x,y)=\mathrm{p.v.}\int_{|z|\le 2\xi}K_\beta(z)\big(u(x-z)-u(y-z)-\omega(\xi)\big)\mathrm{d}z,$$

and recalling that $D_\alpha(x,y)$ has expression formula (2.11), we use kernel estimate (2.6) to obtain that for some $B>0$ chosen later,

$$I(x,y)+B\xi^{\alpha-\beta}D_\alpha(x,y)$$

$$
\begin{aligned}
&= \int_{|z|\le 2\xi}\Big(-K_\beta(z)-B\xi^{\alpha-\beta}\frac{C_\alpha}{|z|^{1+\alpha}}\Big)\big(\omega(\xi)+u(y-z)-u(x-z)\big)\mathrm{d}z \\
&\quad -\int_{|z|\ge 2\xi}\frac{C_\alpha}{|z|^{1+\alpha}}\big(\omega(\xi)+u(y-z)-u(x-z)\big)\mathrm{d}z \\
&\le \int_{|z|\le 2\xi}\Big(C_\beta\frac{|z|^{\alpha-\beta}}{\xi^{\alpha-\beta}}-C_\alpha B\Big)\frac{\xi^{\alpha-\beta}}{|z|^{1+\alpha}}\big(\omega(\xi)+u(y-z)-u(x-z)\big)\mathrm{d}z \\
&\le \int_{|z|\le 2\xi}\big(2^{\alpha-\beta}C_\beta-C_\alpha B\big)\frac{\xi^{\alpha-\beta}}{|z|^{1+\alpha}}\big(\omega(\xi)+u(y-z)-u(x-z)\big)\mathrm{d}z.
\end{aligned}
$$

Thus by choosing $B=\frac{4C_\beta}{C_\alpha}$, we immediately get

$$
|I(x,y)|\le -B\xi^{\alpha-\beta}D_\alpha(x,y). \tag{2.18}
$$

Besides, by starting from formula (2.16), and using estimates (2.7), (2.6) and the concavity property of $\omega(\eta)$, we also have

$$
|I(x,y)|\le C_0\int_{|z|\le 2\xi}|K_\beta(z)|\omega(|z|)\mathrm{d}z\le C_\beta\int_0^{2\xi}\frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta\le C_\beta\int_0^{\xi}\frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta. \tag{2.19}
$$

For $II(x,y)$, since $K_\beta(z)=K_\beta(|z|)\mathrm{sgn}(z)$ is an odd function, by denoting $\tilde{x}=\frac{x+y}{2}$ and in a similar argument as [33, Lemma], we deduce that

$$
\begin{aligned}
|II(x,y)|&=\Big|\int_{|x-z|\ge 2\xi}K_\beta(x-z)(u(z)-u(\tilde{x}))\mathrm{d}z-\int_{|y-z|\ge 2\xi}K_\beta(y-z)(u(z)-u(\tilde{x}))\mathrm{d}z\Big| \\
&\le \int_{|\tilde{x}-z|\ge 3\xi}|K_\beta(x-z)-K_\beta(y-z)||u(z)-u(\tilde{x})|\mathrm{d}z \\
&\quad +\int_{\frac{3}{2}\xi\le|\tilde{x}-z|\le 3\xi}\big(|K_\beta(x-z)|+|K_\beta(y-z)|\big)|u(z)-u(\tilde{x})|\mathrm{d}z \\
&\le \int_{|\tilde{x}-z|\ge 3\xi}\frac{C_\beta\xi}{|\tilde{x}-z|^{2+\beta}}\omega(|\tilde{x}-z|)\mathrm{d}z+\int_{\frac{3}{2}\xi\le|\tilde{x}-z|\le 3\xi}\frac{C_\beta}{|\tilde{x}-z|^{1+\beta}}\omega(|\tilde{x}-z|)\mathrm{d}z \\
&\le C_\beta\xi\int_\xi^\infty\frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta+C_\beta\xi^{-\beta}\omega(\xi),
\end{aligned} \tag{2.20}
$$

where in the fourth line we have used estimate (2.6) and the fact that $|K_\beta(x-z)-K_\beta(y-z)|\le\frac{C_\beta\xi}{|\tilde{x}-z|^{2+\beta}}$, $\forall z\in\mathbb{R}\setminus B_{2\xi}(\tilde{x})$. Combining estimate (2.18) with estimate (2.20) leads to inequality (2.13). Note that $\int_0^\xi\frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta\ge\frac{\xi^{-\beta}}{1-\beta}\omega(\xi)$, thus combining estimate (2.19) with estimate (2.20) leads to inequality (2.14), as desired. ∎

2.3. Some auxiliary lemmas. We have the following product result used in Subsection 4.1.

Lemma 2.5. *Assume that $m\in\mathbb{N}$, $p\in]1,\infty[$, $f\in C_c^\infty(\mathbb{R}^d)$ and $g\in W^{-m,p}(\mathbb{R}^d)$. Then we have*

$$
\|fg\|_{W^{-m,p}(\mathbb{R}^d)}\le C\|f\|_{W^{m,\infty}(\mathbb{R}^d)}\|g\|_{W^{-m,p}(\mathbb{R}^d)}, \tag{2.21}
$$

where $C>0$ is a constant depending only on m and d.

Proof. Since $W^{m,q}(\mathbb{R}^d)$ ($q=\frac{p}{p-1}$ is the dual number of p) is the dual space of $W^{-m,p}(\mathbb{R}^d)$, we get

$$\begin{aligned}\|fg\|_{W^{-m,p}(\mathbb{R}^d)} &= C \sup_{\|h\|_{W^{m,q}(\mathbb{R}^d)}\le 1} \left|\int_{\mathbb{R}^d} fgh\,\mathrm{d}x\right| \\ &\le C \sup_{\|h\|_{W^{m,q}(\mathbb{R}^d)}\le 1} \|g\|_{W^{-m,p}(\mathbb{R}^d)}\|fh\|_{W^{m,q}(\mathbb{R}^d)} \\ &\le C\|g\|_{W^{-m,p}(\mathbb{R}^d)} \sup_{\|h\|_{W^{m,q}(\mathbb{R}^d)}\le 1} \left(\|f\|_{W^{m,\infty}(\mathbb{R}^d)}\|h\|_{W^{m,q}(\mathbb{R}^d)}\right) \\ &\le C\|g\|_{W^{-m,p}(\mathbb{R}^d)}\|f\|_{W^{m,\infty}(\mathbb{R}^d)}.\end{aligned}$$

□

The next lemma is concerned with the energy estimate and L^∞-estimate of smooth solution for the considered dissipative dispersive equation.

Lemma 2.6. *Let $u(x,t)\in C([0,T^*[;H^s(\mathbb{R}))\cap C^\infty(]0,T^*[\times\mathbb{R})$, $s>3/2$ be a smooth solution to the dissipative dispersive Burgers Equation (1.7) with $\alpha\in]0,2]$, $\beta\in[0,1[$. Then we get*

$$\|u(t)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{\dot H^{\frac{\alpha}{2}}}^2\,\mathrm{d}\tau \le \|u_0\|_{L^2}^2, \quad \textit{for all } t\in[0,T^*[, \tag{2.22}$$

and if $\alpha\in]\beta,2]$, we also have

$$\begin{cases}\sup_{t\in[0,T^*[}\|u(t)\|_{L^\infty} \le C(\alpha,\beta,\mu,\nu)\|u_0\|_{L^2\cap L^\infty}, & \textit{for } \alpha\in]\beta,2[, \\ \sup_{t\in[0,T]}\|u(t)\|_{L^\infty} \le C(\beta,\mu,\nu,T)\|u_0\|_{L^2\cap L^\infty}, & \textit{for } \alpha=2 \textit{ and } T\in]0,T^*[.\end{cases} \tag{2.23}$$

Proof. Noticing that the function $m_\beta(\zeta)$ in equality (1.8) is an odd function, we find $\int_{\mathbb{R}} L_\beta u(x)\,u(x)\mathrm{d}x = \int_{\mathbb{R}} -im_\beta(\zeta)|\hat u(\zeta)|^2\mathrm{d}\zeta = 0$, thus the L^2-energy estimate (2.22) can be deduced in the usual way.

Now we show L^∞-estimate (2.23). For the case $\alpha=2$, estimate (2.23) is a consequence of L^2-estimate (2.22) and inequality (6.15) below: indeed, by using assumption (A1), for every $T\in]0,T^*[$ we have

$$\begin{aligned}\sup_{t\in[0,T]}\|u(t)\|_{L^\infty} &\le C\|u_0\|_{L^\infty} + CT^{\frac14}\|L_\beta u\|_{L^2_T(L^2)} \\ &\le C\|u_0\|_{L^\infty} + CT^{\frac14}\|u\|_{L^2_T(L^2\cap \dot H^1)} \le C(\beta,\mu,\nu,T)\|u_0\|_{L^2\cap L^\infty}.\end{aligned}$$

Next we prove estimate (2.23) for the case $\alpha\in]\beta,2[$ by applying an argument from [31]. For $t\in]0,T^*[$ fixed, assume that $x\in\mathbb{R}$ is the spatial point at which $u(x,t)$ attains its maximum $M=M(t)=\sup_{\mathbb{R}} u(\cdot,t)$. Then by virtue of formulas (2.5) and (2.15), at the maximum point we have

$$\begin{aligned}\partial_t u(x,t) &= -\nu\Lambda^\alpha u(x,t) - \mu L_\beta u(x,t) \\ &= -C_\alpha\nu\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{M(t)-u(x-y,t)}{|y|^{1+\alpha}}\mathrm{d}y + \mu\,\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(y)\big(M(t)-u(x-y,t)\big)\mathrm{d}y \\ &:= J_1(x,t)+J_2(x,t),\end{aligned} \tag{2.24}$$

where (by using the fact that $K_\beta(y)$ is an odd function)

$$\begin{aligned}J_1(x,t) &:= \mathrm{p.v.}\int_{|y|\le r}\left(-\frac{C_\alpha\nu}{|y|^{1+\alpha}} + \mu K_\beta(y)\right)(M(t)-u(x-y,t))\mathrm{d}y, \\ J_2(x,t) &:= -C_\alpha\nu\int_{|y|\ge r}\frac{M(t)-u(x-y,t)}{|y|^{1+\alpha}}\mathrm{d}y - \mu\int_{|y|\ge r} K_\beta(y)u(x-y,t)\mathrm{d}y,\end{aligned}$$

with $r>0$ some constant chosen later. From kernel estimate (2.6), we obtain $-\frac{C_\alpha \nu}{|y|^{1+\alpha}}+\mu K_\beta(y)\leq -\frac{C_\alpha\nu}{|y|^{1+\alpha}}+\frac{|\mu|C_\beta}{|y|^{1+\beta}}$, thus by choosing $r:=\left(\frac{C_\alpha\nu}{2|\mu|C_\beta}\right)^{\frac{1}{\alpha-\beta}}$ so that $\frac{|\mu|C_\beta}{|y|^{1+\beta}}\leq \frac{C_\alpha\nu}{2|y|^{1+\alpha}}$ for every $0<|y|\leq r$, it is obvious that

$$J_1(x,t)\leq -\frac{C_\alpha\nu}{2}\mathrm{p.v.}\int_{|y|\leq r}\frac{M(t)-u(x-y,t)}{|y|^{1+\alpha}}\mathrm{d}y. \tag{2.25}$$

Thanks to the rearrangement inequality (e.g. see [37, Chapter 3]), the right-hand side of (2.25) is maximal by replacing $u(x-y,t)$ with its symmetric decreasing rearrangement $u^*(x-y,t)$, and from the property of $u^*(x-y,t)$ and energy estimate (2.22) we see that

$$\left|\left\{y\in\mathbb{R}: u^*(x-y,t)\geq \frac{M(t)}{2}\right\}\right| = \left|\left\{y\in\mathbb{R}: |u(x-y,t)|\geq \frac{M(t)}{2}\right\}\right| \leq \frac{4\|u(t)\|_{L^2}^2}{M(t)^2}\leq \frac{4\|u_0\|_{L^2}^2}{M(t)^2}.$$

Thus by setting $\tau=\frac{2\|u_0\|_{L^2}^2}{M(t)^2}$ and letting $M(t)$ be suitably large enough so that $\tau\leq\frac{r}{2}$ (i.e. $M(t)\geq \frac{2\|u_0\|_{L^2}}{\sqrt{r}}$), we have $\{y\in\mathbb{R}:u^*(x-y,t)\geq M(t)/2\}\subseteq[-\tau,\tau](\subseteq[-\frac{r}{2},\frac{r}{2}])$, and also

$$\begin{aligned}
J_1(x,t)&\leq -\frac{C_\alpha\nu M(t)}{4}\int_{[-r,r]\setminus[-\tau,\tau]}\frac{1}{|y|^{1+\alpha}}\mathrm{d}y\\
&\leq -\frac{C_\alpha\nu M(t)}{2\alpha}(\tau^{-\alpha}-r^{-\alpha})\\
&\leq -\frac{C_\alpha\nu}{\alpha 2^{\alpha+1}\|u_0\|_{L^2}^{2\alpha}}M(t)^{1+2\alpha}+\frac{C_\alpha\nu}{2\alpha}r^{-\alpha}M(t).
\end{aligned} \tag{2.26}$$

For $J_2(x,t)$, since the first term on the right-hand side is negative, we directly use the Hölder inequality and estimates (2.6), (2.22) to get

$$J_2(x,t)\leq |\mu|C_\beta\left(\int_{\mathbb{R}\setminus[-r,r]}\frac{1}{|y|^{2+2\beta}}\mathrm{d}y\right)^{1/2}\|u(t)\|_{L^2}\leq 2|\mu|C_\beta\|u_0\|_{L^2}r^{-\frac12-\beta}. \tag{2.27}$$

Inserting inequalities (2.26), (2.27) into equality (2.24) yields

$$\partial_t u(x,t)\leq -\frac{C_\alpha\nu}{\alpha 2^{\alpha+1}\|u_0\|_{L^2}^{2\alpha}}M(t)^{1+2\alpha}+\frac{C_\alpha\nu}{2\alpha}r^{-\alpha}M(t)+2|\mu|C_\beta\|u_0\|_{L^2}r^{-\frac12-\beta}.$$

As long as $M(t)\geq 2^{\frac12+\frac1\alpha}r^{-\frac12}\|u_0\|_{L^2}$ and $M(t)\geq (\frac{|\mu|\alpha C_\beta 2^{\alpha+4}}{C_\alpha\nu})^{\frac{1}{1+2\alpha}}r^{-\frac{1+2\beta}{2(1+2\alpha)}}\|u_0\|_{L^2}$, the negative contribution dominates on the right-hand side, and we get

$$\partial_t u(x,t)\leq -\frac12\frac{C_\alpha\nu}{\alpha 2^{\alpha+1}\|u_0\|_{L^2}^{2\alpha}}M(t)^{1+2\alpha}<0.$$

Thus by setting

$$M_0:=\max\left\{2^{\frac12+\frac1\alpha}\left(\frac{2|\mu|C_\beta}{C_\alpha\nu}\right)^{\frac{1}{2(\alpha-\beta)}},\left(\frac{\alpha 2^{\alpha+4}|\mu|C_\beta}{C_\alpha\nu}\right)^{\frac{1}{1+2\alpha}}\left(\frac{2|\mu|C_\beta}{C_\alpha\nu}\right)^{\frac{1+2\beta}{2(\alpha-\beta)(1+2\alpha)}}\right\},$$

we can infer that $\frac{\mathrm{d}}{\mathrm{d}t}M(t)\leq\partial_t u(x,t)<0$ for every $t\in]0,T^*[$ satisfying $M(t)\geq M_0\|u_0\|_{L^2}$ (e.g. see [14, Theorem 4.1]). Hence we conclude that $M(t)\leq\max\{M_0\|u_0\|_{L^2},\|u_0\|_{L^\infty}\}$ for any $t\in[0,T^*[$ and the desired estimate (2.23) follows. $\square$

We recall the following uniform-in-ϵ estimates of the ϵ-regularized transport-diffusion equation, and for the proof one can see Theorem 1.2 and Remark 1.3 of [50] (in fact a more general dissipation term is considered there).

LEMMA 2.7. *Consider the following ϵ-regularized drift-diffusion equation*

$$\partial_t u + b_\epsilon \cdot \nabla u + \nu \Lambda^\alpha u - \epsilon \Delta u = f_\epsilon, \quad u|_{t=0} = u_{0,\epsilon} = \phi_\epsilon * (u_0 1_{B_{1/\epsilon}(0)}), \tag{2.28}$$

where $\alpha \in]0,1]$, $b_\epsilon = \phi_\epsilon * b$, $f_\epsilon = \phi_\epsilon * f$, $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$ and ϕ is the standard mollifier. Let $u_0 \in C_0(\mathbb{R}^d)$ with $C_0(\mathbb{R}^d)$ being the space of continuous functions which decay at infinity. Suppose that for any given $T>0$, the functions b and f satisfy

$$b \in L^\infty([0,T]; C^\delta(\mathbb{R}^d)), \quad \text{and} \quad f \in L^\infty([0,T]; C^\delta \cap L^2(\mathbb{R}^d)), \quad \text{for some } \delta \in]1-\alpha, 1[, \tag{2.29}$$

then the solutions $u^{(\epsilon)}$ of the regularized drift-diffusion Equation (2.28) uniformly-in-ϵ belong to

$$L^\infty([0,T]; C_0(\mathbb{R}^d)) \cap L^\infty((0,T], C^{1,\varrho}(\mathbb{R}^d)) \quad \text{for any } \varrho \in]0, \delta+\alpha-1[.$$

More precisely, for any $t' \in]0,T[$, we have

$$\|u^{(\epsilon)}\|_{L^\infty([t',T]; C^{1,\varrho}(\mathbb{R}^d))} \leq C t'^{-\frac{\varrho+1}{\alpha}} \Big(\|u_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta} \Big), \tag{2.30}$$

where C is a positive constant depending only on ν, α, d, δ and $\|b\|_{L^\infty_T \dot{C}^\delta}$ and is independent of ϵ.

If assumption (2.29) holds for some $\delta > 1-\alpha$ without the restriction $\delta<1$, then we also have, uniformly in ϵ,

$$u^{(\epsilon)} \in \begin{cases} L^\infty(]0,T]; C^{\delta+\alpha-1,\varrho}), \ \forall \varrho \in]0,1[, & \text{if } \delta+\alpha \in \mathbb{N}^+, \\ L^\infty(]0,T]; C^{[\delta+\alpha],\varrho}), \ \forall \varrho \in]0, \delta+\alpha-[\delta+\alpha][, & \text{if } \delta+\alpha \notin \mathbb{N}^+, \end{cases} \tag{2.31}$$

with the corresponding uniform-in-ϵ bounds analogous with (2.30). Here, $[a]$ denotes the integer part of the real number a.

Note that in Lemma 2.7, the decaying property of the assumption $u_0 \in C_0(\mathbb{R}^d)$, the cutoff function $1_{B_{1/\epsilon}}$ in the definition of $u_{0,\epsilon}$, as well as the assumption $f \in L^\infty([0,T]; L^2(\mathbb{R}^d))$ are only used to show that the solutions $u^{(\epsilon)}$ are smooth functions having the spatial decay, and these assumptions are not virtual for the uniform estimate (2.30). Thus as a direct consequence of Lemma 2.7 we have the following a priori estimates.

COROLLARY 2.1. *Assume that $u \in C([0,T^*[; H^s(\mathbb{R}^d)) \cap C^\infty(\mathbb{R}^d \times]0,T^*[)$, $s > 1+\frac{d}{2}$ is a smooth solution to the following drift-diffusion equation*

$$\partial_t u + b \cdot \nabla u + \nu \Lambda^\alpha u = f, \quad u|_{t=0} = u_0, \quad \alpha \in]0,1]. \tag{2.32}$$

Suppose that for any given $T \in]0,T^*[$, the functions b and f satisfy

$$b \in L^\infty([0,T]; C^\delta(\mathbb{R}^d)), \quad \text{and} \quad f \in L^\infty([0,T]; C^\delta(\mathbb{R}^d)), \quad \text{for some } \delta \in]1-\alpha, 1[, \tag{2.33}$$

then we have that for any $t' \in]0,T[$,

$$\|u\|_{L^\infty([t',T]; C^{1,\varrho}(\mathbb{R}^d))} \leq C t'^{-\frac{\varrho+1}{\alpha}} \Big(\|u_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta} \Big), \tag{2.34}$$

where C is a positive constant depending only on ν, α, d, δ and $\|b\|_{L^\infty_T \dot{C}^\delta}$. Moreover, if $\delta > 1-\alpha$ in assumption (2.33), we have

$$u \in \begin{cases} L^\infty(]0,T];C^{\delta+\alpha-1,\varrho}), \ \forall \varrho \in]0,1[, & \text{if } \delta+\alpha \in \mathbb{N}^+, \\ L^\infty(]0,T];C^{[\delta+\alpha],\varrho}), \ \forall \varrho \in]0,\delta+\alpha-[\delta+\alpha][, & \text{if } \delta+\alpha \notin \mathbb{N}^+, \end{cases} \tag{2.35}$$

with the upper bounds analogous to (2.34).

REMARK 2.1. In the application of Corollary 2.1 to Equation (1.7), we usually view the dispersive term $\mu L_\beta u$ as the forcing term f.

3. Proof of Theorem 1.1

At first, we have the local well-posedness result for the considered Equation (1.7), whose proof is placed in the Appendix.

PROPOSITION 3.1. *Let $\mu \neq 0$, $\nu > 0$, $\alpha \in]0,2]$, $\beta \in [0,1[$, and $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there is a time $T > 0$ depending on s and $\|u_0\|_{H^s(\mathbb{R})}$ such that the dissipative dispersive Burgers Equation (1.7) admits a unique local solution $u \in C([0,T];H^s(\mathbb{R})) \cap L^2([0,T];H^{s+\frac{\alpha}{2}}(\mathbb{R})) \cap C^\infty(\mathbb{R}\times]0,T])$ with $s > \frac{3}{2}$.*

We also have the classical blowup criterion: let $T^* > 0$ be the maximal existence time of the above constructed solution, then

$$\text{if } T^* < \infty \Rightarrow \int_0^{T^*} \|\partial_x u(t)\|_{L^\infty(\mathbb{R})} \mathrm{d}t = \infty. \tag{3.1}$$

Next, in the following two subsections, we prove Theorem 1.1 for the subcritical case $\alpha \in]1,2]$ and the critical case $\alpha = 1$ respectively.

3.1. Global well-posedness for Equation (1.7) with subcritical dissipation $\alpha \in]1,2]$ and smooth data. Assume that T^* is the maximal time of existence for solution u to Equation (1.7) in $C([0,T^*[,H^s(\mathbb{R})) \cap C^\infty(\mathbb{R}\times]0,T^*[)$ with $s > \frac{3}{2}$. Let $T \in]0,T^*[$ be given.

According to (2.23), for every $\alpha \in]1,2]$ and $T \in]0,T^*[$, we know that

$$\|u\|_{L^\infty_T(L^\infty)} \le B_\alpha(T), \tag{3.2}$$

where $B_\alpha(T)$ is the upper bound in (2.23) depending only on α,β,μ,ν,T and $\|u_0\|_{L^2\cap L^\infty}$.

In the sequel, in order to derive the upper bound of the Lipschitz norm of u on the time period $[0,T]$, we shall prove that for some stationary modulus of continuity

$$\omega_\lambda(\xi) := \lambda^{\alpha-1}\omega(\lambda\xi), \quad \lambda \in]0,\infty[, \tag{3.3}$$

where

$$\omega(\xi) = \begin{cases} \xi - \xi^{\frac{1+\alpha}{2}}, & \text{if } \ 0 < \xi \le \delta, \\ \delta - \delta^{\frac{1+\alpha}{2}}, & \text{if } \ \xi > \delta, \end{cases} \tag{3.4}$$

with some $0 < \delta < 1$ chosen later, such an $\omega_\lambda(\xi)$ is preserved by the evolution of Equation (1.7). Clearly, ω_λ is a modulus of continuity, moreover, it satisfies $\omega_\lambda(0+) = 0$, $\omega_\lambda'(0+) = \lambda$ and $\omega_\lambda''(0+) = -\infty$.

First notice that by choosing λ as

$$\lambda := \max\left\{ \left(\frac{4B_\alpha(T)}{\delta/2 - (\delta/2)^{(1+\alpha)/2}} \right)^{\frac{1}{\alpha-1}}, \frac{\delta\|\partial_x u_0\|_{L^\infty}}{2\|u_0\|_{L^\infty}}, 1 \right\}, \tag{3.5}$$

we find that $u_0(x)$ obeys this $\omega_\lambda(\xi)$ for λ sufficiently large. Indeed, from $|u_0(x)-u_0(y)|\leq 2\|u_0\|_{L^\infty}$ and $|u_0(x)-u_0(y)|\leq \|\partial_x u_0\|_{L^\infty}|x-y|$, it suffices to ensure that $\min\{2\|u_0\|_{L^\infty},\|\partial_x u_0\|_{L^\infty}\xi\}<\omega_\lambda(\xi)$. Hence, by setting $a_0:=\frac{2\|u_0\|_{L^\infty}}{\|\nabla u_0\|_{L^\infty}}$, and using the concavity of $\omega_\lambda(\xi)$, we see that it only needs to show

$$\omega_\lambda(a_0)=\lambda^{\alpha-1}\omega(\lambda a_0)>2\|u_0\|_{L^\infty}. \tag{3.6}$$

Therefore, to prove inequality (3.6), we let λ large enough so that (recalling $B_\alpha(T)\geq \|u_0\|_{L^\infty}$ from estimate (3.2))

$$\omega_\lambda(a_0)>\omega_\lambda\Big(\frac{\delta}{2\lambda}\Big)=\lambda^{\alpha-1}\omega\Big(\frac{\delta}{2}\Big)>2B_\alpha(T), \tag{3.7}$$

that is, $\lambda a_0>\frac{\delta}{2}$ and $\omega\big(\frac{\delta}{2}\big)>\frac{2B_\alpha(T)}{\lambda^{\alpha-1}}$, hence, we can choose λ as formula (3.5) and this proves the claim.

Note that by inequality (3.7) and the choice of λ in formula (3.5), we have $\omega_\lambda(\frac{\delta}{2\lambda})\geq 4B_\alpha(T)$, which implies that $\omega_\lambda^{-1}(4B_\alpha(T))\leq \frac{\delta}{2\lambda}$ with $\omega_\lambda^{-1}(\cdot)$ the inverse function of $\omega_\lambda(\xi)$.

Then in order to prove that the solution $u(x,t)$ obeys the MOC $\omega_\lambda(\xi)$ for all $t\in[0,T]$, according to Proposition 2.1 and Lemma 2.4, and noting that

$$\begin{aligned}\partial_t(u(x,t)-u(y,t))=&-\big(u\partial_x u(x,t)-u\partial_x u(y,t)\big)-\mu\big(\Lambda^\beta\mathcal{H}u(x,t)-\Lambda^\beta\mathcal{H}u(y,t)\big)\\&+\nu\big(-\Lambda^\alpha u(x,t)+\Lambda^\alpha u(y,t)\big),\end{aligned} \tag{3.8}$$

it remains to check that for all $t\in]0,T]$, $x\neq y\in\mathbb{R}$ satisfying (2.7) (with $\omega(\xi,t)=\omega_\lambda(\xi)$) and $0<\xi\in\{\xi:\omega_\lambda(\xi)\leq 2B_\alpha(T)\}$,

$$\omega_\lambda(\xi)\omega_\lambda'(\xi)+|\mu|\Phi_\beta(x,y,t)+\nu D_\alpha(x,y,t)<0, \tag{3.9}$$

where $\omega_\lambda(\xi)\omega_\lambda'(\xi)=\lambda^{2\alpha-1}\omega(\lambda\xi)\omega'(\lambda\xi)$, $\Phi_\beta(x,y,t)$ and $D_\alpha(x,y,t)$ respectively satisfies (from estimates (2.14), (2.10) and (2.12))

$$\Phi_\beta(x,y,t)\leq C_2'\int_0^\xi\frac{\omega_\lambda(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta+C_2'\xi\int_\xi^\infty\frac{\omega_\lambda(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta\leq\lambda^{\alpha+\beta-1}\Phi_\beta(\lambda\xi),$$

with $\Phi_\beta(\xi):=C_2'\int_0^\xi\frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta+C_2'\xi\int_\xi^\infty\frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta$, and

$$\begin{aligned}&D_\alpha(x,y,t)\\&\leq\begin{cases}2\omega_\lambda''(\xi), & \text{if }\alpha=2,\\ C_1\Big(\int_0^{\frac{\xi}{2}}\frac{\omega_\lambda(\xi+2\eta)+\omega_\lambda(\xi-2\eta)-2\omega_\lambda(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta+\int_{\frac{\xi}{2}}^\infty\frac{\omega_\lambda(2\eta+\xi)-\omega_\lambda(2\eta-\xi)-2\omega_\lambda(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta\Big), & \text{if }\alpha\in]1,2[,\end{cases}\\&\leq\lambda^{2\alpha-1}D_\alpha(\lambda\xi),\end{aligned}$$

with

$$D_\alpha(\xi)=\begin{cases}2\omega''(\xi), & \text{if }\alpha=2,\\ C_1\Big(\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta+\int_{\frac{\xi}{2}}^\infty\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta\Big), & \text{if }\alpha\in]1,2[.\end{cases} \tag{3.10}$$

Note that $\xi\in\{\xi>0:\omega_\lambda(\xi)\leq 2B_\alpha(T)\}\subset]0,\frac{\delta}{2\lambda}]$ (using inequality (3.7)) and $\lambda\geq 1$ from our choice, thus it suffices to prove that

$$\lambda^{2\alpha-1}\big(\omega\omega'+|\mu|\Phi_\beta+\nu D_\alpha\big)(\lambda\xi)<0,\quad\text{for all }\xi\in\Big]0,\frac{\delta}{2\lambda}\Big].$$

Hence, our aim is to show that, the modulus of continuity $\omega(\xi)$ defined by (3.4) verifies

$$\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(\xi)+\nu D_\alpha(\xi)<0,\quad \text{for all}\quad \xi\in]0,\delta/2],$$

that is, for $\alpha=2$,

$$\omega(\xi)\omega'(\xi)+C_2'|\mu|\int_0^\xi \frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta+C_2'|\mu|\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta+2\nu\omega''(\xi)<0,\quad \text{for all}\quad \xi\in]0,\delta/2], \tag{3.11}$$

and for $\alpha\in]1,2[$,

$$\begin{aligned}C_1\nu\int_0^{\frac{\xi}{2}} &\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta+C_1\nu\int_{\frac{\xi}{2}}^\infty \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta\\ &+\omega(\xi)\omega'(\xi)+C_2'|\mu|\int_0^\xi \frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta+C_2'|\mu|\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta<0,\quad \text{for all}\quad \xi\in]0,\delta/2], \end{aligned}\tag{3.12}$$

with $C_1=C_1(\alpha)$ and $C_2'=C_2'(\beta)$ being the constants appearing in Lemma 2.4.

We first justify (3.11) for the case $\alpha=2$. Since $\omega(\xi)=\xi-\xi^{3/2}$ for every $\xi\in]0,\delta]$, we get $\omega'(\xi)=1-\frac{3}{2}\xi^{1/2}$, $\omega''(\xi)=-\frac{3}{4}\xi^{-1/2}$, and $\omega(\xi)\omega'(\xi)\le\xi$. It is also easy to see that $\int_0^\xi \frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta\le\int_0^\xi \frac{1}{\eta^\beta}\mathrm{d}\eta\le\frac{\xi^{1-\beta}}{1-\beta}$, and

$$\begin{aligned}\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta&\le\xi\int_\xi^\delta \frac{1}{\eta^{1+\beta}}\mathrm{d}\eta+\xi\delta\int_\delta^\infty \frac{1}{\eta^{2+\beta}}\mathrm{d}\eta\\ &\le\begin{cases}\frac{1}{\beta}\xi^{1-\beta}+\frac{\xi\delta^{-\beta}}{1+\beta}, & \text{for}\quad \beta\in]0,1[,\\ \xi\log\frac{\delta}{\xi}+\xi, & \text{for}\quad \beta=0\end{cases}\\ &\le\bar{c}_\beta\delta^{1-\beta},\end{aligned}$$

where $\bar{c}_\beta=\frac{2}{\beta}$ for $\beta\in]0,1[$ and $\bar{c}_\beta=2$ for $\beta=0$. Gathering the above estimates, we have that for all $\xi\in]0,\delta/2]$,

$$\begin{aligned}\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(\xi)+\nu D_\alpha(\xi)&\le\xi+C_2'\frac{1}{1-\beta}|\mu|\xi^{1-\beta}+C_2'\bar{c}_\beta|\mu|\delta^{1-\beta}-\nu\frac{3}{2}\xi^{-\frac{1}{2}}\\ &\le\xi^{-\frac{1}{2}}\Big(\delta^{\frac{3}{2}}+C_2'\Big(\frac{1}{1-\beta}+\bar{c}_\beta\Big)|\mu|\delta^{\frac{3}{2}-\beta}-\frac{3}{2}\nu\Big)<0,\end{aligned}$$

where the last inequality holds by choosing $\delta>0$ small enough (i.e. $\delta<\min\{1,(\frac{\nu}{2})^{2/3},\big(\frac{(1-\beta)\nu}{2C_2'(1+\bar{c}_\beta)|\mu|}\big)^{\frac{2}{3-2\beta}}\}$).

We next turn to the proof of inequality (3.12) for the case $\alpha\in]1,2[$. Since $\omega(\xi)=\xi-\xi^{\frac{1+\alpha}{2}}$ for every $\xi\in]0,\delta]$, we get $\omega'(\xi)=1-\frac{1+\alpha}{2}\xi^{\frac{\alpha-1}{2}}$, $\omega''(\xi)=-\frac{(1+\alpha)(\alpha-1)}{4}\xi^{\frac{\alpha-3}{2}}$, and $\omega(\xi)\omega'(\xi)\le\xi$. Similarly as above, we also get $\int_0^\xi \frac{\omega(\eta)}{\eta^{1+\beta}}\mathrm{d}\eta\le\frac{\xi^{1-\beta}}{1-\beta}$ and $\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta\le \bar{c}_\beta\delta^{1-\beta}$. Due to the concavity of $\omega(\xi)$, both integrals in formula (3.10) are negative, and from the following estimate

$$\begin{aligned}\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)&=4\eta^2\int_0^1\int_{-1}^1 s\omega''(\xi+2s\tau\eta)\,\mathrm{d}\tau\mathrm{d}s\\ &\le 4\eta^2\int_0^1\int_{-1}^0 s\omega''(\xi)\,\mathrm{d}\tau\mathrm{d}s\le\omega''(\xi)\eta^2,\end{aligned}\tag{3.13}$$

we directly get

$$
\begin{aligned}
D_\alpha(\xi) &\le C_1 \int_0^{\frac{\xi}{2}} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+\alpha}}\mathrm{d}\eta \\
&\le -\frac{C_1(1+\alpha)(\alpha-1)}{4}\xi^{\frac{\alpha-3}{2}}\int_0^{\frac{\xi}{2}}\eta^{1-\alpha}\mathrm{d}\eta \le -\frac{C_1(\alpha-1)}{8(2-\alpha)}\xi^{-\frac{\alpha-1}{2}}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(\xi)+\nu D_\alpha(\xi) &\le \xi + \frac{C_2'|\mu|}{1-\beta}\xi^{1-\beta} + C_2'|\mu|\bar{c}_\beta\delta^{1-\beta} - \frac{C_1\nu(\alpha-1)}{8(2-\alpha)}\xi^{-\frac{\alpha-1}{2}} \\
&\le \xi^{-\frac{\alpha-1}{2}}\Big(\delta^{\frac{\alpha+1}{2}} + \frac{C_2'|\mu|(1+\bar{c}_\beta)}{1-\beta}\delta^{\frac{\alpha+1}{2}-\beta} - \frac{C_1\nu(\alpha-1)}{8(2-\alpha)}\Big) < 0,
\end{aligned}
$$

where the last inequality is guaranteed by letting $\delta>0$ be a fixed constant sufficiently small (that is, $\delta<\min\big\{1,(\frac{C_1\nu(\alpha-1)}{16(2-\alpha)})^{\frac{2}{\alpha+1}},(\frac{C_1(1-\beta)(\alpha-1)\nu}{16C_2'(1+\bar{c}_\beta)|\mu|})^{\frac{2}{3-2\beta}}\big\}$).

Therefore, for any given $T\in]0,T^*[$, and for every $\alpha\in]1,2]$, $\beta\in[0,1[$, the solution $u(x,t)$ to Equation (1.7) obeys the modulus of continuity $\omega_\lambda(\xi)$ with λ given by formula (3.5) for all $t\in[0,T]$, which implies that $\sup_{t\in[0,T]}\|\nabla u(\cdot,t)\|_{L^\infty}\le \omega_\lambda'(0+)=\lambda$. Since $T\in]0,T^*[$ is any given value, thanks to the blowup criterion 3.1, we conclude $T^*=+\infty$, and thus Theorem 1.1 associated with Equation (1.7) for the subcritical case $\alpha\in]1,2]$ is proved.

3.2. Global well-posedness for Equation (1.7) with critical dissipation $\alpha=1$ and smooth data. First, we have the following more refined blowup criterion than criterion (3.1).

LEMMA 3.1. *Under the assumptions of Proposition 3.1, if $T^*<\infty$ and $\alpha\in]\beta,1]$, then necessarily,*

$$
\|u\|_{L^\infty([0,T^*[;C^\sigma(\mathbb{R}))}=\infty, \quad \textit{for every } \sigma\in]\beta+1-\alpha,1[. \tag{3.14}
$$

Proof. If $T^*<\infty$ and $u\in L^\infty([0,T^*[;C^\sigma(\mathbb{R}))$ with $\sigma\in]\beta+1-\alpha,1[$, by using Lemma 2.2, Bernstein's inequality and the fact that $m_\beta(\zeta)$ is bounded on the interval $[-1,1]$, we have that for every $t\in[0,T^*[$,

$$
\begin{aligned}
\|L_\beta u(t)\|_{C^{\sigma-\beta}(\mathbb{R})} &\le C_0\|\Delta_{-1}L_\beta u(t)\|_{L^\infty(\mathbb{R})} + C_0\sup_{j\in\mathbb{N}}2^{j(\sigma-\beta)}\|\Delta_j L_\beta u(t)\|_{L^\infty(\mathbb{R})} \\
&\le C\|\Delta_{-1}L_\beta u(t)\|_{L^2(\mathbb{R})} + C\sup_{j\in\mathbb{N}}2^{j\sigma}\|\Delta_j u(t)\|_{L^\infty(\mathbb{R})} \\
&\le C\|u\|_{L^\infty([0,T^*[;L^2(\mathbb{R}))} + C\|u\|_{L^\infty([0,T^*[;C^\sigma(\mathbb{R}))} \\
&\le C\|u_0\|_{L^2(\mathbb{R})} + C\|u\|_{L^\infty([0,T^*[;C^\sigma(\mathbb{R}))} < \infty,
\end{aligned} \tag{3.15}
$$

where Δ_{-1} and Δ_j are Littlewood-Paley operators defined in formula (2.3). Since $u(t)$ and $\partial_t u$ for any $t\in]0,T^*[$ are already smooth functions with the spatial decay, according to Corollary 2.1 (or Lemma 2.7), we find that for every $t'\in]0,T^*[$,

$$
\begin{aligned}
\|u\|_{L^\infty([t',T^*[;C^{1,\varrho}(\mathbb{R}))} &\le Ct'^{-\frac{\varrho+1}{\alpha}}\big(\|u_0\|_{L^\infty(\mathbb{R})} + \|L_\beta u\|_{L^\infty([0,T^*[;C^{\sigma-\beta})}\big) \\
&\le Ct'^{-\frac{\varrho+1}{\alpha}}\big(\|u_0\|_{L^2\cap L^\infty(\mathbb{R})} + \|u\|_{L^\infty([0,T^*[;C^\sigma)}\big),
\end{aligned} \tag{3.16}
$$

where $\varrho\in]0,\sigma+\alpha-1-\beta[$ and C depends only on μ,ν,σ and $\|u\|_{L^\infty([0,T^*[;\dot C^{\sigma-\beta})}$, then let $T>0$ depending only on $\|u_0\|_{H^s(\mathbb{R})}$ be some existence time (see Proposition 3.1), we can choose $t'=\frac{T}{2}$ so that we can prove that $\partial_x u\in L^\infty(\mathbb{R}\times[0,T^*[)$, which clearly contradicts with blowup criterion (3.1). Hence, the desired blowup criterion (3.14) is followed. □

Thanks to Lemma 2.6, we first have the L^2-estimate that

$$\|u(t)\|_{L^2}^2+\int_0^t \|u(\tau)\|_{\dot H^{\frac{\alpha}{2}}}^2\,\mathrm{d}\tau\le \|u_0\|_{L^2}^2,\quad \forall t>0, \tag{3.17}$$

and the L^∞-estimate that

$$\sup_{t>0}\|u(t)\|_{L^\infty}\le B_0 \tag{3.18}$$

with B_0 a fixed constant depending only on μ,ν, β and $\|u_0\|_{L^2\cap L^\infty(\mathbb{R})}$.

In the following, we consider the dissipative dispersive Burgers Equation (1.7) with $\beta\in[0,1[$ and critical dissipation $\alpha=1$, and we shall apply the method of modulus of continuity to show that

$$\sup_{t\in[0,T^*[}\|u(t)\|_{\dot C^\sigma(\mathbb{R})}<C,\quad \text{for some } \sigma\in]\beta,1[, \tag{3.19}$$

with some $C>0$ depending only on μ,ν, β and $\|u_0\|_{H^s(\mathbb{R})}$.

To this end, it suffices to show the following lemma.

Lemma 3.2. *Let $u\in L^\infty([0,T^*[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,T^*[)$, $s>3/2$ be the maximal lifespan solution to the dissipative dispersive Burgers Equation (1.7). For every $\sigma\in]\beta,1[$, define the following unbounded function*

$$\omega(\xi)=\begin{cases}\kappa\delta^{-\sigma}\xi^\sigma, & \text{for } 0<\xi\le\delta,\\ \kappa+\gamma\log\frac{\xi}{\delta}, & \text{for } \xi>\delta,\end{cases} \tag{3.20}$$

with $\gamma,\kappa,\delta>0$. Then provided that the positive constants γ,κ,δ are sufficiently small (κ,γ are independent of δ, see formulas (3.41)-(3.42) below), the function $\omega(\xi)$ is a modulus of continuity (see Definition 2.1) and the solution $u(x,t)$ preserves MOC $\omega(\xi)$ on the whole time interval $[0,T^*[$.

In fact, with such a result at our disposal, and by using property (3.25) below, we deduce that

$$\sup_{t\in[0,T^*[}\|u(t)\|_{\dot C^\sigma(\mathbb{R})}=\sup_{t\in[0,T^*[}\sup_{x,y\in\mathbb{R},x\neq y}\frac{|u(x,t)-u(y,t)|}{|x-y|^\sigma}\le\sup_{x,y\in\mathbb{R},x\neq y}\frac{\omega(|x-y|)}{|x-y|^\sigma}\le\kappa\delta^{-\sigma}, \tag{3.21}$$

which is as desired. Hence, together with the blowup criterion (3.14), we show that $T^*=\infty$ and thus conclude Theorem 1.1.

We also remark that different from the above subcritical case, here we need to verify inequality (3.28) at all scales (instead of only small scales) and also the MOC $\omega(\xi)$ should satisfy $\omega^{-1}(3\|u\|_{L^\infty_T L^\infty})<\infty$, thus the chosen modulus of continuity has to be an unbounded one.

Proof. **(Proof of Lemma 3.2.)** We first show that $\omega(\xi)$ is indeed a MOC satisfying some needed properties. Clearly,

$$\omega(0+)=0,\quad \text{and}\quad \omega'(0+)=\kappa\sigma\delta^{-\sigma}\lim_{\xi\to0+}\xi^{\sigma-1}=\infty. \tag{3.22}$$

Observe that for every $0<\xi<\delta$,

$$\omega'(\xi)=\kappa\sigma\delta^{-\sigma}\xi^{\sigma-1}>0, \quad \text{and} \quad \omega''(\xi)=-\kappa\sigma(1-\sigma)\delta^{-\sigma}\xi^{\sigma-2}<0, \tag{3.23}$$

and for every $\xi>\delta$,

$$\omega'(\xi)=\gamma\xi^{-1}>0, \quad \text{and} \quad \omega''(\xi)=-\gamma\xi^{-2}<0, \tag{3.24}$$

and for $\xi=\delta$,

$$\omega'(\delta-)=\kappa\sigma\delta^{-1}, \quad \text{and} \quad \omega'(\delta+)=\gamma\delta^{-1},$$

thus if $\gamma<\kappa\sigma$, we infer that ω is increasing and concave for all $\xi>0$. We also find that

$$\text{the mapping } \xi\mapsto\frac{\omega(\xi)}{\xi^\sigma} \text{ is non-increasing for } \xi\in]0,\infty[. \tag{3.25}$$

Indeed, if $\xi\in]0,\delta]$, property (3.25) is an obvious consequence of formula (3.20); while if $\xi>\delta$, we have $\left(\frac{\omega(\xi)}{\xi^\sigma}\right)'=\frac{\xi\omega'(\xi)-\sigma\omega(\xi)}{\xi^{\sigma+1}}$, and noticing that by estimate (3.24), $\sigma>\beta$ and $\gamma<\sigma\kappa$,

$$(\xi\omega'(\xi)-\sigma\omega(\xi))'=\omega'(\xi)+\xi\omega''(\xi)-\sigma\omega'(\xi)<-\sigma\gamma\xi^{-1}<0,$$

and

$$\delta\omega'(\delta+)-\sigma\omega(\delta)=\gamma-\sigma\kappa<0,$$

we deduce that $\frac{d}{d\xi}(\frac{\omega(\xi)}{\xi^\sigma})<0$, which implies property (3.25) in the range $\xi>\delta$.

Now we prove that the initial data u_0 obeys some MOC $\omega(\xi)$ defined by formula (3.20). Indeed, owing to $|u_0(x)-u_0(y)|\le 2\|u_0\|_{L^\infty}$ and $|u_0(x)-u_0(y)|\le\|u_0\|_{\dot C^\sigma}|x-y|^\sigma$, it only needs to be shown that $\min\{2\|u_0\|_{L^\infty},\|u_0\|_{\dot C^\sigma}|x-y|^\sigma\}<\omega(|x-y|)$; then from property (3.25), and by denoting $a_1:=\left(\frac{2\|u_0\|_{L^\infty}}{\|u_0\|_{\dot C^\sigma}}\right)^{1/\sigma}$, it moreover suffices to show that

$$\omega(a_1)>2\|u_0\|_{L^\infty}. \tag{3.26}$$

But without loss of generality assuming $a_1>\delta$, we see that $\omega(a_1)>\gamma\log\frac{a_1}{\delta}$, thus by choosing $\delta>0$ small enough, that is,

$$\delta\le a_1 e^{-2\gamma^{-1}\|u_0\|_{L^\infty}}, \tag{3.27}$$

we conclude that such a MOC $\omega(\xi)$ is obeyed by the data $u_0(x)$.

Next, for our purpose, according to Proposition 2.1, equality (3.8) and Lemma 2.4, it suffices to prove that for all $t\in]0,T^*[$, $x\neq y\in\mathbb{R}$ satisfying (2.7) (with $\omega(\xi,t)=\omega(\xi)$ given by (3.20)) and $0<\xi\in\{\xi:\omega(\xi)\le 2B_0\}$,

$$\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(x,y,t)+\nu D_1(x,y,t)<0, \tag{3.28}$$

where $D_\alpha(x,y,t)$ and $\Phi_\beta(x,y,t)$ respectively satisfy (from estimates (2.12) and (2.13))

$$\Phi_\beta(x,y,t)\le -C_2\xi^{1-\beta}D_1(x,y,t)+C_2\xi\int_\xi^\infty\frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta+C_2\xi^{-\beta}\omega(\xi), \tag{3.29}$$

and

$$D_1(x,y,t)\le C_1\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\mathrm{d}\eta$$
$$+C_1\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\mathrm{d}\eta, \tag{3.30}$$

where $C_1=C_1(\alpha),C_2=C_2(\alpha,\beta)>0$ are the constants appearing in Lemma 2.4.

We also remark that from estimate (3.18) and formula (3.20), the scope of $\xi>0$ satisfying $\omega(\xi)\le 2B_0$ is contained in the following range

$$0<\xi\le\Xi:=\delta e^{2\gamma^{-1}B_0}, \tag{3.31}$$

so that we only need to justify inequality (3.28) for all $\xi\in]0,\Xi]$.

In order to prove inequality (3.28), we divide the proof into two cases.

Case 1: $0<\xi\le\delta$**.**

In this case, we have $\omega(\xi)=\kappa\delta^{-\sigma}\xi^{\sigma}$, and $\omega'(\xi)=\kappa\sigma\delta^{-\sigma}\xi^{\sigma-1}$, thus,

$$\omega(\xi)\omega'(\xi)=\kappa^2\sigma\delta^{-2\sigma}\xi^{2\sigma-1};$$

and by the property (3.25) and $\sigma\in]\beta,1[$, we see that

$$\xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta=\xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^{\sigma}}\frac{1}{\eta^{2+\beta-\sigma}}\mathrm{d}\eta\le\kappa\delta^{-\sigma}\xi\int_{\xi}^{\infty}\eta^{\sigma-2-\beta}\mathrm{d}\eta\le\frac{\kappa}{\beta}\delta^{-\sigma}\xi^{\sigma-\beta}.$$

Then we find that for every $|\mu|\delta^{1-\beta}<\frac{\nu}{2C_2}$,

$$|\mu|\Phi_\beta(x,y,t)\le-\frac{1}{2}\nu D_1(x,y,t)+\frac{2C_2}{\beta}|\mu|\kappa\delta^{-\sigma}\xi^{\sigma-\beta}. \tag{3.32}$$

For the contribution from the dissipation term, by virtue of estimates (3.13) and (3.23), we get

$$D_1(x,y,t)\le C_1\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\mathrm{d}\eta\le-\frac{C_1}{2}\sigma(1-\sigma)\kappa\delta^{-\sigma}\xi^{\sigma-1}. \tag{3.33}$$

Hence we infer that for all $\xi\in]0,\delta]$,

$$\begin{aligned}&\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(x,y,t)+\nu D_1(x,y,t)\\ \le&\kappa\delta^{-\sigma}\xi^{\sigma-1}\left(\sigma\kappa\left(\frac{\xi}{\delta}\right)^{\sigma}+\frac{2C_2}{\beta}|\mu|\delta^{1-\beta}-\frac{C_1\nu\sigma(1-\sigma)}{4}\right)\\ \le&\kappa\delta^{-\sigma}\xi^{\sigma-1}\left(\sigma\kappa+\frac{2C_2}{\beta}|\mu|\delta^{1-\beta}-\frac{C_1\nu\sigma(1-\sigma)}{4}\right)<0,\end{aligned} \tag{3.34}$$

where the last inequality is through choosing δ and κ so that

$$|\mu|\delta^{1-\beta}<\min\left\{\frac{\nu}{2C_2},\frac{C_1\beta\nu\sigma(1-\sigma)}{16C_2}\right\},\quad \kappa<\frac{C_1\nu(1-\sigma)}{8}. \tag{3.35}$$

Case 2: $\delta<\xi\le\Xi$**.**

In this case, we obviously have

$$\omega(\xi)\omega'(\xi)=\gamma\omega(\xi)\xi^{-1}.$$

Taking advantage of property (3.25) again, we see

$$\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{2+\beta}}\mathrm{d}\eta=\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^\sigma}\frac{1}{\eta^{2+\beta-\sigma}}\mathrm{d}\eta\le\frac{\omega(\xi)}{\xi^{\sigma-1}}\int_\xi^\infty\frac{1}{\eta^{2+\beta-\sigma}}\mathrm{d}\eta\le\frac{1}{1+\beta-\sigma}\frac{\omega(\xi)}{\xi^\beta}\le\frac{1}{\beta}\frac{\omega(\xi)}{\xi^\beta}.$$

Thus from estimate (3.29), we obtain that by choosing $|\mu|\Xi^{1-\beta}<\frac{\nu}{2C_2}$,

$$|\mu|\Phi_\beta(x,y,t)\le-\frac{1}{2}\nu D_1(\xi,t)+\frac{2C_2}{\beta}|\mu|\omega(\xi)\xi^{-\beta}. \tag{3.36}$$

For $D_1(x,y,t)$, noticing that $\omega(2\eta+\xi)-\omega(2\eta-\xi)\le\omega(2\xi)<2\omega(\xi)$, we get

$$D_1(x,y,t)\le C_1\big(\omega(2\xi)-2\omega(\xi)\big)\int_{\frac{\xi}{2}}^\infty\frac{1}{\eta^2}\mathrm{d}\eta\le 2C_1\big(\omega(2\xi)-2\omega(\xi)\big)\xi^{-1}. \tag{3.37}$$

Next we claim that for γ small enough (i.e. $\gamma<\frac{\kappa}{2}$), we have

$$\omega(2\xi)\le\frac{3}{2}\omega(\xi),\quad \forall\xi>\delta. \tag{3.38}$$

Indeed, for $\xi=\delta$, we see that $\omega(\delta)=\kappa$ and $\omega(2\delta)=\kappa+\gamma\log 2$, which further yields that $\omega(2\delta)\le\frac{3}{2}\omega(\delta)$ for all $\gamma<\frac{\kappa}{2}$; whereas for $\xi>\delta$, considering an auxiliary function $h(\xi):=\omega(2\xi)-\frac{3}{2}\omega(\xi)$, and noting that

$$h'(\xi)\le 2\omega'(2\xi)-\frac{3}{2}\omega'(\xi)=2\gamma(2\xi)^{-1}-\frac{3}{2}\gamma\xi^{-1}=-\frac{1}{2}\gamma\xi^{-1}\le 0,$$

we deduce $h(\xi)\le h(\delta)\le 0$ for all $\xi\ge\delta$, which implies claim (3.38). Hence, plugging inequality (3.38) into estimate (3.37) yields

$$D_1(x,y,t)\le -2C_1(2-3/2)\omega(\xi)\xi^{-1}=-C_1\omega(\xi)\xi^{-1}.$$

Collecting the above estimates leads to that for all $\xi\in]\delta,\Xi]$,

$$\omega(\xi)\omega'(\xi)+|\mu|\Phi_\beta(x,y,t)+\nu D_1(x,y,t)\le\left(\frac{2C_2}{\beta}|\mu|\Xi^{1-\beta}+\gamma-\frac{C_1\nu}{2}\right)\omega(\xi)\xi^{-1}<0, \tag{3.39}$$

where the last inequality is guaranteed as long as $|\mu|,\gamma$ are satisfying

$$|\mu|\Xi^{1-\beta}<\min\left\{\frac{\nu}{2C_2},\frac{C_1\beta\nu}{8C_2}\right\},\quad \gamma<\min\left\{\frac{\kappa}{2},\frac{C_1\nu}{4}\right\}. \tag{3.40}$$

In sum, by recalling formulas (3.27), (3.31) and gathering inequalities (3.35), (3.40), we can choose

$$\kappa=\frac{C_1\nu(1-\sigma)}{16},\quad \gamma=\frac{C_1\nu(1-\sigma)}{64}, \tag{3.41}$$

and

$$\delta=\min\left\{\left(\frac{2\|u_0\|_{L^\infty}}{\|u_0\|_{\dot C^\sigma}}\right)^{\frac{1}{\sigma}}e^{-\frac{2\|u_0\|_{L^\infty}}{\gamma}},\left(\frac{\widetilde C_1\beta\nu}{16C_2|\mu|}\right)^{\frac{1}{1-\beta}}e^{-\frac{2B_0}{\gamma}},\left(\frac{\widetilde C_1\nu\beta\sigma(1-\sigma)}{32C_2|\mu|}\right)^{\frac{1}{1-\beta}}\right\}, \tag{3.42}$$

with $\widetilde C_1:=\min\{C_1,1\}$, so that all the requirements are fulfilled, and then we conclude inequality (3.28) for $\omega(\xi)$ in formula (3.20) equipped with such constants, which moreover implies $\omega(\xi)$ is preserved by the solution u for all time $t\in[0,T^*[$, as desired. □

4. Proof of Theorem 1.2

We consider the approximate dissipative dispersive Burgers equation with regularized data

$$\partial_t u + u\partial_x u + \mu L_\beta u + \nu \Lambda u = 0, \quad u|_{t=0}(x) = u_0^\epsilon(x) := \phi_\epsilon * u_0(x), \tag{4.1}$$

where $u_0 \in L^2\cap L^\infty(\mathbb{R})$, $\phi_\epsilon = \epsilon^{-1}\phi(\epsilon^{-1}x)$, $\epsilon>0$ and $\phi\in C_c^\infty(\mathbb{R})$ is the standard mollifier, i.e., $\phi\in C^\infty(\mathbb{R})$ with $\mathrm{supp}\,\phi\subset[-1,1]$, $\phi\geq 0$ and $\int_{\mathbb{R}}\phi \mathrm{d}x = 1$.

Thanks to Theorem 1.1, for every $\epsilon>0$, we know that the approximate Equation (4.1) respectively generates a unique global smooth solution $u^\epsilon \in C([0,\infty[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,\infty[)$ with any $s>3/2$.

Then the proof of Theorem 1.2 is divided into two steps. In the first step, we show that by passing ϵ to 0, Equation (1.7) admits a global weak solution $u\in L^\infty([0,\infty[;L^2(\mathbb{R}))\cap L^2([0,\infty[;\dot H^{1/2}(\mathbb{R}))$, and this is placed in Subsection 4.1. As the second step, and in Subsection 4.2, we are devoted to proving that the approximate solutions u^ϵ are C^∞-regular on $\mathbb{R}\times[t',\infty[$ uniformly in ϵ with any $t'>0$, and thus conclude Theorem 1.2 by sending to the limit.

4.1. Global existence of weak solutions for Equation (1.7).

We first recall the definition of weak solution for Equation (1.7).

Definition 4.1. *Let $u_0\in L^2(\mathbb{R})$. We call a solution $u:\mathbb{R}\times[0,\infty[\to\mathbb{R}$ a weak solution to the dissipative dispersive Burgers Equation (1.7), provided that it satisfies the following properties.*

(1) u satisfies Equation (1.7) in the distributional sense, that is, for any $\widetilde\chi\in C_c^\infty(\mathbb{R}\times[0,\infty[)$,

$$\int_0^\infty\int_{\mathbb{R}}\Big(u\partial_t\widetilde\chi + \frac{u^2}{2}\partial_x\widetilde\chi + \mu u L_\beta\widetilde\chi - \nu u\Lambda^\alpha\widetilde\chi\Big)\mathrm{d}x\mathrm{d}t = -\int_{\mathbb{R}} u_0(x)\widetilde\chi(x,0)\mathrm{d}x. \tag{4.2}$$

(2) The following energy inequality holds

$$\|u(t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|u(\tau)\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R})}^2\mathrm{d}\tau \leq \|u_0\|_{L^2(\mathbb{R})}^2, \quad \forall t>0. \tag{4.3}$$

Then we show the global existence of a weak solution for the dissipative dispersive Whitham Equation (1.7).

Proposition 4.1. *Let $\mu\neq 0$, $\nu>0$, $\alpha\in]0,1]$, $\beta\in[0,1[$ and $u_0\in L^2(\mathbb{R})$. Then there exists a global weak solution $u\in L^\infty([0,\infty[;L^2(\mathbb{R}))\cap L^2([0,\infty[;\dot H^{\alpha/2}(\mathbb{R}))$ to the dissipative dispersive Burgers Equation (1.7).*

Proof. We first consider the following approximate dissipative dispersive Whitham equation

$$\partial_t u + u\partial_x u + \mu L_\beta u + \nu\Lambda^\alpha u + \epsilon\Lambda u = 0, \quad u|_{t=0} = u_0^\epsilon := \phi_\epsilon * u_0, \tag{4.4}$$

with $\epsilon>0$, $\phi_\epsilon = \epsilon^{-1}\phi(\frac{\cdot}{\epsilon})$ and ϕ a standard mollifier. For every $\epsilon>0$, according to Theorem 1.1, there exists a unique global smooth solution $u^\epsilon\in C([0,\infty[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,\infty[)$ with $s>3/2$ for the Equation (4.4). By virtue of Lemma 2.6, we have

$$\|u^\epsilon(t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|u^\epsilon(\tau)\|_{\dot H^{\frac{\alpha}{2}}(\mathbb{R})}^2\mathrm{d}\tau \leq \|u_0\|_{L^2(\mathbb{R})}^2, \quad \forall t\geq 0, \tag{4.5}$$

which means that u^ϵ belongs to $L^\infty([0,\infty[;L^2(\mathbb{R}))\cap L^2([0,\infty[;\dot{H}^{\alpha/2}(\mathbb{R}))$ uniformly with respect to ϵ.

Owing to the weak convergence lemmas, the solution sequence u^ϵ, up to a subsequence (still denoting by u^ϵ), weakly converges to a function u in $L^\infty([0,\infty[;L^2(\mathbb{R}))\cap L^2([0,\infty[;\dot{H}^{\alpha/2}(\mathbb{R}))$ (weakly-* converges in the L^∞-topology). From the lower semicontinuity of weak convergence, we can derive the corresponding inequality (4.5) for the limiting function u.

Since equality (4.2) holds with u^ϵ in place of u, by passing to the limit $\epsilon\to 0$, and from $\|L_\beta\widetilde{\chi}\|_{L^2(\mathbb{R}\times[0,\infty[)}\leq C\|\widetilde{\chi}\|_{L^2([0,\infty[;H^\beta(\mathbb{R}))}$, one can show that the limiting function u satisfies equality (4.2) except for the following convergence

$$\int_0^\infty\int_{\mathbb{R}}(u^\epsilon)^2\partial_x\widetilde{\chi}\mathrm{d}x\mathrm{d}t\to\int_0^\infty\int_{\mathbb{R}}u^2\partial_x\widetilde{\chi}\mathrm{d}x\mathrm{d}t,\quad\forall\widetilde{\chi}\in C_c^\infty(\mathbb{R}\times[0,\infty[).\tag{4.6}$$

Moreover, we claim that up to a subsequence and as $\epsilon\to 0$,

$$u^\epsilon\to u\quad\text{strongly in}\, L^2_{\mathrm{loc}}([0,\infty[;L^2_{\mathrm{loc}}(\mathbb{R})).\tag{4.7}$$

Indeed, let $\psi\in C_c^\infty(\mathbb{R}\times[0,\infty[)$, from inequality (4.5) and the following estimate that

$$\|fg\|_{\dot{H}^{\alpha/2}(\mathbb{R})}\leq C\|f\|_{L^\infty}\|g\|_{\dot{H}^{\alpha/2}}+C\|f\|_{\dot{W}^{\alpha/2,\infty}}\|g\|_{L^2}\leq C\|f\|_{W^{\alpha/2,\infty}(\mathbb{R})}\|g\|_{H^{\alpha/2}(\mathbb{R})},$$

we know that

$$\psi u^\epsilon\in L^2([0,\infty[;H^{\alpha/2}(\mathbb{R}))\quad\text{uniformly in}\,\epsilon.\tag{4.8}$$

In order to show convergence (4.7), according to the Aubin-Lions compactness lemma, we shall prove that

$$\partial_t(\psi u^\epsilon)\in L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R})),\quad\text{uniformly in}\,\epsilon.\tag{4.9}$$

Due to the fact that u^ϵ solves Equation (4.4) in the pointwise sense, we see that

$$\begin{aligned}\partial_t(\psi u^\epsilon)&=(\partial_t\psi)u^\epsilon+\psi(\partial_t u^\epsilon)\\&=(\partial_t\psi)u^\epsilon-\psi u^\epsilon\,\partial_x u^\epsilon-\mu\psi L_\beta u^\epsilon-\nu\psi\Lambda^\alpha u^\epsilon-\epsilon\psi\Lambda u^\epsilon;\end{aligned}$$

from inequality (4.5), the interpolation inequality and Sobolev embedding, we get $u^\epsilon\in L^4([0,\infty[;L^{\frac{4}{2-\alpha}}(\mathbb{R}))$, and also thanks to Lemma 2.5 and the L^p-boundedness ($p\in]1,\infty[$) of zero-order pseudo-differential operator (see [44], and the operator $\partial_x(\mathrm{Id}-\Delta)^{-\frac12}$ is of the zero-order symbol, i.e., $|\partial_\zeta^n(\zeta(1+|\zeta|^2)^{-\frac12})|\leq C_n(1+|\zeta|)^{-n}$ for every $n\in\mathbb{N}$),

$$\begin{aligned}&\|\psi\partial_x(u^\epsilon)^2\|_{L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R}))}\\\leq&C\|\psi\|_{L^\infty([0,\infty[;W^{1,\infty}(\mathbb{R}))}\|\partial_x(u^\epsilon)^2\|_{L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R}))}\\\leq&C\|\psi\|_{L^\infty([0,\infty[;W^{1,\infty}(\mathbb{R}))}\|(u^\epsilon)^2\|_{L^2([0,\infty[;L^{\frac{2}{2-\alpha}}(\mathbb{R}))}\leq C\|u_0\|^2_{L^2(\mathbb{R})};\end{aligned}$$

since the test function ψ has a compact support on $\mathbb{R}\times[0,\infty[$, and using the continuous embedding $L^2(I)\hookrightarrow L^{\frac{2}{2-\alpha}}(I)\hookrightarrow W^{-1,\frac{2}{2-\alpha}}(I)$ for any compact interval $I\subset\mathbb{R}$, we find

$$\|(\partial_t\psi)u^\epsilon\|_{L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R}))}\leq C\|(\partial_t\psi)u^\epsilon\|_{L^\infty([0,\infty[;L^2)}\leq C\|u_0\|_{L^2(\mathbb{R})};$$

and also in light of Lemma 2.5 and Lemma 2.2, we infer that

$$\begin{aligned}\|\psi L_\beta u^\epsilon\|_{L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R}))} &\le C\|\psi L_\beta u^\epsilon\|_{L^2([0,\infty[;H^{-1}(\mathbb{R}))}\\ &\le C\|\psi\|_{L^\infty([0,\infty[;W^{1,\infty}(\mathbb{R}))}\|L_\beta u^\epsilon\|_{L^\infty([0,\infty[;H^{-1}(\mathbb{R}))}\\ &\le C\|\psi\|_{L^\infty([0,\infty[;W^{1,\infty}(\mathbb{R}))}\|u_0\|_{L^2(\mathbb{R}))},\end{aligned}$$

where in the second line we have used that $\|L_\beta u^\epsilon(t)\|_{H^{-1}(\mathbb{R})}\le C\|m_\beta(\zeta)(1+|\zeta|^2)^{-\frac{1}{2}}\widehat{u^\epsilon}(\zeta,t)\|_{L^2(\mathbb{R})}\le\|u^\epsilon(t)\|_{L^2(\mathbb{R})}$; finally, the dissipative terms can be estimated in a similar manner: for $\alpha\in]0,1]$,

$$\begin{aligned}&\|\psi(\Lambda^\alpha u^\epsilon)\|_{L^2([0,\infty[;W^{-1,\frac{2}{2-\alpha}}(\mathbb{R}))}\\ \le& C\|\psi(\Lambda^\alpha u^\epsilon)\|_{L^2([0,\infty[;H^{-1}(\mathbb{R}))}\\ \le& C\|\psi\|_{L^\infty([0,\infty[;W^{1,\infty}(\mathbb{R}))}\|\Lambda^\alpha u^\epsilon\|_{L^\infty([0,\infty[;H^{-1}(\mathbb{R}))}\le C\|u_0\|_{L^2(\mathbb{R}))};\end{aligned}$$

thus gathering the above estimates leads to estimate (4.9). Hence, by applying the Aubin-Lions lemma (e.g. see [45, Theorem 2.1]) and estimates (4.8)-(4.9), we conclude the desired assertion (4.7).

It is clear that inequality (4.7), strong convergence (4.5) and Hölder's inequality ensure convergence (4.6). Therefore, we conclude the global existence of a weak solution for the dissipative dispersive Burgers Equation (1.7). □

4.2. Global C^∞-smoothness of the constructed weak solutions to Equation (1.7) for the critical case $\alpha=1$. For approximate Equation (4.1), since $u_0\in L^2\cap L^\infty(\mathbb{R})$, we have $\|u_0^\epsilon\|_{L^2\cap L^\infty}\le\|u_0\|_{L^2\cap L^\infty}$, and $\|u_0^\epsilon\|_{H^s}\lesssim_{\epsilon,s}\|u_0\|_{L^2}$ for every $s>0$, so that thanks to Theorem 1.1, Equation (4.1) admits a unique global smooth solution $u^\epsilon\in C([0,\infty[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,\infty[)$, $s>3/2$, and also by Lemma 2.6 we have the uniform energy estimate (4.5) and the uniform L^∞-bound

$$\sup_{t\ge0}\|u^\epsilon(t)\|_{L^\infty(\mathbb{R})}\le B_0 \tag{4.10}$$

with B_0 a fixed constant depending only on μ,ν, β and $\|u_0\|_{L^2\cap L^\infty(\mathbb{R})}$.

In the following, we intend to show the uniform estimate $u^\epsilon\in C^\infty(\mathbb{R}\times[t_*,\infty[)$ uniformly in ϵ, with any $t_*>0$. We first have the following regularity criterion in terms of the uniform Hölder estimates.

Lemma 4.1. *For any $0<T_1<T_2<\infty$, if the solution u^ϵ for the Equation (4.1) with $\alpha\in]\beta,1]$ satisfies that*

$$\textit{for some } \sigma\in]1+\beta-\alpha,1[,\quad u^\epsilon\in L^\infty([T_1,T_2];\dot C^\sigma(\mathbb{R}))\ \textit{uniformly in } \epsilon, \tag{4.11}$$

then we have $u^\epsilon\in C^\infty(]T_1,T_2]\times\mathbb{R})$ uniformly in ϵ.

Proof. Combined with inequality (4.10), we get $u^\epsilon\in L^\infty([0,T^*[;C^\sigma(\mathbb{R}))$ uniformly in ϵ, then similar to obtaining estimates (3.15) and (3.16), we have

$$\|L_\beta u^\epsilon\|_{L^\infty([T_1,T_2],C^{\sigma-\beta}(\mathbb{R}))}\le C\|u_0\|_{L^2(\mathbb{R})}+C\|u^\epsilon\|_{L^\infty([T_1,T_2];C^\sigma(\mathbb{R}))}<\infty,$$

and for any $t_1'\in]T_1,T_2[$,

$$\|u^\epsilon\|_{L^\infty([t_1',T_2[;C^{1,\varrho}(\mathbb{R}))}\le C(t_1'-T_1)^{-\frac{\varrho+1}{\alpha}}\left(\|u_0\|_{L^2\cap L^\infty(\mathbb{R})}+\|u^\epsilon\|_{L^\infty([T_1,T_2[;C^\sigma(\mathbb{R}))}\right),$$

where $\varrho\in]0,\sigma+\alpha-1-\beta[$ and C depends only on σ and $\|u^\epsilon\|_{L^\infty([T_1,T_2[;\dot C^{\sigma-\beta}(\mathbb{R}))}$; which specially implies that

$$u^\epsilon\in L^\infty([t_1',T_2];C^{\sigma+\frac{\alpha-\beta}{2}}(\mathbb{R})),\quad \text{uniformly in } \epsilon.$$

By arguing as estimate (3.15) again, we deduce that

$$\|L_\beta u^\epsilon\|_{L^\infty([t_1',T_2];C^{\sigma-\beta+\frac{\alpha-\beta}{2}}(\mathbb{R}))}\le C_0\|u_0\|_{L^2}+C_0\|u^\epsilon\|_{L^\infty([t_1',T_2];C^{\sigma+\frac{\alpha-\beta}{2}}(\mathbb{R}))},$$

and taking advantage of estimate (2.35) (with $\delta=\sigma-\beta+\frac{\alpha-\beta}{2}$ and $\alpha\in]\beta,1]$), we get that for any $t_2'\in]t_1',T_2[$,

$$u^\epsilon\in L^\infty([t_2',T_2];C^{\sigma+\alpha-\beta}(\mathbb{R})),\quad \text{uniformly in } \epsilon.$$

We further get that $L_\beta u^\epsilon\in L^\infty([t_2',T_2];C^{\sigma+\alpha-2\beta}(\mathbb{R}))$, so that by using estimate (2.35) again (with $\delta=\sigma+\alpha-2\beta$ and $\alpha\in]\beta,1]$), it leads to that for any $t_3'\in]t_2',T_2[$,

$$u^\epsilon\in L^\infty([t_3',T_2];C^{\sigma+\frac{3(\alpha-\beta)}{2}}(\mathbb{R})),\quad \text{uniformly in } \epsilon.$$

By repeating the above process for more times, we obtain that $u^\epsilon\in L^\infty([t_n',T_2];C^{\sigma+\frac{n(\alpha-\beta)}{2}}(\mathbb{R}))$ uniformly in ϵ with any $T_1<t_1'<t_2'\cdots<t_n'<T_2$ and $n\in\mathbb{N}^+$, which guarantees the uniform C^∞_x-smoothness of u^ϵ on the spacetime domain $\mathbb{R}\times]T_1,T_2]$. The uniform C^∞-smoothness in $t\in]T_1,T_2]$ can be derived by using Equation (4.1). Hence, we prove the assertion under condition (4.11). □

Now, our main target is to prove uniform estimate (4.11). We first observe that if the MOC $\omega(\xi)$ defined in formula (3.20) is initially uniformly-in-ϵ preserved by solution u^ϵ at some time t_1, then the justification of inequality (3.28) can be naturally applied to show solution u^ϵ will uniformly-in-ϵ obey such a MOC for all $t>t_1$. The key issue is that the initial data $u_0^\epsilon=\phi_\epsilon*u_0$ with rough assumption $u_0\in L^2\cap L^\infty(\mathbb{R})$ will not necessarily uniformly-in-ϵ obey such a MOC $\omega(\xi)$ given by formula (3.20). The idea to overcome this difficulty is as follows (see [30]): we intend to choose a family of time-dependent moduli of continuity $\omega(\xi,t)$ which gradually becomes $\omega(\xi)$ given by formula (3.20) after a short time, and we choose $\omega(0+,0)>0$ large enough so that initially $u_0^\epsilon(x)$ uniformly-in-ϵ obeys $\omega(\xi,0)$, then we moreover show that the MOC $\omega(\xi,t)$ is uniformly-in-ϵ obeyed by the evolution of $u^\epsilon(x,t)$ which finally yields that solution $u^\epsilon(x,t_1)$ at some short time t_1 preserves MOC $\omega(\xi)$ defined by formula (3.20), as desired.

For this purpose, we consider the following family of moduli of continuity for $\xi_0>\delta$,

$$\omega(\xi,\xi_0)=\begin{cases}(1-\sigma)\kappa+\gamma\log\frac{\xi_0}{\delta}-\gamma\xi_0^{-1}(\xi_0-\delta)+\sigma\kappa\delta^{-1}\xi, & \text{for } 0<\xi\le\delta,\\ \kappa+\gamma\log\frac{\xi_0}{\delta}-\gamma+\gamma\xi_0^{-1}\xi, & \text{for } \delta<\xi\le\xi_0,\\ \kappa+\gamma\log\frac{\xi}{\delta}, & \text{for } \xi>\xi_0,\end{cases}\tag{4.12}$$

and for $\xi_0\le\delta$,

$$\omega(\xi,\xi_0)=\begin{cases}(1-\sigma)\kappa\delta^{-\sigma}\xi_0^\sigma+\sigma\kappa\delta^{-\sigma}\xi_0^{\sigma-1}\xi, & \text{for } 0<\xi\le\xi_0,\\ \kappa\delta^{-\sigma}\xi^\sigma, & \text{for } \xi_0<\xi\le\delta,\\ \kappa+\gamma\log\frac{\xi}{\delta}, & \text{for } \xi>\delta,\end{cases}\tag{4.13}$$

where $\sigma\in]\beta,1[$, and κ,γ,δ are positive constants chosen later. In formulas (4.12)-(4.13), ξ_0 is a decreasing function of t which goes to 0 as t tending to some time, more precisely, $\xi_0=\xi_0(t)$ can be chosen as

$$\xi_0(t)=\Xi_0-\rho t, \tag{4.14}$$

with ρ and Ξ_0 some positive constants fixed later. Note that when $\xi_0\equiv 0$, $\omega(\xi,0)=\omega(\xi,0+)$ reduces to the MOC $\omega(\xi)$ defined by (3.20). The construction of $\omega(\xi,\xi_0)$ is, motivated by [30], through taking a tangent line at $\xi=\xi_0$ to $\omega(\xi)$ given by (3.20) and replacing $\omega(\xi)$ with this tangent line at the range $0<\xi\le\xi_0$. But since the one-sided derivatives of $\omega(\xi)$ at the point $\xi=\delta$ do not coincide, we thus make a modification in the case $\xi_0>\delta$, that is, the tangent line mentioned above at the range $\delta\le\xi\le\xi_0$ is still adopted, but at the range $0<\xi\le\delta$ it is replaced by a straight line crossing $\omega(\delta+,\xi_0)$ with the larger slope $\omega'(\delta-)=\sigma\kappa\delta^{-1}$.

Clearly,

$$\omega(0+,\xi_0)>0, \quad \text{for all } \xi_0>0. \tag{4.15}$$

Similarly as $\omega(\xi)$ defined by formula (3.20), $\omega(\xi,\xi_0)$ is also an increasing and concave function for all $\xi>0$ and $\xi_0>0$. For $\xi_0=\Xi_0>\delta$, we get

$$\omega(0+,\Xi_0)=(1-\sigma)\kappa+\gamma\log\frac{\Xi_0}{\delta}-\gamma\Xi_0^{-1}(\Xi_0-\delta)\ge\big((1-\sigma)\kappa-\gamma\big)+\gamma\log\frac{\Xi_0}{\delta}, \tag{4.16}$$

By assuming $\gamma<(1-\sigma)\kappa$, and using inequality (4.10), the initial data u_0^ϵ uniformly-in-ϵ obeys the MOC $\omega(\xi,\Xi_0)$ provided that

$$2B_0\le\gamma\log\frac{\Xi_0}{\delta}. \tag{4.17}$$

Note also that under condition (4.17), we have

$$\omega(\Xi_0,\xi_0)\ge\omega(\Xi_0,0+)=\omega(\Xi_0)\ge\omega(0+,\Xi_0)>2B_0, \quad \text{for any } \xi_0\ge 0. \tag{4.18}$$

We have the following key lemma, whose proof is placed at the end of this subsection.

Lemma 4.2. *Suppose that the initial data u_0^ϵ uniformly-in-ϵ obeys the MOC $\omega(\xi,\Xi_0)$ given by (4.12). Then for some positive constants δ, κ, γ, ρ small enough, the solution $u^\epsilon(x,t)$ of approximate Equation (4.1) unformly-in-ϵ obeys the MOC $\omega(\xi,\xi_0(t))$ for all t such that $\xi_0(t)=\Xi_0-\rho t>0$.*

Now with Lemma 4.2 at our disposal, we see that from (4.17), we can choose Ξ_0 to be

$$\Xi_0=\delta e^{2\gamma^{-1}B_0}, \qquad \delta>0 \text{ is chosen later}, \tag{4.19}$$

and by letting

$$t_*=\frac{\Xi_0}{\rho}=\frac{1}{\rho}e^{2\gamma^{-1}B_0}\delta, \tag{4.20}$$

we get $\xi_0(t_*)=0$ from formula (4.14), thus thanks to the time continuity of u^ϵ and $\omega(\xi,\xi_0(t))$, we have

$$|u^\epsilon(x',t_*)-u^\epsilon(y',t_*)|\le\lim_{t\to t_*}\omega(|x'-y'|,\xi_0(t))=\omega(|x'-y'|), \quad \forall x',y'\in\mathbb{R}, \tag{4.21}$$

where $\omega(\xi)$ is given by formula (3.20), and κ,γ,ρ,μ are fixed positive constants satisfying condition (4.65) below with $\sigma\in]\beta,1[$. By setting the MOC

$$\widetilde{\omega}(\xi):=2\omega(\xi)=\begin{cases}2\kappa\delta^{-\sigma}\xi^{\sigma}, & \text{for } 0<\xi\leq\delta,\\ 2\kappa+2\gamma\log\frac{\xi}{\delta}, & \text{for } \xi>\delta,\end{cases}$$

inequality (4.21) implies that solution $u^{\epsilon}(x,t_*)$ obeys the MOC $\widetilde{\omega}(\xi)$, and in a same argument as Lemma 3.2, one can show that such a MOC $\widetilde{\omega}(\xi)$ will be preserved by the solution u^{ϵ} on the time interval $]t_*,\infty[$, provided that

$$|\mu|\Xi_0^{1-\beta}<\min\left\{\frac{\nu}{2C_2},\frac{C_1\nu\beta\sigma(1-\sigma)}{16C_2}\right\},\quad 0<\kappa<\frac{C_1\nu(1-\sigma)}{16},\quad 0<\gamma<\min\left\{\frac{\kappa}{2},\frac{C_1\nu}{8}\right\}. \tag{4.22}$$

In combination with inequality (4.65) below, and by setting $\sigma=\frac{\beta+1}{2}\in]\beta,1[$, we can choose

$$\rho=\frac{\nu(1-\beta)}{C},\ \kappa=\frac{\nu(1-\beta)}{C},\ \gamma=\frac{\nu(1-\beta)^2}{C},\ |\mu|\Xi_0^{1-\beta}=|\mu|\delta^{1-\beta}e^{2(1-\beta)\gamma^{-1}B_0}\leq\frac{\nu\beta(1-\beta)}{C}, \tag{4.23}$$

with some constant $C>0$ depending on $C_1,C_2=C_2(\beta)$.

Hence, for every $|\mu|\in\mathbb{R}^+$ and $t'>0$, by choosing δ to be

$$\delta=\min\left\{\frac{1}{2}\rho e^{-2B_0\gamma}t',\left(\frac{\nu\beta(1-\beta)}{2C|\mu|}\right)^{1/(1-\beta)}e^{-2\gamma^{-1}B_0}\right\}, \tag{4.24}$$

we obtain that

$$\sup_{t\in[\frac{t'}{2},\infty[}\|u^{\epsilon}(t)\|_{C^{\frac{\beta+1}{2}}(\mathbb{R})}\leq B_0+2\kappa\delta^{-\frac{\beta+1}{2}}, \tag{4.25}$$

where B_0 is given in (4.10).

Therefore, we get the uniform estimate (4.25) with respect to ϵ for any $t'>0$, and in view of Lemma 4.1 and Proposition 4.1, we can pass $\epsilon\to 0$ to conclude Theorem 1.2.

Finally, we give the details of proving Lemma 4.2.

Proof. **(Proof of Lemma 4.2.)** Since $u_0^{\epsilon}(x)$ obeys the MOC $\omega(\xi,\xi_0(0))=\omega(\xi,\Xi_0)$ by assumption, and $\omega(\xi,\xi_0(t))$ is the MOC satisfying the needed properties, according to Proposition 2.1, equality (3.8) and Lemma 2.4, it suffices to prove that for every $t>0$ such that $\xi_0(t)>0$, $x\neq y\in\mathbb{R}$ satisfying (2.7) (with $\omega(\xi,t)=\omega(\xi,\xi_0(t))$ given by formulas (4.12)-(4.13)), and $0<\xi\in\{\xi:\omega(\xi,\xi_0(t))\leq 2B_0\}$,

$$-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0(t)+\omega(\xi,\xi_0)\partial_{\xi}\omega(\xi,\xi_0)+|\mu|\Phi_{\beta}(x,y,t)+\nu D_1(x,y,t)<0, \tag{4.26}$$

where $\xi_0(t)=\Xi_0-\rho t$ is abbreviated as ξ_0 below, $\omega(\xi,\xi_0)$ is given by formulas (4.12)-(4.13) and

$$\begin{aligned}D_1(x,y,t)\leq\,& C_1\int_0^{\frac{\xi}{2}}\frac{\omega(\xi+2\eta,\xi_0)+\omega(\xi-2\eta,\xi_0)-2\omega(\xi,\xi_0)}{\eta^2}\mathrm{d}\eta\\ &+C_1\int_{\frac{\xi}{2}}^{\infty}\frac{\omega(2\eta+\xi,\xi_0)-\omega(2\eta-\xi,\xi_0)-2\omega(\xi,\xi_0)}{\eta^2}\mathrm{d}\eta,\end{aligned} \tag{4.27}$$

and

$$\Phi_\beta(x,y,t) \le -C_2\xi^{1-\beta}D_1(x,y,t) + C_2\xi\int_\xi^\infty \frac{\omega(\eta,\xi_0)}{\eta^{2+\beta}}\mathrm{d}\eta + C_2\xi^{-\beta}\omega(\xi,\xi_0). \tag{4.28}$$

In inequality (4.26), if $\partial_{\xi_0}\omega(\xi,\xi_0)$ or $\partial_\xi\omega(\xi,\xi_0)$ does not exist, the larger value of the one-sided derivative should be taken.

We also note that in view of inequality (4.18), the scope of ξ considered in (4.26) belongs to $]0,\Xi_0]$.

According to the values of ξ_0 and ξ, we divide the proof into several cases to justify inequality (4.26).

Case 1: $\xi_0 > \delta$, $0 < \xi \le \delta$.

From $\omega(\xi,\xi_0) = (1-\sigma)\kappa + \gamma\log\frac{\xi_0}{\delta} - \gamma\xi_0^{-1}(\xi_0-\delta) + \sigma\kappa\delta^{-1}\xi$ in this case, we have

$$\partial_{\xi_0}\omega(\xi,\xi_0) = \gamma\xi_0^{-1} - \gamma\xi_0^{-2}\delta \le \gamma\xi_0^{-1}, \quad \text{and} \quad \partial_\xi\omega(\xi,\xi_0) = \sigma\kappa\delta^{-1}, \tag{4.29}$$

and

$$\omega(\xi,\xi_0) \ge \omega(0+,\xi_0) = (1-\sigma)\kappa + \gamma\log\frac{\xi_0}{\delta} - \gamma\xi_0^{-1}(\xi_0-\delta), \tag{4.30}$$

and

$$\omega(\xi,\xi_0) - \omega(0+,\xi_0) \le \omega(\delta,\xi_0) - \omega(0+,\xi_0) = \sigma\kappa. \tag{4.31}$$

Thus by using equalities (4.14) and (4.29), we get

$$-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0(t) \le \rho\gamma\xi_0^{-1}. \tag{4.32}$$

Owing to the integration by parts and the formula of $\omega(\eta,\xi_0)$ in formula (4.12), we obtain

$$\begin{aligned}
&\xi\int_\xi^\infty \frac{\omega(\eta,\xi_0)}{\eta^{2+\beta}}\mathrm{d}\eta = \frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta} + \frac{1}{1+\beta}\xi\int_\xi^\infty \frac{\partial_\eta\omega(\eta,\xi_0)}{\eta^{1+\beta}}\mathrm{d}\eta \\
&= \frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta} + \frac{1}{1+\beta}\xi\int_\xi^\delta \frac{\kappa\sigma\delta^{-1}}{\eta^{1+\beta}}\mathrm{d}\eta + \frac{1}{1+\beta}\xi\int_\delta^{\xi_0}\frac{\gamma\xi_0^{-1}}{\eta^{1+\beta}}\mathrm{d}\eta + \frac{1}{1+\beta}\xi\int_{\xi_0}^\infty \frac{\gamma}{\eta^{2+\beta}}\mathrm{d}\eta \\
&\le \frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta} + \frac{\kappa\sigma}{(1+\beta)\beta}\delta^{-1}\xi(\xi^{-\beta}-\delta^{-\beta}) + \frac{\gamma}{(1+\beta)\beta}\xi_0^{-1}\xi\big(\delta^{-\beta}-\xi_0^{-\beta}\big) + \frac{\gamma\xi_0^{-(1+\beta)}\xi}{(1+\beta)^2}, \\
&\le \frac{\omega(\xi,\xi_0)}{\xi^\beta} + \frac{1}{\beta}\sigma\kappa\delta^{-\beta} + \frac{1}{\beta}\gamma\xi_0^{-\beta}.
\end{aligned} \tag{4.33}$$

Thus by applying estimates (4.28), (4.31) and (4.33), we have that for $|\mu|\delta^{1-\beta} \le \frac{\nu}{4C_2}$,

$$\begin{aligned}
|\mu|\Phi_\beta(x,y,t) &\le -\frac{\nu}{4}D_1(x,y,t) + 2C_2|\mu|\omega(\xi,\xi_0)\xi^{-\beta} + \frac{C_2}{\beta}|\mu|\sigma\kappa\delta^{-\beta} + \frac{C_2}{\beta}|\mu|\gamma\xi_0^{-\beta} \\
&\le -\frac{\nu}{4}D_1(x,y,t) + 2C_2|\mu|\omega(0+,\xi_0)\xi^{-\beta} + \frac{3C_2}{\beta}\sigma\kappa|\mu|\xi^{-\beta} + \frac{C_2}{\beta}|\mu|\gamma\xi_0^{-\beta},
\end{aligned} \tag{4.34}$$

and also,

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0) \le \sigma\kappa\delta^{-1}\left(\omega(0+,\xi_0)+\kappa\sigma\right) \le \sigma\kappa\delta^{-1}\omega(0+,\xi_0) + \sigma^2\kappa^2\delta^{-1}. \tag{4.35}$$

For the contribution from the diffusion term, since the function $\omega(\eta,\xi_0)-\omega(0+,\xi_0)$ is still concave, we infer that

$$D_1(x,y,t)\le -2C_1\omega(0+,\xi_0)\int_{\frac{\xi}{2}}^{\infty}\frac{1}{\eta^2}\mathrm{d}\eta \le -4C_1\omega(0+,\xi_0)\xi^{-1}. \tag{4.36}$$

Thus by setting $|\mu|\delta^{1-\beta}\le \frac{C_1\nu}{4C_2}$ and $\kappa\le\frac{C_1\nu}{4\sigma}$ so that

$$\begin{aligned}&2C_2|\mu|\omega(0+,\xi_0)\xi^{-\beta}\le \left(2C_2|\mu|\delta^{1-\beta}\right)\omega(0+,\xi_0)\xi^{-1}\le -\frac{1}{8}\nu D_1(x,y,t),\\ &\sigma\kappa\delta^{-1}\omega(0+,\xi_0)\le \frac{C_1\nu}{2}\omega(0+,\xi_0)\xi^{-1}\le -\frac{1}{8}\nu D_1(x,y,t),\end{aligned} \tag{4.37}$$

we get

$$|\mu|\Phi_\beta(x,y,t)+\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)\le -\frac{\nu D_1(x,y,t)}{2}+\frac{3C_2}{\beta}\sigma\kappa|\mu|\xi^{-\beta}+\frac{C_2}{\beta}|\mu|\gamma\xi_0^{-\beta}+\frac{\sigma^2\kappa^2}{\delta}. \tag{4.38}$$

If $\xi_0\ge 9\delta$, we see that

$$\omega(0+,\xi_0)\ge (1-\sigma)\kappa+(\log 9-1)\gamma\ge (1-\sigma)\kappa+\gamma, \tag{4.39}$$

and inserting the above estimate into inequality (4.36) leads to

$$D_1(x,y,t)\le -4C_1(1-\sigma)\kappa\xi^{-1}-4C_1\gamma\xi^{-1}. \tag{4.40}$$

Thus for $\xi_0\ge 9\delta$, by collecting estimates (4.32), (4.36), (4.38) and (4.40), we deduce that

$$\begin{aligned}&\text{L.H.S. of (4.26)}\\ &\le \kappa\xi^{-1}\left(\frac{3C_2}{\beta}\sigma\delta^{1-\beta}|\mu|+\sigma^2\kappa-2C_1\nu(1-\sigma)\right)+\gamma\xi^{-1}\left(\rho+\frac{C_2}{\beta}\delta^{1-\beta}|\mu|-2C_1\nu\right)<0,\end{aligned}$$

where L.H.S. denotes the left-hand side and the last inequality is guaranteed as long as ρ, κ, $|\mu|$ satisfy

$$\rho<C_1\nu,\quad |\mu|\delta^{1-\beta}\le\left\{\frac{C_1\nu\beta(1-\sigma)}{3C_2},\frac{\nu}{4C_2}\right\},\quad \kappa<\frac{C_1\nu(1-\sigma)}{4\sigma^2}. \tag{4.41}$$

If $\xi_0\le 9\delta$, the positive contribution which is treated by estimates (4.32) and (4.38) can be bounded by

$$\begin{aligned}&-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0+|\mu|\Phi_\beta(x,y,t)+\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)\\ &\le -\frac{1}{2}D_1(x,y,t)+\kappa\xi^{-1}\left(\rho\frac{\gamma}{\kappa}+\sigma^2\kappa+\frac{3C_2}{\beta}\sigma\delta^{1-\beta}|\mu|+\frac{C_2}{\beta}\delta^{1-\beta}|\mu|\frac{\gamma}{\kappa}\right).\end{aligned}$$

For the negative contribution from the diffusion term, from formula (4.30) and inequality (4.36), we directly get that by letting $\gamma\le\frac{1-\sigma}{2}\kappa$,

$$D_1(x,y,t)\le -4C_1\xi^{-1}\left((1-\sigma)\kappa-\gamma\right)\le -2C_1(1-\sigma)\kappa\xi^{-1}. \tag{4.42}$$

Hence for every $\xi_0 \le 9\delta$, we have

$$\text{L.H.S. of (4.26)} \le \kappa\xi^{-1}\left(\rho+\sigma^2\kappa+\frac{3C_2}{\beta}\delta^{1-\beta}|\mu|-C_1\nu(1-\sigma)\right)<0,$$

where the last inequality is ensured if we set

$$\rho<\frac{C_1\nu(1-\sigma)}{3}, \quad |\mu|\delta^{1-\beta}<\left\{\frac{C_1\nu\beta(1-\sigma)}{9C_2},\frac{\nu}{4C_2}\right\}, \quad \kappa<\frac{C_1\nu(1-\sigma)}{3\sigma^2}, \quad \gamma<\frac{(1-\sigma)}{2}\kappa. \tag{4.43}$$

Case 2: $\xi_0>\delta$, $\delta<\xi\le\xi_0$.
From $\omega(\xi,\xi_0)=\kappa+\gamma\log\frac{\xi_0}{\delta}-\gamma+\gamma\xi_0^{-1}\xi$ in this case, we have

$$\partial_{\xi_0}\omega(\xi,\xi_0)=\gamma\xi_0^{-2}(\xi_0-\xi)\le\gamma\xi_0^{-1}, \quad \text{and} \quad \partial_\xi\omega(\xi,\xi_0)=\gamma\xi_0^{-1},$$

and

$$\omega(\xi,\xi_0)\ge\omega(\delta,\xi_0)=\kappa+\gamma\log\frac{\xi_0}{\delta}-\gamma\xi_0^{-1}(\xi_0-\delta)=\omega(0+,\xi_0)+\sigma\kappa,$$

and

$$\omega(\xi,\xi_0)-\omega(0+,\xi_0)\le\omega(\xi_0,\xi_0)-\omega(0+,\xi_0)=\gamma\xi_0^{-1}(\xi_0-\delta)+\sigma\kappa\le\gamma+\sigma\kappa. \tag{4.44}$$

Thus by using formula (4.14), we get

$$-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0(t)\le\rho\gamma\xi_0^{-1}. \tag{4.45}$$

Thanks to the following estimate

$$\begin{aligned}
\xi\int_\xi^\infty\frac{\omega(\eta,\xi_0)}{\eta^{2+\beta}}\mathrm{d}\eta &= \frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{1}{1+\beta}\xi\int_\xi^{\xi_0}\frac{\gamma\xi_0^{-1}}{\eta^{1+\beta}}\mathrm{d}\eta+\frac{1}{1+\beta}\xi\int_{\xi_0}^\infty\frac{\gamma}{\eta^{2+\beta}}\mathrm{d}\eta\\
&= \frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^{1/2}}+\frac{\gamma}{(1+\beta)\beta}\xi_0^{-1}\xi\left(\xi^{-\beta}-\xi_0^{-\beta}\right)+\frac{\gamma}{(1+\beta)^2}\xi_0^{-(1+\beta)}\xi\\
&\le \frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{\gamma}{\beta}\xi_0^{-\beta},
\end{aligned}$$

we see that for all $|\mu|\xi_0^{1-\beta}\le\frac{\nu}{4C_2}$,

$$\begin{aligned}
|\mu|\,\Phi_\beta(x,y,t) &\le -\frac{1}{4}\nu D_1(x,y,t)+2C_2|\mu|\omega(\xi,\xi_0)\xi^{-\beta}+\frac{C_2}{\beta}|\mu|\gamma\xi_0^{-\beta}\\
&\le -\frac{1}{4}\nu D_1(x,y,t)+2C_2|\mu|\omega(0+,\xi_0)\xi^{-\beta}+2C_2|\mu|(\gamma+\sigma\kappa)\xi^{-\beta}+\frac{C_2}{\beta}|\mu|\gamma\xi_0^{-\beta},
\end{aligned} \tag{4.46}$$

and also by using inequality (4.44),

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)\le\gamma\xi_0^{-1}\omega(0+,\xi_0)+\gamma^2\xi_0^{-1}+\sigma\gamma\kappa\xi_0^{-1}. \tag{4.47}$$

For the contribution from the diffusion term, we also have estimate (4.36). If $\xi_0\ge9\delta$, by using inequality (4.39) and setting $|\mu|\xi_0^{1-\beta}<\frac{C_1\nu}{4C_2}$ and $\gamma<\frac{C_1\nu}{2}$, we deduce that

$$2C_2|\mu|\omega(0+,\xi_0)\xi^{-\beta}+\gamma\xi_0^{-1}\omega(0+,\xi_0)\le-\frac{1}{4}\nu D_1(x,y,t),$$

and thus

$$\begin{aligned}&\text{L.H.S. of (4.26)}\\ &\leq \kappa\xi^{-1}\Big(2\sigma C_2|\mu|\xi_0^{1-\beta}+\sigma\gamma-2C_1\nu(1-\sigma)\Big)+\gamma\xi^{-1}\left(\rho+\frac{3C_2}{\beta}|\mu|\xi_0^{1-\beta}+\gamma-C_1\nu\right)<0,\end{aligned}$$

where the last inequality is guaranteed as long as

$$\rho<\frac{C_1\nu}{3},\quad |\mu|\xi_0^{1-\beta}<\min\left\{\frac{\nu}{4C_2},\frac{C_1\nu\beta(1-\sigma)}{9C_2}\right\},\quad \gamma<\frac{C_1\nu(1-\sigma)}{4}. \tag{4.48}$$

If $\xi_0\leq 9\delta$, the positive contribution treated by estimates (4.45), (4.46) and (4.47) can be bounded as

$$\begin{aligned}&-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0(t)+|\mu|\Phi_\beta(x,y,t)+\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)\\ &\leq -\frac{1}{2}D_1(x,y,t)+\kappa\xi^{-1}\left(\rho\frac{\gamma}{\kappa}+C_2|\mu|\xi_0^{1-\beta}\left(2\sigma+\frac{3}{\beta}\frac{\gamma}{\kappa}\right)+\sigma\gamma+\frac{\gamma^2}{\kappa}\right).\end{aligned}$$

For the negative contribution from diffusion, we obtain estimate (4.42) for all $\gamma\leq\frac{1-\sigma}{2}\kappa$. Hence for $\xi_0\leq 9\delta$, we have

$$\text{L.H.S. of (4.26)}\leq\kappa\xi^{-1}\left(\rho+\frac{3C_2}{\beta}|\mu|\xi_0^{1-\beta}+2\gamma-C_1\nu(1-\sigma)\right)<0,$$

where the last inequality is ensured if we set

$$\rho<\frac{C_1\nu(1-\sigma)}{3},\quad |\mu|\xi_0^{1-\beta}<\min\left\{\frac{\nu}{4C_2},\frac{C_1\nu\beta(1-\sigma)}{9C_2}\right\},\quad \gamma\leq\min\left\{\frac{1-\sigma}{2}\kappa,\frac{C_1\nu(1-\sigma)}{6}\right\}. \tag{4.49}$$

Case 3: $\xi_0>\delta$, $\xi_0<\xi\leq\Xi_0$.

In this case, from $\omega(\xi,\xi_0)=\kappa+\gamma\log\frac{\xi}{\delta}$, we see that $\partial_{\xi_0}\omega(\xi,\xi_0)=0$, $\partial_\xi\omega(\xi,\xi_0)=\gamma\xi^{-1}$, and

$$\xi\int_\xi^\infty\frac{\omega(\eta,\xi_0)}{\eta^{2+\beta}}\mathrm{d}\eta=\frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{1}{1+\beta}\xi\int_\xi^\infty\frac{\gamma}{\eta^{2+\beta}}\mathrm{d}\eta\leq\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\gamma\xi^{-\beta}.$$

Thus thanks to inequality (4.28), we get

$$|\mu|\Phi_\beta(x,y,t)\leq-\frac{C_2|\mu|}{\nu}\xi^{1-\beta}D_1(x,y,t)+2C_2|\mu|\omega(\xi,\xi_0)\xi^{-\beta}+C_2|\mu|\gamma\xi^{-\beta}. \tag{4.50}$$

and

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)=\gamma\omega(\xi,\xi_0)\xi^{-1}. \tag{4.51}$$

For the contribution from the diffusion term, since $\omega(2\eta+\xi,\xi_0)-\omega(2\eta-\xi,\xi_0)\leq\omega(2\xi,\xi_0)<2\omega(\xi,\xi_0)$, we obtain

$$D_1(x,t)\leq C_1\big(\omega(2\xi,\xi_0)-2\omega(\xi,\xi_0)\big)\int_{\frac{\xi}{2}}^\infty\frac{1}{\eta^2}\mathrm{d}\eta\leq 2C_1\big(\omega(2\xi,\xi_0)-2\omega(\xi,\xi_0)\big)\xi^{-1}. \tag{4.52}$$

Observing that $\omega(2\xi,\xi_0)-\omega(\xi,\xi_0)=\gamma\log\frac{2\xi}{\delta}-\gamma\log\frac{\xi}{\delta}=\gamma\log 2$ and $\omega(\xi,\xi_0)\geq\gamma\log\frac{\xi}{\delta}$, thus if ξ satisfies that $\xi\geq 4\delta$, we find $\omega(\xi,\xi_0)\geq 2\gamma\log 2=2(\omega(2\xi,\xi_0)-\omega(\xi,\xi_0))$, and then

$$\omega(2\xi,\xi_0)-2\omega(\xi,\xi_0)\leq-\frac{1}{2}\omega(\xi,\xi_0). \tag{4.53}$$

Hence if $\xi \ge 4\delta$, by gathering the above estimates, using the fact that $\omega(\xi,\xi_0) \ge \gamma$ and setting $|\mu|\Xi_0^{1-\beta} < \frac{\nu}{4C_2}$, we deduce that

$$|\mu|\Phi_\beta(x,y,t) + \omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0) \le -\frac{1}{2}\nu D_1(x,y,t) + 3C_2|\mu|\omega(\xi,\xi_0)\xi^{-\beta} + \gamma\omega(\xi,\xi_0)\xi^{-1},$$

and

$$D_1(x,y,t) \le -C_1\omega(\xi,\xi_0)\xi^{-1},$$

and so

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0) + |\mu|\Phi_\beta(x,y,t) + \nu D_1(x,y,t) \le \Big(3C_2|\mu|\Xi_0^{1-\beta} + \gamma - \frac{C_1\nu}{2}\Big)\omega(\xi,\xi_0)\xi^{-1} < 0,$$

where the last inequality is ensured if we set

$$|\mu|\Xi_0^{1-\beta} < \min\left\{\frac{\nu}{4C_2}, \frac{C_1\nu}{12C_2}\right\}, \quad \gamma < \frac{C_1\nu}{4}. \tag{4.54}$$

On the other hand, if ξ satisfies $\xi \le 4\delta$, since $\omega(\xi,\xi_0) - \omega(0+,\xi_0)$ is concave and $\omega(0+,\xi_0) \ge (1-\sigma)\kappa$, we get

$$D_1(x,y,t) \le -2C_1\omega(0+,\xi_0)\int_{\frac{\xi}{2}}^{\infty} \frac{1}{\eta^2}\mathrm{d}\eta \le -4C_1(1-\sigma)\kappa\xi^{-1}, \tag{4.55}$$

and also by setting $\gamma \le \kappa$,

$$\omega(\xi,\xi_0) = \kappa + \gamma\log\frac{\xi}{\delta} \le \kappa + \gamma\log 4 \le 3\kappa. \tag{4.56}$$

Hence if $\xi \le 4\delta$, by collecting the above estimates and letting $|\mu|\Xi_0^{1-\beta} < \frac{\nu}{4C_2}$, we obtain

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0) + |\mu|\Phi_\beta(x,y,t) \le -\frac{1}{4}\nu D_1(x,y,t) + 7C_2|\mu|\Xi_0^{1-\beta}\kappa\xi^{-1} + 3\gamma\kappa\xi^{-1},$$

and thus

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0) + |\mu|\Phi_\beta(x,y,t) + \nu D_1(x,y,t) \le \Big(7C_2|\mu|\Xi_0^{1-\beta} + 3\gamma - 2C_1\nu(1-\sigma)\Big)\kappa\xi^{-1},$$

where the last inequality is ensured by letting

$$|\mu|\Xi_0^{1-\beta} < \min\left\{\frac{C_1\nu(1-\sigma)}{8C_2}, \frac{\nu}{4C_2}\right\}, \quad \gamma < \min\left\{\frac{C_1\nu(1-\sigma)}{4}, \kappa\right\}. \tag{4.57}$$

Case 4: $0 < \xi_0 \le \delta$**,** $0 < \xi \le \xi_0$**.**

In this case $\omega(\xi,\xi_0) = (1-\sigma)\kappa\delta^{-\sigma}\xi_0^\sigma + \sigma\kappa\delta^{-\sigma}\xi_0^{\sigma-1}\xi$, and thus

$$\partial_{\xi_0}\omega(\xi,\xi_0) = \sigma(1-\sigma)\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-1}\left(1 - \frac{\xi}{\xi_0}\right), \quad \text{and} \quad \partial_\xi\omega(\xi,\xi_0) = \sigma\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-1},$$

and

$$\omega(\xi,\xi_0) \ge \omega(0+,\xi_0) = (1-\sigma)\kappa\delta^{-\sigma}\xi_0^\sigma, \quad \omega(\xi,\xi_0) \le \omega(\delta,\xi_0) \le \kappa\delta^{-\sigma}\xi_0^\sigma, \tag{4.58}$$

and

$$-\partial_{\xi_0}\omega(\xi,\xi_0)\dot{\xi}_0(t)\leq\rho\sigma(1-\sigma)\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-1}. \tag{4.59}$$

By virtue of the integration by parts and formula (4.13), we see that

$$\begin{aligned}
&\xi\int_\xi^\infty\frac{\omega(\eta,\xi_0)}{\eta^{2+\beta}}\mathrm{d}\eta=\frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{1}{1+\beta}\xi\int_\xi^\infty\frac{\partial_\eta\omega(\eta,\xi_0)}{\eta^{1+\beta}}\mathrm{d}\eta\\
&\leq\frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{1}{1+\beta}\xi\int_\xi^{\xi_0}\frac{\sigma\kappa\delta^{-\sigma}\xi_0^{\sigma-1}}{\eta^{1+\beta}}\mathrm{d}\eta+\frac{1}{1+\beta}\xi\int_{\xi_0}^\infty\frac{\sigma\kappa\delta^{-\sigma}\eta^{\sigma-1}}{\eta^{1+\beta}}\mathrm{d}\eta\\
&\leq\frac{1}{1+\beta}\frac{\omega(\xi,\xi_0)}{\xi^{1/2}}+\frac{\sigma\kappa}{(1+\beta)\beta}\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-1}\xi^\beta+\frac{1}{1+\beta}\frac{\sigma\kappa}{1+\beta-\sigma}\xi\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-(1+\beta)}\\
&\leq\frac{\omega(\xi,\xi_0)}{\xi^\beta}+\frac{2\sigma\kappa}{\beta}\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-\beta},
\end{aligned}$$

then gathering the above estimates and inequality (4.28) leads to that for all $|\mu|\delta^{1-\beta}<\frac{\nu}{4C_2}$,

$$\begin{aligned}
|\mu|\Phi_\beta(x,y,t)&\leq-\frac{\nu}{4}D_1(x,y,t)+2C_2|\mu|\omega(\xi,\xi_0)\xi^{-\beta}+\frac{2C_2}{\beta}|\mu|\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi_0^{-\beta}\\
&\leq-\frac{\nu}{4}D_1(x,y,t)+\frac{4C_2}{\beta}|\mu|\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi^{-\beta},
\end{aligned}\tag{4.60}$$

and

$$\omega(\xi,\xi_0)\partial_\xi\omega(\xi,\xi_0)=\sigma\kappa\xi_0^{-1}\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\omega(\xi,\xi_0)\leq\sigma\kappa^2\left(\frac{\delta}{\xi_0}\right)^{-2\sigma}\xi_0^{-1}. \tag{4.61}$$

For the contribution from the diffusion term, by arguing as estimate (4.36) and using inequality (4.58), we obtain

$$D_1(x,y,t)\leq-2C_1\omega(0+,\xi_0)\int_{\frac{\xi}{2}}^\infty\frac{1}{\eta^2}\mathrm{d}\eta\leq-4(1-\sigma)C_1\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi^{-1}. \tag{4.62}$$

Collecting the estimates (4.59), (4.61) and (4.62), we find that

$$\begin{aligned}
&\text{L.H.S. of (4.26)}\\
&\leq\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi^{-1}\left(\rho\sigma(1-\sigma)\frac{\xi}{\xi_0}+\frac{4C_2}{\beta}|\mu|\delta^{1-\beta}+\kappa\frac{\xi}{\xi_0}\left(\frac{\delta}{\xi_0}\right)^{-\sigma}-2C_1\nu(1-\sigma)\right)\\
&\leq\kappa\left(\frac{\delta}{\xi_0}\right)^{-\sigma}\xi^{-1}\left(\rho\sigma(1-\sigma)+\frac{4C_2}{\beta}|\mu|\delta^{1-\beta}+\kappa-2C_1\nu(1-\sigma)\right),
\end{aligned}$$

which leads to the desired inequality (4.26) as long as $\rho,|\mu|,\kappa$ are such that

$$\rho<\frac{C_1\nu}{2\sigma},\quad |\mu|\delta^{1-\beta}<\min\left\{\frac{\nu}{4C_2},\frac{C_1\nu\beta(1-\sigma)}{6C_2}\right\},\quad \kappa<\frac{C_1\nu(1-\sigma)}{6}. \tag{4.63}$$

Case 5: $0<\xi_0\leq\delta$, $\xi_0<\xi\leq\Xi_0$.

The proof of this case is almost identical to that of Case 1 and Case 2 in the proof of Lemma 3.2, and we omit the details. Note that the conditions on $\kappa,|\mu|,\gamma$ are given by

$$|\mu|\Xi_0^{1-\beta}<\min\Big\{\frac{\nu}{2C_2},\frac{C_1\nu\beta\sigma(1-\sigma)}{16C_2}\Big\},\quad \kappa<\frac{C_1\nu(1-\sigma)}{8},\quad \gamma<\min\Big\{\frac{\kappa}{2},\frac{C_1\nu}{4}\Big\}. \tag{4.64}$$

Therefore, for the MOC $\omega(\xi,\xi_0)$ defined by formulas (4.12)-(4.13) and $\xi_0=\xi_0(t)=\Xi_0-\rho t$ with $\rho,\kappa,|\mu|,\gamma$ are appropriate constants satisfying conditions (4.41), (4.43), (4.48), (4.49), (4.54), (4.63), (4.64); based on the above analysis, we verify inequality (4.26) for all $\xi>0$ and $t>0$ satisfying $\xi_0(t)>0$, and thus conclude Lemma 4.2. Observing that by suppressing the dependence on the absolute constants and C_1, $C_2=C_2(\beta)$, the conditions on coefficients $\rho,\kappa,\gamma>0$ are as follows

$$\rho\le\frac{\nu(1-\sigma)}{C},\quad \kappa\le\frac{\nu(1-\sigma)}{C},\quad |\mu|\Xi_0^{1-\beta}\le\frac{\nu\beta(1-\sigma)}{C},\quad \gamma\le\frac{\nu(1-\sigma)^2}{C}, \tag{4.65}$$

with $C>0$ some constant independent of σ. ∎

5. Proof of Theorem 1.3

From the local well-posedness result (see Proposition 3.1), we assume that the dissipative dispersive Burgers Equation (1.7) generates a unique smooth solution $u\in C([0,T[;H^s(\mathbb{R}))\cap C^1([0,T[;H^{s-1}(\mathbb{R}))$, $s>\frac{3}{2}$, with any time $T\in]0,\infty[$.

We define the following quantity

$$\mathcal{E}_{p,q}u(x):=\int_{\mathbb{R}} w_{p,q}(x-y)u(y)\mathrm{d}y, \tag{5.1}$$

where

$$w_{p,q}(x):=\begin{cases}\frac{1}{|x|^p}\,\mathrm{sgn}(x), & \text{for } |x|<1,\\ \frac{1}{|x|^q}\,\mathrm{sgn}(x), & \text{for } |x|\ge 1,\end{cases} \tag{5.2}$$

with $p\in]0,1[$, $q\in]2,\infty[$ chosen later. Denote by $X(x,t)$ the flow trajectory given by

$$\frac{\mathrm{d}}{\mathrm{d}t}X(x,t)=u(X(x,t),t),\quad X(x,0)=x\in\mathbb{R}. \tag{5.3}$$

Since $u(\cdot,t)$ on $[0,T[$ is a Lipschitz function of $\mathbb{R}$, we know that $X(\cdot,t)$ for every $t\in[0,T[$ forms a unique one-to-one diffeomorphism. For $t\in[0,T[$, set $\bar{x}(t)=X(0,t)$.

In the sequel, we mainly consider the evolution of the weighted function

$$E(t)=\mathcal{E}_{p,q}u(\bar{x}(t),t)=\int_{\mathbb{R}} w_{p,q}(\bar{x}(t)-y)\,u(y,t)\mathrm{d}y. \tag{5.4}$$

From Equations (1.7) and (5.3), we have

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t}&=\mathcal{E}_{p,q}(\partial_t u)(\bar{x}(t),t)+u(\bar{x}(t),t)\partial_x\big(\mathcal{E}_{p,q}u\big)(\bar{x}(t),t)\\&=-\frac{1}{2}\mathcal{E}_{p,q}(\partial_x u^2)(\bar{x})-\mu\mathcal{E}_{p,q}(L_\beta u)(\bar{x})-\nu\mathcal{E}_{p,q}(\Lambda^\alpha u)(\bar{x})+u(\bar{x})\partial_x\big(\mathcal{E}_{p,q}u\big)(\bar{x}),\end{aligned} \tag{5.5}$$

where in the second line we have suppressed the t-variable in $\bar{x}(t)$ and $u(\cdot,t)$. In light of the integration by parts, we claim that

$$\begin{aligned}-\frac{1}{2}\mathcal{E}_{p,q}(\partial_x u^2)(\bar{x})&=\frac{1}{2}\int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\partial_y\big(u^2(\bar{x})-u^2(y)\big)\mathrm{d}y\\&=-\frac{1}{2}\int_{\mathbb{R}} W_{p,q}(\bar{x}-y)\big(u^2(\bar{x})-u^2(y)\big)\mathrm{d}y,\end{aligned}$$

with

$$W_{p,q}(x) := -\partial_x w_{p,q}(x) = \begin{cases} \frac{p}{|x|^{p+1}}, & \text{for } |x|<1, \\ \frac{q}{|x|^{q+1}}, & \text{for } |x|\geq 1. \end{cases} \tag{5.6}$$

Indeed, the first equality is just from formula (5.1), while the second equality follows from the integration by parts and a limiting argument, and we omit the details here.

We also find

$$\begin{aligned} \partial_x\big(\mathcal{E}_{p,q}u\big)(\bar{x}) &= \int_{\mathbb{R}} w_{p,q}(y)\partial_x u(\bar{x}-y)\mathrm{d}y \\ &= \int_{\mathbb{R}} w_{p,q}(y)\partial_y\big(u(\bar{x})-u(\bar{x}-y)\big)\mathrm{d}y \\ &= \int_{\mathbb{R}} W_{p,q}(y)\big(u(\bar{x})-u(\bar{x}-y)\big)\mathrm{d}y = \int_{\mathbb{R}} W_{p,q}(\bar{x}-y)\big(u(\bar{x})-u(y)\big)\mathrm{d}y. \end{aligned}$$

Then it is obvious to see that

$$-\frac{1}{2}\mathcal{E}_{p,q}(\partial_x u^2)(\bar{x}) + u(\bar{x})\partial_x\big(\mathcal{E}_{p,q}u\big)(\bar{x}) = \frac{1}{2}\int_{\mathbb{R}} W_{p,q}(\bar{x}-y)\big(u(\bar{x})-u(y)\big)^2\mathrm{d}y. \tag{5.7}$$

Taking advantage of formulas (2.15) and (5.1), we write

$$\begin{aligned} &\mathcal{E}_{p,q}(\Lambda^\alpha u)(\bar{x}) = \frac{1}{2}\int_{\mathbb{R}} w_{p,q}(\bar{x}-y)(\Lambda^\alpha u)(y)\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}} w_{p,q}(\bar{x}-z)(\Lambda^\alpha u)(z)\mathrm{d}z \\ &= \frac{C_\alpha}{2}\int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\mathrm{p.v.}\int_{\mathbb{R}} \frac{u(y)-u(z)}{|y-z|^{1+\alpha}}\mathrm{d}z\mathrm{d}y + \frac{C_\alpha}{2}\int_{\mathbb{R}} w_{p,q}(\bar{x}-z)\mathrm{p.v.}\int_{\mathbb{R}} \frac{u(z)-u(y)}{|z-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}z \\ &= \frac{C_\alpha}{2}\mathrm{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(u(y)-u(z)\big)\frac{w_{p,q}(\bar{x}-y)-w_{p,q}(\bar{x}-z)}{|y-z|^{1+\alpha}}\mathrm{d}y\mathrm{d}z \\ &= \frac{C_\alpha}{2}\mathrm{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(u(\bar{x})-u(z)\big)\frac{w_{p,q}(\bar{x}-y)-w_{p,q}(\bar{x}-z)}{|y-z|^{1+\alpha}}\mathrm{d}y\mathrm{d}z \\ &\quad + \frac{C_\alpha}{2}\mathrm{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(u(\bar{x})-u(y)\big)\frac{w_{p,q}(\bar{x}-z)-w_{p,q}(\bar{x}-y)}{|z-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}z \\ &= -\frac{1}{2}\int_{\mathbb{R}} \big(u(\bar{x})-u(z)\big)\big(\Lambda^\alpha w_{p,q}\big)(\bar{x}-z)\mathrm{d}z - \frac{1}{2}\int_{\mathbb{R}} \big(u(\bar{x})-u(y)\big)\big(\Lambda^\alpha w_{p,q}\big)(\bar{x}-y)\mathrm{d}y \\ &= -\int_{\mathbb{R}} \big(u(\bar{x})-u(y)\big)\big(\Lambda^\alpha w_{p,q}\big)(\bar{x}-y)\mathrm{d}y. \end{aligned} \tag{5.8}$$

In a similar argument as that in [11, Pg. 2844] and by formula (2.5), we get

$$\begin{aligned} &\mathcal{E}_{p,q}(L_\beta u)(\bar{x}) = \int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|y-z|)\mathrm{sgn}(y-z)\big(u(z)-u(y)\big)\mathrm{d}z\mathrm{d}y \\ &= \int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|y-z|)\mathrm{sgn}(y-z)u(z)\mathrm{d}z\mathrm{d}y \\ &= -\int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|y-z|)\mathrm{sgn}(y-z)\big(u(\bar{x})-u(z)\big)\mathrm{d}z\mathrm{d}y \\ &= -\int_{\mathbb{R}} \big(u(\bar{x})-u(z)\big)\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|y-z|)\mathrm{sgn}(y-z)w_{p,q}(\bar{x}-y)\mathrm{d}y\mathrm{d}z \\ &= -\int_{\mathbb{R}} \big(u(\bar{x})-u(z)\big)\mathrm{p.v.}\int_{\mathbb{R}} K_\beta(|y-z|)\mathrm{sgn}(y-z)\big(w_{p,q}(\bar{x}-y)-w_{p,q}(\bar{x}-z)\big)\mathrm{d}y\mathrm{d}z \end{aligned}$$

$$=-\int_{\mathbb{R}}\big(u(\bar{x})-u(z)\big)\big(L_\beta w_{p,q}\big)(\bar{x}-z)\mathrm{d}z. \tag{5.9}$$

Inserting equalities (5.7)-(5.9) into Equation (5.5) leads to that

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t}=&\frac{1}{2}\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)^2W_{p,q}(\bar{x}-y)\mathrm{d}y+\mu\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)\big(L_\beta w_{p,q}\big)(\bar{x}-y)\mathrm{d}y\\&+\nu\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)\big(\Lambda^\alpha w_{p,q}\big)(\bar{x}-y)\mathrm{d}y.\end{aligned} \tag{5.10}$$

In order to estimate the last two terms on the right-hand side of Equation (5.10), we need to use the following lemma, whose proof will be postponed to the end of this section.

LEMMA 5.1. *Let* $p\in]0,1[$, $q\in]2,\infty[$, $\alpha\in]0,1[$, $\beta\in]0,1[$.

(1) Let $J_{p,q}(x)$ *be the following function*

$$J_{p,q}(x):=L_\beta w_{p,q}(x)=\mathrm{p.v.}\int_{\mathbb{R}}K_\beta(|x-y|)\mathrm{sgn}(x-y)(w_{p,q}(y)-w_{p,q}(x))\mathrm{d}y, \tag{5.11}$$

with $K_\beta(x)=K_\beta(|x|)\mathrm{sgn}(x)$ *the kernel of the operator* L_β *(defined by formula (2.5)), then we have*

$$|J_{p,q}(x)|\le\begin{cases}\frac{C}{|x|^{p+\beta}}, & \text{for } 0<|x|\le 1,\\ \frac{C}{|x|^{2+\beta}}, & \text{for } 1\le|x|<\infty,\end{cases} \tag{5.12}$$

where $C>0$ *is a constant depending only on* β,p,q.

(2) Let $H_{p,q}(x)$ *be defined as*

$$H_{p,q}(x):=\Lambda^\alpha w_{p,q}(x)=C_\alpha\,\mathrm{p.v.}\int_{\mathbb{R}}\frac{w_{p,q}(x)-w_{p,q}(y)}{|x-y|^{1+\alpha}}\mathrm{d}y, \tag{5.13}$$

with C_α *some constant depending on* α*, then we have*

$$|H_{p,q}(x)|\le\begin{cases}\frac{C}{|x|^{p+\alpha}}, & \text{for } 0<|x|\le 1,\\ \frac{C}{|x|^{2+\alpha}}, & \text{for } 1\le|x|<\infty,\end{cases} \tag{5.14}$$

where $C>0$ *is a constant depending on* α,p,q.

Then, by virtue of estimate (5.12), Hölder's inequality and Young's inequality, we infer that

$$\begin{aligned}\left|\mu\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)\big(L_\beta w_{p,q}\big)(\bar{x}-y)\mathrm{d}y\right|&\le|\mu|\int_{\mathbb{R}}|u(\bar{x})-u(y)||J_{p,q}(\bar{x}-y)|\mathrm{d}y\\&\le|\mu|\left(\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2W_{p,q}(\bar{x}-y)\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{R}}\frac{|J_{p,q}(x)|^2}{W_{p,q}(x)}\mathrm{d}x\right)^{1/2}\\&\le|\mu|\left(\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2W_{p,q}(\bar{x}-y)\mathrm{d}y\right)^{1/2}C(p,q,\beta)\\&\le\frac{1}{8}\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2W_{p,q}(\bar{x}-y)\mathrm{d}y+|\mu|^2C(p,q,\beta),\end{aligned}$$

where in the third line we let $0<p<\min\{1,2-2\beta\}$, $2<q<2+2\beta$ and used the following fact that

$$\frac{|J_{p,q}(x)|^2}{W_{p,q}(x)} \leq \begin{cases} \frac{C}{|x|^{p+2\beta-1}}, & \text{for } 0<|x|\leq 1, \\ \frac{C}{|x|^{3+2\beta-q}}, & \text{for } 1\leq |x|<\infty. \end{cases}$$

Similarly, due to that

$$\frac{|H_{p,q}(x)|^2}{W_{p,q}(x)} \leq \begin{cases} \frac{C}{|x|^{p+2\alpha-1}}, & \text{for } 0<|x|\leq 1, \\ \frac{C}{|x|^{3+2\alpha-q}}, & \text{for } 1\leq |x|<\infty, \end{cases}$$

and by choosing p,q such that $0<p<\min\{1,2-2\alpha\}$ and $2<q<2+2\alpha$, we obtain

$$\left|\nu\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)\big(\Lambda^\alpha w_{p,q}\big)(\bar{x}-y)\mathrm{d}y\right| \leq \frac{1}{8}\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2 W_{p,q}(\bar{x}-y)\mathrm{d}y+\nu^2 C(p,q,\alpha).$$

Gathering the above estimates yields

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \geq \frac{1}{4}\int_{\mathbb{R}}\big(u(\bar{x})-u(y)\big)^2 W_{p,q}(\bar{x}-y)\mathrm{d}y-(|\mu|^2+\nu^2)C(p,q,\alpha,\beta). \tag{5.15}$$

From expression (5.4) and Hölder's inequality, we also see that

$$\begin{aligned} E(t) &= -\int_{\mathbb{R}} w_{p,q}(\bar{x}-y)\big(u(\bar{x})-u(y)\big)\mathrm{d}y \\ &\leq \left(\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2 W_{p,q}(\bar{x}-y)\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{R}}\frac{|w_{p,q}(x)|^2}{W_{p,q}(x)}\mathrm{d}x\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2 W_{p,q}(\bar{x}-y)\mathrm{d}y\right)^{1/2} C(p,q), \end{aligned}$$

where in the third line we have used the fact that (for $0<p<1$ and $q>2$),

$$\frac{|w_{p,q}(x)|^2}{W_{p,q}(x)} \leq \begin{cases} \frac{C}{|x|^{p-1}}, & \text{for } 0<|x|\leq 1, \\ \frac{C}{|x|^{q-1}}, & \text{for } 1\leq |x|<\infty, \end{cases}$$

which moreover ensures that

$$\int_{\mathbb{R}}|u(\bar{x})-u(y)|^2 W_{p,q}(\bar{x}-y)\mathrm{d}y \geq c(p,q)E(t)^2. \tag{5.16}$$

Hence, for every p,q satisfying $0<p<\min\{1,2-2\alpha,2-2\beta\}$ and $2<q<\min\{3,2+2\alpha,2+2\beta\}$, we deduce

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \geq \frac{c(p,q)}{4}E(t)^2-(|\mu|^2+\nu^2)C(p,q,\alpha,\beta). \tag{5.17}$$

But as long as we choose the initial data u_0 such that $\frac{c(p,q)}{4}E(0)^2-(|\mu|^2+\nu^2)C(p,q,\alpha,\beta)>0$, that is, (noting that $\bar{x}(0)=0$)

$$|E(0)| = \left|\int_{\mathbb{R}} w_{p,q}(y)u_0(y)\mathrm{d}y\right| \geq \left(\frac{4(|\mu|^2+\nu^2)C(p,q,\alpha,\beta)+1}{c(p,q)}\right)^{1/2}, \tag{5.18}$$

we have that the quantity $E(t)$ defined by formula (5.4) blows up at finite time, which is a contradiction with the fact that for every $0<p<\min\{1/2,2-2\alpha,2-2\beta\}$, $2<q<\min\{3,2+2\alpha,2+2\beta\}$ and for all $t\in[0,T[$,

$$|E(t)|=\left|\int_{\mathbb{R}} w_{p,q}(y)\,u(\bar{x}(t)-y,t)\mathrm{d}y\right|\le\|w_{p,q}\|_{L^2}\|u(t)\|_{L^2}\le C\|u_0\|_{L^2(\mathbb{R})}<\infty.$$

Therefore, the initial assumption does not hold, so that there exists a finite time $T\in]0,\infty[$ and Equation (1.7) generates a unique solution $u\in C([0,T[;H^s(\mathbb{R}))\cap C^1([0,T[;H^{s-1}(\mathbb{R}))$, $s>3/2$, with

$$\sup_{t\in[0,T[}\|u(t)\|_{H^s(\mathbb{R})}=\limsup_{t\to T}\|u(t)\|_{H^s(\mathbb{R})}=\infty.$$

Now that the time T is the maximal existence time on the space $C([0,T[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,T[)$, according to blowup criterion (3.1), we thus get scenario (1.10) and conclude Theorem 1.3.

Finally, we give the details of proving Lemma 5.1.

Proof. **(Proof of Lemma 5.1.)** We mainly use the method of [11, Lemma 3.3] or [18, Lemma 2.1].

(1) Since $J_{p,q}(x)$ is an even function, it only needs to consider the case $x>0$. We first consider the case $0<x<\frac{1}{2}$. By change of variables, we have the following splitting

$$\begin{aligned}
J_{p,q}(x)=&\int_{|y|\le1}K_\beta(|x-y|)\operatorname{sgn}(x-y)\Big(\frac{\operatorname{sgn}(y)}{|y|^p}-\frac{1}{x^p}\Big)\mathrm{d}y\\
&+\int_{|y|\ge1}K_\beta(|x-y|)\operatorname{sgn}(x-y)\Big(\frac{\operatorname{sgn}(y)}{|y|^q}-\frac{1}{x^p}\Big)\mathrm{d}y\\
=&\int_0^1\Big(K_\beta(|x-y|)\operatorname{sgn}(x-y)\Big(\frac{1}{y^p}-\frac{1}{x^p}\Big)-K_\beta(x+y)\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\Big)\mathrm{d}y\\
&+\int_1^\infty\Big(-K_\beta(y-x)\Big(\frac{1}{y^q}-\frac{1}{x^p}\Big)-K_\beta(x+y)\Big(\frac{1}{y^q}+\frac{1}{x^p}\Big)\Big)\mathrm{d}y\\
:=&J^1_{p,q}(x)+J^2_{p,q}(x),
\end{aligned}\tag{5.19}$$

where we have omitted the notation of principle value for brevity. For $J^1_{p,q}(x)$, we further decompose it as

$$\begin{aligned}
J^1_{p,q}(x)=&\int_0^x\Big(K_\beta(x-y)\Big(\frac{1}{y^p}-\frac{1}{x^p}\Big)-K_\beta(x+y)\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\Big)\mathrm{d}y\\
&+\int_x^{\frac{3}{2}x}\Big(-K_\beta(y-x)\Big(\frac{1}{y^p}-\frac{1}{x^p}\Big)-K_\beta(x+y)\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\Big)\mathrm{d}y\\
&+\int_{\frac{3}{2}x}^1\Big(-K_\beta(y-x)\Big(\frac{1}{y^p}-\frac{1}{x^p}\Big)-K_\beta(x+y)\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\Big)\mathrm{d}y\\
:=&J^{11}_{p,q}(x)+J^{12}_{p,q}(x)+J^{13}_{p,q}(x).
\end{aligned}$$

According to (2.6), the term $J^{11}_{p,q}(x)$ can be directly treated as follows

$$|J^{11}_{p,q}(x)|\le C\int_0^{\frac{x}{2}}\frac{\frac{1}{y^p}-\frac{1}{x^p}}{(x-y)^{1+\beta}}\mathrm{d}y+C\int_{\frac{x}{2}}^x\frac{\frac{1}{y^p}-\frac{1}{x^p}}{(x-y)^{1+\beta}}\mathrm{d}y+C\int_0^x\frac{\frac{1}{y^p}+\frac{1}{x^p}}{(x+y)^{1+\beta}}\mathrm{d}y$$

$$\le \frac{C}{x^{1+\beta}}\int_0^{\frac{x}{2}}\Big(\frac{1}{y^p}-\frac{1}{x^p}\Big)\mathrm{d}y+\frac{C}{x^{p+1}}\int_{\frac{x}{2}}^{x}\frac{1}{(x-y)^\beta}\mathrm{d}y+\frac{C}{x^{1+\beta}}\int_0^x\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\mathrm{d}y\le\frac{C}{x^{p+\beta}}.$$

The term $J^{12}_{p,q}(x)$ can be similarly estimated as

$$\begin{aligned}|J^{12}_{p,q}(x)|&\le C\int_x^{\frac32 x}\frac{\frac{1}{x^p}-\frac{1}{y^p}}{(y-x)^{1+\beta}}\mathrm{d}y+C\int_x^{\frac32 x}\frac{\frac{1}{y^p}+\frac{1}{x^p}}{(x+y)^{1+\beta}}\mathrm{d}y\\&\le\frac{C}{x^{p+1}}\int_x^{\frac32 x}\frac{1}{(y-x)^\beta}\mathrm{d}y+\frac{C}{x^{1+\beta}}\int_x^{\frac32 x}\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\mathrm{d}y\le\frac{C}{x^{p+\beta}}.\end{aligned}$$

For $J^{13}_{p,q}(x)$, since $y-x\ge\frac13 y$ for every $y\in[\frac32 x,1]$, we use formula (2.6) to deduce that

$$\begin{aligned}|J^{13}_{p,q}(x)|&\le C\int_{\frac32 x}^1\frac{1}{(y-x)^{1+\beta}}\Big(\frac{1}{x^p}-\frac{1}{y^p}\Big)\mathrm{d}y+C\int_{\frac32 x}^1\frac{1}{(x+y)^{1+\beta}}\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\mathrm{d}y\\&\le\frac{C}{x^p}\int_{\frac32 x}^1\frac{1}{y^{1+\beta}}\mathrm{d}y\le\frac{C}{x^{p+\beta}}.\end{aligned}$$

Next we consider $J^2_{p,q}(x)$, by using formula (2.6) again, and due to the fact that $y-x\approx y\approx y+x$ for every $0<x<\frac12$ and $y\ge1$, we have

$$\begin{aligned}|J^2_{p,q}(x)|&\le\frac{1}{x^p}\int_1^\infty\big(|K_\beta(y-x)|+|K_\beta(y+x)|\big)\mathrm{d}y+\int_1^\infty\big(|K_\beta(y-x)|+|K_\beta(x+y)|\big)\frac{1}{y^q}\mathrm{d}y\\&\le\frac{C}{x^p}\int_1^\infty y^{-(1+\beta)}\mathrm{d}y+C\int_1^\infty\frac{1}{y^{q+1+\beta}}\mathrm{d}y\le\frac{C}{x^p}.\end{aligned}$$

Gathering the above estimates leads to the desired estimate (5.12) for the case $0<|x|<\frac12$.

For the case $\frac12\le x\le1$, we also have decomposition (5.19), and by applying the argument as above and the fact that $|\frac{1}{x^p}-\frac{1}{y^p}|\le C|x-y|$ for all y near x, e.g. $\frac14\le y\le\frac54$, we can show that $|J_{p,q}(x)|\le C$.

Now we turn to the case $x>2$. By using the change of variables, we have the following decomposition

$$\begin{aligned}J_{p,q}(x)=&\int_{|y|\le1}K_\beta(|x-y|)\,\mathrm{sgn}(x-y)\Big(\frac{\mathrm{sgn}(y)}{|y|^p}-\frac{1}{x^q}\Big)\mathrm{d}y\\&+\int_{|y|\ge1}K_\beta(|x-y|)\,\mathrm{sgn}(x-y)\Big(\frac{\mathrm{sgn}(y)}{|y|^q}-\frac{1}{x^q}\Big)\mathrm{d}y\\=&\int_0^1\Big(K_\beta(x-y)\Big(\frac{1}{y^p}-\frac{1}{x^q}\Big)-K_\beta(x+y)\Big(\frac{1}{y^p}+\frac{1}{x^q}\Big)\Big)\mathrm{d}y\\&+\int_1^\infty\Big(K_\beta(|x-y|)\,\mathrm{sgn}(x-y)\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big)-K_\beta(x+y)\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\Big)\mathrm{d}y\\=:&J^3_{p,q}(x)+J^4_{p,q}(x).\end{aligned}\tag{5.20}$$

For $J^3_{p,q}(x)$, due to that for every $0<y\le1$ and $x>2$, it yields $x-y\approx x\approx x+y$ and $|K_\beta(x-y)-K_\beta(x+y)|\le Cyx^{-(2+\beta)}$ (recalling kernel estimate (2.6)), we obtain

$$|J^3_{p,q}(x)|\le\int_0^1|K_\beta(x-y)-K_\beta(x+y)|\frac{1}{y^p}\mathrm{d}y+\frac{1}{x^q}\int_0^1\big(|K_\beta(x-y)|+|K_\beta(x+y)|\big)\mathrm{d}y$$

$$\leq \frac{C}{x^{2+\beta}} \int_0^1 y^{1-p} \mathrm{d}y + C\frac{1}{x^q}\int_0^1 \frac{1}{|x-y|^{1+\beta}}\mathrm{d}y \leq \frac{C}{x^{2+\beta}} + C\frac{C}{x^{q+1+\beta}} \leq \frac{C}{x^{2+\beta}}.$$

For $J^4_{p,q}(x)$, we further split it as follows

$$\begin{aligned} J^4_{p,q}(x) =& \int_1^x \left(K_\beta(x-y)\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big) - K_\beta(x+y)\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\right)\mathrm{d}y \\ &+ \int_x^{\frac{3}{2}x} \left(-K_\beta(y-x)\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big) - K_\beta(x+y)\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\right)\mathrm{d}y \\ &+ \int_{\frac{3}{2}x}^\infty \left(-K_\beta(y-x)\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big) - K_\beta(x+y)\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\right)\mathrm{d}y \\ =&: J^{41}_{p,q}(x) + J^{42}_{p,q}(x) + J^{43}_{p,q}(x). \end{aligned}$$

By using estimate (2.6) again and the following fact that $\left|\frac{1}{x^q}-\frac{1}{y^q}\right| \leq C\frac{|x-y|}{x^{q+1}}$ for all $\frac{2}{3}x \leq y \leq \frac{3}{2}x$, and in a similar argument as the treating of $J^3_{p,q}(x)$, we estimate $J^{41}_{p,q}(x)$ and $J^{42}_{p,q}(x)$ to get that

$$\begin{aligned} |J^{41}_{p,q}(x)| \leq& \int_1^{\frac{2}{3}x} |K_\beta(x-y) - K_\beta(x+y)|\frac{1}{y^q}\mathrm{d}y + \frac{1}{x^q}\int_1^{\frac{2}{3}x} (|K_\beta(x-y)| + |K_\beta(x+y)|)\mathrm{d}y \\ &+ \int_{\frac{2}{3}x}^x |K_\beta(x-y)|\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big)\mathrm{d}y + \int_{\frac{2}{3}x}^x |K_\beta(x+y)|\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\mathrm{d}y \\ \leq& \frac{C}{x^{2+\beta}}\int_1^{\frac{2}{3}x} y^{1-q}\mathrm{d}y + \frac{C}{x^q}\int_1^{\frac{2}{3}x}\frac{1}{|x-y|^{1+\beta}}\mathrm{d}y + \frac{C}{x^{q+1}}\int_{\frac{2}{3}x}^x \frac{1}{(x-y)^\beta}\mathrm{d}y + \frac{C}{x^{q+\beta}} \\ \leq& \frac{C}{x^{2+\beta}} + \frac{C}{x^{q+\beta}} \leq \frac{C}{x^{2+\beta}}, \end{aligned}$$

and

$$\begin{aligned} |J^{42}_{p,q}(x)| \leq& \int_x^{\frac{3}{2}x} |K_\beta(x-y)|\Big(\frac{1}{y^q}-\frac{1}{x^q}\Big)\mathrm{d}y + \int_x^{\frac{3}{2}x} |K_\beta(x+y)|\Big(\frac{1}{y^q}+\frac{1}{x^q}\Big)\mathrm{d}y \\ \leq& \frac{C}{x^{q+1}}\int_x^{\frac{3}{2}x}\frac{1}{(y-x)^\beta}\mathrm{d}y + \frac{C}{x^{q+\beta}} \leq \frac{C}{x^{q+\beta}}. \end{aligned}$$

For $J^{43}_{p,q}(x)$, noting that $y-x \approx y \approx y+x$ for all $y \geq \frac{3}{2}x$, we directly infer that

$$|J^{43}_{p,q}(x)| \leq \frac{C}{x^q}\int_{\frac{3}{2}x}^\infty \big(|K_\beta(x-y)| + |K_\beta(x+y)|\big)\mathrm{d}y \leq \frac{C}{x^q}\int_{\frac{3}{2}x}^\infty \frac{1}{y^{1+\beta}}\mathrm{d}y \leq \frac{C}{x^{q+\beta}}.$$

Hence, collecting the above estimates yields the desired estimate (5.12) for all $2 < |x| < \infty$.

For the remaining case $1 \leq x \leq 2$, decomposition (5.20) also holds, and we can estimate similarly as above (in a simpler way) to show that $|J_{p,q}(x)| \leq C$.

Therefore, based on the above analysis, the desired estimate (5.12) follows.

(2) Since $H_{p,q}(x)$ is an odd function, we only need to consider the case $x > 0$. We first consider the case $0 < x < \frac{1}{2}$, and we have the following splitting

$$H_{p,q}(x) = C_\alpha \int_{|y|\leq 1} \frac{1}{|x-y|^{1+\alpha}}\Big(\frac{1}{x^p} - \frac{\mathrm{sgn}(y)}{|y|^p}\Big)\mathrm{d}y + C_\alpha \int_{|y|\geq 1} \frac{1}{|x-y|^{1+\alpha}}\Big(\frac{1}{x^p} - \frac{\mathrm{sgn}(y)}{|y|^q}\Big)\mathrm{d}y$$

$$
\begin{aligned}
=&C_\alpha\int_0^x\left(\frac{1}{(x-y)^{1+\alpha}}\Big(\frac{1}{x^p}-\frac{1}{y^p}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^p}+\frac{1}{y^p}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_x^{\frac32 x}\left(\frac{1}{(y-x)^{1+\alpha}}\Big(\frac{1}{x^p}-\frac{1}{y^p}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^p}+\frac{1}{y^p}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_{\frac32 x}^1\left(\frac{1}{(y-x)^{1+\alpha}}\Big(\frac{1}{x^p}-\frac{1}{y^p}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^p}+\frac{1}{y^p}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_1^\infty\left(\frac{1}{(y-x)^{1+\alpha}}\Big(\frac{1}{x^p}-\frac{1}{y^q}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^p}+\frac{1}{y^q}\Big)\right)\mathrm{d}y.
\end{aligned}
$$

In a similar way as estimating $J^{11}_{p,q}(x)$ - $J^{13}_{p,q}(x)$ and $J^2_{p,q}(x)$, we get

$$
\begin{aligned}
|H_{p,q}(x)|\le&\frac{C}{x^{1+\alpha}}\int_0^{\frac{x}{2}}\Big(\frac{1}{x^p}-\frac{1}{y^p}\Big)\mathrm{d}y+\frac{C}{x^{p+1}}\int_{\frac{x}{2}}^x\frac{1}{(x-y)^\alpha}\mathrm{d}y+\frac{C}{x^{1+\alpha}}\int_0^x\Big(\frac{1}{x^p}+\frac{1}{y^p}\Big)\mathrm{d}y\\
&+\frac{C}{x^{p+1}}\int_x^{\frac32 x}\frac{1}{(y-x)^\alpha}\mathrm{d}y+\frac{C}{x^{1+\alpha}}\int_x^{\frac32 x}\Big(\frac{1}{y^p}+\frac{1}{x^p}\Big)\mathrm{d}y+\frac{C}{x^p}\int_{\frac32 x}^1\frac{1}{y^{1+\alpha}}\mathrm{d}y\\
&+\frac{C}{x^p}\int_1^\infty y^{-(1+\alpha)}\mathrm{d}y+\frac{C}{x^p}\int_1^\infty y^{-(1+\alpha)}\mathrm{d}y\le\frac{C}{x^{p+\alpha}},
\end{aligned}
$$

which corresponds to estimate (5.14) for the case $0<|x|<1/2$.

While for the case $x>2$, we have the following decomposition

$$
\begin{aligned}
H_{p,q}(x)=&C_\alpha\int_{|y|\le1}\frac{1}{|x-y|^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{\mathrm{sgn}(y)}{|y|^p}\Big)\mathrm{d}y+C_\alpha\int_{|y|\ge1}\frac{1}{|x-y|^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{\mathrm{sgn}(y)}{|y|^q}\Big)\mathrm{d}y\\
=&C_\alpha\int_0^1\left(\frac{1}{(x-y)^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{1}{y^p}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^q}+\frac{1}{y^p}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_1^x\left(\frac{1}{(x-y)^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{1}{y^q}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^q}+\frac{1}{y^q}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_x^{\frac32 x}\left(\frac{1}{(y-x)^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{1}{y^q}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^q}+\frac{1}{y^q}\Big)\right)\mathrm{d}y\\
&+C_\alpha\int_{\frac32 x}^\infty\left(\frac{1}{(y-x)^{1+\alpha}}\Big(\frac{1}{x^q}-\frac{1}{y^q}\Big)+\frac{1}{(x+y)^{1+\alpha}}\Big(\frac{1}{x^q}+\frac{1}{y^q}\Big)\right)\mathrm{d}y.
\end{aligned}
$$

By arguing as estimating $J^3_{p,q}(x)$ and $J^{41}_{p,q}(x)$ - $J^{43}_{p,q}(x)$, we obtain

$$
\begin{aligned}
|H_{p,q}(x)|\le&C\int_0^1\left(\frac{1}{(x-y)^{1+\alpha}}-\frac{1}{(x+y)^{1+\alpha}}\right)\frac{\mathrm{d}y}{y^p}+\frac{C}{x^q}\int_0^1\left(\frac{1}{(x-y)^{1+\alpha}}+\frac{1}{(x+y)^{1+\alpha}}\right)\mathrm{d}y\\
&+\frac{C}{x^{2+\alpha}}\int_1^{\frac23 x}y^{1-q}\mathrm{d}y+\frac{C}{x^q}\int_1^{\frac23 x}\frac{1}{|x-y|^{1+\alpha}}\mathrm{d}y+\frac{C}{x^{q+1}}\int_{\frac23 x}^x\frac{1}{(x-y)^\alpha}\mathrm{d}y+\frac{C}{x^{q+\alpha}}\\
&+\frac{C}{x^{q+1}}\int_x^{\frac32 x}\frac{1}{(y-x)^\alpha}\mathrm{d}y+\frac{C}{x^{q+\alpha}}+\frac{C}{x^q}\int_{\frac32 x}^\infty\frac{1}{(y-x)^{1+\alpha}}\mathrm{d}y\le\frac{C}{x^{2+\alpha}},
\end{aligned}
$$

which proves estimate (5.14) for the case $|x|\ge2$.

For the remaining cases $1/2\le x\le2$, we can estimate in a similar and simpler way as above to show that $|H_{p,q}(x)|\le C$. Therefore, estimate (5.14) is proved relying on the above deduction. ∎

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Appendix. We first recall the following commutator estimate (for the proof of $\gamma=0$ case see e.g. [3, Eq. (3.15)], and general γ can be similarly treated).

LEMMA 6.1. *Let $s>0$, $\gamma\geq 0$ and $1\leq p,r\leq\infty$. Let v be a vector field of $\mathbb{R}^d$ and t be a variable independent of the spatial variables. Then there exists a constant C depending only on s,p,r,d so that*

$$\|\{2^{qs}\|[\Delta_q,v\cdot\nabla]t^\gamma v\}_{q\geq -1}\|_{L^p}\|_{\ell^r}\leq C\|\nabla v\|_{L^\infty}\|t^\gamma v\|_{B^s_{p,r}},\tag{6.1}$$

where for operators A,B the commutator operator $[A,B]$ corresponds to $AB-BA$.

Now we turn to the proof of Proposition 3.1 concerning the local well-posedness result for Equation (1.7).

Proof. **(Proof of Proposition 3.1.)** The proof is divided into several steps.

Step 1: A priori estimates.

Assume that $u\in C^1([0,\infty[,H^\infty(\mathbb{R}))$ is a smooth solution to Equation (1.7). According to estimate (2.22), we easily get the energy estimate

$$\|u(t)\|^2_{L^2}+\int_0^t\|u(\tau)\|^2_{\dot H^{\frac{\alpha}{2}}}\mathrm{d}\tau\leq\|u_0\|^2_{L^2},\quad \text{for all } t>0,\tag{6.2}$$

Then for every $q\geq -1$, we apply the dyadic block operator Δ_q to Equation (1.7) to get

$$\partial_t(\Delta_q u)+u\partial_x(\Delta_q u)+\mu L_\beta(\Delta_q u)+\nu\Lambda^\alpha(\Delta_q u)=-[\Delta_q,u\partial_x]u,$$

then by taking the inner product of the above equation with $\Delta_q u$ and using the fact that (as $m_\beta(\zeta)$ is an odd function) $\int_{\mathbb{R}}(L_\beta\Delta_q u)(\Delta_q u)\mathrm{d}x=\int_{\mathbb{R}}im_\beta(\zeta)|\widehat{\Delta_q u}(\zeta)|^2\mathrm{d}\zeta=0$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q u\|^2_{L^2}+\nu\|\Lambda^{\frac{\alpha}{2}}\Delta_q u\|^2_{L^2}\leq\Big|\int_{\mathbb{R}}\partial_x u|\Delta_q u|^2\mathrm{d}x\Big|+\|[\Delta_q,u\partial_x]u\|_{L^2}\|\Delta_q u\|_{L^2}.$$

Integrating on the time interval $[0,t]$ yields

$$\begin{aligned}&\|\Delta_q u(t)\|^2_{L^2}+2\nu\int_0^t\|\Lambda^{\frac{\alpha}{2}}\Delta_q u(\tau)\|^2_{L^2}\mathrm{d}\tau\\ \leq&\|\Delta_q u_0\|^2_{L^2}+2\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\|\Delta_q u(\tau)\|^2_{L^2}\mathrm{d}\tau+2\int_0^t\|[\Delta_q,u\partial_x]u(\tau)\|_{L^2}\|\Delta_q u(\tau)\|_{L^2}\mathrm{d}\tau.\end{aligned}$$

Multiplying both sides of the above inequality with 2^{2qs} and summing over all $q\geq -1$, it follows from estimate (6.1) (with $\gamma=0$) that

$$\begin{aligned}&\|u(t)\|^2_{B^s_{2,2}}+2\nu\int_0^t\|\Lambda^{\frac{\alpha}{2}}u(\tau)\|^2_{B^s_{2,2}}\mathrm{d}\tau\\ \leq&\|u_0\|^2_{B^s_{2,2}}+2\int_0^t\|\partial_x u\|_{L^\infty}\|u\|^2_{B^s_{2,2}}\mathrm{d}\tau+2\int_0^t\|\{2^{qs}\|[\Delta_q,u\partial_x]u\|_{L^2}\}_{q\geq -1}\|_{\ell^2}\|u\|_{B^s_{2,2}}\mathrm{d}\tau\end{aligned}$$

$$\leq \|u_0\|_{B^s_{2,2}}^2 + C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\|u(\tau)\|_{B^s_{2,2}}^2 \mathrm{d}\tau. \tag{6.3}$$

Denoting by $X(t)=\|u\|^2_{L^\infty_t(B^s_{2,2})}+\nu\int_0^t\|\Lambda^{\frac{\alpha}{2}}u(\tau)\|^2_{B^s_{2,2}}\mathrm{d}\tau$, and by using the Sobolev embedding $B^s_{2,2}(\mathbb{R})\hookrightarrow W^{1,\infty}(\mathbb{R})$ $(s>\frac{3}{2})$, we obtain

$$X(t)\leq \|u_0\|^2_{B^s_{2,2}}+C\int_0^t X(\tau)^{\frac{3}{2}}\mathrm{d}\tau \leq \|u_0\|^2_{B^s_{2,2}}+CtX(t)^{\frac{3}{2}}.$$

By applying the continuity method, we infer that for all $T\leq \frac{1}{6C\|u_0\|_{B^s_{2,2}}}$,

$$\|u\|^2_{L^\infty_T(B^s_{2,2})}+\nu\int_0^T\|\Lambda^{\frac{\alpha}{2}}u(\tau)\|^2_{B^s_{2,2}}\mathrm{d}\tau\leq 2\|u_0\|^2_{B^s_{2,2}}. \tag{6.4}$$

Moreover, thanks to high-low frequency decomposition, energy estimate (2.22) and Bernstein's inequality,

$$\|u\|^2_{L^2([0,T];B^{s+\alpha/2}_{2,2})}\leq C_0\|\Delta_{-1}u\|^2_{L^2_T(L^2)}+C_0\|\Lambda^{\frac{\alpha}{2}}u\|^2_{L^2_T(B^s_{2,2})}\leq 2C_0(1+T)\|u_0\|^2_{B^s_{2,2}}. \tag{6.5}$$

Next based on the a priori uniform bounds (6.4)-(6.5) and (6.2), we show the smoothing estimates; more precisely, we prove that for all $\gamma\in\mathbb{R}^+$ and $t\in[0,T]$,

$$\begin{aligned}&\|t^\gamma u(t)\|^2_{L^\infty_T H^{s+\gamma\alpha}}+\|\Lambda^{\frac{\alpha}{2}}(t^\gamma u(t))\|^2_{L^2_T H^{s+\gamma\alpha}}\\ \leq& C([\gamma]+1)^{2(\gamma-[\gamma])}([\gamma]!)^2(1+T^{2\gamma})\|u_0\|^2_{B^s_{2,2}}e^{CT\gamma\|u_0\|_{B^s_{2,2}}},\end{aligned} \tag{6.6}$$

where C is a constant depending only on α,ν,s. Notice that $t^\gamma u$ $(\gamma>0)$ satisfies the following equation

$$\partial_t(t^\gamma u)+u\partial_x(t^\gamma u)+\mu L_\beta(t^\gamma u)+\nu\Lambda^\alpha(t^\gamma u)=\gamma t^{\gamma-1}u,\quad (t^\gamma u)|_{t=0}=0. \tag{6.7}$$

We first treat the case $\gamma\in\mathbb{Z}^+$. For $\gamma=1$, noting that for $q\in\mathbb{N}$,

$$\begin{aligned}\partial_t\Delta_q(tu)+u\partial_x\Delta_q(tu)+\mu L_\beta\Delta_q(tu)+\nu\Lambda^\alpha\Delta_q(tu)=-[\Delta_q,u\partial_x](tu)+\Delta_q u,\\ \Delta_q(tu)|_{t=0}=0,\end{aligned} \tag{6.8}$$

we take the dot product of the above equation with $\Delta_q(tu)$ and integrate on the time variable to get

$$\begin{aligned}&\frac{1}{2}\|\Delta_q(tu(t))\|^2_{L^2}+\nu\int_0^t\|\Lambda^{\frac{\alpha}{2}}\Delta_q(\tau u)\|^2_{L^2}\mathrm{d}\tau\leq\int_0^t\|\partial_x u\|_{L^\infty}\|\Delta_q(\tau u)\|^2_{L^2}\mathrm{d}\tau\\ &+\int_0^t\|[\Delta_q,u\partial_x](\tau u)\|_{L^2}\|\Delta_q(\tau u)\|_{L^2}\mathrm{d}\tau+C_0 2^{-q\frac{\alpha}{2}}\int_0^t\|\Delta_q u\|_{L^2}\|\Lambda^{\frac{\alpha}{2}}\Delta_q(\tau u)\|_{L^2}\mathrm{d}\tau;\end{aligned}$$

by multiplying both sides of the above inequality with $2^{2q(s+\alpha)}$, summing over all $q\in\mathbb{N}$, using Young's inequality and Lemma 6.1, we obtain

$$\begin{aligned}&\sum_{q\in\mathbb{N}}2^{2q(s+\alpha)}\|\Delta_q(tu(t))\|^2_{L^2}+\nu\int_0^t\Big(\sum_{q\in\mathbb{N}}2^{2q(s+\alpha)}\|\Lambda^{\frac{\alpha}{2}}\Delta_q(\tau u)\|^2_{L^2}\Big)\mathrm{d}\tau\\ \leq& C\int_0^t\|\partial_x u\|_{L^\infty}\|\tau u(\tau)\|^2_{B^{s+\alpha}_{2,2}}\mathrm{d}\tau+\frac{C}{\nu}\int_0^t\Big(\sum_{q\in\mathbb{N}}2^{2q(s+\frac{\alpha}{2})}\|\Delta_q u(\tau)\|^2_{L^2}\Big)\mathrm{d}\tau\end{aligned}$$

$$\le C\int_0^t \|\partial_x u\|_{L^\infty}\|\tau u(\tau)\|^2_{B^{s+\alpha}_{2,2}}\mathrm{d}\tau+\frac{C}{\nu}\int_0^t \|\Lambda^{\frac{\alpha}{2}}u(\tau)\|^2_{B^s_{2,2}}\mathrm{d}\tau; \tag{6.9}$$

while from Equation (6.8) with $q=-1$ and estimate (6.1), we deduce that

$$\begin{aligned}
&\frac{1}{2}\|\Delta_{-1}(tu(t))\|^2_{L^2}+\nu\int_0^t \|\Lambda^{\frac{\alpha}{2}}\Delta_{-1}(\tau u(\tau))\|^2_{L^2}\mathrm{d}\tau\\
\le&\int_0^t \|\partial_x u\|_{L^\infty}\|\Delta_{-1}(\tau u)\|^2_{L^2}\mathrm{d}\tau+\int_0^t \big(\|[\Delta_{-1},u\partial_x](\tau u)\|_{L^2}+\|\Delta_{-1}u\|_{L^2}\big)\|\Delta_{-1}(\tau u)\|_{L^2}\mathrm{d}\tau\\
\le& C\int_0^t \|\partial_x u\|_{L^\infty}\|\tau u(\tau)\|^2_{B^{s+\alpha}_{2,2}}\mathrm{d}\tau+\|u_0\|^2_{L^2}t^2;
\end{aligned} \tag{6.10}$$

combining estimate (6.9) with estimate (6.10) leads to

$$\begin{aligned}
&\|tu(t)\|^2_{B^{s+\alpha}_{2,2}}+\nu\|\Lambda^{\frac{\alpha}{2}}(tu(t))\|^2_{L^2_T(B^{s+\alpha}_{2,2})}\\
\le& C\int_0^T \|\partial_x u(t)\|_{L^\infty}\|tu(t)\|^2_{B^{s+\alpha}_{2,2}}\mathrm{d}t+\frac{C}{\nu}\|\Lambda^{\frac{\alpha}{2}}u\|^2_{L^2_T(B^s_{2,2})}+\|u_0\|^2_{L^2}T^2;
\end{aligned}$$

Grönwall's inequality yields that

$$\|tu(t)\|^2_{L^\infty_T(B^{s+\alpha}_{2,2})}+\nu\|\Lambda^{\frac{\alpha}{2}}(tu(t))\|^2_{L^2_T(B^{s+\alpha}_{2,2})}\le C(1+T^2)\|u_0\|^2_{B^s_{2,2}}e^{CT\|u_0\|_{B^s_{2,2}}}; \tag{6.11}$$

thus the desired estimate (6.6) with $\gamma=1$ follows. Now suppose that estimate (6.6) holds for $\gamma=N$, we shall consider the case $N+1$. We use Equation (6.7) with $\gamma=N+1$, and similar to obtaining estimate (6.11), we have

$$\begin{aligned}
&\|t^{N+1}u(t)\|^2_{H^{s+(N+1)\alpha}}+\|\Lambda^{\frac{\alpha}{2}}(t^{N+1}u(t))\|^2_{L^2_T(H^{s+(N+1)\alpha})}\\
\le& Ce^{CT\|u_0\|_{B^s_{2,2}}}\Big((N+1)^2\|\Lambda^{\frac{\alpha}{2}}(t^N u(t))\|^2_{L^2_T(H^{s+N\alpha})}+(N+1)^2\|t^N u(t)\|^2_{L^2_T(L^2)}\Big)\\
\le& Ce^{CT\|u_0\|_{B^s_{2,2}}}\Big(((N+1)!)^2(1+T^{2N})\|u_0\|^2_{B^s_{2,2}}e^{C(1+T)N\|u_0\|_{B^s_{2,2}}}+(N+1)^2T^{2N+2}\|u_0\|^2_{L^2}\Big)\\
\le& C((N+1)!)^2(1+T^{2N+2})\|u_0\|^2_{B^s_{2,2}}e^{CT(N+1)\|u_0\|_{B^s_{2,2}}},
\end{aligned}$$

where in the second line we have used the following estimation $\|t^N u(t)\|_{L^2_T(L^2)}\|t^{N+1}u(t)\|_{L^2_T(L^2)}\le T^{2N+2}\|u\|^2_{L^\infty_T(L^2)}\le T^{2N+2}\|u_0\|^2_{L^2}$. Thus the induction method ensures the estimate (6.6) for all $\gamma\in\mathbb{Z}^+$. Note that estimate (6.4) corresponds to inequality (6.6) with $\gamma=0$, hence we obtain estimate (6.6) for all $\gamma\in\mathbb{N}$. For the general $\gamma\ge 0$, we see $[\gamma]\le\gamma<[\gamma]+1$ with $[\gamma]$ the integer part of γ, and we use the interpolation inequality in Sobolev spaces to get

$$\begin{aligned}
\|t^\gamma u\|^2_{L^\infty_T(B^{s+\gamma\alpha}_{2,2})}\le& C\|t^{[\gamma]}u\|^{2([\gamma]+1-\gamma)}_{L^\infty_T(B^{s+[\gamma]\alpha}_{2,2})}\|t^{[\gamma]+1}u\|^{2(\gamma-[\gamma])}_{L^\infty_T(B^{s+([\gamma]+1)\alpha}_{2,2})}\\
\le& C([\gamma]+1)^{2(\gamma-[\gamma])}([\gamma]!)^2(1+T^{2\gamma})\|u_0\|^2_{B^s_{2,2}}e^{CT\gamma\|u_0\|_{B^s_{2,2}}}.
\end{aligned}$$

Similar estimate holds for $\|\Lambda^{\frac{\alpha}{2}}(t^\gamma u(t))\|^2_{L^2_T(B^{s+\gamma\alpha}_{2,2})}$.

Step 2: Existence.

Denoting the frequency cutoff operator $\mathcal{J}_\epsilon: L^2(\mathbb{R})\to H^\infty(\mathbb{R})$, $\epsilon>0$ by $(\mathcal{J}_\epsilon f)(x)=\mathcal{F}^{-1}(\hat{f}(\cdot)1_{B_{1/\epsilon}}(\cdot))(x)$, then we regularize the dissipative dispersive Burgers Equation (1.7) as follows

$$\partial_t u^\epsilon+\mathcal{J}_\epsilon\big((\mathcal{J}_\epsilon u^\epsilon)\partial_x(\mathcal{J}_\epsilon u^\epsilon)\big)+\mu\mathcal{J}_\epsilon L_\beta u^\epsilon+\nu\mathcal{J}_\epsilon\Lambda^\alpha u^\epsilon=0,\quad u^\epsilon|_{t=0}=\mathcal{J}_\epsilon u_0. \tag{6.12}$$

For every $\epsilon>0$ and $u_0\in L^2$, by the Cauchy-Lipschitz theorem, one easily deduces that there exists a unique solution $u^\epsilon\in C^1([0,T_*[,\mathcal{J}_\epsilon L^2)$ to the regularized Equation (6.12), with $T_*>0$ the maximal existence time. Moreover, the L^2-energy estimate $\sup_{t\in[0,T_*[}\|u^\epsilon\|_{L^2}\leq\|\mathcal{J}_\epsilon u_0\|_{L^2}\leq\|u_0\|_{L^2}$ can also be derived, which guarantees $T_\epsilon=\infty$ and indeed $u^\epsilon\in C^1([0,\infty[,\mathcal{J}_\epsilon L^2)$.

Then almost paralleling to the proof in the Step 1 and using the equivalence $\|\cdot\|_{H^s}\approx\|\cdot\|_{B^s_{2,2}}$, we know the uniform estimate that for all $T\leq\frac{1}{6C\|u_0\|_{H^s}}$,

$$\sup_{t\in[0,T]}\|u^\epsilon(t)\|^2_{H^s}+\|u^\epsilon\|^2_{L^2_T(H^{s+\alpha/2})}\leq 2C_0(1+T)\|u_0\|^2_{H^s},\tag{6.13}$$

and also estimate (6.6) with u replaced by u^ϵ, where C is a positive constant depending only on ν,α,s. Based on this uniform estimate (6.13), and using the deduction in the uniqueness part below, we can argue as the corresponding part in [40] to prove that $\{u^\epsilon\}$ is a Cauchy sequence in $C([0,T];L^2(\mathbb{R}))$, which implies that it strongly converges to a function $u\in C([0,T];L^2(\mathbb{R}))$. By interpolation and uniform estimate (6.13), we also get that u^ϵ strongly converges to u in $C([0,T];H^m(\mathbb{R}))$ for every $0<m<s$. From the classical method we know that u is a distributional solution of the limiting Equation (1.7), and satisfies $u\in L^\infty([0,T];H^s(\mathbb{R}))\cap L^2([0,T];H^{s+\alpha/2}(\mathbb{R}))$. Moreover, we can show that $u\in C([0,T],H^s(\mathbb{R}))$; indeed, the deduction in Step 1 guarantees that the formula (6.4) remains true by replacing $\|u\|_{L^\infty_t B^s_{2,2}}$ with $\|u\|_{\widetilde{L}^\infty_t(B^s_{2,2})}$, where $\|u\|^2_{\widetilde{L}^\infty_T(B^s_{2,2})}:=\sum_{q\geq-1}2^{2qs}\|\Delta_q u\|^2_{L^\infty_T(L^2)}$, thus by this fact and a standard process (e.g. see [40]), we can prove the continuity-in-time issue. If $\alpha\in]0,1]$, from the relation $\partial_t u=-u\partial_x u-\mu L_\beta u-\nu\Lambda^\alpha u$, we also get $u\in C^1([0,T];H^{s-1}(\mathbb{R}))$, and thus u is a classical solution to the Equation (1.7).

Using Fatou's Lemma, and from estimate (6.6) (with u replaced by u^ϵ), we have $t^\gamma u(t)\in L^\infty([0,T],H^{s+\gamma\alpha}(\mathbb{R}))$, $\gamma\in\mathbb{R}^+$, which combined with Equation (1.7) implies that $u\in C^\infty(\mathbb{R}\times]0,T])$.

Step 3: Uniqueness.

Let u^1, $u^2\in C([0,T],H^s(\mathbb{R}))$, $s>\frac{3}{2}$ be two solutions to the dissipative dispersive Whitham Equation (1.7) with the same initial data. Denote by $\delta u:=u^1-u^2$, $i=1,2$, then we write the difference equation as

$$\partial_t\delta u+u^1\partial_x\delta u+\mu L_\beta\delta u+\nu\Lambda^\alpha\delta u=-\delta u\,\partial_x u^2,\quad \delta u|_{t=0}=\delta u_0\equiv 0.$$

By using the standard energy method, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta u(t)\|^2_{L^2}+\nu\|\Lambda^{\frac{\alpha}{2}}\delta u(t)\|^2_{L^2}\leq\big(\|\partial_x u^1(t)\|_{L^\infty}+\|\partial_x u^2(t)\|_{L^\infty}\big)\|\delta u(t)\|^2_{L^2}.$$

Grönwall's inequality ensures that

$$\sup_{t\in[0,T]}\|\delta u(t)\|^2_{L^2}\leq\|\delta u_0\|^2_{L^2}\exp\Big\{C_0T\big(\|u^1\|_{L^\infty_T(H^s)}+\|u^2\|_{L^\infty_T(H^s)}\big)\Big\}\equiv 0,$$

that is, $u^1\equiv u^2$ on $[0,T]\times\mathbb{R}$, as desired.

Step 4: Blowup criteria.

Let $T^*>0$ be the maximal existence time of the above constructed solution, then firstly we have a natural blowup criterion: if $T^*<\infty$ then necessarily $\|u\|_{L^\infty([0,T^*[,H^s(\mathbb{R}))}=\infty$; since otherwise from the local result, the solution will continue past T^*.

In the same way as obtaining estimate (6.3), we get

$$\|u(t)\|_{H^s}^2+\nu\|\Lambda^{\frac{\alpha}{2}}u\|_{L^2_t(H^s)}^2\le C_0\|u_0\|_{H^s}^2+C\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\|u(\tau)\|_{H^s}^2\mathrm{d}\tau,$$

with $C=C(s)$ a positive constant. Grönwall's inequality leads to that for every $T<T^*$,

$$\|u\|_{L^\infty_T(H^s)}^2+\nu\|\Lambda^{\frac{\alpha}{2}}u\|_{L^2([0,T],H^s)}^2\le C_0\|u_0\|_{H^s}^2\exp\Big\{C\int_0^T\|\partial_x u(t)\|_{L^\infty}\mathrm{d}t\Big\}.$$

Thus, if $T^*<\infty$ and the integral $\int_0^{T^*}\|\partial_x u(t)\|_{L^\infty}\mathrm{d}t<\infty$, then we directly have $\sup_{0\le t<T^*}\|u(t)\|_{H^s}<\infty$, and this contradicts the above natural blowup criterion. Hence, the desired blowup criterion (3.1) is followed. □

Finally we present an L^∞-estimate for the viscous Burgers equation with forcing term.

PROPOSITION 6.1. *Let $u\in C([0,T^*[;H^s(\mathbb{R}))\cap C^\infty(\mathbb{R}\times]0,\infty[)$, $s>3/2$ be a smooth solution of the following viscous Burgers equation with forcing*

$$\partial_t u+u\partial_x u-\nu\partial_{xx}u=f,\quad u|_{t=0}(x)=u_0(x), \tag{6.14}$$

with $u_0\in L^\infty(\mathbb{R})$ and $f\in L^2_T(L^2)$ with $T>0$. Then there exists a constant $C>0$ depending only on ν such that for every $t\in[0,T]$ and $T\in]0,T^*[$,

$$\|u(t)\|_{L^\infty}\le C\|u_0\|_{L^\infty}+CT^{\frac14}\|f\|_{L^2_T(L^2)}. \tag{6.15}$$

Proof. We shall use a DeGiorgi-Nash-Moser's iterative method to prove inequality (6.15) (one can see [49] for a similar argument). Let M be a positive number chosen later, and $M_k:=M(1-2^{-k-1})$ for all $k\in\mathbb{N}$. From a pointwise positivity inequality that for every Φ convex function, $-\Phi'(u)\partial_{xx}u\ge-\partial_{xx}\Phi(u)$, we know that $-1_{\{u\ge M_k\}}\partial_{xx}u\ge-\partial_{xx}(u-M_k)_+$, thus we have

$$\partial_t(u-M_k)_++(u-M_k)_+\partial_x(u-M_k)_+-\nu\partial_{xx}(u-M_k)_+\le f\,1_{\{u\ge M_k\}}.$$

Multiplying the above equation with $(u-M_k)_+$ and integrating over the spatial variable, we see that,

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|(u-M_k)_+(t)\|_{L^2}^2+\nu\|(u-M_k)_+(t)\|_{\dot H^1}^2\le\Big|\int_{\mathbb{R}^n}f(x,t)(u-M_k)_+(x,t)\mathrm{d}x\Big|, \tag{6.16}$$

which leads to

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|(u-M_k)_+(t)\|_{L^2}^2+\nu\|(u-M_k)_+(t)\|_{\dot H^1}^2\le\|f(t)1_{\{u(t)\ge M_k\}}\|_{\dot H^{-\frac14}}\|(u-M_k)_+(t)\|_{\dot H^{\frac14}}.$$

Denoting by

$$U_k:=\|(u-M_k)_+\|_{L^\infty_T(L^2)}^2+2\nu\|(u-M_k)_+\|_{L^2_T(\dot H^1)}^2,$$

and integrating in the time interval $[0,T]$, and by setting $M>2\|u_0\|_{L^\infty}$ (so that $\|(u_0-M_k)_+\|_{L^2}\equiv0$ for every $k\in\mathbb{N}$), we get

$$U_k\le2\int_0^T\|f(t)1_{\{u(t)\ge M_k\}}\|_{\dot H^{-\frac14}}\|(u-M_k)_+(t)\|_{\dot H^{\frac14}}\mathrm{d}t.$$

By virtue of the continuous embedding (see [3, Corollary 1.39]) $L^{\frac{2}{1+2s}}(\mathbb{R})\hookrightarrow \dot H^{-s}(\mathbb{R})$ ($s\in]0,\frac12[$), and the following interpolation inequality that $\nu^{\frac14}\|(u-M_k)_+\|^2_{L^8_T(\dot H^{\frac14})}\le C_0 U_k$, we obtain

$$
\begin{aligned}
U_k &\le C_0\|f(t)1_{\{u(t)\ge M_k\}}\|_{L^{\frac87}_T(L^{\frac43})}\|(u-M_k)_+\|_{L^8_T(\dot H^{\frac14})}\\
&\le C_0\nu^{-\frac18}\|f(t)1_{\{u(t)\ge M_k\}}\|_{L^{\frac87}_T(L^{\frac43})}U_k^{1/2}.
\end{aligned}
$$

The Young inequality and the Hölder inequality yield

$$
\begin{aligned}
U_k &\le C_0\nu^{-\frac14}\|f(t)1_{\{u(t)\ge M_k\}}\|^2_{L^{\frac87}_T(L^{\frac43})}\\
&\le C_0\nu^{-\frac14}\|f\|^2_{L^2_T(L^2)}\|1_{\{u(t)\ge M_k\}}\|^2_{L^{\frac83}_T L^4}\\
&\le C_0\nu^{-\frac14}\|f\|^2_{L^2_T(L^2)}\Big(\int_0^T|\{u(t)\ge M_k\}|^{\frac23}\mathrm{d}t\Big)^{\frac34},
\end{aligned}\tag{6.17}
$$

where $|\{u(t)\ge M_k\}|$ denotes the Lebesgue measure of the set $\{x: u(x,t)\ge M_k\}\subset\mathbb{R}$. Noting that $u(x,t)-M_{k-1}\ge M2^{-k-1}$ for all $u(x,t)\ge M_k$, we have that for every $\delta\ge1$,

$$
1_{\{u(t)\ge M_k\}}\le\Big(\frac{(u(t)-M_{k-1})_+}{M2^{-k-1}}\Big)^{\delta},
$$

and

$$
|\{u(t)\ge M_k\}|\le\frac{2^{(k+1)\delta}}{M^\delta}\|(u-M_{k-1})_+(t)\|^\delta_{L^\delta}.
$$

Hence, inserting the above estimate into estimate (6.17) leads to

$$
U_k\le C_0\nu^{-\frac14}\|f\|^2_{L^2_T(L^2)}\Big(\frac{2^{k+1}}{M}\Big)^{\frac\delta2}\Big(\int_0^T\|(u-M_{k-1})_+(t)\|^{\frac{2\delta}{3}}_{L^\delta}\mathrm{d}t\Big)^{\frac34}.\tag{6.18}
$$

Since $\dot H^s(\mathbb{R})\hookrightarrow L^{\frac{2}{1-2s}}(\mathbb{R})$ for every $s\in[0,\frac12[$, and from the interpolation inequality, we know that for every $\delta\in[2,\infty[$,

$$
\|(u-M_{k-1})_+\|^2_{L_T^{\frac{4\delta}{\delta-2}}(L^\delta)}\le C_0\|(u-M_{k-1})_+\|^2_{L_T^{\frac{4\delta}{\delta-2}}(\dot H^{\frac{\delta-2}{2\delta}})}\le C_0\nu^{-\frac{\delta-2}{2\delta}}U_{k-1}.
$$

In the following it requires $\delta\in[2,\infty[$ satisfies $\frac23\delta\le\frac{4\delta}{\delta-2}$ and $\frac\delta2>2$ simultaneously, that is, $\delta\in]4,8]$, thus we can fix $\delta=8$, and (6.18) reduces to

$$
U_k\le C_0\nu^{-\frac14}\|f\|^2_{L^2_T(L^2)}2^{4k}M^{-4}\|(u-M_{k-1})_+\|^4_{L_T^{\frac{16}{3}}(L^8)}\le C_0\nu^{-1}\|f\|^2_{L^2_T(L^2)}2^{4k}M^{-4}U^2_{k-1}.\tag{6.19}
$$

We also need to estimate U_0. From (6.16), we obtain that

$$
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|(u-M/2)_+(t)\|^2_{L^2}+\nu\|(u-M/2)_+(t)\|^2_{\dot H^1}\le\|f(t)\|_{L^2}\|(u-M/2)_+(t)\|_{L^2},
$$

which yields

$$
\frac12\frac{\mathrm{d}}{\mathrm{d}t}F(t)\le\|f(t)\|_{L^2}F(t)^{1/2},
$$

with $F(t)=\|(u-M/2)_+(t)\|_{L^2}^2+2\nu\int_0^t\|(u-M/2)_+(\tau)\|_{\dot{H}^1}^2\mathrm{d}\tau$, thus by setting $\frac{M}{2}>\|u_0\|_{L^\infty}$, we derive

$$U_0\le\|f\|_{L_T^2(L^2)}^2T. \tag{6.20}$$

Thanks to [49, Lemma 2.6], we can choose $M>0$ satisfying

$$\|f\|_{L_T^2(L^2)}^2T\le 2^{-4}C_0^{-1}\nu\|f\|_{L_T^2(L^2)}^{-2}M^4 \quad \text{and} \quad M>2\|u_0\|_{L^\infty},$$

equivalently,

$$M=C\|u_0\|_{L^\infty}+C\nu^{-\frac14}T^{\frac14}\|f\|_{L_T^2(L^2)}, \tag{6.21}$$

so that we have $U_k\to0$ as $k\to\infty$, which implies $\|(u-M)_+\|_{L_T^\infty L^2}=0$. Hence, for a.e. $(x,t)\in\mathbb{R}\times[0,T]$, $u(x,t)\le M$.

Applying the above deduction to $-u$, we also get $u(x,t)\ge -M$ for a.e. $(x,t)\in\mathbb{R}\times[0,T]$. Clearly, the desired estimate (6.15) follows. ∎

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