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ON THE WELL-POSEDNESS OF A 2D NONLINEAR AND NONLOCAL SYSTEM ARISING FROM THE DISLOCATION DYNAMICS

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In this paper we consider a 2D nonlinear and nonlocal model describing the dynamics of the dislocation densities. We prove the local well-posedness of strong solution in the suitable functional framework, and we show the global well-posedness for some dissipative cases by the method of nonlocal maximum principle.

Keywords: Nonlinear transport equations; nonlocal transport equations; dissipation; nonlocal modulus of continuity; maximum principle; dynamics of dislocation densities.

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1. Introduction

In the materials science, dislocations are termed as certain defects shown by real crystals in the organization of their crystalline structure. They were considered as the principal explanation of plastic deformation at the microscopic scale of materials. Dislocations can move under the effect of an exterior stress. In a particular case where the defects are parallel line in the three-dimensional space, dislocations can be viewed as points in a plane by considering their cross-sections. These dislocations are called "edge dislocations" which move in the direction of "Burgers vector" that has a fixed direction (cf. [18] for more physical description).

In this paper we focus on the following nonlinear and nonlocal system on $\mathbb{R}^2$ which arises from modeling the edge dislocations

$$\begin{cases} \partial_t \rho^+ + u \cdot \nabla \rho^+ + \kappa |D|^\alpha \rho^+ = 0, & \alpha \in]0,2], \\ \partial_t \rho^- - u \cdot \nabla \rho^- + \kappa |D|^\alpha \rho^- = 0, \\ u = (\mathcal{R}_1^2 \mathcal{R}_2^2 (\rho^+ - \rho^-), 0), \\ \rho^+|_{t=0} = \rho_0^+, \quad \rho^-|_{t=0} = \rho_0^-, \end{cases} \tag{1.1}$$

where $\kappa \geq 0$ is the viscosity coefficient, $\mathcal{R}_i \triangleq \partial_i / |D|$ ($i = 1, 2$, $\partial_i \triangleq \partial_{x_i}$) is the usual Riesz transform and $|D|^\alpha$ is defined via the Fourier transform

$$\widehat{|D|^\alpha f}(\zeta) = |\zeta|^\alpha \hat{f}(\zeta).$$

The inviscid case (i.e. $\kappa = 0$) of (1.1) is the model introduced by Groma and Balogh in [16, 17] where they consider two types of dislocations in the plane (x_1, x_2). Typically for a given velocity field, the dislocations of type (+) propagate in the direction $+b$, with $b = (1, 0)$ the Burgers vector, while those of type (−) propagate in the direction $-b$. The terms $\rho^\pm$ are the plastic deformations in the material. The velocity vector field u is the shear stress in the material, which solves the equation of elasticity (cf. [5, Sec. 2]). Another closely related physical quantities are the derivatives of $\rho^\pm$ in the x_1-direction $\partial_1 \rho^\pm$, denoting by $\theta^\pm$, which represent the dislocation densities of type (±). Physically, $\theta^\pm$ are non-negative functions. In terms of $\theta^\pm$, one can formally rewrite the system (1.1) as follows:

$$\begin{cases} \partial_t \theta^+ + \partial_1 (u_1 \theta^+) + \kappa |D|^\alpha \theta^+ = 0, & \alpha \in]0,2], \\ \partial_t \theta^- - \partial_1 (u_1 \theta^-) + \kappa |D|^\alpha \theta^- = 0, \\ u_1 = \mathcal{R}_1 \mathcal{R}_2^2 |D|^{-1} (\theta^+ - \theta^-), \\ \theta^+|_{t=0} = \theta_0^+, \quad \theta^-|_{t=0} = \theta_0^-. \end{cases} \tag{1.2}$$

In [5], Cannone *et al.* considered the inviscid system (1.1) with the initial data

$$\rho^\pm(t = 0, x_1, x_2) = \rho_0^\pm(x_1, x_2) = \bar{\rho}_0^\pm(x_1, x_2) + L x_1, \quad L \geq 0, \tag{1.3}$$

where $\bar{\rho}_0^\pm(x_1, x_2) = \rho_0^{\pm,\mathrm{per}}(x_1, x_2)$ and $\rho^{\pm,\mathrm{per}}$ is a 1-periodic function in $x = (x_1, x_2)$, and by exploiting a fundamental entropy estimate satisfied by the dislocation densities, the authors can show the global existence of a weak solution. In [15], El Hajj proved that the inviscid model (1.1) has a unique local-in-time solution with the initial data (1.3) prescribed on $\mathbb{R}^2$ and $\bar{\rho}_0^\pm(x_1, x_2) \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ with $r > 1$ and $p \in]1, \infty[$. Note that L may be chosen large enough so that $\partial_1 \rho_0^\pm \geq 0$. For the study of more general dynamics of dislocation lines, we refer to the works [2, 3] and references therein for some existence and uniqueness results.

In this article, in contrast with [15], we start with studying the system (1.2) about the dislocation densities, and then from the relation between $\theta^\pm$ and $\rho^\pm$, we go back to the system (1.1). The first result is the local well-posedness for the system (1.2).

Theorem 1.1. *Let $\kappa \geq 0$, $\alpha \in]0,2]$, $p \in]1,2[$, $m > 2$ and $(\theta_0^+, \theta_0^-) \in H^m(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ be composed of real scalar functions. Then there exists $T > 0$ depending only on $\|\theta_0^\pm\|_{H^m \cap L^p}$ such that the system* (1.2) *has a unique solution $(\theta^+, \theta^-) \in C([0,T]; H^m \cap L^p)$. Moreover, we have $(\theta^+, \theta^-) \in C^1([0,T]; H^{m_0})$ with $m_0 = \min\{m-1, m-\alpha\}$.*

Besides, let $T^* > 0$ be the maximal existence time of $(\theta^+, \theta^-) \in C([0,T^*[; H^m \cap L^p)$, then if $T^* < \infty$, we necessarily have

$$\int_0^{T^*} \|(\theta^+, \theta^-)(t)\|_{L^\infty} dt = \infty, \tag{1.4}$$

where we have used the notation that $\|(f,g)\|_X \triangleq \|f\|_X + \|g\|_X$ for some $f, g \in X$.

We also have some further properties of the solution.

Proposition 1.2. *Let $\kappa \geq 0$, $\alpha \in]0,2]$, $p \in]1,2[$, $m > 4$. Suppose that $(\theta^+, \theta^-) \in C([0,T^*[; H^m \cap L^p)$ is the corresponding maximal lifespan solution of the system* (1.2) *obtained in Theorem* 1.1. *Then the following statements hold true.*

(1) *If $\theta_0^\pm$ are non-negative, then $\theta^\pm(t)$ are also non-negative for all $]0,T^*[$.*
(2) *Assume that $\kappa = 0$ or $\kappa > 0$ and $\alpha \in]\frac{1}{2}, 2]$. If $\theta_0^\pm \in L^{\infty,1}_{x_2,x_1}(\mathbb{R}^2)$ (for definition see the next section) are non-negative, then $\theta^\pm \in L^\infty([0,T^*[; L^{\infty,1}_{x_2,x_1})$ satisfies that*

$$\|\theta^\pm(t)\|_{L^{\infty,1}_{x_2,x_1}} \leq \|\theta_0^\pm\|_{L^{\infty,1}_{x_2,x_1}}, \quad \forall t \in [0,T^*[. \tag{1.5}$$

Besides, the expression

$$\rho^\pm(t,x_1,x_2) \triangleq \int_{-\infty}^{x_1} \theta^\pm(t,\tilde{x}_1,x_2) d\tilde{x}_1, \quad \forall t \in [0,T^*[, (x_1,x_2) \in \mathbb{R}^2 \tag{1.6}$$

is well-defined and $\rho^\pm$ are the mild solutions to the system (1.1).
(3) *If the conditions of* (2) *are supposed, and we moreover assume that for each $k = 1,2,3$, $\partial_2^k \rho_0^\pm \in L^\infty_x(\mathbb{R}^2)$ and $\lim_{x_1 \to -\infty} \partial_2^k \rho_0^\pm(x) = 0$ for every $x_2 \in \mathbb{R}$, then*

$$\rho^\pm \in L^\infty([0,T^*[, W^{3,\infty}) \cap C([0,T^*[; W^{1,\infty}),$$

and (ρ^+, ρ^-) satisfies the system (1.1) *in the classical pointwise sense.*
(4) *Under the assumption of* (3), *then for every $\epsilon > 0$ and $t \in]0,T^*[$, there exists $R > 0$ depending on κ, ϵ, t and $\|\theta^\pm\|_{L^\infty_t(H^m \cap L^p)}$ such that*

$$\|\nabla \rho^\pm\|_{L^\infty([0,t]; L^\infty_x(B_R^c))} \leq \|\nabla \rho_0^\pm\|_{L^\infty_x} + \epsilon, \tag{1.7}$$

where $B_R \triangleq \{x \in \mathbb{R}^2; |x| < R\}$ and B_R^c is its complement.

Remark 1.3. Under the conditions of Proposition 1.2(3), the solutions $\theta^{\pm}$ are locally well-posed, and the related quantities $\rho^{\pm}$ classically solve the system (1.1). Compared with the local result in [15], the initial data here are of different type, and this result may have more advantage in guaranteeing the extension from the local solution to the global solution (which can be convinced in some dissipative cases below). We also note that these assumptions can do admit a class of initial data; for instance, the data of the form $\theta_0^{\pm}(x) = f^{\pm}(x_1)g^{\pm}(x_2)$ such that $f^{\pm} \in H^m(\mathbb{R}) \cap L^1(\mathbb{R})$, $g^{\pm} \in H^m(\mathbb{R}) \cap L^p(\mathbb{R})$ $(m > 4, p \in]1, 2[)$.

Next we shall consider the dissipative cases to show some global results. From Theorem 1.1, in order to show the global well-posedness of the system (1.2), one should prove that for every $T \in]0, T^*[$, there is an upper bound of the quantity $\int_0^T \|(\theta^+, \theta^-)(t)\|_{L^\infty} dt$, or equivalently, $\int_0^T \|(\partial_1\rho^+, \partial_1\rho^-)(t)\|_{L^\infty} dt$. It seems very hard to obtain such a bound directly from the system (1.2), thus we shall turn to take advantage of the system (1.1) to derive the desired bound.

Observe that for $\theta_0^- \equiv 0$, from the uniqueness issue in Theorem 1.1 and the fact that zero solution is a solution to the equation of θ^-, we have that $\theta^-(t) = \rho^-(t) \equiv 0$ for all $t \in [0, T^*[$. By setting $\rho \triangleq \rho^+ - \rho^- = \rho^+$, we obtain

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \kappa |D|^\alpha \rho = 0, & \alpha \in]0, 2], \\ u = (\mathcal{R}_1^2 \mathcal{R}_2^2 \rho, 0), & \rho|_{t=0} = \rho_0. \end{cases} \tag{1.8}$$

Equation (1.8) is reminiscent of the surface quasi-geostrophic (SQG) equation

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \kappa |D|^\alpha \rho = 0, & \alpha \in]0, 2], \\ u = (-\mathcal{R}_2 \rho, \mathcal{R}_1 \rho), & \rho|_{t=0} = \rho_0, \end{cases} \tag{1.9}$$

which arises from the geostrophic study of strongly rotating fluids (see [7]) and has been intensely studied in recent years (cf. [4, 6, 9–11, 23, 21, 26] and references therein). For the dissipative (i.e. $\kappa > 0$) SQG equation, so far we only know that the cases of $\alpha \in [1, 2]$ are global well-posed in various functional spaces, and whether the supercritical cases of $\alpha \in]0, 1[$ finite time blowup or not remains an outstanding open problem. We here briefly recall some remarkable results. For the subcritical cases (i.e. $\alpha \in]1, 2]$), it has been known that the SQG equation has global strong solutions since the works [26, 9]. For the subtle critical case (i.e. $\alpha = 1$), the issue of global regularity was independently settled by [23, 4] almost at the same time. Kiselev *et al.* in [23] proved the global well-posedness with the periodic smooth data by developing a new method called the "nonlocal maximum principle", whose idea is to show that a family of suitable moduli of continuity are preserved by the evolution. From a totally different direction, Caffarelli and Vasseur in [4] established the global regularity of weak solutions by deeply exploiting the De Giorgi's iteration method. We also refer to [22, 8] for another two delicate and still quite different proofs.

Compared to the SQG equation, the main disadvantage of the simplified model (1.8) is that the velocity field u in (1.8) is not divergence-free. This deficiency often leads to some difficulty in the application of the existing methods (like Caffarelli–Vasseur's method), thus despite its possible advantage, we here do not expect to obtain better well-posed results than the SQG equation. Hence, we hope that the coupling system (1.1) in the cases of $\kappa > 0$ (setting $\kappa = 1$ for brevity) and $\alpha \in [1, 2]$ can generate a unique global strong solution and there is an upper bound of the quantity $\int_0^T \|(\partial_1\rho^+, \partial_1\rho^-)(t)\|_{L^\infty} dt$ for every $T \in]0, T^*[$. We find that the method of nonlocal maximum principle originated in [23] is not sensitive to the divergence-free condition of the velocity field, and by applying this method, we indeed can prove the global results for the system (1.1) in the cases $\alpha \in [1, 2]$. More precisely, we have the following theorem.

Theorem 1.4. *Let* $\kappa = 1$, $\alpha \in [1, 2]$, (θ_0^+, θ_0^-) *be composed of non-negative real functions which belong to* $H^m(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \cap L^{\infty,1}_{x_2,x_1}(\mathbb{R}^2)$ *with* $m > 4$, $p \in]1, 2[$. *Assume* $\rho_0^\pm(x_1, x_2) = \int_{-\infty}^{x_1} \theta_0^\pm(\tilde{x}_1, x_2) d\tilde{x}_1$ *satisfy that for each* $k = 1, 2, 3$, $\partial_2^k \rho_0^\pm \in L^\infty_x(\mathbb{R}^2)$ *and* $\lim_{x_1\to-\infty} \partial_2^k \rho_0^\pm(x) = 0$ *for every* $x_2 \in \mathbb{R}$. *Then there exists a unique global solution*

$$(\theta^+, \theta^-) \in C([0, \infty[; H^m \cap L^p) \cap L^\infty([0, \infty[; L^{\infty,1}_{x_2,x_1})$$

to the system (1.2). *Moreover,* $(\rho^+, \rho^-) \in L^\infty([0, \infty[; W^{3,\infty}) \cap C([0, \infty[; W^{1,\infty})$ *solves the system* (1.1) *in the classical pointwise sense.*

Compared with the application of nonlocal maximum principle method to the SQG equation, there are another two noticeable different points: the first is that what we considered here is a coupling system instead of a single equation, and the second is that (ρ^+, ρ^-) does not have the spatial decay property that $\|(\nabla\rho^+, \nabla\rho^-)\|_{L^\infty([0,t]; L^\infty_x(B^c_R))} \to 0$ as $R \to \infty$ for each $t \in]0, T^*[$. Notice that in the works [1, 12, 25], this spatial decay property is needed when applying the method of [23] to the whole-space SQG-type equation. For the first point, we find that by proper modification of the scheme, the nonlocal maximum principle method can still be suited to the system (1.1). While for the second point, we observe that we indeed do not need such a strong decay property, and what we need is that the Lipschitz norm of (ρ^+, ρ^-) does not grow rapidly near spatial-infinity (cf. (5.18)), which just can be implied by Proposition 1.2(4).

In the proof of Theorem 1.4, statements (2) and (3) of Proposition 1.2 will also play an important role. Since in the program of the nonlocal maximum principle method, we need that (ρ^+, ρ^-) satisfies the system (1.1) in the pointwise sense and it also has the suitable smoothness property.

Remark 1.5. From the direction of showing the regularity of weak solutions to the system (1.1), so far there is no direct result implying the global regularity, due to that the velocity field $u = (\mathcal{R}_1^2\mathcal{R}_2^2(\rho^+ - \rho^-), 0)$ is neither divergence-free nor belonging to $L^\infty_{t,x}$. The main obstacle lies on the improvement from the bounded solution to the Hölder continuous solution; indeed, for the drift–diffusion equation $\partial_t\rho + u \cdot \nabla\rho + |D|^\alpha\rho = 0$ with $\alpha \in [1, 2[$ and general velocity field, the best result by now is given by Silvestre [27], which calls for $u \in L^\infty_{t,x}$ to ensure this improvement.

Remark 1.6. The procedure in showing the global part of Theorem 1.4 can be applied to the Groma–Balogh model with generalized dissipation, and we shall sketch it in the Appendix.

The paper is organized as follows. In Sec. 2, we present some preparatory results including the auxiliary lemmas and some facts about the modulus of continuity. We prove Theorem 1.1, Proposition 1.2 and Theorem 1.4 in Secs. 3–5 respectively.

Throughout this paper, C stands for a constant which may be different from line to line. For two quantities X and Y, we sometimes use $X \lesssim Y$ instead of $X \leq CY$, and we use $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. Denote by $\hat{f}$ the Fourier transform of f, i.e. $\hat{f}(\zeta) = \int_{\mathbb{R}^2} e^{ix\cdot\zeta} f(x) d\zeta$.

2. Preliminaries

In this preparatory section, we compile the definitions of functional spaces used in this paper, some auxiliary lemmas and some facts related to the modulus of continuity.

2.1. *Functional spaces and auxiliary lemmas*

For $q \in [1, \infty]$, $L^q_x = L^q_x(\mathbb{R}^2)$, $L^q_{x_i} = L^q_{x_i}(\mathbb{R})$ $(i = 1, 2)$ denote the usual Lebesgue spaces, and we sometimes abbreviate $L^q_x(\mathbb{R}^2)$ by L^q. For $(q, r) \in [1, \infty]^2$, denote $L^{q,r}_{x_2,x_1} = L^{q,r}_{x_2,x_1}(\mathbb{R}^2)$ the set of the tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ satisfying that

$$\|f\|_{L^{q,r}_{x_2,x_1}} \triangleq \|\|f(x)\|_{L^r_{x_1}}\|_{L^q_{x_2}} < \infty.$$

Similarly we can define the space $L^{q,r}_{x_1,x_2} = L^{q,r}_{x_1,x_2}(\mathbb{R}^2)$. Note that in general $L^{q,r}_{x_2,x_1} \neq L^{q,r}_{x_1,x_2}$.

For $s \in \mathbb{N}$, $q \in [1, \infty]$, $W^{s,q} = W^{s,q}(\mathbb{R}^2)$ denotes the usual Sobolev space:

$$W^{s,q} \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{W^{s,q}} \triangleq \sum_{|\beta| \leq s} \|\partial^\beta_x f\|_{L^q} < \infty \right\}.$$

When $q = 2$, we also write $W^{s,2} = H^s = H^s(\mathbb{R}^2)$ with the norm $\|\cdot\|_{H^s}$. For general $s \in \mathbb{R}$, we can define the L^2-based Sobolev space of fractional power $H^s = H^s(\mathbb{R}^2)$ via the Fourier transform, i.e.

$$H^s \triangleq \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{H^s} \triangleq \|(1 + |\zeta|^s)\hat{f}(\zeta)\|_{L^2_\zeta} < \infty\}.$$

In order to define the Besov spaces, we need the following dyadic partition of unity. Let $\chi \in C^\infty(\mathbb{R}^2)$ be a radial function taking values in $[0, 1]$, supported on the ball $B_{4/3}$ and $\chi \equiv 1$ on B_1. Define $\varphi(\zeta) = \chi(\zeta/2) - \chi(\zeta)$ for all $\zeta \in \mathbb{R}^2$, then φ is a smooth radial function supported on the shell $\{\zeta \in \mathbb{R}^2 : 1 \leq |\zeta| \leq \frac{8}{3}\}$. Clearly,

$$\chi(\zeta) + \sum_{j\geq 0} \varphi(2^{-j}\zeta) = 1, \ \ \forall\, \zeta \in \mathbb{R}^2; \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\zeta) = 1, \ \ \forall\, \zeta \neq 0.$$

Then for all $f \in \mathcal{S}'(\mathbb{R}^2)$, define the following nonhomogeneous Littlewood–Paley operators

$$\Delta_{-1} f \triangleq \chi(D) f; \qquad \Delta_j f \triangleq \varphi(2^{-j} D) f, \ \ \forall j \in \mathbb{N},$$

and thus $\sum_{j\geq -1} \Delta_j f = f$. While for all $f \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}(\mathbb{R}^2)$ with $\mathcal{S}'/\mathcal{P}$ the quotient space of tempered distributions up to polynomials, define the homogeneous Littlewood–Paley operator

$$\dot{\Delta}_j f \triangleq \varphi(2^{-j} D) f, \quad \forall j \in \mathbb{Z},$$

and thus $\sum_{j\in\mathbb{Z}} \dot{\Delta}_j f = f$.

Now for $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, we define the nonhomogeneous Besov space as follows:

$$B^s_{p,r} \triangleq \{f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{B^s_{p,r}} \triangleq \|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j\geq -1}\|_{\ell^r} < \infty\}.$$

We point out that for all $s \in \mathbb{R}$, $B^s_{2,2} = H^s$. We also introduce the space–time Besov space $L^\sigma([0, T], B^s_{p,r})$, abbreviated by $L^\sigma_T B^s_{p,r}$, which is the set of tempered distributions f satisfying

$$\|f\|_{L^\sigma_T B^s_{p,r}} \triangleq \|\|\{2^{qs} \|\Delta_q f\|_{L^p_x}\}_{q\geq -1}\|_{\ell^r}\|_{L^\sigma_T} < \infty.$$

Bernstein's inequality is fundamental in the analysis involving frequency localized functions.

Lemma 2.1. *Let $1 \leq p \leq q \leq \infty$, $0 < a < b < \infty$, $k \geq 0$, $\lambda > 0$ and $f \in L^p(\mathbb{R}^2)$. Then,*

$$\text{if } \operatorname{supp} \hat{f} \subset \{\zeta : |\zeta| \leq \lambda b\}, \Rightarrow \||D|^k f\|_{L^q(\mathbb{R}^2)} \lesssim \lambda^{k+2(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^2)};$$

and

$$\text{if } \operatorname{supp} \hat{f} \subset \{\zeta : a\lambda \leq |\zeta| \leq b\lambda\}, \Rightarrow \||D|^k f\|_{L^p(\mathbb{R}^2)} \approx \lambda^k \|f\|_{L^p(\mathbb{R}^2)}.$$

We shall use the following lemma in the proof of the local part.

Lemma 2.2. *Let f be a smooth real function on $\mathbb{R}^2$ and u be a smooth vector field of $\mathbb{R}^2$. Then the following assertions hold.*

(1) *For every $s \geq 0$, we have*

$$\sum_{j\geq 0} 2^{2js} \left| \int_{\mathbb{R}^2} \Delta_j (\nabla \cdot (uf))(x) \Delta_j f(x) dx \right| \lesssim \|\nabla u\|_{L^\infty} \|f\|^2_{H^s} + \|f\|_{L^\infty} \|\nabla u\|_{H^s} \|f\|_{H^s}. \tag{2.1}$$

(2) *If $u = (\mathcal{R}_1 \mathcal{R}_2^2 |D|^{-1} g, 0)$, we have that for every $s > 1$ and $p \in]1, 2[$,*

$$\sum_{j\in\mathbb{N}} \|\Delta_j (\nabla \cdot (uf))\|_{L^p} \lesssim \|g\|_{H^s \cap L^p} \|f\|_{H^s}. \tag{2.2}$$

Proof. (1) The proof of (2.1) essentially follows from the proof of [24, Lemma 2.4] with proper modification, and here we omit the details.
(2) By Bony's decomposition, we get

$$\begin{aligned}\sum_{j\in\mathbb{N}}\|\Delta_j(\nabla\cdot(uf))\|_{L^p} = &\sum_{j\in\mathbb{N};|k-j|\le 4}\|\Delta_j(\nabla\cdot(S_{k-1}u\Delta_k f))\|_{L^p}\\ &+\sum_{j\in\mathbb{N}}\|\Delta_j(\nabla\cdot(\Delta_{-1}uS_1 f))\|_{L^p}\\ &+\sum_{j\in\mathbb{N};k\ge j-4,k\in\mathbb{N}}\|\Delta_j(\nabla\cdot(\Delta_k uS_{k+2}f))\|_{L^p}\\ \triangleq{}& A_1+A_2+A_3,\end{aligned}$$

where $S_k=\sum_{-1\le k'\le k-1}\Delta_{k'}$ for every $k\in\mathbb{N}$. For A_1, from Bernstein's inequality, Hölder's inequality and Hardy–Littlewood–Sobolev's inequality, we obtain

$$\begin{aligned}A_1 &\lesssim \sum_{j\in\mathbb{N};|k-j|\le 4} 2^j\|S_{k-1}u\|_{L^{2p/(2-p)}}\|\Delta_k f\|_{L^2}\\ &\lesssim \|g\|_{L^p}\sum_{j\in\mathbb{N}}2^{j(1-s)}2^{ks}\|\Delta_k f\|_{L^2}\lesssim\|g\|_{L^p}\|f\|_{H^s}.\end{aligned}$$

For A_2, since $\Delta_j(\Delta_{-1}uS_1f)=0$ for $j\ge 3$, we get

$$A_2\lesssim\sum_{0\le j\le 2}\|\Delta_j(\Delta_{-1}uS_1f)\|_{L^p}\lesssim\|g\|_{L^p}\|f\|_{L^2}.$$

For A_3, from Bernstein's inequality, Hölder's inequality and Young's inequality, we have

$$\begin{aligned}A_3 &\lesssim\sum_{j\in\mathbb{N}}\sum_{k\ge j-4,k\in\mathbb{N}}2^j\|\Delta_k u\|_{L^{2p/(2-p)}}\|S_{k+2}f\|_{L^2}\\ &\lesssim\|f\|_{L^2}\sum_{j\in\mathbb{N}}\sum_{k\ge j-4,k\in\mathbb{N}}2^j2^{-k}2^{k\frac{2(p-1)}{p}}\|\Delta_k g\|_{L^2}\\ &\lesssim\|f\|_{L^2}\sum_{k\in\mathbb{N}}2^{k\frac{2(p-1)}{p}}\|\Delta_k g\|_{L^2}\lesssim\|f\|_{L^2}\|g\|_{H^s}.\end{aligned}$$

Gathering the upper estimates leads to (2.2). □

The following logarithmic inequality will be used to show a refined blowup criterion.

Lemma 2.3. *Let $f\in H^m(\mathbb{R}^2)$ with $m>1$. Suppose that $S\in C^\infty(\mathbb{R}^2\backslash\{0\})$ is a zeroth-order homogeneous function and $\mathcal{T}$ is the operator on $\mathbb{R}^2$ with S as the symbol. Then we have*

$$\|\mathcal{T}f\|_{L^\infty(\mathbb{R}^2)}\le C+C\|f\|_{L^\infty(\mathbb{R}^2)}\log(e+\|f\|_{H^m(\mathbb{R}^2)}).$$

Proof. By a high–low frequency decomposition, and from Bernstein's inequality and Calderǿn–Zygmund's theorem, we have that for some $J \in \mathbb{N}$ determined later,

$$\begin{aligned}\|\mathcal{T}f\|_{L^\infty} &\le \left(\sum_{j\le -J} + \sum_{-J<j<J} + \sum_{j\ge J}\right)(\|\dot{\Delta}_j\mathcal{T}f\|_{L^\infty}) \\ &\lesssim \sum_{j\le -J} 2^j\|\dot{\Delta}_j\mathcal{T}f\|_{L^2} + \sum_{-J<j<J}\|\dot{\Delta}_j\mathcal{T}f\|_{L^\infty} + \sum_{j\ge J} 2^{j(1-m)}2^{jm}\|\dot{\Delta}_j\mathcal{T}f\|_{L^2} \\ &\lesssim 2^{-J}\|f\|_{L^2} + J\|f\|_{L^\infty} + 2^{-J(m-1)}\|f\|_{H^m} \\ &\le C2^{-Ja}\|f\|_{H^m} + CJ\|f\|_{L^\infty},\end{aligned}$$

where $a \triangleq \min\{1, m-1\}$. Thus in order to make $2^{-Ja}\|f\|_{H^m} \approx 1$, we can choose

$$J \triangleq \left[\log\frac{(e+\|f\|_{H^m})}{a}\right] + 1$$

with $[x]$ denoting the integer part of a real number x, and the desired estimate follows. □

We have the following integral formula of the operator $|D|^\alpha$ $(\alpha \in]0,2[)$ (cf. [13, Theorem 1]).

Lemma 2.4. *Let $\alpha \in]0,2[$, $r > 0$ and $f \in C_b^2(\mathbb{R}^2)$ $(= C^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2))$. Then for every $x \in \mathbb{R}^2$,*

$$|D|^\alpha f(x) = -c_\alpha\left(\int_{B_r}\frac{f(x+y)-f(x)-y\cdot\nabla f(x)}{|y|^{2+\alpha}}dy + \int_{B_r^c}\frac{f(x+y)-f(x)}{|y|^{2+\alpha}}dy\right),$$

where $c_\alpha = \frac{\alpha\Gamma(1+\alpha/2)}{2\pi^{1+\alpha}\Gamma(1-\alpha/2)}$ and Γ is the usual Euler's function.

The following positivity lemma is also useful (cf. [24, Lemma 2.7]).

Lemma 2.5. *Let $\kappa \ge 0$, $\alpha \in]0,2]$, $p \in [1,\infty[$ and $T > 0$. Denote $U_T \triangleq]0,T] \times \mathbb{R}^2$, and $C_{t,x}^{i,j}(U_T) \triangleq C_t^i(]0,T]; C_x^j(\mathbb{R}^2))$, $i,j \in \mathbb{N}$. Assume that $u \in C_{t,x}^{0,1}(U_T)$ is a real vector field of $\mathbb{R}^2$, $\theta_0 \in C(\mathbb{R}^2)$ is a real scalar and*

$$\theta \in C_{t,x}^{1,0}(U_T) \cap C_{t,x}^{0,2}(U_T) \cap C_{t,x}^0(\overline{U}_T) \cap L^p(U_T)$$

is a real scalar function satisfying the following pointwise inequality:

$$\begin{cases}\partial_t\theta + \nabla\cdot(u\theta) \ge -\kappa|D|^\alpha\theta, & (t,x) \in U_T,\\ \theta(0,x) = \theta_0(x), & x \in \mathbb{R}^2.\end{cases}$$

We also suppose that there is a positive constant $C < \infty$ such that

$$\sup_{\overline{U}_T}|\theta| + \sup_{U_T}(|\partial_t\theta| + |\nabla\theta| + |\nabla^2\theta|) + \sup_{U_T}|\mathrm{div}\, u| \le C.$$

Then if $\theta_0 \ge 0$, we have $\theta \ge 0$ in $\overline{U}_T$.

2.2. *Modulus of continuity*

We begin with introducing some terminology.

Definition 2.6. A function $\omega : [0,\infty[\mapsto [0,\infty[$ is called a modulus of continuity (MOC) if ω is continuous on $[0,\infty[$, increasing, concave, and piecewise C^2 with one-sided derivatives defined at each point in $[0,\infty[$ (maybe infinite at $\xi = 0$). We call that a function $f : \mathbb{R}^2 \to \mathbb{R}$ has (or obeys) the MOC ω if $|f(x)-f(y)| \leq \omega(|x-y|)$ for every $x, y \in \mathbb{R}^2$. We also say that f strictly obeys the MOC if the above inequality is strict for $x \neq y$.

We first have the lemma concerning the action of the zeroth-order pseudo-differential operator like $\mathcal{R}_1^2\mathcal{R}_2^2$ on the function obeying MOC.

Lemma 2.7. *Let $f, g : \mathbb{R}^2 \mapsto \mathbb{R}$ obey the MOC ω and the vector field $u = (\mathcal{R}_1^2\mathcal{R}_2^2(f-g), 0)$. Then the following assertions hold.*

(1) *u obeys the following MOC*

$$\Omega(\xi) = A_1\omega(\xi) + A_2\left(\int_0^\xi \frac{\omega(\eta)}{\eta}d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}d\eta\right), \tag{2.3}$$

where A_1 and A_2 are positive absolute constants.

(2) *If f do not strictly have the MOC ω and there exist two separate points $x, y \in \mathbb{R}^2$ satisfying $f(x) - f(y) = \omega(\xi)$ with $\xi = |x-y|$, then,*

$$|u\cdot\nabla f(x) - u\cdot\nabla f(y)| \leq \Omega(\xi)\omega'(\xi). \tag{2.4}$$

Proof. (1) Since $m(\zeta) = \frac{\zeta_1^2\zeta_2^2}{|\zeta|^4}$ is the symbol of $\mathcal{R}_1^2\mathcal{R}_2^2$ satisfying that it is a zeroth-order homogeneous function belonging to $C^\infty(\mathbb{R}^2\backslash\{0\})$, by virtue of [14, Lemma 4.13], and denoting $\mathbb{S}^1$ the unit circle, we know that there exist $H \in C^\infty(\mathbb{S}^1)$ with zero average and two positive constants $a_1 = \frac{1}{2\pi}\int_{\mathbb{S}^1} m(\zeta)d\zeta$, $a_2 > 0$ such that

$$\mathcal{R}_1^2\mathcal{R}_2^2(f-g) = a_1(f-g) + a_2\left(\text{p.v.}\,\frac{H(x')}{|x|^2}\right) * (f-g),$$

with $x' \in \mathbb{S}^1$. Based on this expression and the fact that $f-g$ has the MOC 2ω, the desired result follows from the deduction in [23] treating the corresponding point.

(2) We refer to [23] for the proof of this point. □

We also need a special action of the dissipation operator $|D|^\alpha$ on the function having MOC.

Lemma 2.8. *Let $\alpha \in]0,2]$, the real scalar function $f \in C_b^2(\mathbb{R}^2)$ obey the MOC ω but do not strictly obey it. Assume that there are two separate points $x, y \in \mathbb{R}^2$ such* *that* $f(x) - f(y) = \omega(\xi)$ *with* $\xi = |x-y|$. *Then we have*

$$[-|D|^\alpha f](x) - [-|D|^\alpha f](y) \leq \Psi_\alpha(\xi),$$

where

$$\Psi_\alpha(\xi) = \begin{cases} B_\alpha \displaystyle\int_0^{\xi/2} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \\ \qquad + B_\alpha \displaystyle\int_{\xi/2}^{\infty} \frac{\omega(\xi+2\eta) - \omega(2\eta-\xi) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta, \quad \alpha \in]0,2[, \\ 2\omega''(\xi), \quad \alpha = 2, \end{cases} \tag{2.5}$$

and $B_\alpha > 0$.

The proof is essentially contained in [23, 21], and we omit the details here.

At last, we state a simple lemma concerning the function having MOC.

Lemma 2.9. *Let* ω *be a MOC which in addition satisfies that*

$$\omega(0) = 0, \quad \omega'(0) < \infty, \quad \text{and} \quad \omega''(0+) = -\infty.$$

If the real scalar function $f \in C_b^2(\mathbb{R}^2)$ *obeys the MOC* ω, *then for every* $r \in]0, \infty[$, *we have*

$$\|\nabla f\|_{L^\infty(B_r)} < \omega'(0).$$

Proof. The proof is similar to that in [23]. Indeed, since $|\nabla f|$ is a continuous function on B_r, we suppose that it attains the maximum at $x \in \overline{B}_r$. Let $y = x + \xi\ell$ with $\xi > 0$ and $\ell = \frac{\nabla f(x)}{|\nabla f(x)|}$, and by definition we have $f(y) - f(x) \leq \omega(\xi)$. According to the Taylor formula, the left-hand side of the inequality is bounded from below by $|\nabla f(x)|\xi - \frac{1}{2}\|\nabla^2 f\|_{L^\infty}\xi^2$, while the right-hand side is bounded from above by $\omega'(0)\xi + g(\xi)\xi^2$ with $g(\xi) \to -\infty$ as $\xi \to 0+$. Thus $|\nabla f(x)| \leq \omega'(0) + \xi(g(\xi) + \frac{1}{2}\|\nabla^2 f\|_{L^\infty})$, and as ξ small enough the assertion follows. □

3. Proof of Theorem 1.1

Denote $\theta = \theta^+ - \theta^-$ and we rewrite the system (1.2) as follows:

$$\begin{cases} \partial_t \theta^\pm + \partial_1(u_1^\pm \theta^\pm) + \kappa |D|^\alpha \theta^\pm = 0, & \alpha \in]0,2], \ \kappa \geq 0, \\ u_1^\pm = \pm \mathcal{R}_1 \mathcal{R}_2^2 |D|^{-1}\theta, & \theta^\pm|_{t=0} = \theta_0^\pm, \end{cases} \tag{3.1}$$

where the equation of $\theta^\pm$ should be understood as two equations of θ^+ and θ^- respectively.

3.1. *A priori estimates*

In this section, we, *a priori*, suppose that $\theta^\pm \in C(\mathbb{R}^+; H^m \cap L^p)$ and $\theta \in C(\mathbb{R}^+; H^m \cap L^p)$ with $m > 2$, $p \in]1,2[$ are independent functions and they satisfy (3.1).

We first obtain the L^q estimate of $\theta^{\pm}$ with $q \in [p, \infty[$. Let χ be the cut-off function introduced in Sec. 2.1 and $\chi_R(\cdot) \triangleq \chi(\frac{\cdot}{R})$ for $R > 0$. Multiplying the equations of $\theta^{\pm}$ by $|\theta^{\pm}|^{q-2}\theta^{\pm}\chi_R$ and integrating over the spatial variable, we get

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\theta^{\pm}|^q(t,x)\chi_R(x)dx\right) &= -\int_{\mathbb{R}^2}\partial_1(u_1^{\pm}\theta^{\pm})(t,x)|\theta^{\pm}|^{q-2}\theta^{\pm}(t,x)\chi_R(x)dx \\ &\quad - \kappa\int_{\mathbb{R}^2}|D|^{\alpha}\theta^{\pm}(t,x)|\theta^{\pm}|^{q-2}\theta^{\pm}(t,x)\chi_R(x)dx \\ &\triangleq \mathrm{I}^{\pm}(t) + \mathrm{II}^{\pm}(t).\end{aligned}$$

For $\mathrm{I}^{\pm}(t)$, from the integration by parts, we have

$$\begin{aligned}\mathrm{I}^{\pm}(t) &= -\int_{\mathbb{R}^2}(\partial_1 u_1^{\pm})|\theta^{\pm}|^q\chi_R(x)dx - \int_{\mathbb{R}^2}u_1^{\pm}\partial_1\theta^{\pm}|\theta^{\pm}|^{q-2}\theta^{\pm}\chi_R(x)dx \\ &= -\left(1-\frac{1}{q}\right)\int_{\mathbb{R}^2}(\partial_1 u_1^{\pm})|\theta^{\pm}|^q\chi_R(x)dx + \left(\frac{1}{q}\right)R^{-1}\int_{\mathbb{R}^2}u_1^{\pm}|\theta^{\pm}|^q\partial_1\chi\left(\frac{x}{R}\right)dx \\ &\le \left(1-\frac{1}{q}\right)\|\partial_1 u_1^{\pm}(t)\|_{L^\infty}\|\theta^{\pm}(t)\|_{L^q}^q + (qR)^{-1}\|\partial_1\chi\|_{L^\infty}\|u_1^{\pm}(t)\|_{L^\infty}\|\theta^{\pm}(t)\|_{L^q}^q \\ &\lesssim (\|\partial_1 u_1^{\pm}(t)\|_{L^\infty} + (qR)^{-1}\|\theta(t)\|_{L^p\cap H^m})\|\theta^{\pm}(t)\|_{L^q}^q,\end{aligned}$$

where in the last line we have used the following estimation

$$\begin{aligned}\|u_1^{\pm}(t)\|_{L_x^\infty} &\le \|\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\Delta_{-1}\theta(t)\|_{L_x^\infty} + \sum_{j\ge 0}\|\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\Delta_j\theta(t)\|_{L_x^\infty} \\ &\lesssim \||D|^{-1}\theta(t)\|_{L^{2p/(2-p)}} + \sum_{j\ge 0}2^{-jm}(2^{jm}\|\Delta_j\theta(t)\|_{L^2}) \\ &\lesssim \|\theta(t)\|_{L^p\cap H^m}. \end{aligned} \tag{3.2}$$

For $\mathrm{II}^{\pm}(t)$, by virtue of the following pointwise inequality (cf. [20, Proposition 3.3])

$$|f(x)|^{\beta}f(x)(|D|^{\alpha}f)(x) \ge \frac{1}{\beta+2}(|D|^{\alpha}|f|^{\beta+2})(x), \quad \forall\,\alpha\in[0,2],\ \beta\in[-1,\infty[,$$

we have

$$\begin{aligned}\mathrm{II}^{\pm}(t) &\le -\frac{\kappa}{q}\int_{\mathbb{R}^2}(|D|^{\alpha}|\theta^{\pm}|^q)(t,x)\chi_R(x)dx \\ &\le -\frac{\kappa}{q}\int_{\mathbb{R}^2}|\theta^{\pm}|^q(t,x)(|D|^{\alpha}\chi_R)(x)dx \\ &\le \frac{\kappa}{q}R^{-\alpha}\||D|^{\alpha}\chi\|_{L^\infty}\|\theta^{\pm}(t)\|_{L^q}^q.\end{aligned}$$

Integrating in time and gathering the upper results, and from the support property of χ, we get

$$\begin{aligned}\int_{|x|\le R}|\theta^\pm(t,x)|^q dx &\le \int_{\mathbb{R}^2}|\theta^\pm(t,x)|^q\chi_R(x)dx\\ &\le \|\theta_0^\pm\|_{L^q}^q + qC\int_0^t\|\partial_1 u_1^\pm(\tau)\|_{L^\infty}\|\theta^\pm(\tau)\|_{L^q}^q d\tau\\ &\quad + C\int_0^t(R^{-1}\|\theta^\pm(\tau)\|_{L^p\cap H^m}+\kappa R^{-\alpha})\|\theta^\pm(\tau)\|_{L^q}^q d\tau.\end{aligned}$$

According to the monotone convergence theorem and $\theta^\pm\in C(\mathbb{R}^+;H^m\cap L^p)$, and by passing R to ∞, we have that for every $t\in\mathbb{R}^+$ and $q\in[p,\infty[$,

$$\begin{aligned}\int_{\mathbb{R}^2}|\theta^\pm(t,x)|^q dx &\le \|\theta_0^\pm\|_{L^q}^q + qC\int_0^t\|\partial_1 u_1^\pm(\tau)\|_{L^\infty}\|\theta^\pm(\tau)\|_{L^q}^q d\tau\\ &\triangleq F^\pm(t)^q. \end{aligned}\tag{3.3}$$

Since

$$\begin{aligned}qF^\pm(t)^{q-1}\frac{d}{dt}F^\pm(t) = \frac{d}{dt}(F^\pm(t)^q) &= qC\|\partial_1 u_1^\pm(t)\|_{L^\infty}\|\theta^\pm(t)\|_{L^q}^q\\ &\le qC\|\partial_1 u_1^\pm(t)\|_{L^\infty}\|\theta^\pm(t)\|_{L^q}F^\pm(t)^{q-1},\end{aligned}$$

we have

$$F^\pm(t)\le F^\pm(0)+C\int_0^t\|\partial_1 u_1^\pm(\tau)\|_{L^\infty}\|\theta^\pm(\tau)\|_{L^q}d\tau.$$

This implies that for every $t\in\mathbb{R}^+$ and $q\in[p,\infty[$,

$$\|\theta^\pm(t)\|_{L^q}\le\|\theta_0^\pm\|_{L^q}+C\int_0^t\|\partial_1 u_1^\pm(\tau)\|_{L^\infty}\|\theta^\pm(\tau)\|_{L^q}d\tau, \tag{3.4}$$

where C is independent of q.

Next we consider the H^m estimate of $\theta^\pm$ with $m>2$. For every $j\in\mathbb{N}$, we apply the dyadic operator Δ_j to the equations of $\theta^\pm$ in (3.1) to get

$$\partial_t\Delta_j\theta^\pm+\kappa|D|^\alpha\Delta_j\theta^\pm = -\Delta_j\partial_1(u_1^\pm\theta^\pm).$$

Multiplying both sides of the upper equations by $\Delta_j\theta^\pm$ and integrating over the spatial variable, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta^\pm(t)\|_{L^2}^2+\kappa\||D|^{\frac{\alpha}{2}}\Delta_j\theta^\pm(t)\|_{L^2}^2=\int_{\mathbb{R}^2}\Delta_j\partial_1(u_1^\pm\theta^\pm)(t,x)\Delta_j\theta^\pm(t,x)dx.$$

Integrating on the time variable over $[0,t]$ leads to

$$\begin{aligned}&\|\Delta_j\theta^\pm(t)\|_{L^2}^2+2\kappa\||D|^{\frac{\alpha}{2}}\Delta_j\theta^\pm\|_{L_t^2L^2}^2\\ &\quad\le\|\Delta_j\theta_0^\pm\|_{L^2}^2+2\int_0^t\left|\int_{\mathbb{R}^2}\Delta_j\partial_1(u_1^\pm\theta^\pm)(\tau,x)\Delta_j\theta^\pm(\tau,x)dx\right|d\tau.\end{aligned}$$

Then, by multiplying both sides of the above equations by 2^{2jm} and summing over $j \in \mathbb{N}$, and from Lemma 2.2, we find

$$\begin{aligned}\sum_{j\in\mathbb{N}} 2^{2jm}\|\Delta_j\theta^{\pm}(t)\|_{L^2}^2 + 2\kappa\sum_{j\in\mathbb{N}} 2^{2jm}\||D|^{\frac{\alpha}{2}}\Delta_j\theta^{\pm}\|_{L_t^2L^2}^2 \\ \le \sum_{j\in\mathbb{N}} 2^{2jm}\|\Delta_j\theta_0^{\pm}\|_{L^2}^2 \\ + C\int_0^t (\|\nabla u_1^{\pm}\|_{L^\infty}\|\theta^{\pm}\|_{H^m}^2 + \|\theta^{\pm}\|_{L^\infty}\|\nabla u_1^{\pm}\|_{H^m}\|\theta^{\pm}\|_{H^m})(\tau)d\tau. \quad (3.5)\end{aligned}$$

For $j = -1$, from (3.3) and Bernstein's inequality, we directly have

$$\|\Delta_{-1}\theta^{\pm}(t)\|_{L^2}^2 \le \|\theta^{\pm}(t)\|_{L^2}^2 \le \|\theta_0^{\pm}\|_{L^2}^2 + C\int_0^t \|\partial_1 u_1^{\pm}(\tau)\|_{L^\infty}\|\theta^{\pm}(\tau)\|_{L^2}^2 d\tau.$$

Gathering the upper two estimates, and from $\|\cdot\|_{B_{2,2}^m} \approx \|\cdot\|_{H^m}$, we get

$$\begin{aligned}\|\theta^{\pm}(t)\|_{H^m}^2 \le C_0\|\theta_0^{\pm}\|_{H^m}^2 \\ + C\int_0^t (\|\nabla u_1^{\pm}\|_{L^\infty}\|\theta^{\pm}\|_{H^m}^2 + \|\theta^{\pm}\|_{L^\infty}\|\nabla u_1^{\pm}\|_{H^m}\|\theta^{\pm}\|_{H^m})(\tau)d\tau.\end{aligned}$$

In a similar way as obtaining (3.4) from (3.3), we see that

$$\begin{aligned}\|\theta^{\pm}(t)\|_{H^m} \le C_0\|\theta_0^{\pm}\|_{H^m} + C\int_0^t (\|\nabla u_1^{\pm}\|_{L^\infty}\|\theta^{\pm}\|_{H^m} \\ + \|\theta^{\pm}\|_{L^\infty}\|\nabla u_1^{\pm}\|_{H^m})(\tau)d\tau, \quad (3.6)\end{aligned}$$

with $C_0 \ge 1$. From the Sobolev embedding and Calderón–Zygmund theorem, we further deduce

$$\|\theta^{\pm}(t)\|_{H^m} \le C_0\|\theta_0^{\pm}\|_{H^m} + C\int_0^t \|\theta(\tau)\|_{H^m}\|\theta^{\pm}(\tau)\|_{H^m} d\tau. \quad (3.7)$$

Gronwall's inequality ensures that

$$\|\theta^{\pm}(t)\|_{H^m} \le C_0\|\theta_0^{\pm}\|_{H^m} e^{C\int_0^t \|\theta(\tau)\|_{H^m} d\tau}.$$

Now, by combining (3.4) with (3.6), we have

$$\begin{aligned}\|\theta^{\pm}(t)\|_{H^m\cap L^p} \le C_0\|\theta_0^{\pm}\|_{H^m\cap L^p} + C\int_0^t (\|\nabla u_1^{\pm}\|_{L^\infty}\|\theta^{\pm}\|_{H^m\cap L^p} \\ + \|\theta^{\pm}\|_{L^\infty}\|\nabla u_1^{\pm}\|_{H^m})(\tau)d\tau. \quad (3.8)\end{aligned}$$

This estimate also yields

$$\begin{aligned}\|\theta^{+}(t)\|_{H^m\cap L^p} + \|\theta^{-}(t)\|_{H^m\cap L^p} \le C_0(\|\theta_0^{+}\|_{H^m\cap L^p} + \|\theta_0^{-}\|_{H^m\cap L^p}) \\ + C_1\int_0^t \|\theta(\tau)\|_{H^m}(\|\theta^{+}\|_{H^m\cap L^p} \\ + \|\theta^{-}\|_{H^m\cap L^p})(\tau)d\tau. \quad (3.9)\end{aligned}$$

Hence, for every $T > 0$ satisfying that

$$T \leq \frac{1}{4C_0C_1(\|\theta_0^+\|_{H^m\cap L^p} + \|\theta_0^-\|_{H^m\cap L^p})}, \tag{3.10}$$

and θ satisfying that

$$\|\theta\|_{L_T^\infty H^m} \leq 2C_0(\|\theta_0^+\|_{H^m\cap L^p} + \|\theta_0^-\|_{H^m\cap L^p}), \tag{3.11}$$

we have

$$\|\theta^+\|_{L_T^\infty(H^m\cap L^p)} + \|\theta^-\|_{L_T^\infty(H^m\cap L^p)} \leq 2C_0(\|\theta_0^+\|_{H^m\cap L^p} + \|\theta_0^-\|_{H^m\cap L^p}). \tag{3.12}$$

From (3.4) and (3.5), we moreover obtain

$$\kappa\|\theta^+\|_{L_T^2 H^{m+\frac{\alpha}{2}}} + \kappa\|\theta^-\|_{L_T^2 H^{m+\frac{\alpha}{2}}} \lesssim_{T,\|\theta_0^\pm\|_{H^m\cap L^p}} 1.$$

3.2. *Uniqueness*

Assume that $(\theta^{1,+},\theta^{1,-})$ and $(\theta^{2,+},\theta^{2,-})$ belonging to $C([0,T];H^m\cap L^p)$ ($m>2$, $p\in]1,2[$) are two solutions to the system (1.2) with initial data $(\theta_0^{1,+},\theta_0^{1,-})$ and $(\theta_0^{2,+},\theta_0^{2,-})$ respectively. Denote $\delta\theta^\pm \triangleq \theta^{1,\pm} - \theta^{2,\pm}$, $\delta\theta_0^\pm \triangleq \theta_0^{1,\pm} - \theta_0^{2,\pm}$, $\theta^i \triangleq \theta^{i,+} - \theta^{i,-}$, $u_1^{i,\pm} \triangleq \pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\theta^i$ for $i = 1,2$ and $\delta\theta \triangleq \theta^1 - \theta^2 = \delta\theta^+ - \delta\theta^-$, $\delta u_1^\pm \triangleq u_1^{1,\pm} - u_2^{2,\pm} = \pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\delta\theta$. Then we write the equations of $\delta\theta^\pm$ as follows:

$$\partial_t\delta\theta^\pm + \partial_1(u_1^{2,\pm}\delta\theta^\pm) + \kappa|D|^\alpha\delta\theta^\pm = -\partial_1(\delta u_1^\pm\theta^{1,\pm})$$
$$\delta\theta^\pm|_{t=0} = \delta\theta_0^\pm.$$

For $R > 0$, let χ_R be the cut-off function introduced in Sec. 3.1; then we multiply both sides of the upper equations by $|\delta\theta^\pm|^{p-2}\delta\theta^\pm\chi_R$ and integrate on the spatial variable to obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\delta\theta^\pm(t,x)|^p\chi_R(x)dx\right) &= -\int_{\mathbb{R}^2}\partial_1(u_1^{2,\pm}\delta\theta^\pm)(t,x)|\delta\theta^\pm|^{p-2}\delta\theta^\pm(t,x)\chi_R(x)dx \\ &\quad - \kappa\int_{\mathbb{R}^2}|D|^\alpha\delta\theta^\pm(t,x)|\delta\theta^\pm|^{p-2}\delta\theta^\pm(t,x)\chi_R(x)dx \\ &\quad - \int_{\mathbb{R}^2}\partial_1(\delta u_1^\pm\theta^{1,\pm})(t,x)|\delta\theta^\pm|^{p-2}\delta\theta^\pm(t,x)\chi_R(x)dx \\ &\triangleq A_1^\pm(t) + A_2^\pm(t) + A_3^\pm(t).\end{aligned}$$

Similarly as estimating $\mathrm{I}^\pm(t)$ and $\mathrm{II}^\pm(t)$ in Sec. 3.1, we get

$$\begin{aligned}A_1^\pm(t) &\leq C(\|\partial_1 u_1^{2,\pm}(t)\|_{L^\infty} + R^{-1}\|u_1^{2,\pm}(t)\|_{L^\infty})\|\delta\theta^\pm(t)\|_{L^p}^p \\ &\leq C(\|\theta^2(t)\|_{H^m} + R^{-1}\|\theta^2(t)\|_{H^m\cap L^p})\|\delta\theta^\pm(t)\|_{L^p}^p,\end{aligned}$$

and

$$A_2^\pm(t) \leq C\kappa R^{-\alpha}\|\delta\theta^\pm(t)\|_{L^p}^p.$$

For $A_3^\pm(t)$, by virtue of the Hölder inequality, Calderón–Zygmund theorem and Hardy–Littlewood–Sobolev inequality, we find

$$\begin{aligned} A_3^\pm(t) &= -\int_{\mathbb{R}^2}((\partial_1\delta u_1^\pm)\theta^{1,\pm} + \delta u^\pm\partial_1\theta^{1,\pm})(t,x)|\delta\theta^\pm|^{p-2}\delta\theta^\pm(t,x)\chi_R(x)dx \\ &\le (\|\partial_1\delta u_1^\pm\|_{L^p}\|\theta^{1,\pm}\|_{L^\infty} + \|\delta u_1^\pm\|_{L^{\frac{2p}{2-p}}}\|\partial_1\theta^{1,\pm}\|_{L^2})\|\delta\theta^\pm\|_{L^p}^{p-1}\|\chi_R\|_{L^\infty} \\ &\le C\|\theta^{1,\pm}(t)\|_{H^m}\|\delta\theta(t)\|_{L^p}\|\delta\theta^\pm(t)\|_{L^p}^{p-1}. \end{aligned}$$

Collecting the above estimates, and in a similar way as obtaining (3.3), we infer that

$$\begin{aligned} \|\delta\theta^\pm(t)\|_{L^p}^p &\le \|\delta\theta_0^\pm\|_{L^p}^p \\ &\quad + pC\int_0^t(\|\theta^2\|_{H^m}\|\delta\theta^\pm\|_{L^p} + \|\theta^{1,\pm}\|_{H^m}\|\delta\theta\|_{L^p})(\tau)\|\delta\theta^\pm(\tau)\|_{L^p}^{p-1}d\tau. \end{aligned}$$

This estimate implies that

$$\begin{aligned} \|\delta\theta^\pm(t)\|_{L^p} &\le \|\delta\theta_0^\pm\|_{L^p} \\ &\quad + C\int_0^t(\|\theta^2(\tau)\|_{H^m}\|\delta\theta^\pm(\tau)\|_{L^p} + \|\theta^{1,\pm}(\tau)\|_{H^m}\|\delta\theta(\tau)\|_{L^p})d\tau. \end{aligned} \quad (3.13)$$

Hence, summing over the upper estimates of $\delta\theta^+$ and $\delta\theta^-$, we have

$$\begin{aligned} &\|\delta\theta^+(t)\|_{L^p} + \|\delta\theta^-(t)\|_{L^p} \\ &\quad \le \|\delta\theta_0^+\|_{L^p} + \|\delta\theta_0^-\|_{L^p} + \int_0^t C(\tau)(\|\delta\theta^+(\tau)\|_{L^p} + \|\delta\theta^-(\tau)\|_{L^p})d\tau, \end{aligned}$$

where $C(\tau) = C\|\theta^2(\tau)\|_{H^m} + C\|\theta^{1,+}(\tau)\|_{H^m} + C\|\theta^{1,-}(\tau)\|_{H^m}$. Gronwall's inequality yields that for every $t \in [0,T]$,

$$\|\delta\theta^+(t)\|_{L^p} + \|\delta\theta^-(t)\|_{L^p} \le (\|\delta\theta_0^+\|_{L^p} + \|\delta\theta_0^-\|_{L^p})e^{T\|C(t)\|_{L_T^\infty}},$$

and this clearly guarantees the uniqueness.

3.3. *Existence*

We construct the sequences of approximate solutions $\{(\theta^{n,+},\theta^{n,-})\}_{n\in\mathbb{N}}$ as follows. Denote $\theta^{0,\pm}(t,x) = e^{-\kappa t|D|^\alpha}\theta_0^\pm(x)$, and for each $n \in \mathbb{N}$, $(\theta^{n+1,+},\theta^{n+1,-})$ solves the following system

$$\begin{cases} \partial_t\theta^{n+1,\pm} + \partial_1(u_1^{n,\pm}\theta^{n+1,\pm}) + \kappa|D|^\alpha\theta^{n+1,\pm} = 0, \\ u_1^{n,\pm} = \pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^{n,+} - \theta^{n,-}), \\ \theta^{n+1,\pm}|_{t=0} = \theta_0^\pm. \end{cases} \quad (3.14)$$

Since $\theta_0^\pm \in H^m \cap L^p$ with $m > 2$, $p \in]1,2[$, we know that $\theta^{0,\pm} \in C(\mathbb{R}^+; H^m \cap L^p)$. Now assuming that for each $n \in \mathbb{N}$, $\theta^{n,\pm} \in C(\mathbb{R}^+; H^m \cap L^p)$, we further show that $\theta^{n+1,\pm} \in C(\mathbb{R}^+; H^m \cap L^p)$. By a classical process, it is not hard to show that $\theta^{n+1,\pm} \in C(\mathbb{R}^+; H^m)$. To prove that $\theta^{n+1,\pm} \in C(\mathbb{R}^+; L^p)$, we use the Duhamel's formula

$$\theta^{n+1,\pm}(t,x) = e^{-\kappa t|D|^\alpha}\theta_0^\pm(x) + \int_0^t e^{-\kappa(t-\tau)|D|^\alpha} f^{n+1,\pm}(\tau,x)d\tau$$

with $f^{n+1,\pm} = \partial_1(u_1^{n,\pm}\theta^{n+1,\pm})$. By a direct computation, we deduce that for every $t \in [0,\infty[$,

$$\begin{aligned}
\|f^{n+1,\pm}\|_{L_t^\infty L^p} &\le \|\partial_1 u^{n,\pm}\theta^{n+1,\pm}\|_{L_t^\infty L^p} + \|u^{n,\pm}\partial_1\theta^{n+1,\pm}\|_{L_t^\infty L^p} \\
&\le \|\partial_1 u^{n,\pm}\|_{L_t^\infty L^p}\|\theta^{n+1,\pm}\|_{L_t^\infty L^\infty} + \|u^{n,\pm}\|_{L_t^\infty L^{\frac{2p}{2-p}}}\|\partial_1\theta^{n+1,\pm}\|_{L_t^\infty L^2} \\
&\lesssim (\|\theta^{n,+}\|_{L_t^\infty L^p} + \|\theta^{n,-}\|_{L_t^\infty L^p})\|\theta^{n+1,\pm}\|_{L_t^\infty H^m}, \qquad (3.15)
\end{aligned}$$

thus

$$\|\theta^{n+1,\pm}(t)\|_{L^p} \lesssim \|\theta_0^\pm\|_{L^p} + t(\|\theta^{n,+}\|_{L_t^\infty L^p} + \|\theta^{n,-}\|_{L_t^\infty L^p})\|\theta^{n+1,\pm}\|_{L_t^\infty H^m},$$

and this implies that $\theta^{n+1,\pm} \in L^\infty(\mathbb{R}^+; L^p)$. When $\kappa = 0$, in a similar manner we can show that $\theta^{n+1,\pm} \in C(\mathbb{R}^+; L^p)$. When $\kappa > 0$, for every $t, s \in [0,\infty[$, $t > s$, we have

$$\begin{aligned}
\theta^{n+1,\pm}(t,x) - \theta^{n+1,\pm}(s,x) &= (e^{-\kappa t|D|^\alpha} - e^{-\kappa s|D|^\alpha})\theta_0^\pm(x) \\
&\quad + \int_s^t e^{-\kappa(t-\tau)|D|^\alpha} f^{n+1,\pm}(\tau,x)d\tau \\
&\quad + \int_0^s (e^{-\kappa(t-\tau)|D|^\alpha} - e^{-\kappa(s-\tau)|D|^\alpha}) f^{n+1,\pm}(\tau,x)d\tau \\
&\triangleq B_1(t,s,x) + B_2(t,s,x) + B_3(t,s,x).
\end{aligned}$$

It is obvious that

$$\lim_{t\to s}(\|B_1(t,s,x)\|_{L_x^p} + \|B_2(t,s,x)\|_{L_x^p}) = 0.$$

For B_3, by Bernstein's inequality, Fubini's theorem, Young's inequality and the following estimate (cf. [19, Proposition 2.2]) that

$$\|e^{-h|D|^\alpha}\Delta_j f\|_{L^p} \le Ce^{-ch2^{j\alpha}}\|\Delta_j f\|_{L^p}, \quad \forall j \in \mathbb{N},\ p \in [1,\infty],\ h > 0,$$

we find that

$$\begin{aligned}
\|B_3(t,s,x)\|_{L_x^p} &\le \kappa\int_0^s\int_{s-\tau}^{t-\tau} \|e^{-\kappa\tau'|D|^\alpha}|D|^\alpha f^{n+1,\pm}(\tau,x)\|_{L_x^p}d\tau' d\tau \\
&\le \kappa\int_0^s\int_{s-\tau}^{t-\tau} \|e^{-\kappa\tau'|D|^\alpha}|D|^\alpha \Delta_{-1} f^{n+1,\pm}(\tau,x)\|_{L_x^p}d\tau' d\tau \\
&\quad + \kappa\int_0^s\int_{s-\tau}^{t-\tau}\left(\sum_{j\in\mathbb{N}} \|e^{-\kappa\tau'|D|^\alpha}|D|^\alpha \Delta_j f^{n+1,\pm}(\tau,x)\|_{L_x^p}\right)d\tau' d\tau
\end{aligned}$$

$$\lesssim \kappa s(t-s)\|f^{n+1,\pm}\|_{L^\infty_s L^p_x}$$
$$+\kappa(t-s)\sum_{j\in\mathbb{N}}\int_0^s e^{-c(s-\tau)2^{j\alpha}}2^{j\alpha}\|\Delta_j f^{n+1,\pm}(\tau)\|_{L^p_x}d\tau$$
$$\lesssim \kappa s(t-s)\|f^{n+1,\pm}\|_{L^\infty_s L^p_x}+\kappa(t-s)\sum_{j\in\mathbb{N}}\|\Delta_j f^{n+1,\pm}\|_{L^1_s L^p_x}.$$

Combining the upper estimate with (3.15) and (2.2) yields

$$\|B_3(t,s,x)\|_{L^p_x}\le C\kappa s(t-s)$$

with C depending only on $\|\theta^{n,\pm}\|_{L^\infty_s(H^m\cap L^p)}$ and $\|\theta^{n+1,\pm}\|_{L^\infty_s H^m}$. Hence $\theta^{n+1,\pm}\in C(\mathbb{R}^+;H^m\cap L^p)$. By induction, we have $\theta^{n,\pm}\in C(\mathbb{R}^+;H^m\cap L^p)$ for every $n\in\mathbb{N}$.

We also show that $\{(\theta^{n,+},\theta^{n,-})\}_{n\in\mathbb{N}}$ are n-uniformly bounded in $C([0,T];H^m\cap L^p)$ with T defined by (3.10), that is,

$$\|\theta^{n,+}\|_{L^\infty_T(H^m\cap L^p)}+\|\theta^{n,-}\|_{L^\infty_T(H^m\cap L^p)}\le 2C_0(\|\theta_0^+\|_{H^m\cap L^p}+\|\theta_0^-\|_{H^m\cap L^p}). \quad (3.16)$$

Indeed, from (3.10)–(3.12), it reduces to prove that (3.11) is satisfied for every $n\in\mathbb{N}$. This can be seen from the estimate that $\|\theta^{0,+}-\theta^{0,-}\|_{L^\infty_T H^m}\le\|\theta_0^+-\theta_0^-\|_{H^m}\le 2C_0(\|\theta_0^+\|_{H^m\cap L^p}+\|\theta_0^-\|_{H^m\cap L^p})$ and the induction method.

Next we show that $\{\theta^{n,\pm}\}_{n\in\mathbb{N}}$ are convergent in $C([0,T'];L^p)$ with some $T'\in]0,T]$ fixed later. For $n,k\in\mathbb{N}$, $n>k$, denote $\theta^{n,k,\pm}\triangleq\theta^{n+1,\pm}-\theta^{k+1,\pm}$, and the difference equations write

$$\begin{cases}\partial_t\theta^{n,k,\pm}+\partial_1(u_1^{n+1,\pm}\theta^{n,k,\pm})+\kappa|D|^\alpha\theta^{n,k,\pm}=-\partial_1(u_1^{n,k,\pm}\theta^{k+1,\pm}),\\ u_1^{n,k,\pm}\triangleq\pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^{n-1,k-1,+}-\theta^{n-1,k-1,-}),\\ \theta^{n,k,\pm}|_{t=0}=0.\end{cases}$$

In a similar way as obtaining (3.13), we get

$$\|\theta^{n,k,\pm}(t)\|_{L^p}\le C\int_0^t\|(\theta^{n+1,+}-\theta^{n+1,-})(\tau)\|_{H^m}\|\theta^{n,k,\pm}(\tau)\|_{L^p}d\tau$$
$$+C\int_0^t\|\theta^{k+1,\pm}(\tau)\|_{H^m}\|(\theta^{n-1,k-1,+}-\theta^{n-1,k-1,-})(\tau)\|_{L^p}d\tau.$$

Denoting $\Theta^{n,k}(t)\triangleq\|\theta^{n,k,+}(t)\|_{L^p}+\|\theta^{n,k,-}(t)\|_{L^p}$ for every $t\in[0,T]$, we further have

$$\Theta^{n,k}(t)\le\int_0^t h_n(\tau)\Theta^{n,k}(\tau)d\tau+\int_0^t h_k(\tau)\Theta^{n-1,k-1}(\tau)d\tau,$$

where $h_i(\tau)=C\|\theta^{i+1,+}(\tau)\|_{H^m}+C\|\theta^{i+1,-}(\tau)\|_{H^m}$, $i=n,k$ satisfies the uniform estimate $\|h_i(\tau)\|_{L^\infty_T}\le CM$ with M an upper bound from (3.16). Hence, Gronwall's inequality leads to that for every $t \in [0,T]$

$$\begin{aligned}\Theta^{n,k}(t) &\le e^{\int_0^t h_n(\tau)d\tau}\int_0^t h_k(\tau)\Theta^{n-1,k-1}(\tau)d\tau \\ &\le e^{CMt}tCM\Theta^{n-1,k-1}(t).\end{aligned}$$

By choosing t small enough, i.e. for $t \in [0,T']$ (noting that T' still only depends on $\|\theta_0^\pm\|_{H^m\cap L^p}$), then there exists a constant $\mu < 1$ such that

$$\Theta^{n,k}(t) \le \mu\Theta^{n-1,k-1}(t), \quad \forall\, t \in [0,T'].$$

From iteration, we find that for every $n,k \in \mathbb{N}$, $n > k$,

$$\begin{aligned}\Theta^{n,k}(t) &\le \mu^{k+1}(\|\theta^{n-k,+} + \theta^{0,+}\|_{L_t^\infty L^p} + \|\theta^{n-k,-} + \theta^{0,-}\|_{L_t^\infty L^p}) \\ &\le CM\mu^{k+1}.\end{aligned}$$

This ensures that $\{\theta^{n,\pm}\}_{n\in\mathbb{N}}$ are Cauchy sequences in $C([0,T'];L^p)$. Therefore there exist $\theta^\pm \in C([0,T'];L^p)$ such that $\theta^{n,\pm} \to \theta^\pm$ strongly in $C([0,T'];L^p)$.

Now we consider more properties of the limiting functions $\theta^\pm$. From (3.16) and interpolation, we have that for every $\tilde{m} \in [0,m[$,

$$\begin{aligned}\|\theta^{n,\pm} - \theta^\pm\|_{L_{T'}^\infty H^{\tilde{m}}(\mathbb{R}^2)} &\lesssim \|\theta^{n,\pm} - \theta^\pm\|^\gamma_{L_{T'}^\infty L^p(\mathbb{R}^2)}\|\theta^{n,\pm} - \theta^\pm\|^{1-\gamma}_{L_{T'}^\infty H^m(\mathbb{R}^2)} \\ &\lesssim M^{1-\gamma}\|\theta^{n,\pm} - \theta^\pm\|^\gamma_{L_{T'}^\infty L^p(\mathbb{R}^2)},\end{aligned}$$

where $\gamma = \frac{m-\tilde{m}}{m+2/p-1}$. Hence $\theta^{n,\pm} \to \theta^\pm$ strongly in $C([0,T'];H^{\tilde{m}})$ with $\tilde{m} \in [0,m[$. By a classical argument, we know that $\theta^\pm$ solve the limiting equations (1.2), and if $m > 3$, they satisfy the equations in the classical sense. From Fatou's lemma, we get $\theta^\pm \in L^\infty([0,T'];H^m)$.

Similarly as proving the corresponding point in [24, Theorem 1.1], we can also show that $\theta^\pm \in C([0,T'];H^m) \cap C^1([0,T'[;H^{m_0})$ with $m_0 = \min\{m-1, m-\alpha\}$.

3.4. *Blowup criterion*

First we know that the system (1.2) has a natural blowup criterion: if $T^* < \infty$, then necessarily

$$\|\theta^+\|_{L^\infty([0,T^*[;H^m\cap L^p)} + \|\theta^-\|_{L^\infty([0,T^*[;H^m\cap L^p)} = \infty.$$

Otherwise the solution will go beyond the time T^*.

Next, from (3.8) and the Calderón–Zygmund theorem, we find

$$\begin{aligned}\|\theta^\pm(t)\|_{H^m\cap L^p} \le C_0\|\theta_0^\pm\|_{H^m\cap L^p} + C\int_0^t (&\|\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\nabla\theta\|_{L^\infty}\|\theta^\pm\|_{H^m\cap L^p} \\ &+ \|\theta^\pm\|_{L^\infty}\|\theta\|_{H^m})(\tau)d\tau.\end{aligned}$$

Denote $G(t) = \|\theta^+(t)\|_{H^m\cap L^p} + \|\theta^-(t)\|_{H^m\cap L^p}$ for every $t \in [0,T^*[$, then from Lemma 2.3 and $\theta = \theta^+ - \theta^-$, we get

$$G(t) \le C_0 G(0) + C\int_0^t (\|\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}\nabla\theta(\tau)\|_{L^\infty} + \|\theta^+(\tau)\|_{L^\infty} + \|\theta^-(\tau)\|_{L^\infty})G(\tau)d\tau$$
$$\le C_0G(0) + C\int_0^t (1 + \|\theta^+(\tau)\|_{L^\infty} + \|\theta^-(\tau)\|_{L^\infty})\log(e+G(\tau))G(\tau)d\tau.$$

Direct computation yields that for every $t\in[0,T^*[$,

$$G(t) \le (C_0G(0)+e)^{\exp\{Ct + C\int_0^t(\|\theta^+(\tau)\|_{L^\infty} + \|\theta^-(\tau)\|_{L^\infty})d\tau\}}.$$

Therefore, if $T^* < \infty$, we necessarily need that $\int_0^{T^*}(\|\theta^+(t)\|_{L^\infty} + \|\theta^-(t)\|_{L^\infty})\, dt = \infty$.

4. Proof of Proposition 1.2

Throughout this section, we assume that $(\theta^+,\theta^-) \in C([0,T^*[;H^m\cap L^p) \cap C^1([0,T^*[;H^{m_0})$ with $m > 4$, $p\in]1,2[$, $m_0 = \min\{m-1, m-\alpha\}$ is the corresponding maximal lifespan solution obtained in Theorem 1.1.

4.1. *Proof of Proposition* 1.2(1): *the non-negativity of the solutions*

For every $T\in]0,T^*[$, denote $U_T =]0,T]\times\mathbb{R}^2$. According to the Sobolev embedding, we infer that

$$\theta^\pm \in C^{1,0}_{t,x}(U_T)\cap C^{0,2}_{t,x}(U_T)\cap C^0_{t,x}(\overline{U}_T)\cap L^2(U_T)$$

satisfies

$$\sup_{U_T}(|\partial_t\theta^\pm| + |\nabla\theta^\pm| + |\nabla^2\theta^\pm|) + \sup_{\overline{U}_T}|\theta^\pm| \lesssim_{\|\theta^\pm\|_{L^\infty_T(H^m\cap L^p)}} 1,$$

and $\theta^\pm$ solve the following equations pointwise

$$\begin{cases} \partial_t\theta^\pm + \nabla\cdot(u^\pm\theta^\pm) = \kappa|D|^\alpha\theta^\pm, \\ u^\pm = \pm(\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^+-\theta^-),0), \\ \theta^\pm(0,x) = \theta_0^\pm(x). \end{cases}$$

To show that $u^\pm \in C^{0,1}_{t,x}(U_T)$, noticing $\nabla u^\pm = \nabla(\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^+ - \theta^-),0)$, it suffices to prove that $\theta^\pm \in C(]0,T];B^0_{\infty,1})$, and this turns out to be a consequence of $\partial_t\theta^\pm \in C([0,T];H^{m_0})$ with $m_0 = \min\{m-1,m-\alpha\}$ and Sobolev's embedding. It is also clear to see that $\theta_0^\pm \in C^2(\mathbb{R}^2)$ and

$$\sup_{U_T}|\mathrm{div}\, u^\pm| = \sup_{U_T}|\mathcal{R}_1^2\mathcal{R}_2^2(\theta^+ - \theta^-)| \lesssim \|\theta^+\|_{L^\infty_T H^m} + \|\theta^-\|_{L^\infty_T H^m}.$$

Hence by virtue of Lemma 2.5, and from $\theta_0^\pm \ge 0$, we have $\theta^\pm \ge 0$ in U_T. Since $T\in]0,T^*[$ is arbitrary, this implies $\theta^\pm\ge 0$ for all $[0,T^*[\times\mathbb{R}^2$.

4.2. *Proof of Proposition* 1.2(2)

Let $T\in]0,T^*[$ be arbitrary, $\phi\in C^\infty(\mathbb{R})$ be an even cut-off function satisfying that

$$0\le\phi\le 1,\quad \operatorname{supp}\phi\subset]-2,2[,\quad \phi\equiv 1\ \text{ on } [-1,1].$$

Denote $\phi_R(\cdot)=\phi(\frac{\cdot}{R})$ for $R>0$. Multiplying both sides of the equations of $\theta^\pm$ by $\phi_R(x_1)$ and integrating over the x_1-variable, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\theta^\pm(t,x)\phi_R(x_1)dx_1 &= -\int_{\mathbb{R}}\partial_1(u_1^\pm\theta^\pm)(t,x)\phi_R(x_1)dx_1\\ &\quad-\kappa\int_{\mathbb{R}}|D|^\alpha\theta^\pm(t,x)\phi_R(x_1)dx_1\\ &\triangleq \mathrm{I}^\pm(t,x_2)+\mathrm{II}^\pm(t,x_2),\end{aligned}$$

with $u_1^\pm\triangleq\pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^+-\theta^-)$. For $\mathrm{I}^\pm$, from the integration by parts and Hölder's inequality, we obtain that for every $(t,x_2)\in[0,T]\times\mathbb{R}$

$$\begin{aligned}\mathrm{I}^\pm(t,x_2)&=\frac{1}{R}\int_{\mathbb{R}}u_1^\pm(t,x)\theta^\pm(t,x)(\partial_1\phi)\left(\frac{x_1}{R}\right)dx_1\\ &\le\frac{1}{R^{1/2}}\|u_1^\pm\|_{L^\infty_TL^\infty_x}\|\theta^\pm\|_{L^\infty_TL^{\infty,2}_{x_2,x_1}}\|\nabla\phi\|_{L^2}.\end{aligned}$$

From (3.2), we see

$$\|u_1^\pm\|_{L^\infty_TL^\infty_x}\lesssim\|\theta^+\|_{L^\infty_T(H^m\cap L^p)}+\|\theta^-\|_{L^\infty_T(H^m\cap L^p)}. \tag{4.1}$$

By the Sobolev embedding, we also find that

$$\|\theta^\pm\|_{L^\infty_TL^{\infty,2}_{x_2,x_1}}\lesssim\|(\mathrm{Id}+|D_2|)\theta^\pm\|_{L^\infty_TL^2_x}\lesssim\|\theta^\pm\|_{L^\infty_TH^m}.$$

Thus

$$\|\mathrm{I}^\pm\|_{L^\infty_TL^\infty_{x_2}}\lesssim\frac{1}{R^{1/2}}(\|\theta^+\|^2_{L^\infty_T(H^m\cap L^p)}+\|\theta^-\|^2_{L^\infty_T(H^m\cap L^p)})\|\nabla\phi\|_{L^2}. \tag{4.2}$$

We can rewrite $\mathrm{II}^\pm$ as follows:

$$\begin{aligned}\mathrm{II}^\pm(t,x_2)&=-\kappa\int_{\mathbb{R}}|D_2|^\alpha\theta^\pm(t,x)\phi_R(x_1)dx_1-\kappa\int_{\mathbb{R}}(|D|^\alpha-|D_2|^\alpha)\theta^\pm(t,x)\phi_R(x_1)dx_1\\ &\triangleq\mathrm{II}_1^\pm(t,x_2)+\mathrm{II}_2^\pm(t,x_2).\end{aligned}$$

It is obvious to see

$$\mathrm{II}_1^\pm(t,x_2)=-\kappa|D_2|^\alpha\left(\int_{\mathbb{R}}\theta^\pm(t,x)\phi_R(x_1)dx_1\right).$$

For $\mathrm{II}_2^\pm$, observe that

$$\begin{aligned}\mathrm{II}_2^\pm(t,x_2)&=-\kappa\int_{\mathbb{R}}\left(\frac{|D|^\alpha-|D_2|^\alpha}{|D_1|^\alpha}\theta^\pm\right)(t,x)|D_1|^\alpha(\phi_R)(x_1)dx_1\\ &=-\kappa R^{-\alpha}\int_{\mathbb{R}}\left(\frac{|D|^\alpha-|D_2|^\alpha}{|D_1|^\alpha}\theta^\pm\right)(t,x)(|D_1|^\alpha\phi)\left(\frac{x_1}{R}\right)dx_1.\end{aligned}$$

If $\alpha \in]1/2, 2]$, from Hölder's inequality and the fact that $|\zeta|^\alpha - |\zeta_2|^\alpha \le |\zeta_1|^\alpha$ for all $\alpha \in]0, 2]$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, we obtain

$$\begin{aligned}\|\mathrm{II}_2^\pm\|_{L^\infty_T L^\infty_{x_2}} &\le \kappa R^{-\alpha} \left\| \frac{|D|^\alpha - |D_2|^\alpha}{|D_1|^\alpha} \theta^\pm \right\|_{L^\infty_T L^{\infty,2}_{\tilde{x}_2,x_1}} \left\| (|D_1|^\alpha \phi)\left(\frac{x_1}{R}\right) \right\|_{L^2_{x_1}} \\ &\lesssim \kappa R^{-(\alpha-\frac{1}{2})} \left\| (\mathrm{Id} + |D_2|) \frac{|D|^\alpha - |D_2|^\alpha}{|D_1|^\alpha} \theta^\pm \right\|_{L^\infty_T L^2_x} \||D|^\alpha \phi\|_{L^2} \\ &\lesssim \kappa R^{-(\alpha-\frac{1}{2})} \|\theta^\pm\|_{L^\infty_T H^m} \||D|^\alpha \phi\|_{L^2},\end{aligned}$$

where we also have used the estimate that $\|f\|_{L^{\infty,2}_{x_2,x_1}} \le \|f\|_{L^{2,\infty}_{x_1,x_2}} \lesssim \|(\mathrm{Id}+|D_2|)f\|_{L^2_x}$.

Since

$$\frac{d}{dt}\int_{\mathbb{R}} \theta^\pm(t,x)\phi_R(x_1)dx_1 + \kappa|D_2|^\alpha \left(\int_{\mathbb{R}} \theta^\pm(t,x)\phi_R(x_1)dx_1 \right) = \mathrm{I}^\pm(t,x_2) + \mathrm{II}_2^\pm(t,x_2),$$

we get

$$\begin{aligned}\left\| \int_{|x_1|\le R} \theta^\pm(t,x)dx_1 \right\|_{L^\infty_T L^\infty_{x_2}} &\le \left\| \int_{\mathbb{R}} \theta^\pm(t,x)\phi_R(x_1)dx_1 \right\|_{L^\infty_T L^\infty_{x_2}} \\ &\le \left\| \int_{\mathbb{R}} \theta_0^\pm(x)\phi_R(x_1)dx_1 \right\|_{L^\infty_{x_2}} \\ &\quad + T(\|\mathrm{I}^\pm\|_{L^\infty_T L^\infty_{x_2}} + \|\mathrm{II}_2^\pm\|_{L^\infty_T L^\infty_{x_2}}) \\ &\le \|\theta_0^\pm\|_{L^{\infty,1}_{x_2,x_1}} + CT(R^{-\frac{1}{2}} + R^{-(\alpha-\frac{1}{2})}),\end{aligned}$$

where C is a positive constant depending on κ, $\|\theta^\pm\|_{L^\infty_T(H^m\cap L^p)}$ and ϕ. From $\theta^\pm(t) \ge 0$ for all $t \in [0,T]$ and the monotone convergence theorem, and by passing R to infinity, we find

$$\|\theta^\pm\|_{L^\infty_T L^{\infty,1}_{x_2,x_1}} \le \|\theta_0^\pm\|_{L^{\infty,1}_{x_2,x_1}}.$$

Hence this estimate combined with the fact that $T \in]0, T^*[$ is arbitrary leads to (1.5).

Now, since $\theta^\pm \in C([0,T^*[; H^m \cap L^p)$ with $m > 4$ and $p \in]1, 2[$, we have

$$\lim_{x_1\to-\infty} \left(\theta^\pm(t,x) + \sum_{k=1,2,3} |\nabla^k \theta^\pm(t,x)| \right) = 0, \quad \forall\, (t,x_2) \in [0,T^*[\times \mathbb{R}, \quad (4.3)$$

thus we moreover deduce that for every $t \in [0, T^*[$,

$$\begin{aligned}\|\rho^\pm(t,x)\|_{L^\infty_x} &\le \left\| \int_{-\infty}^{x_1} \theta^\pm(t,\tilde{x}_1,x_2)d\tilde{x}_1 \right\|_{L^\infty_x} \\ &\le \left\| \int_{\mathbb{R}} \theta^\pm(t,x)dx_1 \right\|_{L^\infty_{x_2}} \le \|\theta_0^\pm\|_{L^{\infty,1}_{x_2,x_1}}, \quad (4.4)\end{aligned}$$

and

$$\lim_{x_1\to-\infty}\rho^{\pm}(t,x)=\lim_{x_1\to-\infty}\int_{-\infty}^{x_1}\theta^{\pm}(t,\tilde{x}_1,x_2)d\tilde{x}_1=0,\quad \forall\,(t,x_2)\in[0,T^*[\,\times\mathbb{R}. \tag{4.5}$$

Next we shall justify that $\rho^{\pm}$ are the mild solutions of the system (1.1) for $(t,x)\in[0,T^*[\,\times\mathbb{R}^2$. From Theorem 1.1, we know that

$$\theta^{\pm}(t,x)=e^{-\kappa t|D|^{\alpha}}\theta_0^{\pm}(x)-\int_0^t e^{-\kappa(t-\tau)|D|^{\alpha}}\partial_1(u^{\pm}\theta^{\pm})(\tau,x)d\tau,$$
$$\forall\,(t,x)\in[0,T^*[\,\times\mathbb{R}^2,$$

with

$$u_1^{\pm}=\pm\mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^+-\theta^-)=\pm\mathcal{R}_1^2\mathcal{R}_2^2(\rho^+-\rho^-).$$

Taking advantage of the relation $\theta^{\pm}=\partial_1\rho^{\pm}$, we get

$$\begin{aligned}\rho^{\pm}(t,x)&=\int_{-\infty}^{x_1}\theta^{\pm}(t,\tilde{x}_1,x_2)d\tilde{x}_1\\&=\int_{-\infty}^{x_1}e^{-\kappa t|D|^{\alpha}}\partial_1\rho_0^{\pm}(\tilde{x}_1,x_2)d\tilde{x}_1\\&\quad-\int_{-\infty}^{x_1}\int_0^t e^{-\kappa(t-\tau)|D|^{\alpha}}\partial_1(u_1^{\pm}\partial_1\rho^{\pm})(\tau,\tilde{x}_1,x_2)d\tau d\tilde{x}_1\\&=e^{-\kappa t|D|^{\alpha}}\rho_0^{\pm}(x)-\int_0^t e^{-\kappa(t-\tau)|D|^{\alpha}}(u_1^{\pm}\partial_1\rho^{\pm})(\tau,x)d\tau+E(t,x_2),\end{aligned}$$

with

$$\begin{aligned}E(t,x_2)&=E_1(t,x_2)+E_2(t,x_2)\\&\triangleq-\lim_{\tilde{x}_1\to-\infty}e^{-\kappa t|D|^{\alpha}}\rho_0^{\pm}(\tilde{x}_1,x_2)\\&\quad+\lim_{\tilde{x}_1\to-\infty}\int_0^t e^{-\kappa(t-\tau)|D|^{\alpha}}(u_1^{\pm}\theta^{\pm})(\tau,\tilde{x}_1,x_2)d\tau.\end{aligned}$$

When $\kappa=0$, by virtue of (4.5), (4.1) and (4.3), it just reduces to

$$\rho^{\pm}(t,x)=\rho_0^{\pm}(x)-\int_0^t(u_1^{\pm}\partial_1\rho^{\pm})(\tau,x)d\tau.$$

When $\kappa>0$, noticing that

$$e^{-\kappa t|D|^{\alpha}}\rho_0^{\pm}(x)=\int_{\mathbb{R}^2}K_{\alpha}(\kappa t,y)\rho_0^{\pm}(x-y)dy,\quad \alpha\in\,]0,2], \tag{4.6}$$

where $K_\alpha(\kappa t, y) = (\kappa t)^{-2/\alpha} K_\alpha(y/(\kappa t)^{1/\alpha})$ and $K_\alpha(y) = \mathcal{F}^{-1}(e^{-|\zeta|^\alpha})(y)$ $(\alpha \in]0,2])$ satisfies

$$K_\alpha \geq 0, \quad \begin{cases} K_\alpha(y) \approx \dfrac{1}{(1+|y|^2)^{(2+\alpha)/2}}, & y \in \mathbb{R}^2, \ \alpha \in]0,2[, \\ K_2(y) = \dfrac{1}{4} e^{-|y|^2/4}, & y \in \mathbb{R}^2, \ \alpha = 2, \end{cases} \tag{4.7}$$

thus from (4.4), (4.5) and the dominated convergence theorem, we find $E_1(t,x_2) = 0$. Similarly, from (4.1) and (4.3), we also get $E_2(t,x_2) = 0$. Hence we have for every $(t,x) \in [0,T^*[\times \mathbb{R}^2$,

$$\rho^\pm(t,x) = e^{-\kappa t|D|^\alpha} \rho_0^\pm(x) - \int_0^t e^{-\kappa(t-\tau)|D|^\alpha} (u_1^\pm \partial_1 \rho^\pm)(\tau,x) d\tau. \tag{4.8}$$

4.3. *Proof of Proposition* 1.2(3)

We first show that for $k = 1,2,3$, $\nabla^k \rho^\pm(t) \in L_x^\infty$ for all $t \in [0,T^*[$ under some appropriate assumptions of $\rho_0^\pm$. Clearly, since $\nabla^{k-1}\partial_1 \rho^\pm(t) = \nabla^{k-1}\theta^\pm(t) \in L_x^\infty$ for all $t \in [0,T^*[$, it suffices to consider the case of $\partial_2^k \rho^\pm$. Due to that $\theta^\pm \in C([0,T^*[; H^m \cap L^p)$ with $m > 4$ and $p \in]1,2[$, the nonlinear term satisfies that for every $T \in]0,T^*[$,

$$\begin{aligned} \|\partial_2^k (u_1^\pm \partial_1 \rho^\pm)\|_{L_T^\infty L_x^\infty} &\leq \sum_{0 \leq j \leq k} \|\partial_2^j u_1^\pm \partial_2^{k-j} \theta^\pm\|_{L_T^\infty L_x^\infty} \\ &\leq \sum_{0 \leq j \leq k} \|\partial_2^j u_1^\pm\|_{L_T^\infty L_x^\infty} \|\partial_2^{k-j}\theta^\pm\|_{L_T^\infty L_x^\infty} \\ &\lesssim (\|\theta^+\|_{L_T^\infty(H^m \cap L^p)} + \|\theta^-\|_{L_T^\infty(H^m\cap L^p)}) \|\theta^\pm\|_{L_T^\infty H^m}. \end{aligned} \tag{4.9}$$

Thus, from (4.8) and $\partial_2^k \rho_0^\pm \in L_x^\infty$, we have

$$\partial_2^k \rho^\pm(t,x) = e^{-\kappa t|D|^\alpha} \partial_2^k \rho_0^\pm(x) - \int_0^t e^{-\kappa(t-\tau)|D|^\alpha} \partial_2^k (u_1^\pm \partial_1 \rho^\pm)(\tau,x) d\tau, \tag{4.10}$$

and

$$\|\partial_2^k \rho^\pm\|_{L_T^\infty L_x^\infty} \leq \|\partial_2^k \rho_0^\pm\|_{L_x^\infty} + CT,$$

with C depending on $\|\theta^\pm\|_{L_T^\infty(H^m\cap L^p)}$, which implies that $\partial_2^k \rho^\pm(t) \in L_x^\infty$ for all $t \in [0,T^*[$. Moreover, thanks to $\lim_{x_1 \to -\infty} \partial_2^k \rho_0^\pm(x) = 0$ for every $x_2 \in \mathbb{R}$, (4.3) and the dominated convergence theorem, we also have

$$\lim_{x_1 \to -\infty} \partial_2^k \rho^\pm(t,x) = 0, \quad \forall\, (t,x_2) \in [0,T^*[\times \mathbb{R}. \tag{4.11}$$

Next we show that $\rho^\pm$ solve the system (1.1) in the classical pointwise sense. Since $\theta^\pm$ are the classical solutions to the system (1.2) and $\partial_1 \rho^\pm = \theta^\pm$, we have that for every $(t,x)\in]0,T^*[\times\mathbb{R}^2$,

$$\begin{aligned}\partial_t\rho^\pm(t,x) &= \int_{-\infty}^{x_1}\partial_t\theta^\pm(t,\tilde{x}_1,x_2)d\tilde{x}_1\\ &= -\int_{-\infty}^{x_1}\partial_1(u_1^\pm\theta^\pm)(t,\tilde{x}_1,x_2)d\tilde{x}_1-\kappa\int_{-\infty}^{x_1}|D|^\alpha\theta^\pm(t,\tilde{x}_1,x_2)d\tilde{x}_1\\ &= -\int_{-\infty}^{x_1}\partial_1(u_1^\pm\partial_1\rho^\pm)(t,\tilde{x}_1,x_2)d\tilde{x}_1-\kappa\int_{-\infty}^{x_1}\partial_1|D|^\alpha\rho^\pm(t,\tilde{x}_1,x_2)d\tilde{x}_1\\ &= -u_1^\pm\partial_1\rho^\pm(t,x)-\kappa|D|^\alpha\rho^\pm(t,x)+\tilde{E}^\alpha(t,x_2),\end{aligned}$$

where

$$\tilde{E}^\alpha(t,x_2)=\lim_{x_1\to-\infty}|D|^\alpha\rho^\pm(t,x),$$

and in the last line we have used (4.1) and (4.3). When $\alpha=2$, from (4.3) and (4.11), we directly get $\tilde{E}^2(t,x)=0$. When $\alpha\in]0,2[$, due to $\rho^\pm\in L^\infty([0,T^*[;C_b^2(\mathbb{R}^2))$, from Lemma 2.4 we have

$$\begin{aligned}|D|^\alpha\rho^\pm(t,x) &= -c_\alpha\Bigg(\int_{B_1}\frac{\rho^\pm(t,x+y)-\rho^\pm(t,x)-\nabla\rho^\pm(t,x)\cdot y}{|y|^{2+\alpha}}dy\\ &\quad+\int_{B_1^c}\frac{\rho^\pm(t,x+y)-\rho^\pm(t,x)}{|y|^{2+\alpha}}dy\Bigg)\\ &= -c_\alpha\Bigg(\int_{B_1}\int_0^1\int_0^1\frac{(y\cdot\nabla^2\rho^\pm(t,x+s\tau y))\cdot y}{|y|^{2+\alpha}}\tau ds d\tau dy\\ &\quad+\int_{B_1^c}\frac{\rho^\pm(t,x+y)-\rho^\pm(t,x)}{|y|^{2+\alpha}}dy\Bigg),\end{aligned}$$

and by the dominated convergence theorem, (4.3) and (4.11), we find $\tilde{E}^\alpha(t,x_2)=0$.

Similarly, we can prove that $\nabla\rho^\pm$ solve the equations in the classical pointwise sense

$$\partial_t(\nabla\rho^\pm)+u_1^\pm\partial_1(\nabla\rho^\pm)+\kappa|D|^\alpha(\nabla\rho^\pm)=-\nabla u_1^\pm\partial_1\rho^\pm,\quad \nabla\rho^\pm|_{t=0}=\nabla\rho_0^\pm,$$

and $\partial_t(\nabla\rho^\pm)\in L^\infty([0,T^*[;L^\infty)$ which implies that $\nabla\rho^\pm\in C([0,T^*[;L^\infty)$.

4.4. *Proof of Proposition* 1.2(4)

Since $\theta^\pm\in C([0,T^*[;H^m(\mathbb{R}^2))$ with $m>4$, then for every $t\in[0,T^*[$, there exists a constant $R_1>0$ (that may depend on t) such that

$$\|\partial_1\rho^\pm\|_{L^\infty([0,t];L^\infty(B_{R_1}^c))}=\|\theta^\pm\|_{L^\infty([0,t];L^\infty(B_{R_1}^c)}\le\|\partial_1\rho_0^\pm\|_{L^\infty}.$$

For $\partial_2\rho^\pm$, from (4.10), and by denoting $f^\pm(t,x) = \partial_2(u_1^\pm\partial_1\rho^\pm)(t,x)$, we infer that for every $t \in]0,T^*[$ and for some constant $R_2 > 0$ chosen later,

$$\|\partial_2\rho^\pm\|_{L^\infty([0,t];L^\infty(B^c_{R_2}))} \leq \|\partial_2\rho_0^\pm\|_{L^\infty} + \int_0^t \|e^{-\kappa(t-\tau)|D|^\alpha} f^\pm(\tau,\cdot)\|_{L^\infty(B^c_{R_2})} d\tau.$$

Let χ be the cut-off function in Sec. 2.1, and denote $\psi(x) \triangleq 1 - \chi(x)$ for every $x \in \mathbb{R}^2$. Clearly, $\psi(x) \in C^\infty(\mathbb{R}^2)$ satisfies that

$$0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subset B_1^c, \ \ \psi \equiv 1 \ \text{ on } \overline{B}^c_{\frac{4}{3}},$$

thus we get

$$\begin{aligned}\int_0^t \|e^{-\kappa(t-\tau)|D|^\alpha} f^\pm(\tau,\cdot)\|_{L^\infty(B^c_{R_2})} d\tau &\leq \int_0^t \left\| e^{-\kappa(t-\tau)|D|^\alpha} f^\pm(\tau,x)\psi\left(\frac{2x}{R_2}\right)\right\|_{L^\infty_x} d\tau \\ &\triangleq \Gamma^\pm(t).\end{aligned}$$

We divide it into several cases

$$\begin{aligned}\Gamma^\pm(t) &\leq \int_0^t \left\| e^{-\kappa(t-\tau)|D|^\alpha}\left(f^\pm(\tau,\cdot)\psi\left(\frac{\cdot}{\frac{R_2}{2}}\right)\right)(x)\right\|_{L^\infty_x} d\tau \\ &\quad + \int_0^t \left\| \left(\left[e^{-\kappa(t-\tau)|D|^\alpha}, \psi\left(\frac{\cdot}{\frac{R_2}{2}}\right)\right] f^\pm(\tau,\cdot)\right)(x)\right\|_{L^\infty_x} d\tau \\ &\triangleq \Gamma_1^\pm(t) + \Gamma_2^\pm(t),\end{aligned}$$

where $[X,Y] \triangleq XY - YX$ is the commutator. For $\Gamma_1^\pm$, noticing that as $r \to \infty$,

$$\begin{aligned}\|f^\pm\|_{L^\infty_t L^\infty(B^c_r)} &\lesssim \|(\theta^+ - \theta^-)\|_{L^\infty_t(H^m\cap L^p)}(\|\theta^\pm\|_{L^\infty_t(L^\infty(B^c_r))} + \|\partial_2\theta^\pm\|_{L^\infty_t(L^\infty(B^c_r))}) \\ &\to 0,\end{aligned}$$

we can choose R_2 large enough so that for every $t \in]0,T^*[$,

$$\Gamma_1^\pm(t) \leq Ct\left\| f^\pm\psi\left(\frac{2x}{R_2}\right)\right\|_{L^\infty_t L^\infty_x} \leq Ct\|f^\pm\|_{L^\infty_t(L^\infty(B^c_{R_2/2}))} \leq \frac{\epsilon}{2}.$$

From (4.6), we can rewrite $\Gamma_2^\pm$ as

$$\begin{aligned}\Gamma_2^\pm(t) &= \int_0^t \left\| \int_{\mathbb{R}^2} K_\alpha(\kappa(t-\tau),y) f^\pm(\tau,x-y)\left(\psi\left(\frac{x-y}{\frac{R_2}{2}}\right) - \psi\left(\frac{x}{\frac{R_2}{2}}\right)\right) dy\right\|_{L^\infty_x} d\tau \\ &= \int_0^t \left\| \int_{\mathbb{R}^2} K_\alpha(\kappa(t-\tau),y) f^\pm(\tau,x-y)\left(\chi\left(\frac{x-y}{\frac{R_2}{2}}\right) - \chi\left(\frac{x}{\frac{R_2}{2}}\right)\right) dy\right\|_{L^\infty_x} d\tau.\end{aligned}$$

Thus by using the estimate that

$$|g(z_1) - g(z_2)| \leq \|g\|_{C^{\alpha/2}(\mathbb{R}^2)} |z_1 - z_2|^{\alpha/2}, \quad \forall\, z_1, z_2 \in \mathbb{R}^2,$$

and the Minkowski inequality, (4.9), (4.7), we obtain that

$$\begin{aligned}
\Gamma_2^{\pm}(t) &\leq \|\chi\|_{C^{\alpha/2}} \int_0^t \left\| \int_{\mathbb{R}^2} K_\alpha(\kappa(t-\tau), y) |f^{\pm}(\tau, x-y)| \left(\frac{|y|}{\frac{R_2}{2}} \right)^{\alpha/2} dy \right\|_{L^\infty_x} d\tau \\
&\lesssim \|\chi\|_{C^{\alpha/2}} \|f^{\pm}\|_{L^\infty_t L^\infty_x} \int_0^t \int_{\mathbb{R}^2} (\kappa(t-\tau))^{-\frac{2}{\alpha}} K_\alpha \left(\frac{y}{(\kappa(t-\tau))^{1/\alpha}} \right) \frac{|y|^{\alpha/2}}{R_2^{\alpha/2}} dy d\tau \\
&\lesssim \frac{\|\chi\|_{C^{\alpha/2}}}{R_2^{\alpha/2}} \|\theta^{\pm}\|^2_{L^\infty_t(H^m \cap L^p)} \int_0^t (\kappa(t-\tau))^{\frac{1}{2}} d\tau \int_{\mathbb{R}^2} K_\alpha(y) |y|^{\frac{\alpha}{2}} dy \\
&\lesssim R_2^{-\alpha/2} \|\chi\|_{C^{\alpha/2}} \|\theta^{\pm}\|^2_{L^\infty_t(H^m \cap L^p)} (\kappa t)^{3/2}.
\end{aligned}$$

Thus through choosing R_2 large enough, we also have

$$\Gamma_2^{\pm}(t) \leq \frac{\epsilon}{2}, \quad \forall\, t \in]0, T^*[.$$

Denote $R = \max\{R_1, R_2\}$, then gathering the above estimates leads to (1.7).

5. Proof of Theorem 1.4

From Theorem 1.1 and Proposition 1.2, we assume that $T^* > 0$ is the maximal existence time of the solutions $(\theta^+, \theta^-) \in C([0, T^*[; H^m \cap L^p) \cap L^\infty([0, T^*[; L^{\infty,1}_{x_2,x_1}) \cap C^1([0, T^*[; H^{m_0})$ and $(\rho^+, \rho^-) \in L^\infty([0, T^*[; W^{2,\infty}) \cap C([0, T^*[; W^{1,\infty})$ with $m > 4$, $p \in]1, 2[$ and $m_0 = \min\{m-1, m-\alpha\}$. There is also a blowup criterion: if $T^* < \infty$, we necessarily have

$$\int_0^{T^*} \|(\theta^+, \theta^-)(t)\|_{L^\infty} dt = \int_0^{T^*} \|(\partial_1 \rho^+, \partial_1 \rho^-)(t)\|_{L^\infty} dt = \infty. \tag{5.1}$$

We shall apply the nonlocal maximum principle method to the system (1.1) to show that some appropriate MOC is preserved, which implies that the Lipschitz norm of $(\rho^+(t), \rho^-(t))$ is bounded uniformly in time. Clearly, this combined with (5.1) leads to $T^* = \infty$.

Let $\lambda \in]0, \infty[$ be a real number chosen later, ω be a stationary MOC with its explicit formula shown later. According to the scaling transformation of (1.1), we set

$$\omega_\lambda(\xi) \triangleq \lambda^{\alpha-1} \omega(\lambda \xi), \quad \forall\, \xi \in [0, \infty[. \tag{5.2}$$

First, we show that (ρ_0^+, ρ_0^-) strictly obeys the MOC ω_λ for some λ. From (1.6) and the non-negativity of θ, we know that $\|\rho_0^{\pm}\|_{L^\infty} \leq \|\theta_0^{\pm}\|_{L^{\infty,1}_{x_2,x_1}}$. Denote ω_λ^{-1} and ω^{-1} the inverse functions of ω_λ and ω (if they are multi-valued for some z, we choose the smallest ones as their values), then we need $\omega_\lambda^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}) < \infty$, so that for every x,y satisfying $|x-y| \geq \omega_\lambda^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}})$, we have

$$|\rho_0^\pm(x) - \rho_0^\pm(y)| \leq 2\|\theta_0^\pm\|_{L^{\infty,1}_{x_2,x_1}} \leq \frac{2}{3}\omega_\lambda(|x-y|).$$

For $\alpha \in]1,2]$, with no loss of generality we suppose that there are fixed constants $c_0, \xi_0 > 0$ depending on α such that $\omega(\xi_0) = c_0$, which yields $\omega^{-1}(c_0) \leq \xi_0$. Then we can choose some $\lambda \in]0,\infty[$ such that

$$\lambda^{\alpha-1} > \frac{3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{c_0}, \tag{5.3}$$

and from $\omega_\lambda^{-1}(z) = \frac{1}{\lambda}\omega^{-1}(\frac{z}{\lambda^{\alpha-1}})$, we get

$$\omega_\lambda^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}) \leq \frac{\xi_0}{\lambda} < \infty. \tag{5.4}$$

For $\alpha = 1$, we have to call for that ω is unbounded near infinity, so that $\omega_\lambda^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}) = \lambda^{-1}\omega^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}})$ is meaningful for the large data. Thus for every x,y satisfying $\lambda|x-y| \geq \tilde{C}_0$ with

$$\tilde{C}_0 = \begin{cases} \xi_0 & \text{for } \alpha \in]1,2], \\ \omega^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}) & \text{for } \alpha = 1, \end{cases}$$

we obtain

$$|\rho_0^\pm(x) - \rho_0^\pm(y)| \leq \frac{2}{3}\omega_\lambda(|x-y|). \tag{5.5}$$

The other treatment we can rely on is the mean value theorem, from which we have

$$|\rho_0^\pm(x) - \rho_0^\pm(y)| \leq \|\nabla\rho_0^\pm\|_{L^\infty}|x-y|.$$

Let $0 < \delta_0 < \tilde{C}_0$. Due to the concavity of ω, we infer that for every x,y such that $\lambda|x-y| \leq \delta_0$,

$$\frac{\lambda^{\alpha-1}\omega(\delta_0)}{\delta_0} \leq \frac{\omega_\lambda(|x-y|)}{\lambda|x-y|}.$$

Thus by choosing λ such that

$$\lambda^\alpha > \frac{\delta_0}{\omega(\delta_0)}\|(\nabla\rho_0^+,\nabla\rho_0^-)\|_{L^\infty}, \tag{5.6}$$

we get that for every x,y satisfying $x \neq y$ and $\lambda|x-y| \leq \delta_0$,

$$|\rho_0^\pm(x) - \rho_0^\pm(y)| < \omega_\lambda(|x-y|). \tag{5.7}$$

Finally, we consider the case of x,y satisfying $\delta_0 \leq \lambda|x-y| \leq \tilde{C}_0$. Observe that $|\rho_0^\pm(x) - \rho_0^\pm(y)| \leq \frac{\tilde{C}_0}{\lambda}\|(\nabla\rho_0^+,\nabla\rho_0^-)\|_{L^\infty}$ and $\lambda^{\alpha-1}\omega(\delta_0) \leq \omega_\lambda(|x-y|)$. Thus by choosing λ satisfying

$$\lambda^\alpha > \frac{\tilde{C}_0}{\omega(\delta_0)} \|(\nabla\rho_0^+, \nabla\rho^-)\|_{L^\infty}, \tag{5.8}$$

we obtain that for every x, y satisfying $\delta_0 \leq \lambda|x-y| \leq \tilde{C}_0$,

$$|\rho_0^\pm(x) - \rho_0^\pm(y)| < \omega_\lambda(|x-y|). \tag{5.9}$$

Hence, to fit our purpose, we can choose

$$\lambda \triangleq \begin{cases} \max\left\{\left(\frac{4\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{c_0}\right)^{\frac{1}{\alpha-1}}, \frac{\xi_0}{\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}\|(\nabla\rho_0^+, \nabla\rho_0^-)\|_{L^\infty}\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha \in]1,2], \\ \frac{\omega^{-1}(3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}})}{\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}\|(\nabla\rho_0^+, \nabla\rho_0^-)\|_{L^\infty}, \quad \alpha = 1, \end{cases} \tag{5.10}$$

and $\delta_0 \triangleq \omega^{-1}(2\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}/\lambda^{\alpha-1})$.

Assume that $T_* > 0$ is the first time that the strict MOC ω_λ is lost by $\rho^\pm(t)$, i.e.

$$T_* \triangleq \sup\{T \in [0, T^*[; |\rho^\pm(t,x) - \rho^\pm(t,y)| < \omega_\lambda(|x-y|), \\ \forall\, t \in [0, T[, \forall\, x \neq y \in \mathbb{R}^2\}. \tag{5.11}$$

Then we have the following assertion.

Lemma 5.1. *Let $T_* > 0$ be defined by* (5.11)*. Assume that ω moreover satisfies that*

$$\omega(0) = 0, \quad \omega'(0) < \infty, \quad \omega''(0+) = -\infty. \tag{5.12}$$

Then only three cases can occur:

(i) *ρ^- strictly obeys the MOC ω_λ and there exist two separate points $x^+, y^+ \in \mathbb{R}^2$ such that*

$$\rho^+(T_*, x^+) - \rho^+(T_*, y^+) = \omega_\lambda(\xi^+), \quad \text{with } \xi^+ = |x^+ - y^+|; \tag{5.13}$$

(ii) *ρ^+ strictly obeys the MOC ω_λ and there exist two separate points $x^-, y^- \in \mathbb{R}^2$ such that*

$$\rho^-(T_*, x^-) - \rho^-(T_*, y^-) = \omega_\lambda(\xi^-), \quad \text{with } \xi^- = |x^- - y^-|; \tag{5.14}$$

(iii) *there exist four points $x^\pm, y^\pm \in \mathbb{R}^2$, $x^\pm \neq y^\pm$ such that*

$$\rho^\pm(T_*, x^\pm) - \rho^\pm(T_*, y^\pm) = \omega_\lambda(\xi^\pm), \quad \text{with } \xi^\pm = |x^\pm - y^\pm|. \tag{5.15}$$

Note that all ξ^+ and ξ^- satisfy that $\xi^\pm \leq \omega_\lambda^{-1}(3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}})$.

Proof. It is clear to see that for every $t < T_*$, $\rho^{\pm}(t)$ strictly obeys the MOC ω_λ, and from the time continuity of $\rho^{\pm}(t)$, we have that for every $x, y \in \mathbb{R}^2$,

$$|\rho^{\pm}(T_*, x) - \rho^{\pm}(T_*, y)| \leq \omega_\lambda(|x-y|). \tag{5.16}$$

Then for every $x, y \in \mathbb{R}^2$, $x \neq y$, define

$$F^{\pm}(t,x,y) \triangleq \frac{|\rho^{\pm}(t,x) - \rho^{\pm}(t,y)|}{\omega_\lambda(|x-y|)}, \quad \forall\, t \in]0, T^*[.$$

Obviously, $F^{\pm}(T_*, x, y) \leq 1$. We assume that $F^{\pm}(T_*, x, y) < 1$ for all $x \neq y \in \mathbb{R}^2$, since otherwise the claim follows.

First, denote

$$\overline{C}_0 \triangleq \omega_\lambda^{-1}(3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}) = \lambda^{-1}\omega^{-1}\left(\frac{3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{\lambda^{\alpha-1}}\right),$$

and we find that for every x, y satisfying $|x-y| \geq \overline{C}_0$,

$$2\|\theta_0^{\pm}\|_{L^{\infty,1}_{x_2,x_1}} \leq \frac{2}{3}\omega_\lambda(\overline{C}_0) \leq \frac{2}{3}\omega_\lambda(|x-y|).$$

Thus by (4.4), we have for every $t \in]0, T^*[$ and x, y satisfying $|x-y| \geq \overline{C}_0$,

$$|\rho^{\pm}(t,x) - \rho^{\pm}(t,y)| \leq 2\|\theta_0^{\pm}\|_{L^{\infty,1}_{x_2,x_1}} \leq \frac{2}{3}\omega_\lambda(|x-y|). \tag{5.17}$$

Second, we consider the case of x, y near infinity. From the mean value theorem, we get for every $t \in]0, T^*[$ and for every x, y satisfying that $0 < |x-y| \leq \overline{C}_0$ and x or y belongs to $B^c_{R+\overline{C}_0}$ with $R > 0$ fixed later,

$$|\rho^{\pm}(t,x) - \rho^{\pm}(t,y)| \leq \|\nabla\rho^{\pm}\|_{L^\infty([0,t];L^\infty(B_R^c))}|x-y|.$$

By the concavity of ω and $|x-y| \leq \overline{C}_0$, we find that

$$\lambda\frac{3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{\omega^{-1}\left(\frac{3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{\lambda^{\alpha-1}}\right)} = \frac{\omega_\lambda(\overline{C}_0)}{\overline{C}_0} \leq \frac{\omega_\lambda(|x-y|)}{|x-y|}.$$

In order to make

$$\|\nabla\rho^{\pm}\|_{L^\infty([0,t];L^\infty(B_R^c))} \leq \frac{\lambda}{2}\frac{3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{\omega^{-1}\left(\frac{3\|(\theta_0^+, \theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}{\lambda^{\alpha-1}}\right)},$$

from $\lambda \geq \frac{\tilde{C}_0}{\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}}\|(\nabla\rho_0^+, \nabla\rho_0^-)\|_{L^\infty}$ and (5.3), it suffices to choose R such that

$$\|\nabla\rho^{\pm}\|_{L^\infty([0,t];L^\infty(B_R^c))} \leq \frac{3}{2}\|(\nabla\rho_0^+, \nabla\rho_0^-)\|_{L^\infty}. \tag{5.18}$$

This estimate can be guaranteed by (1.7), and we denote the chosen number by $R(t)$. Thus we obtain that for every x, y satisfying that $0 < |x-y| \leq \overline{C}_0$ and x or y belongs to $B^c_{R(t)+\overline{C}_0}$,

$$|\rho^\pm(t,x)-\rho^\pm(t,y)| \le \frac{1}{2}\omega_\lambda(|x-y|), \quad \forall\, t \in]0,T^*[.$$

In particular, there exists a number $h_1 > 0$ such that for every x, y satisfying that $0 < |x-y| \le \overline{C}_0$ and x or y belongs to $B^c_{R(T_*+h_1)+\overline{C}_0}$,

$$|\rho^\pm(t,x)-\rho^\pm(t,y)| \le \frac{1}{2}\omega_\lambda(|x-y|), \quad \forall\, t \in [T_*, T_*+h_1]. \tag{5.19}$$

Next we reduce to consider the case that $x, y \in B_{R(T_*+h_1)+\overline{C}_0}$ and $0 < |x-y| \le \overline{C}_0$. Since (5.12) and $\rho^\pm(T_*) \in C_b^2(\mathbb{R}^2)$, from Lemma 2.9 we get that

$$\|\nabla\rho^\pm(T_*)\|_{L^\infty(B_{R(T_*+h_1)+\overline{C}_0})} < \omega_\lambda'(0) = \lambda^\alpha\omega'(0).$$

From $\rho^\pm \in C([0,T^*[; W^{1,\infty})$, there exist small constants $h_2, \tilde{\delta} > 0$ such that for every $t \in [T_*, T_*+h_2]$,

$$\|\nabla\rho^\pm(t)\|_{L^\infty(B_{R(T_*+h_1)+\overline{C}_0})} \le (1-\tilde{\delta})\lambda^\alpha\frac{\omega(\tilde{\delta})}{\tilde{\delta}}.$$

Thus for every $x, y \in B_{R(T_*+h_1)+\overline{C}_0}$ satisfying $0 < \lambda|x-y| \le \tilde{\delta}$, from

$$\frac{\lambda^{\alpha-1}\omega(\tilde{\delta})}{\tilde{\delta}} \le \frac{\omega_\lambda(|x-y|)}{\lambda|x-y|},$$

we obtain

$$\begin{aligned}|\rho^\pm(t,x)-\rho^\pm(t,y)| &\le \|\nabla\rho^\pm(t)\|_{L^\infty(B_{R(T_*+h_1)+\overline{C}_0})}|x-y| \\ &\le (1-\tilde{\delta})\omega_\lambda(|x-y|), \quad \forall\, t \in [T_*, T_*+h_2].\end{aligned} \tag{5.20}$$

Now it remains to treat the case that the continuous function $F^\pm(t,x,y)$ on the compact set

$$\mathcal{K} := \{(x,y) \in \mathbb{R}^2\times\mathbb{R}^2; \max\{|x|,|y|\} \le R(T_*+h_1)+\overline{C}_0,\ |x-y| \ge \tilde{\delta}/\lambda\}.$$

By virtue of $F^\pm(T_*,x,y) < 1$ for all $(x,y) \in \mathcal{K}$, we have that there exist small constants $h_3, \bar{\delta} > 0$ such that

$$F^\pm(t,x,y) \le 1-\bar{\delta}, \quad \forall\, (t,x,y) \in [T_*, T_*+h_3]\times\mathcal{K}. \tag{5.21}$$

Set $h \triangleq \min\{h_1,h_2,h_3\} > 0$, then by gathering the above estimates, we know that $\rho^\pm(T_*+h)$ strictly obeys the MOC ω_λ and this clearly contradicts with the definition of T_*. □

Now we shall show that this scenarios (i)–(iii) cannot happen. More precisely, we shall prove

$$\begin{cases}\text{for (i)}, & (f^+)'(T_*) < 0, \quad \text{with } f^+(T_*) = \rho^+(T_*,x^+)-\rho^+(T_*,y^+), \\ \text{for (ii)}, & (f^-)'(T_*) < 0, \quad \text{with } f^-(T_*) = \rho^-(T_*,x^-)-\rho^-(T_*,y^-), \\ \text{for (iii)}, & (f^\pm)'(T_*) < 0, \quad \text{with } f^\pm(T_*) = \rho^\pm(T_*,x^\pm)-\rho^\pm(T_*,y^\pm).\end{cases} \tag{5.22}$$

Clearly, this means that for some $t < T_*$, the strict MOC ω_λ is lost by $\rho^+(t)$ or $\rho^-(t)$, and this contradicts the definition of T_*.

Since $\rho^\pm$ solves (1.1) in the classical pointwise sense, we directly have

$$\begin{aligned}\partial_t\rho^\pm(T_*,x^\pm) - \partial_t\rho^\pm(T_*,y^\pm) &= -u^\pm\cdot\nabla\rho^\pm(T_*,x^\pm) + u^\pm\cdot\nabla\rho^\pm(T_*,y^\pm)\\ &\quad + [-|D|^\alpha]\rho^\pm(T_*,x^\pm) - [-|D|^\alpha]\rho^\pm(T_*,y^\pm)\end{aligned}$$

with

$$u^\pm = \pm(\mathcal{R}_1^2\mathcal{R}_2^2(\rho^+-\rho^-),0).$$

Taking advantage of Lemmas 2.7, 2.8 and the change of variable, we find that

$$\begin{cases}\text{for (i)}, & (f^+)'(T_*) \le \lambda^{2\alpha-1}(\Omega\omega' + \Psi_\alpha)(\lambda\xi^+),\\ \text{for (ii)}, & (f^-)'(T_*) \le \lambda^{2\alpha-1}(\Omega\omega' + \Psi_\alpha)(\lambda\xi^-),\\ \text{for (iii)}, & (f^\pm)'(T_*) \le \lambda^{2\alpha-1}(\Omega\omega' + \Psi_\alpha)(\lambda\xi^\pm),\end{cases}$$

with Ω and Ψ_α defined by (2.3) and (2.5) respectively.

Next we construct appropriate moduli of continuity satisfying (5.12) in the spirit of [23]. Let $0 < \gamma < \delta < 1$ be two absolute constants chosen later; then for every $\alpha \in [1,2]$, we define the following continuous functions that for $\alpha = 1$

$$\text{MOC}_1 \begin{cases}\omega(\xi) = \xi - \xi^{3/2} & \text{for } \xi \in [0,\delta],\\ \omega'(\xi) = \dfrac{\gamma}{\xi\left(4+\log\left(\frac{\xi}{\delta}\right)\right)} & \text{for } \xi \in]\delta,\infty[,\end{cases} \tag{5.23}$$

and for $\alpha \in]1,2]$

$$\text{MOC}_\alpha \begin{cases}\omega(\xi) = \xi - \xi^{3/2} & \text{for } \xi \in [0,\delta],\\ \omega'(\xi) = 0 & \text{for } \xi \in]\delta,\infty[.\end{cases} \tag{5.24}$$

Notice that, for small δ, we have $\omega'(\delta-) \approx 1$, while $\omega'(\delta+) \le \frac{1}{4}$, thus ω is a concave piecewise C^2 function if δ is small enough. Obviously, $\omega(0) = 0$, $\omega'(0) = 1$ and $\omega''(0+) = -\infty$. For $\alpha = 1$, ω is unbounded near infinity, and for $\alpha \in]1,2]$, ω is a bounded function with maximum $\delta - \delta^{3/2}$ (in (5.10), we can choose $c_0 = \delta - \delta^{3/2}$ and $\xi_0 = \delta$).

Then our target is to prove that for suitable MOC given by (5.23) and (5.24),

$$\Omega(\xi)\omega'(\xi) + \Psi_\alpha(\xi) < 0, \tag{5.25}$$

for all $0 < \xi \le \lambda\overline{C}_0 = \lambda\omega_\lambda^{-1}(3\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}})$, more precisely,

$$\left(A_1\omega(\xi) + A_2\int_0^\xi \frac{\omega(\eta)}{\eta}d\eta + A_2\xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}d\eta\right)\omega'(\xi) + \Psi_\alpha(\xi) < 0, \quad \forall\,\xi \in]0,\lambda\overline{C}_0].$$

Note that from (5.4), we know $\lambda\overline{C}_0 \le \delta$ for $\alpha \in]1,2]$.

We divide into two cases.

Case 1: $\alpha \in [1,2]$ and $0 < \xi \le \delta$. Since $\frac{\omega(\eta)}{\eta} \le \omega'(0) = 1$ for all $\eta > 0$, we have $\int_0^\xi \frac{\omega(\eta)}{\eta} d\eta \le \xi$ and $\int_\xi^\delta \frac{\omega(\eta)}{\eta^2} d\eta \le \int_\xi^\delta \frac{1}{\eta} d\eta = \log(\delta/\xi)$.

Further,

$$\begin{cases} \displaystyle\int_\delta^\infty \frac{\omega(\eta)}{\eta^2} d\eta = \frac{\omega(\delta)}{\delta} + \int_\delta^\infty \frac{\gamma}{\eta^2 \left(4 + \log\left(\frac{\eta}{\delta}\right)\right)} d\eta \le 1 + \frac{\gamma}{4\delta} \le 2 & \text{for } \alpha = 1, \\ \displaystyle\int_\delta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \le \int_\delta^\infty \frac{\delta}{\eta^2} d\eta = 1 & \text{for } \alpha \in]1,2]. \end{cases}$$

Obviously $\omega'(\xi) \le \omega'(0) = 1$, so we get that the positive part is bounded by $\xi(A_1 + 3A_2 + A_2 \log(\delta/\xi))$.

For the negative part, we have $\omega''(\xi) = -\frac{3}{4}\xi^{-\frac{1}{2}} < 0$, and

$$\begin{aligned} &\int_0^{\frac{\xi}{2}} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+\alpha}} d\eta \\ &\quad \le \int_0^{\frac{\xi}{2}} \frac{\omega''(\xi) 2\eta^2}{\eta^{1+\alpha}} d\eta \le -\frac{3}{4}\xi^{\frac{3}{2}-\alpha} \quad \text{for } \alpha \in [1,2[. \end{aligned}$$

Hence by choosing δ small enough, we have for all $\xi \in]0,\delta]$,

$$\begin{cases} \xi \left(A_1 + 3A_2 + A_2 \log\left(\frac{\delta}{\xi}\right) - \frac{3B_\alpha}{4}\xi^{\frac{1}{2}-\alpha} \right) < 0 & \text{for } \alpha \in [1,2[, \\ \xi \left(A_1 + 2A_2 + A_2 \log\left(\frac{\delta}{\xi}\right) - \frac{3}{4}\xi^{-\frac{3}{2}} \right) < 0 & \text{for } \alpha = 2. \end{cases}$$

Case 2: $\alpha = 1$ and $\xi \ge \delta$. To show (5.25), this is almost identical to the corresponding part of [23], and it suffices to choose γ small enough; we here omit the details.

Therefore, (5.22) holds, and it implies that $T_* = T^*$. Moreover, for every $t \in [0, T^*[$, we have $\|\nabla\rho^\pm(t)\|_{L^\infty} \le \omega_\lambda'(0) = \lambda^\alpha$ with λ defined by (5.10). This estimate combining with the breakdown criterion (5.1) yields that $T^* = \infty$.

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Appendix

In this section, we consider the Groma–Balogh model with generalized dissipation

$$\begin{cases} \partial_t \rho^+ + u\cdot\nabla\rho^+ + |D|^\alpha \rho^+ = 0, & \alpha\in]0,2], \\ \partial_t \rho^- - u\cdot\nabla\rho^- + |D|^\beta \rho^- = 0, & \beta\in]0,2], \\ u = (\mathcal{R}_1^2\mathcal{R}_2^2(\rho^+ - \rho^-), 0), \\ \rho^+|_{t=0} = \rho_0^+, \quad \rho^-|_{t=0} = \rho_0^-. \end{cases} \tag{A.1}$$

In terms of the dislocation densities $\theta^\pm \triangleq \partial_1\rho^\pm$, we write

$$\begin{cases} \partial_t \theta^+ + \partial_1(u_1\theta^+) + |D|^\alpha \theta^+ = 0, & \alpha\in]0,2], \\ \partial_t \theta^- - \partial_1(u_1\theta^-) + |D|^\beta \theta^- = 0, & \beta\in]0,2], \\ u_1 = \mathcal{R}_1\mathcal{R}_2^2|D|^{-1}(\theta^+ - \theta^-), \\ \theta^+|_{t=0} = \theta_0^+, \quad \theta^-|_{t=0} = \theta_0^-. \end{cases} \tag{A.2}$$

Similarly as Theorem 1.4, we get the following global result in the subcritical regime.

Proposition A.1. *Let* $(\alpha,\beta)\in]1,2]^2$, $\alpha\neq\beta$, $(\theta_0^+,\theta_0^-)\in H^m(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)\cap L^{\infty,1}_{x_2,x_1}(\mathbb{R}^2)$ *with* $m>4$, $p\in]1,2[$ *be composed of non-negative real scalar functions. Assume that* $\rho_0^\pm(x_1,x_2)=\int_{-\infty}^{x_1}\theta_0^\pm(\tilde{x}_1,x_2)d\tilde{x}_1$ *satisfy that for each* $k=1,2,3$, $\partial_2^k\rho_0^\pm\in L^\infty_x(\mathbb{R}^2)$ *and* $\lim_{x_1\to-\infty}\partial_2^k\rho_0^\pm(x)=0$ *for every* $x_2\in\mathbb{R}$. *Then there exists a unique global solution*

$$(\theta^+,\theta^-)\in C([0,\infty[;H^m\cap L^p)\cap L^\infty([0,\infty[;L^{\infty,1}_{x_2,x_1})$$

to (A.2). *Moreover,* $(\rho^+,\rho^-)\in L^\infty([0,\infty[;W^{3,\infty})\cap C([0,\infty[;W^{1,\infty})$ *solves* (A.1) *in the classical pointwise sense.*

Remark A.2. When $1=\alpha<\beta\le 2$ or $1=\beta<\alpha\le 2$, in a similar way we can obtain the same global result under the condition that the norm $\|(\theta_0^+,\theta_0^-)\|_{L^{\infty,1}_{x_2,x_1}}$ is small enough.

Proof of Proposition A.1. Note that Theorem 1.1 and Proposition 1.2 also hold for the systems (A.1) and (A.2), and it remains to show that for every $T\in]0,T^*[$, there is an upper bound of the quantity $\int_0^T \|(\partial_1\rho^+,\partial_1\rho^-)(t)\|_{L^\infty}dt$.

With no loss of generality, we fix $1<\alpha<\beta\le 2$ in the sequel. Let ω be an appropriate MOC chosen later, and denote

$$\omega_\lambda(\xi)=\lambda^{\alpha-1}\omega(\lambda\xi), \quad \forall\,\xi>0.$$

Let $\lambda\ge 1$ be defined by (5.10) with $\alpha\in]1,2]$ (if the quantity in (5.10) is less than 1, set $\lambda=1$), similarly as in Sec. 5, we get $\rho_0^\pm$ strictly satisfy the MOC ω_λ. Let T_* be defined by (5.11), we also find that Lemma 5.1 holds true, and it suffices to show that (5.22) is satisfied.

From (A.1) we have

$$(f^+)'(T_*) = -u\cdot\nabla\rho^+(T_*,x^+) + u\cdot\nabla\rho^+(T_*,y^+) + [-|D|^\alpha]\rho^+(T_*,x^+) - [-|D|^\alpha]\rho^+(T_*,y^+),$$

and

$$(f^-)'(T_*) = u\cdot\nabla\rho^-(T_*,x^-) - u\cdot\nabla\rho^-(T_*,y^-) + [-|D|^\beta]\rho^-(T_*,x^-) - [-|D|^\beta]\rho^-(T_*,y^-),$$

with $u = (\mathcal{R}_1^2\mathcal{R}_2^2(\rho^+ - \rho^-), 0)$. By Lemmas 2.7, 2.8 and the change of variable, we obtain that

$$\begin{cases} (f^+)'(T_*) \le \lambda^{2\alpha-1}(\Omega\omega' + \Psi_\alpha)(\lambda\xi^+) & \text{for (i), (iii)},\\ (f^-)'(T_*) \le \lambda^{2\alpha-1}(\Omega\omega' + \lambda^{\beta-\alpha}\Psi_\beta)(\lambda\xi^-) & \\ \qquad\qquad \le \lambda^{2\alpha-1}(\Omega\omega' + \Psi_\beta)(\lambda\xi^-) & \text{for (ii), (iii)}, \end{cases}$$

where Ω is defined by (2.3) corresponding to ω, and Ψ_α, Ψ_β are defined by (2.5).

Next we construct suitable MOC satisfying (5.12). Let $0 < \delta < 1$ be a fixed constant chosen later; then for every $1 < \alpha < \beta \le 2$, we define the following continuous function

$$\text{MOC}\begin{cases} \omega(\xi) = \xi - \xi^{3/2} & \text{for } \xi \in [0,\delta],\\ \omega'(\xi) = 0 & \text{for } \xi \in]\delta,\infty[. \end{cases} \tag{A.3}$$

Then our target is to prove that for the suitable MOC given by (A.3),

$$\Omega(\xi)\omega'(\xi) + \Psi_\alpha(\xi) < 0, \quad \forall\,\xi \in]0,\delta], \tag{A.4}$$

and

$$\Omega(\xi)\omega'(\xi) + \Psi_\beta(\xi) < 0, \quad \forall\,\xi \in]0,\delta]. \tag{A.5}$$

Similarly as proving (5.25), for appropriate positive constants δ that may depend on α, β, we can show (A.4) and (A.5) are satisfied.

Therefore, we have $T_* = T^*$. Moreover, for every $t \in [0, T^*[$, we have $\|\nabla\rho^\pm(t)\|_{L^\infty} \le \lambda^\alpha$. This combining with the breakdown criterion (5.1) yields $T^* = \infty$. □

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