PAPER

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To cite this article: Yatao Li *et al* 2023 *Nonlinearity* **36** 3866

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **36** (2023) 3866–3908 https://doi.org/10.1088/1361-6544/acda75

Global well-posedness for 2D fractional inhomogeneous Navier–Stokes equations with rough density

Yatao Li[1,2], **Qianyun Miao**[3] and **Liutang Xue**[2,*]

[1] Department of Mathematics, Jiangxi University of Finance and Economics, Nanchang 330032, People's Republic of China
[2] Laboratory of Mathematics and Complex Systems (MOE), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China
[3] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People's Republic of China

E-mail: xuelt@bnu.edu.cn

Received 12 April 2022; revised 1 May 2023
Accepted for publication 31 May 2023
Published 13 June 2023

Recommended by Dr Jean-Luc Thiffeault

Abstract

This paper is concerned with the global well-posedness issue of the two-dimensional (2D) incompressible inhomogeneous Navier–Stokes equations with fractional dissipation and rough density. By establishing the $L^q_t(L^p_x)$-maximal regularity estimate for the generalized Stokes system and using the Lagrangian approach, we prove the global existence and uniqueness of regular solutions for the 2D fractional inhomogeneous Navier–Stokes equations with large velocity field, provided that the initial density is sufficiently close to the constant 1 in $L^2 \cap L^\infty$ and in the norm of some multiplier spaces. Moreover, we also consider the associated density patch problem, and show the global persistence of $C^{1,\gamma}$-regularity of the density patch boundary when the piecewise jump of density is small enough.

Keywords: fractional inhomogeneous Navier–Stokes equations, maximal regularity, Lagrangian method, density patch.
Mathematics Subject Classification numbers: Primary 35Q30, 76D05, 35B40

* Author to whom any correspondence should be addressed.

1. Introduction

We consider the two-dimensional (2D) incompressible fractional inhomogeneous Navier–Stokes (abbr. fINS) equations:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho(\partial_t u + u\cdot\nabla u) + \nu\Lambda^{2\alpha}u + \nabla\pi = 0, \\ \mathrm{div}\, u = 0, \\ (\rho, u)|_{t=0}(x) = (\rho_0(x), u_0(x)), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^2$, $\nu > 0$ is the kinematic viscosity coefficient, the scalar ρ is the density, $u = (u^1, u^2)$ represents the velocity field, and π stands for the pressure of the fluid. The fractional Laplacian operator $\Lambda^{2\alpha} := (-\Delta)^\alpha$ for $\alpha \in (0,1)$ is a nonlocal operator that is defined by the Fourier transform via

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform of f. From the viewpoint of the stochastic process, the fractional Laplacian $\Lambda^{2\alpha}$ is an infinitesimal generator of the symmetric 2α-stable Lévy process (e.g. see [1]). When $\rho \equiv 1$ and $\nu = 1$, (1.1) reduces to the 2D incompressible fractional (homogeneous) Navier–Stokes equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \Lambda^{2\alpha}u + \nabla\pi = 0, \\ \mathrm{div}\, u = 0, \quad u|_{t=0}(x) = u_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^2$, $\alpha \in (0,1)$. The system (1.2) was first proposed by Frisch *et al* [23] and later was used in modelling a fluid motion with internal friction interaction [42]. Recently many purely analytic results have dedicated to the mathematical research of (1.2); for example one can see [10, 34, 55, 56] and references therein. Compared with (1.2), the density-dependent system (1.1) can describe the dynamics of flows with variable densities.

In the past decades, there have been a lot of works on the fractionally dissipative systems arising from many physical applications. The fractional Laplacian operators describe various phenomena in hydrodynamics [4, 31, 32], fractional quantum mechanics [35], anomalous diffusion in semiconductor growth [53], physics and chemistry [43, 50]. We also mention a related fractionally dissipative model, known as the Euler-alignment system,

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho(\partial_t u + u\cdot\nabla u) + \mathcal{D}(u,\rho) + \nabla p(\rho) = 0, \\ \mathcal{D}(u,\rho) = \rho\big(u\Lambda^{2\alpha}\rho - \Lambda^{2\alpha}(u\rho)\big) = c_\alpha\,\rho\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{|x-y|^{d+2\alpha}}\rho(y)\mathrm{d}y, \\ (\rho, u)|_{t=0}(x) = (\rho_0(x), u_0(x)), \end{cases} \tag{1.3}$$

where $x \in \mathbb{R}^d$, $d \geqslant 1$, $\alpha \in (0,1)$, $c_\alpha > 0$, ρ is the density, u is the velocity field, and $p(\rho) = \kappa\rho^\gamma$, $\kappa > 0$, $\gamma \geqslant 1$ is the pressure. The Euler-alignment system (1.3) is the hydrodynamic limit model [30] of the Cucker–Smale kinetic model which describes the flocking phenomenon of animal groups, and one can see [6, 9, 13] for the recent mathematical studies. The model (1.3) can be viewed as a compressible Euler system with fractional dissipation, and thus one may formally view system (1.1) as an intermediate model between fractional Navier–Stokes (1.2) and the Euler-alignment system (1.3).

When $\alpha = 1$ and $\nu = 1$, the fINS system (1.1) corresponds to the classical incompressible inhomogeneous Navier–Stokes (abbr. INS) system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{1.4}$$

where $x \in \mathbb{R}^d$, $d = 2, 3$. The INS system (1.4) originates in describing the dynamics of geophysical flows which are incompressible and also have variable densities [41]. When $\rho \equiv 1$, the system (1.4) becomes exactly the classical incompressible Navier–Stokes equations. System (1.4) has been extensively investigated in recent decades. When the density is bounded and (ρ_0, u_0) has finite energy, the global existence of weak solutions with finite energy for INS system (1.4) was obtained in [41, 49]. If the density is bounded and smooth enough (at least continuous with some fractional derivatives in Lebesgue spaces), the global existence and uniqueness results can be obtained for the INS system (1.4) in dimension two with large initial data, and in dimension three under a smallness condition of the velocity (e.g. see [8, 14, 33]). The case of the rough density admitting piecewise constant densities is of much interest, and can be used in modelling a mixture of two fluids. Danchin and Mucha [18] developed a novel Lagrangian approach to address the uniqueness in the rough density case, and proved the global existence and uniqueness of regular solutions to the INS system (1.4) in the critical Besov spaces setting, under a smallness condition on the initial velocity and the jumps of initial density. Huang, Paicu and Zhang [28] in the 2D case removed the smallness condition on the initial velocity in [18] and got the global well-posedness of solutions by assuming that the jumps of initial density is sufficiently small (depending on the size of the velocity). Danchin and Mucha [19] proved the local-in-time existence and uniqueness result of the INS system without the smallness condition on the jumps of initial density, and they also established the global result in dimension two (and in 3D case with additional smallness on velocity) if the density is close to a positive constant. Paicu *et al* [48] moreover showed the global well-posedness of solutions to the INS system (1.4) in dimension two with initial density only being bounded from above and below by some positive constants. For bounded initial densities admitting vacuum states, Danchin and Mucha [20] obtained the global existence and uniqueness result for the INS system (1.4) in either a periodic torus $\mathbb{T}^d$ or a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. Note that all the above articles dealing with the rough density case essentially apply the Lagrangian method to show the uniqueness, and one can also see Constantin *et al* [11, 12] for such a method applied to the related hydrodynamic models.

Another interesting and closely related result on the discontinuous density for the INS system (1.4) is the study of the so-called *density patch problem*, which was first raised by Lions [41] regarding the density patch $\rho_0 = 1_{\Omega_0}$ with $\Omega_0 \subset \mathbb{R}^d$ a smooth simple-connected domain. Since the density solves the transport equation, it formally yields $\rho(t,x) = \rho_0(X_t^{-1}(x))$ with X_t^{-1} the inverse of X_t and $X_t(\cdot)$ the particle-trajectory satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = u(t, X_t(x)), \qquad X_t(x)|_{t=0} = x. \tag{1.5}$$

Thus one has that $\rho(t,x) = 1_{\Omega(t)}(x)$ with $\Omega(t) := X_t(\Omega_0)$. The *density patch problem* asks whether or not the initial smoothness of the density patch boundary persists globally in time. The aforementioned works [18, 19, 28, 48] ensure the global well-posedness of patch solution for INS system (1.4) associated with the density patch initial data in various situations, and showed the global persistence of either C^1- or $C^{1,\gamma}$-regularity of the evolutionary patch boundary. One can also refer to [7, 21, 24, 38–40] concerning the global persistence of higher boundary regularity of the density patch.

When $\nu = 0$, the system (1.1) becomes the density-dependent incompressible Euler equations, for which the classical incompressible Euler system is a special case. If the initial data has enough regularity (at least the initial velocity is Lipschitz continuous and the gradient of initial density is continuous and bounded), the local existence and uniqueness results for the density-dependent Euler equations can be obtained in many kinds of functional spaces, and one can refer to [3, 15, 16] and references therein.

As for the fINS equations (1.1) with $\nu > 0$, the existing articles [22, 52] only deal with the hyper-dissipative case with $\alpha \geqslant \frac{5}{4}$ in dimension three, and the global well-posedness results have been established in this case.

Our main goal in this paper is to show the global-in-time existence and uniqueness result of 2D fINS equations (1.1) with large initial velocity field and rough initial density which admits jump discontinuities. We restrict to the case $\frac{1}{2} < \alpha < 1$, which is reasonable from the maximal regularity of the fractional Laplacian operator $\Lambda^{2\alpha}$ and the need for the velocity to be at least Lipschitz continuous.

We also remark that it seems very hard to generalize the global results of [20, 48] to the 2D fINS equations (1.1) with $\frac{1}{2} < \alpha < 1$ and the density being merely bounded. One reason is the intrinsic difference between the $\alpha = 1$ case and the $\alpha < 1$ case[4]: noting that the 2D fINS equations (1.1) are scale-invariant under the following transformation

$$\rho(x,t) \mapsto \rho(\lambda x, \lambda^{2\alpha} t), \quad u(x,t) \mapsto \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \quad \pi(x,t) \mapsto \lambda^{4\alpha-2} \pi(\lambda x, \lambda^{2\alpha} t),$$

for every $\lambda > 0$, and in combination with the classical L^2-energy estimate, one can view the $\alpha = 1$ case as the energy critical case and the $\alpha < 1$ case as the energy supercritical case; thus the time-weighted energy estimates used in [20, 48] work for the critical $\alpha = 1$ case, but will not directly extend to the supercritical $\alpha < 1$ case.

Inspired by [28, 45–47], here we compare the solution of the 2D fINS equations (1.1) with the large solution of the 2D system (1.2) and study the stability issue. We assume $\nu = 1$ for brevity. More precisely, let $\bar{u}(x,t) = (\bar{u}^1, \bar{u}^2)(x,t)$ be a 2D vector field solving the 2D fractional Navier–Stokes equations (1.2) with initial data u_0, and define

$$a := \rho - 1, \quad w := u - \bar{u}, \quad p := \pi - \bar{\pi}. \tag{1.6}$$

We investigate the following perturbed system

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t w + u \cdot \nabla w + \Lambda^{2\alpha} w + \nabla p = F, \\ \operatorname{div} w = 0, \\ (a, w)|_{t=0} = (\rho_0 - 1, 0), \end{cases} \tag{1.7}$$

where

$$F := -a\partial_t w - a\partial_t \bar{u} - a(u \cdot \nabla w) - a(\bar{u} \cdot \nabla \bar{u}) - \rho(w \cdot \nabla \bar{u}). \tag{1.8}$$

Our main result reads as follows.

[4] This is analogous to the difference of the 2D Navier–Stokes equations (1.2) between $\alpha = 1$ case and $\alpha < 1$ case; however, the system (1.2) has additional uniform bounded quantity, that is, the vorticity $\omega = \operatorname{curl} u$ is uniformly bounded, which makes the L^2-energy supercritical $\alpha < 1$ case be globally well-posed.

Theorem 1.1. *Let* $\frac{1}{2} < \alpha < 1$, $p > \frac{2}{2\alpha-1}$ *and* $u_0 \in H^1 \cap \dot{B}^{\alpha}_{p,2}(\mathbb{R}^2)$, $\rho_0 - 1 \in L^2 \cap L^\infty(\mathbb{R}^2)$. *There exists a generic constant* $c_0 \in (0,1)$ *depending only on* α *and* p *such that if*

$$\|\rho_0 - 1\|_{L^2\cap L^\infty} \leqslant c_0 \exp\Big\{ - c_0^{-1} \|u_0\|^2_{H^1\cap \dot{B}^{\alpha}_{p,2}}\Big\}, \tag{1.9}$$

then the 2D fINS system (1.1) has a global-in-time strong solution $(\rho, u, \nabla\pi)$ *satisfying the estimates*

$$\|\rho - 1\|_{L^\infty(\mathbb{R}_+;L^2\cap L^\infty(\mathbb{R}^2))} \leqslant \|\rho_0 - 1\|_{L^2\cap L^\infty(\mathbb{R}^2)}, \quad \textit{and} \tag{1.10}$$

$$\|u\|_{L^\infty(\mathbb{R}_+;L^2\cap\dot{B}^{\alpha}_{p,2})} + \|\big(\partial_t u, \Lambda^{2\alpha}u, \nabla\pi\big)\|_{L^2(\mathbb{R}_+;L^p)} + \|u\|_{L^2(\mathbb{R}_+;\dot{H}^\alpha)} \leqslant C\left(1 + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right), \tag{1.11}$$

with C > 0 *a constant depending only on* α *and* p.

If we additionally assume that $u_0 \in \dot{B}^{\alpha+s}_{p,2}(\mathbb{R}^2)$ *with* $s \in (0,1)$ *and* $\rho_0 - 1 \in \mathcal{M}(\dot{B}^{s}_{p,2}) \cap \mathcal{M}(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1})$ *satisfying that for a sufficiently small generic constant* $c_* > 0$ *(depending only on* α, p, s*),*

$$\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{s}_{p,2})\cap\mathcal{M}\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} \leqslant c_* \exp\left\{-c_*^{-1}\left(1 + \|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\right\}, \tag{1.12}$$

then the above constructed solution is unique, and u *also satisfies that*

$$\|u\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\alpha+s}_{p,2})} + \|u\|_{L^2(\mathbb{R}_+;\dot{B}^{2\alpha}_{p,1}\cap\dot{B}^{2\alpha+s}_{p,2})} + \|u\|_{L^1(\mathbb{R}_+;\dot{W}^{1,\infty})} \leqslant C\left(1 + \|u_0\|_{\dot{B}^{\alpha+s}_{p,2}} + \|u_0\|^{\frac{8(4\alpha-1)}{(2\alpha-1)^2}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right), \tag{1.13}$$

with C > 0 *a constant depending only on* α, p *and* s.

In the above, $\mathcal{M}(\dot{B}^{s}_{p,r})$ denotes the multiplier space (see definition 2.2 below).

Remark 1.2. We apply the Lagrangian method in the uniqueness part of theorem 1.1. Owing to the nonlocal effect of fractional Laplacian operator $\Lambda^{2\alpha}$, it seems that we need a little bit more regularity of u than the obtained regularity in the existence part. More precisely, in view of (6.51) below, one has to control $\|u\|_{L^2_T(\dot{B}^{2\alpha}_{p,1})}$ (from which $\|v\|_{L^2_T(\dot{B}^{2\alpha}_{p,1})}$ is bounded), which is essentially stronger than the quantity $\|u\|_{L^2_T(\dot{W}^{2\alpha,p})}$ in (1.11). So we additionally assume u_0 is slightly more regular and $\rho_0 - 1$ is small enough in the norm of some multiplier spaces, and we build the refined estimate (1.13). It should be emphasized that, thanks to lemma 2.5 below, the multiplier spaces $\mathcal{M}(\dot{B}^{s}_{p,r})$ for small s contain the elements with piecewise jump discontinuity.

Next, we consider the density patch problem of the 2D fINS equations (1.1). As a direct consequence of theorem 1.1, we can show the global well-posedness result and the global persistence of $C^{1,\gamma}$-patch boundaries, as long as the density jump across the $C^{1,\gamma}$-interface is small enough.

Proposition 1.3. *Let* $\frac{1}{2} < \alpha < 1$, $p > \frac{2}{2\alpha-1}$, $u_0 \in H^1 \cap \dot{B}^{\alpha+s}_{p,2}(\mathbb{R}^2)$ *with* $0 < s < \frac{1}{p}$. *Assume that* Ω_0 *is a bounded simply-connected* $C^{1,\gamma}$*-domain of* $\mathbb{R}^2$ *with* $0 < \gamma \leqslant 2\alpha - 1 + s - \frac{2}{p}$ *and* $\rho_0 = (1+\sigma)1_{\Omega_0} + 1_{\Omega_0^c}$ *where* $\sigma \in \mathbb{R}$ *satisfies*

$$|\sigma| \leqslant c' \exp\left(-\frac{1}{c'}\left(1 + \|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\right), \tag{1.14}$$

with a generic small constant $c' > 0$ depending only on α, p, s and Ω_0, then the 2D fINS system (1.1) has a unique global solution (ρ, u) on $\mathbb{R}^2 \times \mathbb{R}_+$ satisfying the estimates (1.11) and (1.13). The density ρ has the following expression

$$\rho(t) = (1+\sigma)1_{\Omega(t)} + 1_{\Omega(t)^c} \quad \textit{with} \quad \Omega(t) = X_t(\Omega_0), \tag{1.15}$$

and the associated patch boundary $\partial\Omega(t) \in C^{1,\gamma}(\mathbb{R}^2)$ for every $t \in \mathbb{R}_+$.

Proof of proposition 1.3. Thanks to lemma 2.5 and the smallness condition (1.14), $\rho_0 - 1 = \sigma 1_{\Omega_0}$ belongs to $L^2 \cap L^\infty \cap \mathcal{M}(\dot{B}^s_{p,2}) \cap \mathcal{M}(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1})$ and it also fulfills (1.9) and (1.12), so theorem 1.1 guarantees that there is a unique global-in-time regular solution (ρ, u) to the 2D fINS system (1.1). The estimates (1.11), (1.13) and the continuous embedding $\dot{B}^{2\alpha+s}_{p,2} \cap \dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow \dot{C}^{1,\gamma}(\mathbb{R}^2)$ with $0 < \gamma \leqslant 2\alpha - 1 + s - 2/p$ imply that $u \in L^1(\mathbb{R}_+; \dot{W}^{1,\infty}(\mathbb{R}^2)) \cap L^1([0,T]; C^{1,\gamma}(\mathbb{R}^2))$ for any $T > 0$. By the Cauchy–Lipschitz theory, there exists a unique particle-trajectory $X_t(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ for every $t \in \mathbb{R}_+$ which solves (1.5) and is a measure-preserving bi-Lipschitzian homeomorphism with inverse X_t^{-1}. Besides, owing to [2, prop. 3.10], it is easy to see that

$$\|\nabla X_t^{\pm 1}\|_{\dot{C}^\gamma} \leqslant \int_0^t \|\nabla X_\tau^{\pm 1}\|_{\dot{C}^\gamma} \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau + \int_0^t \|\nabla X_\tau^{\pm 1}\|^{1+\gamma}_{L^\infty} \|\nabla u(\tau)\|_{\dot{C}^\gamma} \mathrm{d}\tau,$$

and thus

$$\|\nabla X_t^{\pm 1}\|_{L^\infty_T(\dot{C}^\gamma)} \leqslant \|\nabla X_t^{\pm 1}\|^{1+\gamma}_{L^\infty_T(L^\infty)} \|\nabla u\|_{L^1_T(\dot{C}^\gamma)} e^{\|\nabla u\|_{L^1_T(L^\infty)}} < \infty.$$

The method of characteristics gives that $\rho(x,t) = \rho_0(X_t^{-1}(x))$, which leads to (1.15). Since the initial boundary $\partial\Omega_0 \in C^{1,\gamma}$ and $X_t^{\pm 1} \in L^\infty_T(C^{1,\gamma})$, we conclude that the evolutionary patch boundary $\partial\Omega(t) \in L^\infty_T(C^{1,\gamma})$ with $0 < \gamma \leqslant 2\alpha - 1 + s - 2/p$ and $T > 0$ any given, as desired. □

Let us sketch the proof of theorem 1.1. By applying the technique of vector-valued Calderón–Zygmund operators in Lemarié–Rieusset [36, chapter 7], we first establish the $L^q_t(L^p_x)$ maximal regularity estimates for the generalized Stokes system with fractional dissipation in the whole space $\mathbb{R}^d$, which may be of independent interest. Next, by performing the L^2-energy estimate and the $L^2_t(L^p)$ maximal regularity estimate for the fractional Navier–Stokes equations (1.2) and the perturbed system (1.7), we build the *a priori* estimates for the 2D fINS equations (1.1). Then by an approximation process and the compactness argument, we show the existence part of theorem 1.1.

As for the uniqueness part, due to the hyperbolic property of the density equation $(1.1)_1$ and the low-regularity assumption of the density, we have to adopt the Lagrangian approach from [18, 19]. Due to the nonlocal effect of fractional Laplacian operator $\Lambda^{2\alpha}$, the process is more complicated than that in the INS equations (1.4). We rewrite the system of the difference of two velocities in Lagrangian coordinates as the twisted fractional Stokes system (6.18), and by making full use of the particle-trajectory technique and the finite-difference characterization of homogeneous Besov spaces, we establish the crucial $L^2_t(L^2_x)$-maximal regularity estimate (6.23) for the system (6.18) on a short time interval (one can see remark 6.3 below for some additional comments). Then, by carefully estimating the right-hand terms of (6.23), we can show the uniqueness for small time. Moreover, an iteration argument implies the uniqueness on the whole $\mathbb{R}_+$. Note that the uniqueness part needs the stronger regularity of solutions (as explained in remark 1.2), which is obtained in proposition 4.4 by using the technique of multiplier spaces. To the best of our knowledge, this seems to be the first result where the Lagrangian method is used to tackle the uniqueness issue for a fractionally dissipative system.

The rest of this paper is organized as follows. In section 2, we introduce the definitions of some functional spaces and their related estimates, and also gather several useful auxiliary lemmas. Section 3 focuses on the establishment of proposition 3.1 concerning the $L^q_t(L^p_x)$-maximal regularity estimate for the generalized Stokes system. Sections 4–6 are devoted to the proof of theorem 1.1, corresponding to the proof of the *a priori* estimates, existence, and uniqueness, respectively.

2. Preliminaries

In this section we compile some notations, definitions of some function spaces and auxiliary lemmas used in the paper.

The following notations will be used throughout the paper.

- The notation $a \lesssim b$ means $a \leqslant Cb$, where the constant C may be different from line to line. We sometimes use $C = C(\lambda_1, \lambda_2 \ldots, \lambda_n)$ to indicate the dependence on the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- For every $p, q \in [1, \infty]$, $k \in \mathbb{N}$, $s \in \mathbb{R}$, the function spaces $L^p(\mathbb{R}^d)$, $W^{k,p}(\mathbb{R}^d)$, $\dot{W}^{k,p}(\mathbb{R}^d)$, $\dot{W}^{s,p}(\mathbb{R}^d)$ denote the Lebesgue space, the Sobolev space, the homogeneous Sobolev space and the fractional-order Sobolev space, respectively (e.g. see [2]).
- Denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions (e.g. see [27]).
- We abbreviate $L^q(0,T;X)$ as $L^q_T(X)$, with $X = X(\mathbb{R}^d)$ a spatial function space. We write $\|(f_1, \ldots, f_n)\|_X$ for $\|f_1\|_X + \cdots \|f_n\|_X$ for $n \in \mathbb{Z}_+$.
- For two matrices $A = (a_{ij})_{d\times d}$ and $B = (b_{ij})_{d\times d}$, denote by $A : B$ the quantity $\sum_{1\leqslant i,j\leqslant d} a_{ij} b_{ji}$, and denote by A^{T} the transpose matrix of A.

2.1. Functional spaces and related estimates

We first recall some basic knowledge of the Littlewood–Paley theory. One can choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^d)$ (see [2]) supported respectively in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leqslant \frac{4}{3}\}$ and the annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^d; \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

The homogeneous dyadic operators $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd} h(2^j \cdot) * u, \quad \dot{S}_j u = \chi(2^{-j}D)u = 2^{jd}\widetilde{h}(2^j \cdot) * u,$$

with $h = \mathcal{F}^{-1}\varphi$, $\widetilde{h} = \mathcal{F}^{-1}\chi$ and $\mathcal{F}^{-1}$ the Fourier inverse transform.

Next, we introduce the definitions of the homogeneous Besov space and the related Chemin-Lerner's mixed spacetime space.

Definition 2.1. (1) Let $s \in \mathbb{R}$, $1 \leqslant p, r \leqslant \infty$. Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions and $P(\mathbb{R}^d)$ be the set of all polynomials. The homogeneous Besov space $\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^d)$ is defined as the following quotient space

$$\dot{B}^s_{p,r}(\mathbb{R}^d) = \Big\{ u \in \mathcal{S}'(\mathbb{R}^d)/P(\mathbb{R}^d) : \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \big\| \big\{ 2^{js} \|\dot{\Delta}_j u\|^r_{L^p(\mathbb{R}^d)} \big\}_{j\in\mathbb{Z}} \big\|_{\ell^r} < \infty \Big\},$$

where $\mathcal{S}'(\mathbb{R}^d) \setminus P(\mathbb{R}^d)$ is the quotient space.

(2) Let $s \in \mathbb{R}$, $1 \leqslant p,q,r \leqslant \infty$, and $T \in (0,\infty]$. The Chemin–Lerner's mixed spacetime homogeneous Besov space $\widetilde{L}^q(0,T;\dot{B}^s_{p,r}(\mathbb{R}^d))$, abbreviated as $\widetilde{L}^q_T(\dot{B}^s_{p,r})$, is defined as the set of all tempered distributions u such that

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} := \big\| \big\{ 2^{js} \|\dot{\Delta}_j u\|_{L^q_T(L^p)} \big\}_{j\in\mathbb{Z}} \big\|_{\ell^r} < \infty.$$

The following product estimate in the homogeneous Besov spaces is useful.

Lemma 2.1 ([2], Corollary 2.54). *Let* s > 0*,* $(p,r) \in [1,\infty]^2$*. Then there exists a constant* $C = C(s,d) > 0$ *such that*

$$\|uv\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \leqslant C\big(\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^d)} \big). \tag{2.1}$$

We have the characterization of homogeneous Besov spaces in terms of the fractional heat semigroup $e^{-t\Lambda^{2\alpha}}$.

Lemma 2.2 ([44], Proposition 2.1). *Let* s > 0*,* $(p,r) \in [1,\infty]^2$*. Then, for any* $\varphi \in \dot{B}^{-s}_{p,r}(\mathbb{R}^d)$*, there exists a constant* $C = C(s,p,r,d) \geqslant 1$ *such that*

$$C^{-1}\|\varphi\|_{\dot{B}^{-s}_{p,r}(\mathbb{R}^d)} \leqslant \big\| t^{\frac{s}{2\alpha}} \|e^{-t\Lambda^{2\alpha}}\varphi\|_{L^p(\mathbb{R}^d)} \big\|_{L^r(\mathbb{R}_+;\frac{dt}{t})} \leqslant C\|\varphi\|_{\dot{B}^{-s}_{p,r}(\mathbb{R}^d)}.$$

We also use the following finite-difference characterization of homogeneous Besov spaces.

Lemma 2.3 ([2], Theorem 2.36). *Let* $s \in (0,1)$ *and* $(p,r) \in [1,\infty]^2$*. Then there exists a constant* $C = C(s,p,r,d) \geqslant 1$ *such that*

$$C^{-1}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \leqslant \left\| \frac{\|u(y+\cdot) - u(\cdot)\|_{L^p}}{|y|^s} \right\|_{L^r(\mathbb{R}^d;\frac{dy}{|y|^d})} \leqslant C\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)}. \tag{2.2}$$

The maximal regularity estimate in the framework of Besov spaces for the fractional heat equation is useful in the sequel.

Lemma 2.4 ([54], Theorem 3.2). *Let* $s \in \mathbb{R}$*,* $(p,r) \in [1,\infty]^2$*,* $\alpha \in (0,1)$ *and* $1 \leqslant \rho_1 \leqslant \rho \leqslant \infty$*. Assume that* $f_0 \in \dot{B}^s_{p,r}(\mathbb{R}^d)$*,* $g \in \widetilde{L}^{\rho_1}_T(\dot{B}^{s-2\alpha+\frac{2\alpha}{\rho_1}}_{p,r}(\mathbb{R}^d))$ *and* f *solves the fractional heat equation*

$$\partial_t f + \Lambda^{2\alpha} f = g, \quad f|_{t=0} = f_0.$$

Then there exists a constant $C = C(d,\alpha) > 0$ *such that for every* $T \in (0,\infty]$*,*

$$\|f\|_{\widetilde{L}^\rho_T\left(\dot{B}^{s+\frac{2\alpha}{\rho}}_{p,r}\right)} \leqslant C\left(\|f_0\|_{\dot{B}^s_{p,r}} + \|g\|_{\widetilde{L}^{\rho_1}_T\left(\dot{B}^{s-2\alpha+\frac{2\alpha}{\rho_1}}_{p,r}\right)} \right). \tag{2.3}$$

The (pointwise) multiplier space of a homogeneous Besov space is defined as follows.

Definition 2.2. Let $1 \leqslant p,r \leqslant \infty$, $\sigma \in \mathbb{R}$. The multiplier space $\mathcal{M}(\dot{B}^\sigma_{p,r}(\mathbb{R}^d))$ of $\dot{B}^\sigma_{p,r}(\mathbb{R}^d)$, abbreviated as $\mathcal{M}(\dot{B}^\sigma_{p,r})$, is the set of tempered distributions f such that $f\phi \in \dot{B}^\sigma_{p,r}(\mathbb{R}^d)$ for any $\phi \in \dot{B}^\sigma_{p,r}(\mathbb{R}^d)$, with the norm

$$\|f\|_{\mathcal{M}(\dot{B}^{\sigma}_{p,r})} := \sup_{\|\phi\|_{\dot{B}^{\sigma}_{p,r}(\mathbb{R}^d)} \leqslant 1} \|f\phi\|_{\dot{B}^{\sigma}_{p,r}(\mathbb{R}^d)}.$$

The following lemma states that patches can be elements of the multiplier space $\mathcal{M}(\dot{B}^{\sigma}_{p,r})$.

Lemma 2.5 ([18], Lemma A.7). *Let Ω be the half-space $\mathbb{R}^d_+$ or a bounded domain of $\mathbb{R}^d$ with C^1-boundary. Assume that $s \in \mathbb{R}$ and $p, r \in [1,\infty]$ are such that $-1+\frac{1}{p} < s < \frac{1}{p}$. Then the characteristic function $1_\Omega(x)$ of Ω belongs to the multiplier space $\mathcal{M}(\dot{B}^{s}_{p,r}(\mathbb{R}^d))$.*

We have the following regularity propagation estimates for a composite function involving the particle-trajectory map.

Lemma 2.6. *Let $T \in (0,\infty]$, $(q,r) \in [1,\infty]^2$, and $u \in L^1(0,T;\mathrm{Lip}(\mathbb{R}^d))$ be an incompressible vector field. Let $X_t : \mathbb{R}^d \to \mathbb{R}^d$ be the particle-trajectory map defined by (1.5) with its inverse X_t^{-1}. The following statements hold.*

(1) If $f \in L^q(0,T;\dot{B}^{\sigma}_{p,r}(\mathbb{R}^d))$, $\sigma \in (-1,1)$, then $f \circ X_t^{\pm 1} \in L^q(0,T;\dot{B}^{\sigma}_{p,r}(\mathbb{R}^d))$ with

$$\|f \circ X_t^{\pm 1}\|_{L^q_T(\dot{B}^{\sigma}_{p,r})} \leqslant C\|f\|_{L^q_T(\dot{B}^{\sigma}_{p,r})} e^{C\int_0^T \|\nabla u\|_{L^\infty} \mathrm{d}t}. \tag{2.4}$$

(2) Assume $a_0 \in \mathcal{M}(\dot{B}^{\sigma}_{p,r}(\mathbb{R}^d))$, $\sigma \in (-1,1)$, and $a(t,x)$ is a smooth solution to the free transport equation $\partial_t a + u \cdot \nabla a = 0$ associated with $a|_{t=0} = a_0$. Then $a \in L^\infty(0,T;\mathcal{M}(\dot{B}^{\sigma}_{p,r}))$ with

$$\|a\|_{L^\infty_T(\mathcal{M}(\dot{B}^{\sigma}_{p,r}))} \leqslant C\|a_0\|_{\mathcal{M}(\dot{B}^{\sigma}_{p,r})} e^{C\int_0^T \|\nabla u\|_{L^\infty} \mathrm{d}t}. \tag{2.5}$$

Proof of lemma 2.6. If $q = \infty$, $r = 1$ and $p \in (2,4)$, both inequalities (2.4) and (2.5) have appeared in [28, section 5]. Here we sketch the proof of the slightly generalized cases.

(1) Lemma 2.7 of [2] yields

$$\|\dot{\Delta}_j((\dot{\Delta}_k f) \circ X_t^{\pm 1})\|_{L^p} \leqslant C c_k 2^{-k\sigma} \|f\|_{\dot{B}^{\sigma}_{p,r}} \min\{2^{j-k}, 2^{k-j}\} e^{C\int_0^t \|\nabla u\|_{L^\infty} \mathrm{d}\tau},$$

where $\|c_k\|_{\ell^r(\mathbb{Z})} = 1$. Together with the condition $\sigma \in (-1,1)$ we deduce that

$$\begin{aligned}\|\dot{\Delta}_j(f \circ X_t^{\pm 1})\|_{L^p} &\leqslant \Big(\sum_{k<j} + \sum_{k\geqslant j}\Big) \|\dot{\Delta}_j((\dot{\Delta}_k f) \circ X_t^{\pm 1})\|_{L^p} \\ &\leqslant C\Big(\sum_{k<j} 2^{k-j} c_k 2^{-k\sigma} + \sum_{k\geqslant j} 2^{j-k} c_k 2^{-k\sigma}\Big) \|f\|_{\dot{B}^{\sigma}_{p,r}} e^{C\int_0^T \|\nabla u\|_{L^\infty} \mathrm{d}t} \\ &\leqslant C c_j 2^{-j\sigma} \|f\|_{\dot{B}^{\sigma}_{p,r}} e^{C\int_0^T \|\nabla u\|_{L^\infty} \mathrm{d}t}.\end{aligned}$$

Taking the ℓ^r-norm over $j \in \mathbb{Z}$ and then taking the L^q-norm on $[0,T]$ leads to (2.4), as desired.

(2) Note that $a(t,x) = a_0 \circ X_t^{-1}(x)$. By virtue of the definition of the multiplier space $\mathcal{M}(\dot{B}^{\sigma}_{p,r})$, the measure-preserving property of $X_t^{\pm 1}$ and (2.4), the inequality (2.5) can be easily deduced (e.g. see [28, proposition 5.1]). □

2.2. Auxiliary lemmas

We list some useful results related to the fractional Laplacian Λ^{α}.

Lemma 2.7 ([27], Theorem 7.6.1). *Let $1<r<\infty$ and $1<p_1,p_2,q_1,q_2\leqslant\infty$ satisfy $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}$. Given $s>0$, then there exists a constant $C=C(d,s,r,p_1,p_2,q_1,q_2)>0$ such that for every $f,g\in\mathcal{S}(\mathbb{R}^d)$,*

$$\|\Lambda^s(fg)\|_{L^r(\mathbb{R}^d)}\leqslant C\big(\|\Lambda^s f\|_{L^{p_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}(\mathbb{R}^d)}+\|f\|_{L^{q_1}(\mathbb{R}^d)}\|\Lambda^s g\|_{L^{q_2}(\mathbb{R}^d)}\big). \tag{2.6}$$

Lemma 2.8 ([44], Lemmas 2.1, 2.2). *Let $K(x)$ be the kernel function of the fractional heat semigroup $e^{-\Lambda^{2\alpha}}$, $\alpha\in(0,1)$, that is,*

$$K(x)=\mathcal{F}^{-1}(e^{-|\xi|^{2\alpha}})=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}e^{-|\xi|^{2\alpha}}\,\mathrm{d}\xi. \tag{2.7}$$

Then there exists a positive constant $C=C(d,\alpha)$ such that

$$|K(x)|\leqslant C(1+|x|)^{-d-2\alpha},\quad \forall x\in\mathbb{R}^d; \tag{2.8}$$

and for every $\beta>0$, there exists a positive constant $C=C(d,\alpha,\beta)$ such that

$$|\Lambda^{\beta}K(x)|+|\Lambda^{\beta-1}\nabla K(x)|\leqslant C(1+|x|)^{-d-\beta},\quad \forall x\in\mathbb{R}^d. \tag{2.9}$$

Lemma 2.9. *Let $\alpha\in(0,1)$, then for every $f\in\mathcal{S}(\mathbb{R}^d)$ and $i\in\{1,\ldots,d\}$, we have*

$$\partial_{x_i}\Lambda^{2\alpha-2}f(x)=c_\alpha\,\mathrm{p.v.}\int_{\mathbb{R}^d}\frac{x_i-y_i}{|x-y|^{d+2\alpha}}\big(f(x)-f(y)\big)\mathrm{d}y, \tag{2.10}$$

with $c_\alpha=\frac{(d+2\alpha-2)\Gamma(\frac{d}{2}-1+\alpha)}{\pi^{d/2}2^{2-2\alpha}\Gamma(1-\alpha)}$.

Proof of lemma 2.9. Recalling that (e.g. see [51, section 5.1])

$$\Lambda^{2\alpha-2}f(x)=\bar{c}_\alpha\int_{\mathbb{R}^d}\frac{1}{|y|^{d+2\alpha-2}}f(x-y)\mathrm{d}y,$$

with $\bar{c}_\alpha=\frac{\Gamma(\frac{d}{2}-1+\alpha)}{\pi^{d/2}2^{2-2\alpha}\Gamma(1-\alpha)}$, from the integration by parts, we get

$$\begin{aligned}\partial_{x_i}\Lambda^{2\alpha-2}f(x)&=\bar{c}_\alpha\lim_{\epsilon\to0}\int_{|y|\geqslant\epsilon}\frac{1}{|y|^{d+2\alpha-2}}\partial_{x_i}f(x-y)\mathrm{d}y\\&=\bar{c}_\alpha\lim_{\epsilon\to0}\int_{|y|\geqslant\epsilon}\frac{1}{|y|^{d+2\alpha-2}}\partial_{y_i}\big(f(x)-f(x-y)\big)\mathrm{d}y\\&=c_\alpha\lim_{\epsilon\to0}\int_{|y-x|\geqslant\epsilon}\frac{x_i-y_i}{|x-y|^{d+2\alpha}}\big(f(x)-f(y)\big)\mathrm{d}y+\lim_{\epsilon\to0}R_\epsilon,\end{aligned}$$

where $R_\epsilon:=\bar{c}_\alpha\int_{|y|=\epsilon}\frac{1}{|y|^{d+2\alpha-2}}\big(-\frac{y_i}{|y|}\big)\big(f(x)-f(x-y)\big)\mathrm{d}y$ satisfies that

$$\lim_{\epsilon\to0}|R_\epsilon|\leqslant\|\nabla f\|_{L^\infty}\lim_{\epsilon\to0}\frac{1}{\epsilon^{d+2\alpha-2}}\int_{|y|=\epsilon}|y|\mathrm{d}y\leqslant C\|\nabla f\|_{L^\infty}\lim_{\epsilon\to0}\epsilon^{2-2\alpha}=0.$$

□

In the proof of proposition 3.1 we use the following boundedness result about Calderón–Zygmund operators for vector-valued singular integrals, which can be found in [36, chapter 7].

Lemma 2.10 (Calderón–Zygmund operators). *Let $1 < p, p_1, p_2 < \infty$. Let X, X_1, X_2 be three locally compact σ-compact metric spaces, with regular Borel measures μ, μ_1, μ_2 on those spaces, respectively. Define $E = L^{p_1}(X_1, \mu_1)$ and $F = L^{p_2}(X_2, \mu_2)$. Let $L(x, y; x_1, x_2)$ be a continuous function defined on $(X \times X - \Delta) \times X_1 \times X_2$ (with Δ the diagonal set of $X \times X$), then we define $L(x, y)$ as the operator from $E = L^{p_1}(X_1, \mu_1)$ to $F = L^{p_2}(X_2, \mu_2)$ given by the integral*

$$L(x,y)f(x_2) = \int_{X_1} \mathcal{L}(x,y;x_1,x_2)f(x_1)\mathrm{d}\mu_1(x_1).$$

Whenever $x \notin \mathrm{supp} f$, we define $\mathcal{T}f(x)$ as

$$\begin{aligned}\mathcal{T}f(x,x_2) &= \int_X \int_{X_1} \mathcal{L}(x,y;x_1,x_2)f(y,x_1)\mathrm{d}\mu(y)\mathrm{d}\mu_1(x_1)\\ &= \int_X L(x,y)f(y,x_2)\mathrm{d}\mu(y).\end{aligned}$$

Suppose that the space X is equipped with a quasi-distance d satisfying the quasi-triangular inequality $d(x,y) \leqslant C_d\big(d(x,z) + d(z,y)\big)$ with $C_d > 0$ a generic constant; and there exists positive numbers n and C_μ such that $\mu(B(x,r)) \leqslant C_\mu r^n$ for every $x \in X$ and $\mathrm{r} > 0$. Assume that $\mathcal{T}$ is bounded from $L^p(X, \mu; E)$ to $L^p(X, \mu; F)$ such that

$$\int_{\mathbb{R}^k} \|\mathcal{T}f(x)\|_F^p \mathrm{d}x \leqslant C \int_{\mathbb{R}^k} \|f(x)\|_E^p \mathrm{d}x.$$

Assume that $\mathcal{L}(x,y)$ is continuous from $X \times X - \Delta$ to $\mathcal{L}(E,F)$ and satisfies that for some $\epsilon > 0$:

$$\begin{aligned} &\|L(x,y)\|_{\mathrm{op}(E\mapsto F)} \leqslant C\frac{1}{d(x,y)^n};\\ d(z,y) \leqslant \frac{1}{2}d(x,y) \Longrightarrow &\|L(x,y) - L(x,z)\|_{\mathrm{op}(E\mapsto F)} \leqslant C\frac{d(z,y)^\epsilon}{d(x,y)^{n+\epsilon}};\\ d(x,z) \leqslant \frac{1}{2}d(x,y) \Longrightarrow &\|L(x,y) - L(z,y)\|_{\mathrm{op}(E\mapsto F)} \leqslant C\frac{d(x,z)^\epsilon}{d(x,y)^{n+\epsilon}}.\end{aligned}$$

Then the operator $\mathcal{T}$ is bounded from $L^q(X;E)$ to $L^q(X;F)$ for any $1 < q < \infty$.

The following multiplier theorem can be found in theorem 3, section 4.3.2 of [51].

Lemma 2.11. *Assume that $m : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{C}$ is of class C^k with an integer $k \geqslant [\frac{d}{2}] + 1$ ($[\frac{d}{2}]$ is the integer part of $\frac{d}{2}$), and it satisfies that $|\partial_\xi^\beta m(\xi)| \leqslant C|\xi|^{-|\beta|}$ for every $\xi \neq 0$ and $|\beta| \leqslant k$. Then for any $f \in L^p(\mathbb{R}^d)$ with $1 < p < \infty$, there is a constant C such that*

$$\|m(D)f\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)}.$$

3. The $L_t^q(L^p)$-maximal regularity estimate for the generalized Stokes system

We mainly focus on showing the following result in this section.

Proposition 3.1. *Let $\alpha \in (0,1)$, $1 < p, q < \infty$, $u_0 \in \dot{B}_{p,q}^{2\alpha(1-\frac{1}{q})}(\mathbb{R}^d)$, $f \in L^q(0,T;L^p(\mathbb{R}^d))$, $R \in \dot{W}^{1,q}(0,T;L^p(\mathbb{R}^d))$ with $\mathrm{div}\, R \in L^q(0,T;\dot{W}^{2\alpha-1,p}(\mathbb{R}^d))$. Then the generalized Stokes system*

$$\begin{cases} \partial_t u + \Lambda^{2\alpha} u + \nabla\pi = f, \\ \mathrm{div}\, u = \mathrm{div}\, R, \\ u|_{t=0}(x) = u_0(x), \end{cases} \tag{3.1}$$

has a unique solution $(u, \nabla\pi)$. *Moreover, there exists a generic positive constant* $C = C(d,\alpha,p,q)$ *such that for any* $T \in (0,\infty]$,

$$\begin{aligned} &\|u\|_{L^\infty(0,T;\dot{B}^{2\alpha(1-1/q)}_{p,q}(\mathbb{R}^d))} + \|(\Lambda^{2\alpha}u, \partial_t u, \nabla\pi)\|_{L^q(0,T;L^p(\mathbb{R}^d))} \\ &\quad \leqslant C\Big(\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}(\mathbb{R}^d)} + \|(f,\partial_t R)\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|\operatorname{div} R\|_{L^q(0,T;\dot{W}^{2\alpha-1,p}(\mathbb{R}^d))}\Big). \end{aligned} \tag{3.2}$$

Remark 3.2. When $\alpha = 1$ one can refer to [19] for the $L^q_t(L^p)$-maximal regularity estimate for the usual Stokes system; and one can see [26, 29] concerning more general domains including exterior domains. For $\alpha \in (0,1)$, the $L^p_t(L^p)$ maximal regularity estimate for the fractional Stokes system (3.1) with $R = 0$ was also investigated by Giga *et al* [25, 26] using the abstract semigroup argument and by Cao *et al* [5] using the Fourier multiplier method.

The proof of proposition 3.1 relies on the following result, whose proof is placed below in this section.

Lemma 3.3. *Set*

$$\mathcal{A}_{2\alpha}f(x,t) := \int_0^t e^{-(t-s)\Lambda^{2\alpha}}\Lambda^{2\alpha}f(\cdot,s)\mathrm{d}s. \tag{3.3}$$

Then, for any $T \in (0,\infty]$ *and* $1 < p,q < \infty$, *the operator* $\mathcal{A}_{2\alpha}$ *is continuously bounded from* $L^q(0,T;L^p(\mathbb{R}^d))$ *to* $L^q(0,T;L^p(\mathbb{R}^d))$, *and there exists a constant* $C > 0$ *such that*

$$\|\mathcal{A}_{2\alpha}f\|_{L^q(0,T;L^p(\mathbb{R}^d))} \leqslant C\|f\|_{L^q(0,T;L^p(\mathbb{R}^d))}.$$

Proof of proposition 3.1. The existence and uniqueness of solutions to the system (3.1) are standard, and can be proved as those of the usual inhomogeneous Stokes system. Next we are devoted to proving the regularity estimate (3.2).

Taking the divergence of equation $(3.1)_1$ leads to

$$\Delta\pi = -\partial_t \operatorname{div} u - \Lambda^{2\alpha}\operatorname{div} u + \operatorname{div} f = -\operatorname{div}\partial_t R - \operatorname{div}\Lambda^{2\alpha}R + \operatorname{div} f,$$

thus denoting by $\mathcal{P} := \nabla\Delta^{-1}\operatorname{div}$, we see that

$$\nabla\pi = -\mathcal{P}\partial_t R - \mathcal{P}\Lambda^{2\alpha}R + \mathcal{P}f. \tag{3.4}$$

Rewriting the equation $(3.1)_1$ as

$$\partial_t u + \Lambda^{2\alpha}u = \mathcal{P}\partial_t R + \mathcal{P}\Lambda^{2\alpha}R + (\mathrm{Id} - \mathcal{P})f =: \bar{f}, \qquad u|_{t=0} = u_0, \tag{3.5}$$

we see that Duhamel's formula yields

$$u(x,t) = e^{-t\Lambda^{2\alpha}}u_0(x) + \int_0^t e^{-(t-\tau)\Lambda^{2\alpha}}\bar{f}(x,\tau)\mathrm{d}\tau, \tag{3.6}$$

where the semigroup operator $e^{-t\Lambda^{2\alpha}}$ is given by (3.12) below.

Applying the operator $\Lambda^{2\alpha}$ to the above formula and recalling (3.3) gives

$$\Lambda^{2\alpha}u(x,t) = e^{-t\Lambda^{2\alpha}}\Lambda^{2\alpha}u_0(x) + \mathcal{A}_{2\alpha}\bar{f}(x,t). \tag{3.7}$$

Then by virtue of lemmas 2.2 and 3.3, one finds that

$$\begin{aligned} \|\Lambda^{2\alpha}u\|_{L^q(0,T;L^p(\mathbb{R}^d))} &\leqslant \|e^{-t\Lambda^{2\alpha}}\Lambda^{2\alpha}u_0\|_{L^q(\mathbb{R}_+;L^p(\mathbb{R}^d))} + \|\mathcal{A}_{2\alpha}\bar{f}\|_{L^q(0,T;L^p(\mathbb{R}^d))} \\ &\leqslant C\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}} + C\|\bar{f}\|_{L^q(0,T;L^p)} \\ &\leqslant C\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}} + C\|(\partial_t R, f)\|_{L^q(0,T;L^p)} + C\|\operatorname{div} R\|_{L^q(0,T;\dot{W}^{2\alpha-1,p})}, \end{aligned} \tag{3.8}$$

where in the last line the L^p $(1 < p < \infty)$ boundedness property of the singular integral operators is also used. Thanks to (3.4), and using the Calderón–Zygmund theorem again, we get

$$\begin{aligned}\|\nabla\pi\|_{L^q(0,T;L^p(\mathbb{R}^d))} &\leqslant C\|\big(\mathcal{P}\partial_t R, \mathcal{P}\Lambda^{2\alpha}R, \mathcal{P}f\big)\|_{L^q(0,T;L^p)}\\ &\leqslant C\|(\partial_t R, f)\|_{L^q(0,T;L^p)} + C\|\operatorname{div}R\|_{L^q(0,T;\dot{W}^{2\alpha-1,p})}.\end{aligned} \tag{3.9}$$

We use the equation $(3.1)_1$ and gather the above estimates to infer that

$$\begin{aligned}\|\partial_t u\|_{L^q(0,T;L^p(\mathbb{R}^d))} &\leqslant C\|(\Lambda^{2\alpha}u, \nabla\pi, f)\|_{L^q(0,T;L^p(\mathbb{R}^d))}\\ &\leqslant C\|(\partial_t R, f)\|_{L^q(0,T;L^p)} + C\|\operatorname{div}R\|_{L^q(0,T;\dot{W}^{2\alpha-1,p})}.\end{aligned} \tag{3.10}$$

Noticing that $\|e^{-t\Lambda^{2\alpha}}u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}(\mathbb{R}^d)} \leqslant C\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}(\mathbb{R}^d)}$ (following from (2.8)) and

$$\begin{aligned}\|\mathcal{A}_{2\alpha}\bar{f}(\cdot,t)\|_{\dot{B}^{-2\alpha/q}_{p,q}(\mathbb{R}^d)} &\leqslant C\Big\|\|e^{-\tau'\Lambda^{2\alpha}}\mathcal{A}_{2\alpha}\bar{f}\|_{L^p(\mathbb{R}^d)}\Big\|_{L^q_{\tau'}(\mathbb{R}_+)}\\ &\leqslant C\Big\|\int_0^t e^{-(t+\tau'-\tau)\Lambda^{2\alpha}}\Lambda^{2\alpha}\bar{f}(x,\tau)\,\mathrm{d}\tau\Big\|_{L^q_{\tau'}(\mathbb{R}_+;L^p)}\\ &\leqslant C\Big\|\int_0^{t+\tau'} e^{-(t+\tau'-\tau)\Lambda^{2\alpha}}\Lambda^{2\alpha}\bar{f}(x,\tau)1_{[0,t]}(\tau)\,\mathrm{d}\tau\Big\|_{L^q_{\tau'}(\mathbb{R}_+;L^p)}\\ &\leqslant C\|\bar{f}(x,\tau)1_{[0,t]}(\tau)\|_{L^q_\tau(\mathbb{R}_+;L^p)} \leqslant C\|\bar{f}\|_{L^q(0,t;L^p)},\end{aligned}$$

it yields from (3.7) that

$$\begin{aligned}\|u\|_{L^\infty(0,T;\dot{B}^{2\alpha(1-1/q)}_{p,q})} &\leqslant C\|\Lambda^{2\alpha}u\|_{L^\infty(0,T;\dot{B}^{-2\alpha/q}_{p,q})}\\ &\leqslant C\|e^{-t\Lambda^{2\alpha}}\Lambda^{2\alpha}u_0\|_{L^\infty(0,T;\dot{B}^{-2\alpha/q}_{p,q})} + C\|\mathcal{A}_{2\alpha}\bar{f}\|_{L^\infty(0,T;\dot{B}^{-2\alpha/q}_{p,q})}\\ &\leqslant C\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}} + C\|\bar{f}\|_{L^q(0,T;L^p)}\\ &\leqslant C\|u_0\|_{\dot{B}^{2\alpha(1-1/q)}_{p,q}} + C\|(\partial_t R, f)\|_{L^q(0,T;L^p)} + C\|\operatorname{div}R\|_{L^q(0,T;\dot{W}^{2\alpha-1,p})}.\end{aligned} \tag{3.11}$$

Therefore, collecting estimates (3.8)–(3.11) we conclude that (3.2) holds, as desired. □

It remains to prove lemma 3.3.

Proof of lemma 3.3. The idea is analogous to that of [36, theorem 7.3] given by Lemarié-Rieusset. For every $\varphi \in L^p(\mathbb{R}^d)$, set

$$e^{-t\Lambda^{2\alpha}}\varphi(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}})(x) * \varphi(x) = K_t(x) * \varphi(x), \tag{3.12}$$

where $K_t(x) = t^{-\frac{d}{2\alpha}}K(\frac{x}{t^{1/2\alpha}})$. Noting that $\Lambda^{2\alpha}K_t(x) = t^{-\frac{d}{2\alpha}-1}(\Lambda^{2\alpha}K)(\frac{x}{t^{1/2\alpha}})$, we let $\Omega(x,t) := t^{-\frac{d}{2\alpha}}(\Lambda^{2\alpha}K)(\frac{x}{t^{1/2\alpha}})$, and then it follows that $\frac{1}{t}\Omega(x,t) = \mathcal{F}^{-1}(|\xi|^{2\alpha}e^{-t|\xi|^{2\alpha}})$ and

$$\mathcal{A}_{2\alpha}f(x,t) = \int_0^t\int_{\mathbb{R}^d}\frac{1}{t-s}\Omega(x-y,t-s)f(y,s)\,\mathrm{d}y\mathrm{d}s. \tag{3.13}$$

Without loss of generality, we only need to prove the case of $T = \infty$ because the other case can be reduced to this case by zero extension in time t. Moreover, we denote by $\widetilde{f}(x,t)$, $\widetilde{\Omega}(x,t)$, $\mathcal{A}_{2\alpha}\widetilde{f}(x,t)$ the zero extensions of $f(x,t)$, $\Omega(x,t)$, $\mathcal{A}_{2\alpha}f(x,t)$ to negative values of t, respectively. This is harmless since $\mathcal{A}_{2\alpha}f(x,t)$ depends only on the values of f on $(0,t)\times\mathbb{R}^d$.

Now we compute the Fourier transform of $\frac{1}{t}\widetilde{\Omega}(x,t)$ with respect to spacetime variable (x,t): the Fourier transform in x for every $t>0$ leads to $\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\,\frac{1}{t}\widetilde{\Omega}(x,t)\mathrm{d}x = |\xi|^{2\alpha}e^{-t|\xi|^{2\alpha}}$, and then the Fourier transform in t gives

$$m(\xi,\tau) = \int_0^\infty e^{-it\tau}|\xi|^{2\alpha}e^{-|\xi|^{2\alpha}t}\mathrm{d}t = \frac{|\xi|^{2\alpha}}{i\tau + |\xi|^{2\alpha}}.$$

Since $|m(\xi,\tau)| \leqslant 1$, we immediately get that $\mathcal{A}_{2\alpha}$ is bounded on $L^2(\mathbb{R}\times\mathbb{R}^d)$.

Next, we consider $\mathcal{A}_{2\alpha}$ as a Calderón–Zygmund operator on $\mathbb{R}^d\times\mathbb{R}$ endowed with the Lebesgue measure μ on $\mathbb{R}^{d+1}$ and with the quasi-distance $\rho((x,t),(y,s)) = (|x-y|^{4\alpha} + |t-s|^2)^{\frac{1}{4\alpha}}$. We also have that for any $(x,t)\in\mathbb{R}^d\times\mathbb{R}$ and $r>0$,

$$\mu(B((x,t),r)) = \int_{\rho((y,s),(x,t))\leqslant r}\mathrm{d}y\mathrm{d}s \leqslant C_0 r^{d+2\alpha}.$$

Define

$$L((x,t),(y,s)) = \frac{1_{\{s<t\}}(s)}{t-s}\widetilde{\Omega}(x-y,t-s) = \frac{1_{\{s<t\}}(s)}{(t-s)^{\frac{d}{2\alpha}+1}}(\Lambda^{2\alpha}K)\left(\frac{x-y}{(t-s)^{1/2\alpha}}\right).$$

We claim that $L((x,t),(y,s))$ satisfies the assumptions in lemma 2.10. Indeed, by making use of estimate (2.9), if $|x-y|^{2\alpha}\leqslant|t-s|$, one has

$$|L((x,t),(y,s))| \leqslant \frac{\|\Lambda^{2\alpha}K\|_{L^\infty}}{|t-s|^{\frac{d}{2\alpha}+1}} \leqslant \frac{C}{|t-s|^{\frac{d}{2\alpha}+1}};$$

while if $|x-y|^{2\alpha}\geqslant|t-s|$, one has

$$|L((x,t),(y,s))| \leqslant \frac{\||z|^{d+2\alpha}\Lambda^{2\alpha}K\|_{L^\infty}}{|x-y|^{d+2\alpha}} \leqslant \frac{C}{|x-y|^{d+2\alpha}};$$

thus, we get

$$|L((x,t),(y,s))| \leqslant \frac{C}{d((x,t),(y,s))^{d+2\alpha}}. \tag{3.14}$$

In a similar manner, we can also obtain

$$\left|\frac{\partial}{\partial t}L((x,t),(y,s))\right| = \left|\frac{\partial}{\partial s}L((x,t),(y,s))\right| \leqslant \frac{C}{\rho((x,t),(y,s))^{d+4\alpha}},$$

and for every $j=1,2,\ldots,d$,

$$\left|\frac{\partial}{\partial x_j}L((x,t),(y,s))\right| = \left|\frac{\partial}{\partial y_j}L((x,t),(y,s))\right| \leqslant \frac{C}{\rho((x,t),(y,s))^{d+2\alpha+1}}.$$

Gathering the above estimates with Taylor's formula, we have that for every $j\in\{1,2,\ldots,d\}$ and for any sufficiently small (h,θ),

$$\begin{aligned}|L((x+h,t+\theta),(y,s)) - L((x,t),(y,s))| &\leqslant \frac{C|h|}{\rho((x,t),(y,s))^{d+2\alpha+1}} + \frac{C|\theta|}{\rho((x,t),(y,s))^{d+4\alpha}}\\ &\leqslant C\frac{\rho((x,t),(x+h,t+\theta))}{\rho((x,t),(y,s))^{d+2\alpha+1}} + C\frac{\rho((x,t),(x+h,t+\theta))^{2\alpha}}{\rho((x,t),(y,s))^{d+4\alpha}}\\ &\leqslant C\frac{\rho((x,t),(x+h,t+\theta))}{\rho((x,t),(y,s))^{d+2\alpha+1}},\end{aligned} \tag{3.15}$$

and similarly,

$$|L((x,t),(y+h,s+\theta))-L((x,t),(y,s))|\leqslant C\frac{\rho((y+h,s+\theta),(y,s))}{\rho((x,t),(y,s))^{d+2\alpha+1}}. \quad (3.16)$$

Therefore, together with (3.14)–(3.16) and the L^2-boundedness of $\mathcal{A}_{2\alpha}$, we can apply lemma 2.10 to infer that for every $1<p<\infty$,

$$\|\mathcal{A}_{2\alpha}\widetilde{f}\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}\leqslant C\|\widetilde{f}\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}. \quad (3.17)$$

Next we regard $\mathcal{A}_{2\alpha}$ as a vector-valued Calderón–Zygmund operator on the real line $\mathbb{R}$:

$$\mathcal{A}_{2\alpha}\widetilde{f}(\cdot,t)=\int_{\mathbb{R}}L(t,s)\widetilde{f}(\cdot,s)\mathrm{d}s,$$

where $L(t,s)$ for every $\{(t,s)\in\mathbb{R}^2:t\neq s\}$ is given by the integral

$$L(t,s)\widetilde{f}(x)=\Lambda^{2\alpha}e^{-(t-s)\Lambda^{2\alpha}}\widetilde{f}(x,s)=\int_{\mathbb{R}^d}\frac{1_{\{s<t\}}(s)}{t-s}\widetilde{\Omega}(x-y,t-s)\widetilde{f}(y,s)\mathrm{d}y.$$

In view of (3.17), we have that $\mathcal{A}_{2\alpha}$ is continuously bounded from $L^p(\mathbb{R};L^p(\mathbb{R}^d))$ to $L^p(\mathbb{R};L^p(\mathbb{R}^d))$. For every $0<s<t$, note that the Fourier multipliers of L and ∂_tL satisfy that for every $0\leqslant k\leqslant[\frac{d}{2}]+1$ and $\xi\neq0$,

$$\sup_{|\beta|=k}\left|\partial_\xi^\beta\left(|\xi|^{2\alpha}e^{-(t-s)|\xi|^{2\alpha}}\right)\right|\leqslant\frac{C|\xi|^{-k}}{t-s},\quad\text{and}\quad\sup_{|\beta|=k}\left|\partial_\xi^\beta\left(|\xi|^{4\alpha}e^{-(t-s)|\xi|^{2\alpha}}\right)\right|\leqslant\frac{C|\xi|^{-k}}{(t-s)^2}.$$

Thus lemma 2.11 guarantees that

$$\|L\|_{\mathrm{op}(L^p\mapsto L^p)}\leq\frac{C}{t-s},\quad\text{and}\quad\|\partial_tL\|_{\mathrm{op}(L^p\mapsto L^p)}=\|\partial_sL\|_{\mathrm{op}(L^p\mapsto L^p)}\leqslant\frac{C}{(t-s)^2}.$$

Hence, the assumptions in lemma 2.10 are all fulfilled, so that $\mathcal{A}_{2\alpha}$ is continuously bounded from $L^q(\mathbb{R};L^p(\mathbb{R}^d))$ to $L^q(\mathbb{R};L^p(\mathbb{R}^d))$ for any $1<q,p<\infty$, which completes the proof of proposition 3.3 for $T=\infty$.

□

4. *A priori* estimates

In this section, we shall derive the *a priori* bounds for the vector fields $\bar{u}$, w and u.

4.1. A priori *estimates for $\bar{u}$ solving the 2D fractional Navier–Stokes system*

Proposition 4.1. *Let $\alpha\in(1/2,1)$, $u_0\in H^1\cap\dot{B}^{\alpha}_{p,2}(\mathbb{R}^2)$ with $p>\frac{2}{2\alpha-1}$. Then the smooth solution $(\bar{u},\nabla\bar{\pi})$ of the 2D fractional Navier–Stokes equations (1.2) satisfies the following:*

$$\|\bar{u}\|^2_{L^\infty_t(H^1)}+\|\Lambda^\alpha\bar{u}\|^2_{L^2_t(H^1)}\leqslant C\|u_0\|^2_{H^1}, \quad (4.1)$$

$$\|\bar{u}\|_{L^\infty_t(\dot{B}^\alpha_{p,2})}+\|(\partial_\tau\bar{u},\Lambda^{2\alpha}\bar{u},\nabla\bar{\pi})\|_{L^2_t(L^p)}\leqslant C\|u_0\|_{\dot{B}^\alpha_{p,2}}+C\|u_0\|_{H^1}^{\frac{4\alpha-1}{2\alpha-1}}. \quad (4.2)$$

Proof of proposition 4.1. The L^2-energy estimate gives that for every $t\in(0,\infty]$,

$$\|\bar{u}\|^2_{L^\infty_t(L^2)}+\|\Lambda^\alpha\bar{u}\|^2_{L^2_t(L^2)}\leqslant\|u_0\|^2_{L^2}. \quad (4.3)$$

Let $\bar\omega := \operatorname{curl}\bar u = \partial_{x_1}\bar u^2 - \partial_{x_2}\bar u^1$ be the vorticity of fluid, then it satisfies

$$\begin{cases} \partial_t\bar\omega + \bar u\cdot\nabla\bar\omega + \Lambda^{2\alpha}\bar\omega = 0,\\ \operatorname{div}\bar u = 0,\\ \bar\omega|_{t=0} = \omega_0 = \operatorname{curl} u_0. \end{cases} \tag{4.4}$$

The energy estimate of system (4.4) also leads to

$$\|\nabla\bar u\|^2_{L^\infty_t(L^2)} + \|\Lambda^\alpha\nabla\bar u\|^2_{L^2_t(L^2)} \leqslant C\big(\|\bar\omega\|^2_{L^\infty_t(L^2)} + \|\Lambda^\alpha\bar\omega\|^2_{L^2_t(L^2)}\big) \leqslant C\|\nabla u_0\|^2_{L^2}, \tag{4.5}$$

which combined with (4.3) yields the desired estimate (4.1).

Next, applying proposition 3.1 with $q=2$ to system (1.2), we find that for $\delta_1>0$ small enough,

$$\begin{aligned}
\|\bar u\|_{L^\infty_t(\dot B^\alpha_{p,2})} + \|(\partial_\tau\bar u,\Lambda^{2\alpha}\bar u,\nabla\bar\pi)\|_{L^2_t(L^p)} &\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\|\bar u\cdot\nabla\bar u\|_{L^2_t(L^p)}\\
&\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\Big\|\|\bar u\|_{L^{\frac{p(p+\delta_1)}{\delta_1}}}\|\nabla\bar u\|_{L^{p+\delta_1}}\Big\|_{L^2_t}\\
&\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\Big\|\|\bar u\|_{H^1}\|\nabla\bar u\|^{\theta_1}_{L^2}\|\Lambda^{2\alpha}\bar u\|^{1-\theta_1}_{L^p}\Big\|_{L^2_t}\\
&\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\|\bar u\|_{L^\infty_t(H^1)}\|\nabla\bar u\|^{\theta_1}_{L^2_t(L^2)}\|\Lambda^{2\alpha}\bar u\|^{1-\theta_1}_{L^2_t(L^p)}\\
&\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\|u_0\|^{1/\theta_1}_{H^1}\|\Lambda^\alpha\bar u\|^\alpha_{L^2_t(L^2)}\|\Lambda^{1+\alpha}\bar u\|^{1-\alpha}_{L^2_t(L^2)} + \tfrac12\|\Lambda^{2\alpha}\bar u\|_{L^2_t(L^p)}\\
&\leqslant C\|u_0\|_{\dot B^\alpha_{p,2}} + C\|u_0\|^{\frac{1}{\theta_1}+1}_{H^1} + \tfrac12\|\Lambda^{2\alpha}\bar u\|_{L^2_t(L^p)},
\end{aligned}\tag{4.6}$$

where $\theta_1 = \frac{(2\alpha-1)p^2+(2\alpha-1)\delta_1p-2\delta_1}{(p+\delta_1)(2\alpha p-2)} \in (0,1)$ (due to $p>\frac{2}{2\alpha-1}$). Noting that $\lim_{\delta_1\to0^+}\frac{1}{\theta_1} = \frac{2\alpha p-2}{(2\alpha-1)p}$, we can choose a suitably small $\delta_1>0$ so that $\frac{1}{\theta_1} = \frac{2\alpha}{2\alpha-1}$, and plug it into (4.6) yields estimate (4.2), as desired. □

4.2. A priori *estimates for* w *solving the perturbed system (1.7)*

Proposition 4.2. *Let* $\frac12<\alpha<1, p>\frac{2}{2\alpha-1}$, $u_0\in H^1\cap\dot B^\alpha_{p,2}(\mathbb R^2)$, *and* $\rho_0-1\in L^2\cap L^\infty(\mathbb R^2)$. *Assume that* $\|\rho_0-1\|_{L^\infty}$ *is small enough so that (1.9) is satisfied. Then the smooth solution* $(a,w,p) = (\rho-1,u-\bar u,\pi-\bar\pi)$ *of the system (1.7) satisfies the following estimates:*

$$E_2(w) := \|w\|_{L^\infty_t(L^2)} + \|\Lambda^\alpha w\|_{L^2_t(L^2)} \leqslant C\|a_0\|_{L^2\cap L^\infty} e^{C\|u_0\|^2_{H^1}}, \tag{4.7}$$

and

$$E_p(w) := \|w\|_{L^\infty_t(\dot B^\alpha_{p,2})} + \|(\partial_\tau w,\Lambda^{2\alpha}w,\nabla p)\|_{L^2_t(L^p)} \leqslant C\|a_0\|_{L^2\cap L^\infty} e^{C\|u_0\|^2_{H^1\cap\dot B^\alpha_{p,2}}}, \tag{4.8}$$

where $C>0$ *depending only on* α *and* p.

Proof of proposition 4.2. It follows easily from the equation $(1.7)_1$ that

$$\|a\|_{L^\infty_t(L^2\cap L^\infty)} = \|\rho-1\|_{L^\infty_t(L^2\cap L^\infty)} \leqslant \|\rho_0-1\|_{L^2\cap L^\infty}. \tag{4.9}$$

By taking the inner product of equation $(1.7)_2$ with w and using the condition $\operatorname{div}u=0$, we have

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\|w\|^2_{L^2} + \|\Lambda^\alpha w\|^2_{L^2} = -\frac12\frac{\mathrm d}{\mathrm dt}\|\sqrt{a}w\|^2_{L^2} - \int_{\mathbb R^2} a\partial_t\bar u\cdot w\,\mathrm dx\\
-\int_{\mathbb R^2} a(\bar u\cdot\nabla\bar u)\cdot w\,\mathrm dx - \int_{\mathbb R^2}\rho(w\cdot\nabla\bar u)\cdot w\,\mathrm dx,\end{aligned}\tag{4.10}$$

where we also have used that

$$\begin{aligned}\int_{\mathbb{R}^2}\big(-a\partial_t w-a(u\cdot\nabla w)\big)\cdot w\,\mathrm{d}x&=-\frac{1}{2}\int_{\mathbb{R}^2}a\partial_t|w|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}au\cdot\nabla|w|^2\,\mathrm{d}x\\&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}a|w|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}(\partial_t a+\mathrm{div}(au))|w|^2\,\mathrm{d}x\\&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}a|w|^2\,\mathrm{d}x.\end{aligned}\tag{4.11}$$

By virtue of the inequality $\|\nabla\bar{u}\|_{L^{\frac{2}{\alpha}}}\leqslant C\|\Lambda^\alpha\bar{u}\|_{L^2}^{2\alpha-1}\|\Lambda^{1+\alpha}\bar{u}\|_{L^2}^{2-2\alpha}\leqslant C\|\Lambda^\alpha\bar{u}\|_{H^1}$, and using the Gagliardo–Nirenberg inequality, we obtain that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}a(\bar{u}\cdot\nabla\bar{u})w\,\mathrm{d}x\Big|&\leqslant C\|a\|_{L^\infty}\|\bar{u}\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\bar{u}\|_{L^{\frac{2}{\alpha}}}\|w\|_{L^2}\\&\leqslant C\|a_0\|_{L^\infty}\|\Lambda^\alpha\bar{u}\|_{L^2}\|\Lambda^\alpha\bar{u}\|_{H^1}\|w\|_{L^2}\\&\leqslant C\|\Lambda^\alpha\bar{u}\|_{L^2}^2\|w\|_{L^2}^2+C\|a_0\|_{L^\infty}^2\|\Lambda^\alpha\bar{u}\|_{H^1}^2,\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\rho(w\cdot\nabla\bar{u})\cdot w\,\mathrm{d}x\Big|&\leqslant C\|\rho\|_{L^\infty}\|w\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\bar{u}\|_{L^{\frac{2}{\alpha}}}\|w\|_{L^2}\\&\leqslant C\|\rho_0\|_{L^\infty}\|\Lambda^\alpha w\|_{L^2}\|\Lambda^\alpha\bar{u}\|_{H^1}\|w\|_{L^2}\\&\leqslant C\|\rho_0\|_{L^\infty}^2\|\Lambda^\alpha\bar{u}\|_{H^1}^2\|w\|_{L^2}^2+\tfrac{1}{4}\|\Lambda^\alpha w\|_{L^2}^2.\end{aligned}\tag{4.13}$$

Applying the Leray projection operator $\mathbb{P}:=\mathrm{Id}-\nabla\Delta^{-1}\,\mathrm{div}$ to system (1.2) gives that

$$\|\partial_t\bar{u}\|_{L^2}\leqslant\|\Lambda^{2\alpha}\bar{u}\|_{L^2}+\|\mathbb{P}(\bar{u}\cdot\nabla\bar{u})\|_{L^2}\leqslant\|\Lambda^{2\alpha}\bar{u}\|_{L^2}+\|\bar{u}\cdot\nabla\bar{u}\|_{L^2},$$

which together with Hölder's inequality and (4.12), (4.13) leads to that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}a\partial_t\bar{u}\cdot w\,\mathrm{d}x\Big|&\leqslant\|\partial_t\bar{u}\|_{L^2}\|aw\|_{L^2}\\&\leqslant\|a\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{2\alpha}\bar{u}\|_{L^2}\|w\|_{L^{\frac{2}{1-\alpha}}}+\|a\|_{L^\infty}\|\bar{u}\cdot\nabla\bar{u}\|_{L^2}\|w\|_{L^2}\\&\leqslant C\|a_0\|_{L^2\cap L^\infty}\|\Lambda^\alpha\bar{u}\|_{H^1}\|\Lambda^\alpha w\|_{L^2}+C\|a_0\|_{L^\infty}\|\Lambda^\alpha\bar{u}\|_{L^2}\|\Lambda^\alpha\bar{u}\|_{H^1}\|w\|_{L^2}\\&\leqslant C\|\Lambda^\alpha\bar{u}\|_{L^2}^2\|w\|_{L^2}^2+C\|a_0\|_{L^2\cap L^\infty}^2\|\Lambda^\alpha\bar{u}\|_{H^1}^2+\tfrac{1}{4}\|\Lambda^\alpha w\|_{L^2}^2.\end{aligned}\tag{4.14}$$

Plugging (4.12) and (4.14) into (4.10), and integrating in time, we obtain that

$$\begin{aligned}&\|w(t)\|_{L^2}^2+\int_0^t\|\Lambda^\alpha w\|_{L^2}^2\mathrm{d}\tau\\&\quad\leqslant-\|\sqrt{a}w(t)\|_{L^2}^2+C\|a_0\|_{L^\infty}^2\int_0^t\|\Lambda^\alpha\bar{u}\|_{L^2}^2\mathrm{d}\tau+C\|a_0\|_{L^2\cap L^\infty}^2\int_0^t\|\Lambda^\alpha\bar{u}\|_{H^1}^2\,\mathrm{d}\tau\\&\qquad+C(\|\rho_0\|_{L^\infty}^2+1)\int_0^t\|\Lambda^\alpha\bar{u}\|_{H^1}^2\|w\|_{L^2}^2\mathrm{d}\tau\\&\quad\leqslant C\|a_0\|_{L^2\cap L^\infty}^2\|u_0\|_{H^1}^2+C\big(\|\rho_0\|_{L^\infty}^2+1\big)\int_0^t\|\Lambda^\alpha\bar{u}\|_{H^1}^2\|w\|_{L^2}^2\mathrm{d}\tau.\end{aligned}\tag{4.15}$$

Utilizing Gronwall's inequality together with (4.1) implies (4.7), as desired.

Next let us estimate the L^p-type norm of w. Applying proposition 3.1 to equation (1.7)$_2$ with $w|_{t=0}=0$ yields that

$$E_p(w) \leqslant C_1 \sum_{i=1}^{6} \|F_i\|_{L^2_t(L^p)}, \tag{4.16}$$

where

$$\begin{aligned} &F_1 := -a\partial_t w, && F_2 := -a\partial_t \bar{u}, && F_3 := -\rho(\bar{u}\cdot\nabla w),\\ &F_4 := -\rho(w\cdot\nabla w), && F_5 := -a(\bar{u}\cdot\nabla\bar{u}), && F_6 := -\rho(w\cdot\nabla\bar{u}). \end{aligned}$$

Let $\|a_0\|_{L^\infty} \leqslant \frac{1}{2C_1}$, then it is immediate that

$$\|F_1\|_{L^2_t(L^p)} \leqslant \|a\|_{L^\infty}\|\partial_\tau w\|_{L^2_t(L^p)} \leqslant \tfrac{1}{2C_1}\|\partial_\tau w\|_{L^2_t(L^p)}. \tag{4.17}$$

In view of (4.2), we obtain

$$\|F_2\|_{L^2_t(L^p)} = \|a\partial_\tau\bar{u}\|_{L^2_t(L^p)} \leqslant \|a_0\|_{L^\infty}\|\partial_\tau\bar{u}\|_{L^2_t(L^p)} \leqslant C\Big(1+\|u_0\|_{H^1\cap\dot{B}^\alpha_{p,2}}^{\frac{4\alpha-1}{2\alpha-1}}\Big). \tag{4.18}$$

Making use of Hölder's inequality and interpolation inequality, we get that for some small $\delta_2>0$ to be chosen later,

$$\begin{aligned} \|F_3\|_{L^2_t(L^p)} &\leqslant \|\rho\|_{L^\infty_t(L^\infty)}\|\bar{u}\|_{L^\infty_t(L^{\frac{p(p+\delta_2)}{\delta_2}})}\|\nabla w\|_{L^2_t(L^{p+\delta_2})}\\ &\leqslant C\|\rho_0\|_{L^\infty}\|\bar{u}\|_{L^\infty_t(H^1)}\left(\int_0^t \|\Lambda^\alpha w\|_{L^2}^{2\theta_2}\|\Lambda^{2\alpha}w\|_{L^p}^{2(1-\theta_2)}\mathrm{d}\tau\right)^{\frac12}\\ &\leqslant C\|\rho_0\|_{L^\infty}\|u_0\|_{H^1}\|\Lambda^\alpha w\|_{L^2_t(L^2)}^{\theta_2}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}^{1-\theta_2}\\ &\leqslant C\|\rho_0\|_{L^\infty}^{1/\theta_2}\|u_0\|_{H^1}^{1/\theta_2}\|\Lambda^\alpha w\|_{L^2_tL^2} + \tfrac{1}{4C_1}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}, \end{aligned} \tag{4.19}$$

where $\theta_2 = \frac{2\alpha-1-(\frac{2}{p}-\frac{2}{p+\delta_2})}{\alpha+1-2/p} \in (0,1)$, thus from $\lim\limits_{\delta_2\to 0^+}\frac{1}{\theta_2} = \frac{\alpha+1-2/p}{2\alpha-1}$, we can choose $\delta_2>0$ so that $\frac{1}{\theta_2} = \frac{\alpha+1}{2\alpha-1}$ and

$$\begin{aligned} \|F_3\|_{L^2_t(L^p)} &\leqslant C\|\rho_0\|_{L^\infty}^{\frac{\alpha+1}{2\alpha-1}}\|u_0\|_{H^1}^{\frac{\alpha+1}{2\alpha-1}}\|\Lambda^\alpha w\|_{L^2_tL^2} + \tfrac{1}{4C_1}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}\\ &\leqslant C\|\rho_0\|_{L^\infty}^{\frac{\alpha+1}{2\alpha-1}}\|u_0\|_{H^1}^{\frac{\alpha+1}{2\alpha-1}}\|a_0\|_{L^2\cap L^\infty}e^{C\|u_0\|_{H^1}^2} + \tfrac{1}{4C_1}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}\\ &\leqslant C\|a_0\|_{L^2\cap L^\infty}e^{C\|u_0\|_{H^1}^2} + \tfrac{1}{4C_1}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}, \end{aligned} \tag{4.20}$$

where in the above we have used (4.7) and the fact $\|\rho_0\| \leqslant 1+\|a_0\|_{L^\infty} \leqslant 2$. Similarly, by using the interpolation inequality, Young's inequality and estimate (4.7) again, it follows that

$$\begin{aligned} \|F_4\|_{L^2_t(L^p)} &\leqslant \|\rho\|_{L^\infty_t(L^\infty)}\|w\|_{L^\infty_t(L^\infty)}\|\nabla w\|_{L^2_t(L^p)}\\ &\leqslant C\|\rho_0\|_{L^\infty}\|w\|_{L^\infty_t(L^2)}^{\theta_3}\|w\|_{L^\infty_t(\dot{B}^\alpha_{p,2})}^{1-\theta_3}\|\Lambda^\alpha w\|_{L^2_t(L^2)}^{\theta_4}\|\Lambda^{2\alpha}w\|_{L^2_t(L^p)}^{1-\theta_4}\\ &\leqslant C\|\rho_0\|_{L^\infty}\left(\|w\|_{L^\infty_t(L^2)}+\|\Lambda^\alpha w\|_{L^2_t(L^2)}\right)^{\theta_3+\theta_4}\left(E_p(w)\right)^{2-(\theta_3+\theta_4)}\\ &\leqslant C\|\rho_0\|_{L^\infty}^{\frac{2}{\theta_3+\theta_4}}\left(\|w\|_{L^\infty_t(L^2)}+\|\Lambda^\alpha w\|_{L^2_t(L^2)}\right)^2 + \tfrac{1}{2C_1}\left(E_p(w)\right)^2\\ &\leqslant C\|a_0\|_{L^2\cap L^\infty}^2 e^{C\|u_0\|_{H^1}^2} + \tfrac{1}{2C_1}\left(E_p(w)\right)^2, \end{aligned} \tag{4.21}$$

where $\theta_3 = \frac{\alpha-2/p}{1+\alpha-2/p} \in (0,1)$, $\theta_4 = \frac{2\alpha-1}{1+\alpha-2/p} \in (0,1)$. By arguing as (4.6), we have

$$\begin{aligned}\|F_5\|_{L^2_t(L^p)} &\leqslant \|a\|_{L^\infty_t(L^\infty)}\|\bar{u}\cdot\nabla\bar{u}\|_{L^2_t(L^p)}\\ &\leqslant C\|a\|_{L^\infty_t(L^\infty)}\|\bar{u}\|_{L^\infty_t(H^1)}\|\nabla\bar{u}\|^{\frac{2\alpha-1}{2\alpha}}_{L^2_t(L^2)}\|\Lambda^{2\alpha}\bar{u}\|^{\frac{1}{2\alpha}}_{L^2_t(L^p)}\\ &\leqslant C\|a_0\|_{L^\infty}\|u_0\|_{H^1}\left(\|u_0\|_{H^1\cap\dot{B}^\alpha_{p,2}} + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1}\right).\end{aligned} \tag{4.22}$$

The term F_6 can be estimated in the same manner as F_3 and F_4:

$$\begin{aligned}\|F_6\|_{L^2_t(L^p)} &\leqslant \|\rho_0\|_{L^\infty}\|w\|_{L^\infty_t(L^\infty)}\|\nabla\bar{u}\|_{L^2_t(L^p)}\\ &\leqslant C\|\rho_0\|_{L^\infty}\|w\|^{\theta_3}_{L^\infty_t(L^2)}\|w\|^{1-\theta_3}_{L^\infty_t(\dot{B}^\alpha_{p,2})}\left(\|\nabla\bar{u}\|_{L^2_t(L^2)} + \|\Lambda^{2\alpha}\bar{u}\|_{L^2_t(L^p)}\right)\\ &\leqslant C\|\rho_0\|^{\frac{1}{\theta_3}}_{L^\infty}\left(\|u_0\|_{H^1\cap\dot{B}^\alpha_{p,2}} + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1}\right)^{\frac{1+\alpha-2/p}{\alpha-2/p}}\|w\|_{L^\infty_t(L^2)} + \frac{1}{4C_1}\|w\|_{L^\infty_t(\dot{B}^\alpha_{p,2})}\\ &\leqslant C\|a_0\|_{L^2\cap L^\infty}e^{C\|u_0\|^2_{H^1\cap\dot{B}^\alpha_{p,2}}} + \frac{1}{4C_1}\|w\|_{L^\infty_t(\dot{B}^\alpha_{p,2})}.\end{aligned} \tag{4.23}$$

Collecting the above estimates on F_i, $i = 1,\ldots,6$, we deduce that

$$E_p(w) \leqslant C_2\|a_0\|_{L^2\cap L^\infty}e^{C_2\|u_0\|^2_{H^1\cap\dot{B}^\alpha_{p,2}}} + \left(E_p(w)\right)^2. \tag{4.24}$$

By choosing c_0 in (1.9) so small that $c_0 \leqslant (4C_2)^{-1}$, we conclude the estimate (4.8) by an elementary computation. □

4.3. A priori *estimates for* u *solving the 2D fINS system (1.1)*

Based on the above estimates for $\bar{u}$ and w, we directly have the following bounds for u.

Proposition 4.3. *Let* $\frac{1}{2} < \alpha < 1, p > \frac{2}{2\alpha-1}$ *and* $u_0 \in H^1 \cap \dot{B}^\alpha_{p,2}(\mathbb{R}^2)$. *Let* $\rho_0 - 1 \in L^2 \cap L^\infty(\mathbb{R}^2)$ *be satisfying the condition (1.9) with* $c_0 = c_0(\alpha,p) > 0$ *a sufficiently small constant. Then there exists a constant* $C = C(\alpha,p) > 0$ *such that*

$$\|u\|_{L^\infty(\mathbb{R}_+;L^2\cap\dot{B}^\alpha_{p,2})} + \|(\Lambda^{2\alpha}u,\partial_t u,\nabla\pi)\|_{L^2(\mathbb{R}_+;L^p)} + \|u\|_{L^2(\mathbb{R}_+;\dot{H}^\alpha)} \leqslant C\left(1 + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}^\alpha_{p,2}}\right), \tag{4.25}$$

and for any $T > 0$,

$$\|\nabla u\|_{L^1_T(L^\infty)} \leqslant C\left(1 + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}^\alpha_{p,2}}\right)\sqrt{T}. \tag{4.26}$$

Proof of proposition 4.3. The proof of (4.25) is obvious. As for the proof of (4.26), we use the Sobolev embedding to deduce

$$\|\nabla u\|_{L^1_T(L^\infty)} \leqslant C\sqrt{T}\|u\|_{L^2_T(\dot{H}^\alpha\cap\dot{W}^{2\alpha,p})} \leqslant C\left(1 + \|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}^\alpha_{p,2}}\right)\sqrt{T}, \tag{4.27}$$

as required. □

Under the additional stronger assumptions on (ρ_0,u_0), we can show some more refined *a priori* estimates, which are of use in the uniqueness part.

Proposition 4.4. *Let* $s \in (0,1)$, $u_0 \in H^1 \cap \dot{B}^{\alpha+s}_{p,2}(\mathbb{R}^2)$, $\rho_0 - 1 \in L^2 \cap L^\infty \cap \mathcal{M}(\dot{B}^s_{p,2}) \cap \mathcal{M}(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1})(\mathbb{R}^2))$. *There exists a constant* $c_* > 0$ *depending only on* α, p *and* s *so that, if the condition (1.12) with this* c_* *is satisfied, then we have*

$$\|\nabla u\|_{L^1(\mathbb{R}_+;L^\infty)} \leqslant C\Big(1 + \|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap \dot{B}^\alpha_{p,2}}\Big), \tag{4.28}$$

and

$$\begin{aligned} &\|u\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\alpha+s}_{p,2})} + \|(\Lambda^{2\alpha}u, \partial_t u, \nabla\pi)\|_{L^2(\mathbb{R}_+;\dot{B}^s_{p,2})} + \|u\|_{L^2(\mathbb{R}_+;\dot{B}^{2\alpha}_{p,1})} \\ &\leqslant C\left(1 + \|u_0\|_{\dot{B}^{\alpha+s}_{p,2}} + \|u_0\|^{\frac{8(4\alpha-1)}{(2\alpha-1)^2}}_{H^1\cap \dot{B}^\alpha_{p,2}}\right). \end{aligned} \tag{4.29}$$

Proof of proposition 4.4. We note that the proof of (4.29) below uses the smallness of $\|\rho - 1\|_{L^\infty_T(\mathcal{M}(\dot{B}^s_{p,2}))}$, which according to (2.5) is needed to get the uniform estimate of $\|u\|_{L^1(\mathbb{R}_+;\dot{W}^{1,\infty})}$ (estimate (4.26) in proposition 4.3 is insufficient for $T = \infty$). Thus, the proof of proposition 4.4 is divided into two parts.

(1) First we prove (4.28). Applying the Leray operator $\mathbb{P} := \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$ to the equation $(1.7)_2$ yields

$$\partial_t w + \Lambda^{2\alpha} w = \mathbb{P}F - \mathbb{P}(u\cdot\nabla w), \quad w|_{t=0} = w_0, \tag{4.30}$$

with

$$F := -a\partial_t w - a\partial_t \bar{u} - a(u\cdot\nabla w) - a(\bar{u}\cdot\nabla\bar{u}) - \rho(w\cdot\nabla\bar{u}),$$

and then it follows from lemma 2.4 that

$$\begin{aligned} \|w\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|w\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1}_{p,1}\right)} &\leqslant C\left(\|\mathbb{P}F\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|\mathbb{P}(u\cdot\nabla w)\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\right) \\ &\leqslant C\left(\|F\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|(u\cdot\nabla w)\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\right). \end{aligned} \tag{4.31}$$

Noting that $\nabla p = \nabla\Delta^{-1}\operatorname{div}(F - u\cdot\nabla w)$, we use the equation (4.30) to get

$$\|(\partial_\tau w, \nabla p)\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} \leqslant C\left(\|F\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|u\cdot\nabla w\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\right). \tag{4.32}$$

By virtue of definition 2.2 concerning the multiplier space, we find

$$\begin{aligned} \|F\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} \leqslant{}& C\|a\|_{L^\infty_t\left(\mathcal{M}\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)\right)}\left\|(\partial_\tau w, \partial_\tau\bar{u}, \bar{u}\cdot\nabla\bar{u}, u\cdot\nabla w, w\cdot\nabla\bar{u})\right\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} \\ &+ C\|w\cdot\nabla\bar{u}\|_{L^1_t\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}. \end{aligned} \tag{4.33}$$

Utilizing the divergence-free condition of u and the interpolation inequality gives that

$$\begin{aligned}\|u\cdot\nabla w\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} &\leqslant \|u\otimes w\|_{L^1_t\left(\dot B^{\frac{2}{p}+2-2\alpha}_{p,1}\right)}\\ &\leqslant C\int_0^t\left(\|u\|_{L^\infty}\|w\|_{\dot B^{\frac{2}{p}+2-2\alpha}_{p,1}}+\|u\|_{\dot B^{\frac{2}{p}+2-2\alpha}_{p,1}}\|w\|_{L^\infty}\right)\mathrm{d}\tau\\ &\leqslant C\Big(\|u\|_{L^2_t(L^\infty)}+\|u\|_{L^2_t\left(\dot B^{2/p+2-2\alpha}_{p,1}\right)}\Big)\left(\|w\|_{L^2_t(L^\infty)}+\|w\|_{L^2_t\left(\dot B^{2/p+2-2\alpha}_{p,1}\right)}\right).\end{aligned}$$

The interpolation inequality and Sobolev/Besov embedding ensure that

$$\|u\|_{L^\infty}\leqslant C\|u\|^{\frac{2\alpha-2/p}{1+\alpha-2/p}}_{L^{\frac{2}{1-\alpha}}}\|u\|^{\frac{1-\alpha}{1+\alpha-2/p}}_{\dot W^{2\alpha,p}}\leqslant C\big(\|u\|_{\dot H^\alpha}+\|u\|_{\dot W^{2\alpha,p}}\big),$$

$$\|u\|_{\dot B^{\frac{2}{p}+2-2\alpha}_{p,1}}\leqslant C\|u\|^{\frac{2(2\alpha-1)-2/p}{1+\alpha-2/p}}_{\dot B^{2/p+\alpha-1}_{p,\infty}}\|u\|^{\frac{3(1-\alpha)}{1+\alpha-2/p}}_{\dot B^{2\alpha}_{p,\infty}}\leqslant C\big(\|u\|_{\dot H^\alpha}+\|u\|_{\dot W^{2\alpha,p}}\big),$$

thus we arrive at

$$\|u\cdot\nabla w\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\leqslant C\|u\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}\|w\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}. \tag{4.34}$$

Repeating the above procedure analogously for $w\cdot\nabla\bar u$ and $\bar u\cdot\nabla\bar u$ yields

$$\|w\cdot\nabla\bar u\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\leqslant C\|\bar u\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}\|w\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}, \tag{4.35}$$

$$\|\bar u\cdot\nabla\bar u\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\leqslant C\|\bar u\|^2_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}. \tag{4.36}$$

For the estimate of $\bar u$ solving the equation (1.2), note that by interpolation and Besov embedding,

$$\|u_0\|_{\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}}\leqslant C\|u_0\|^{\frac{3\alpha-1-2/p}{1+\alpha-2/p}}_{\dot B^{2/p-1}_{p,\infty}}\|u_0\|^{\frac{2-2\alpha}{1+\alpha-2/p}}_{\dot B^{\alpha}_{p,\infty}}\leqslant C\|u_0\|_{L^2\cap\dot B^\alpha_{p,2}}.$$

Similarly to the proof of (4.31) and (4.32), we obtain

$$\begin{aligned}&\|\bar u\|_{\widetilde L^\infty_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}+\|\bar u\|_{L^1_t\left(\dot B^{\frac{2}{p}+1}_{p,1}\right)}+\|(\partial_\tau\bar u,\nabla\bar\pi)\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\\ &\leqslant C\left(\|u_0\|_{\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}}+\|\bar u\cdot\nabla\bar u\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\right)\leqslant C\Big(\|u_0\|_{L^2\cap\dot B^\alpha_{p,2}}+\|\bar u\|^2_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}\Big).\end{aligned} \tag{4.37}$$

Gathering (4.31)–(4.37) together leads to the estimate

$$\begin{aligned}&\|w\|_{\widetilde L^\infty_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}+\|w\|_{L^1_t\left(\dot B^{\frac{2}{p}+1}_{p,1}\right)}+\|(\partial_\tau w,\nabla p)\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}\\ &\leqslant C\|a\|_{L^\infty_t\left(\mathcal M\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)\right)}\left(\|\partial_\tau w\|_{L^1_t\left(\dot B^{\frac{2}{p}+1-2\alpha}_{p,1}\right)}+\|u_0\|_{L^2\cap\dot B^\alpha_{p,2}}+\|(\bar u,w)\|^2_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}\right)\\ &\quad+C\|(\bar u,w)\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}\|w\|_{L^2_t(\dot H^\alpha\cap\dot W^{2\alpha,p})}.\end{aligned} \tag{4.38}$$

Observe that owing to lemma 2.6 and inequality (4.37),

$$\begin{aligned}
&\|a\|_{L_t^\infty\left(\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)\right)}\\
&\leqslant C\|a_0\|_{\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)} e^{C\int_0^t\|\nabla u\|_{L^\infty}\mathrm{d}\tau}\\
&\leqslant C\|a_0\|_{\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)} e^{C\|w\|_{L_t^1\left(\dot{B}_{p,1}^{2/p+1}\right)}+C\|\bar{u}\|_{L_t^1\left(\dot{B}_{p,1}^{2/p+1}\right)}}\\
&\leqslant C\|a_0\|_{\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)} e^{C\|w\|_{L_t^1\left(\dot{B}_{p,1}^{2/p+1}\right)}} \exp\left\{C\left(\|u_0\|_{L^2\cap\dot{B}_{p,2}^\alpha}+\|\bar{u}\|^2_{L_t^2(\dot{H}^\alpha\cap\dot{W}^{2\alpha,p})}\right)\right\}.
\end{aligned} \tag{4.39}$$

Recalling that propositions 4.1 and 4.2 guarantee that

$$\|\bar{u}\|_{L_t^2(\dot{H}^\alpha\cap\dot{W}^{2\alpha,p})}\leqslant C\left(1+\|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}_{p,2}^\alpha}\right),\quad \|w\|_{L_t^2(\dot{H}^\alpha\cap\dot{W}^{2\alpha,p})}\leqslant C\|a_0\|_{L^2\cap L^\infty}e^{C\|u_0\|^2_{H^1\cap\dot{B}_{p,2}^\alpha}},$$

we insert the above estimates and (4.39) into (4.38) to get

$$\begin{aligned}
&\|w\|_{\widetilde{L}_t^\infty\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)}+\|w\|_{L_t^1\left(\dot{B}_{p,1}^{\frac{2}{p}+1}\right)}+\|(\partial_\tau w,\nabla p)\|_{L_t^1\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)}\\
&\leqslant \overline{C}\|a_0\|_{\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)} e^{\overline{C}\left(1+\|u_0\|^2_{H^1\cap\dot{B}_{p,2}^\alpha}+\|u_0\|_{H^1}^{\frac{8\alpha-2}{2\alpha-1}}\right)}\cdot e^{\overline{C}\|w\|_{L_t^1\left(\dot{B}_{p,1}^{2/p+1}\right)}}\left(\|\partial_\tau w\|_{L_t^1\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)}+1\right)\\
&\quad+\overline{C}\|a_0\|_{L^2\cap L^\infty}e^{\overline{C}\|u_0\|^2_{H^1\cap\dot{B}_{p,2}^\alpha}},
\end{aligned} \tag{4.40}$$

where $\overline{C}>0$ is a constant depending only on α and p.

According to proposition 4.2 and the continuous embedding $\dot{W}^{2\alpha,p}\cap\dot{H}^\alpha\hookrightarrow\dot{B}_{p,1}^{2/p+1}$ and $\dot{B}_{2,\infty}^0\cap\dot{B}_{p,\infty}^\alpha\hookrightarrow\dot{B}_{p,1}^{2/p+1-2\alpha}$, we know that $w\in\widetilde{L}_T^\infty(\dot{B}_{p,1}^{2/p+1-2\alpha})\cap L_T^1\left(\dot{B}_{p,1}^{2/p+1}\right)$ for any $T>0$. Since $w\in\widetilde{L}_T^\infty(\dot{B}_{p,1}^{2/p+1-2\alpha})$, by a high-low frequency decomposition argument, one easily deduces that $w\in C([0,T];\dot{B}_{p,1}^{2/p+1-2\alpha})$. Let $T^*>0$ be the maximal existence time such that $w\in C([0,T^*);\dot{B}_{p,1}^{2/p+1-2\alpha})\cap L^1([0,T^*);\dot{B}_{p,1}^{2/p+1})$. Denote by T' as

$$T':=\sup\left\{t<T^*:\|w\|_{\widetilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha})}+\|w\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}\leqslant 1\right\}. \tag{4.41}$$

We claim $T'=\infty$. Indeed, by choosing a_0 small enough so that

$$\overline{C}e^{\overline{C}}\|a_0\|_{\mathcal{M}\left(\dot{B}_{p,1}^{\frac{2}{p}+1-2\alpha}\right)}\exp\left\{\overline{C}(1+\|u_0\|^2_{H^1\cap\dot{B}_{p,2}^\alpha}+\|u_0\|_{H^1}^{\frac{8\alpha-2}{2\alpha-1}})\right\}\leqslant\tfrac{1}{4}, \tag{4.42}$$

and

$$\overline{C}\|a_0\|_{L^2\cap L^\infty}e^{\overline{C}\|u_0\|^2_{H^1\cap\dot{B}_{p,2}^\alpha}}\leqslant\tfrac{1}{4}, \tag{4.43}$$

we infer from (4.40) that

$$\|w\|_{\widetilde{L}^\infty_{T'}\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|w\|_{L^1_{T'}\left(\dot{B}^{\frac{2}{p}+1}_{p,1}\right)} + \frac{1}{2}\|\partial_t w\|_{L^1_{T'}\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} + \|\nabla p\|_{L^1_{T'}\left(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1}\right)} \leqslant \frac{1}{2}. \tag{4.44}$$

If $T' < \infty$, then the above shows that the solution can be continued past $T^{'}$, which contradicts with the maximality of $T^{'}$ defined by (4.41). Hence we conclude that $T' = T^* = \infty$.

By combining the estimates (4.37) and (4.44), we get the desired estimate (4.28):

$$\|\nabla u\|_{L^1(\mathbb{R}_+;L^\infty)} \leqslant C\|(\bar{u}, w)\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{2}{p}+1}_{p,1})} \leqslant C\Big(1 + \|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^\alpha_{p,2}}\Big).$$

(2) Next we show (4.29). Applying the fractional Laplacian operator Λ^s to equation $(1.1)_2$ yields that

$$\partial_t\Lambda^s u + \Lambda^{2\alpha+s}u + \nabla\Lambda^s\pi = -\Lambda^s\big(a\,\partial_t u\big) - \Lambda^s\big(\rho u\cdot\nabla u\big) =: F_s,$$

with $a = \rho - 1$ and $\operatorname{div} u = 0$. By using the Leray operator $\mathbb{P} := \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$, we get

$$\partial_t(\Lambda^s u) + \Lambda^{2\alpha}(\Lambda^s u) = \mathbb{P}F_s, \quad (\Lambda^s u)|_{t=0} = \Lambda^s u_0.$$

Similarly as deriving (4.31) and (4.32), we obtain that for every $T \in (0,\infty]$,

$$\begin{aligned} \|\Lambda^s u\|_{\widetilde{L}^\infty_T(\dot{B}^\alpha_{p,2})} + \|\Lambda^s u\|_{L^2_T(\dot{B}^{2\alpha}_{p,2})} + \|(\nabla\pi, \partial_t u)\|_{L^2_T(\dot{B}^s_{p,2})} &\leqslant C\big(\|\Lambda^s u_0\|_{\dot{B}^\alpha_{p,2}} + \|\mathbb{P}F_s\|_{L^2_T(\dot{B}^0_{p,2})}\big) \\ &\leqslant C\big(\|u_0\|_{\dot{B}^{\alpha+s}_{p,2}} + \|F_s\|_{L^2_T(\dot{B}^0_{p,2})}\big). \end{aligned} \tag{4.45}$$

By using definition 2.2, we infer that

$$\begin{aligned} \|F_s\|_{L^2_T(\dot{B}^0_{p,2})} &\leqslant \|a\,\partial_t u\|_{L^2_T(\dot{B}^s_{p,2})} + \|(1+a)(u\cdot\nabla u)\|_{L^2_T(\dot{B}^s_{p,2})} \\ &\leqslant C\|a\|_{L^\infty_T(\mathcal{M}(\dot{B}^s_{p,2}))}\|\partial_t u\|_{L^2_T(\dot{B}^s_{p,2})} + C(1 + \|a\|_{L^\infty_T(\mathcal{M}(\dot{B}^s_{p,2}))})\|u\cdot\nabla u\|_{L^2_T(\dot{B}^s_{p,2})}. \end{aligned} \tag{4.46}$$

Utilizing the interpolation inequality and (4.25) leads to

$$\begin{aligned} \|u\cdot\nabla u\|_{L^2_T(\dot{B}^s_{p,2})} &\leqslant C\|u\|_{L^\infty_T(L^\infty)}\|u\|_{L^2_T(\dot{B}^{1+s}_{p,2})} \\ &\leqslant C\|u\|_{L^\infty_T(L^2\cap\dot{B}^\alpha_{p,2})}\|u\|^{\theta_5}_{L^2_T(\dot{H}^\alpha)}\|u\|^{1-\theta_5}_{L^2_T(\dot{B}^{2\alpha+s}_{p,2})} \\ &\leqslant C_\varepsilon\|u\|^{1/\theta_5}_{L^\infty_T(L^2\cap\dot{B}^\alpha_{p,2})}\|u\|_{L^2_T(\dot{H}^\alpha)} + \varepsilon\|u\|_{L^2_T(\dot{B}^{2\alpha+s}_{p,2})} \\ &\leqslant C_\varepsilon\left(1 + \|u_0\|^{\frac{8(4\alpha-1)}{(2\alpha-1)^2}}_{H^1\cap\dot{B}^\alpha_{p,2}}\right) + \varepsilon\|u\|_{L^2_T(\dot{B}^{2\alpha+s}_{p,2})}, \end{aligned} \tag{4.47}$$

where $\theta_5 = \frac{2\alpha-1}{\alpha+1+s-2/p}$ and $\varepsilon > 0$ is a constant chosen later. By virtue of (2.5) and (4.28), we get

$$\|a\|_{L^\infty_T(\mathcal{M}(\dot{B}^s_{p,2}))} \leqslant \|a_0\|_{\mathcal{M}(\dot{B}^s_{p,2})}e^{C\|\nabla u\|_{L^1_T(L^\infty)}} \leqslant C\|a_0\|_{\mathcal{M}(\dot{B}^s_{p,2})}\exp\left\{C\left(1 + \|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^\alpha_{p,2}}\right)\right\}. \tag{4.48}$$

Collecting the estimates (4.45)–(4.48) and using the interpolation inequality $\dot{B}^{2\alpha+s}_{p,2}\cap\dot{H}^\alpha \hookrightarrow \dot{B}^{2\alpha}_{p,1}$ together with (4.25), we see that for every $T \in (0,\infty]$,

$$
\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T(\dot{B}^{\alpha+s}_{p,2})}+\|u\|_{L^2_T(\dot{B}^{2\alpha+s}_{p,2})}+\|(\nabla\pi,\partial_t u)\|_{L^2_T(\dot{B}^{s}_{p,2})}+\|u\|_{L^2_T(\dot{B}^{2\alpha}_{p,1})}\\
&\quad\leqslant C\|u_0\|_{\dot{B}^{\alpha+s}_{p,2}}+C_\varepsilon\left(1+\|u_0\|^{\frac{8(4\alpha-1)}{(2\alpha-1)^2}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\\
&\qquad+C\|a_0\|_{\mathcal{M}(\dot{B}^{s}_{p,2})}\exp\left\{C\left(1+\|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\right\}\|\partial_t u\|_{L^2_T(\dot{B}^{s}_{p,2})}\\
&\qquad+C\varepsilon\left(1+\|a_0\|_{\mathcal{M}(\dot{B}^{s}_{p,2})}\exp\left\{C\left(1+\|u_0\|^{\frac{8\alpha-2}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\right\}\right)\|u\|_{L^2_T(\dot{B}^{2\alpha+s}_{p,2})},
\end{aligned}
$$

so that by letting $c_*>0$ in (1.12) small enough and then $\varepsilon>0$ small enough, we conclude the desired estimate (4.29). □

5. Proof of theorem 1.1: the existence part

In subsection 5.1, as a first step we show the global well-posedness of strong solution to the 2D fINS system (1.1) with the additional regularity assumption $\nabla\rho_0\in L^{\frac{2}{\alpha}}(\mathbb{R}^2)$, and then in subsection 5.2 by the compactness argument we prove the global existence of solution to the 2D fINS system (1.1) with rough density.

5.1. Global well-posedness result for the 2D fINS system with regular density

Our main result in this subsection is the following proposition.

Proposition 5.1. *Let $\frac{1}{2}<\alpha<1$, $p>\frac{2}{2\alpha-1}$ and $u_0\in H^1\cap\dot{B}^{\alpha}_{p,2}(\mathbb{R}^2)$, $\rho_0-1\in L^2\cap L^\infty(\mathbb{R}^2)$ be with the smallness condition (1.9). In addition, assume $\nabla\rho_0\in L^{\frac{2}{\alpha}}(\mathbb{R}^2)$. Then there exists a unique global-in-time solution $(\rho,u,\nabla\pi)$ to the 2D fINS system (1.1) which satisfies the estimates (1.10) and (1.11). In addition, it holds that for any $T>0$,*

$$
\|\nabla\rho\|_{L^\infty_T(L^{\frac{2}{\alpha}})}\leqslant\|\nabla\rho_0\|_{L^{\frac{2}{\alpha}}}\exp\left\{C\left(1+\|u_0\|^{\frac{4\alpha-1}{2\alpha-1}}_{H^1\cap\dot{B}^{\alpha}_{p,2}}\right)\sqrt{T}\right\}. \tag{5.1}
$$

Proof of proposition 5.1. Since $u_0\in H^1\cap\dot{B}^{\alpha}_{p,2}(\mathbb{R}^2)$ with $\alpha\in(\frac{1}{2},1)$ and $p>\frac{2}{2\alpha-1}$, according to proposition 4.1 and the standard compactness theory, there exists a unique global-in-time strong solution $\bar{u}\in C(\mathbb{R}_+;H^1\cap\dot{B}^{\alpha}_{p,2})\cap L^2(\mathbb{R}_+;H^{1+\alpha})$ to the 2D fractional Navier–Stokes system (1.2) which satisfies estimates (4.1) and (4.2).

Now, it suffices to treat the global existence issue of the perturbed system (1.7). This is more-or-less a standard process. We here only give a sketch of the proof, and for the details one can see the arXiv version [37].

We first consider (w^{n+1},a^{n+1}) $(n\in\mathbb{N})$ as the solutions to the following approximate perturbed system

$$
\begin{cases}
\partial_t a^{n+1}+u^n\cdot\nabla a^{n+1}=0,\\
\partial_t w^{n+1}+u^n\cdot\nabla w^{n+1}+\Lambda^{2\alpha}w^{n+1}+\nabla p^{n+1}=\sum\limits_{i=1}^{5}{F^n}_i,\\
\operatorname{div}w^{n+1}=0,\\
(a^{n+1},w^{n+1})|_{t=0}=(a_0,0),
\end{cases}
\tag{5.2}
$$

where $u^n=w^n+\bar{u}$ and

$$
\begin{aligned}
&{F^n}_1=-a^n\partial_t w^{n+1},\quad {F^n}_2=-a^n(\partial_t\bar{u}),\quad F^n_3=-a^n(u^{n-1}\cdot\nabla w^{n+1}),\\
&F^n_4=-a^n\big(\bar{u}\cdot\nabla\bar{u}\big),\quad F^n_5=-(1+a^n)\big(w^{n+1}\cdot\nabla\bar{u}\big).
\end{aligned}
\tag{5.3}
$$

We also set $u^{-1}(t,x) \equiv 0$, $w^0(t,x) \equiv 0$, $a^0(t,x) \equiv a_0(x)$, $u^0(t,x) \equiv u_0(x)$. It can admit a unique solution of system (5.2) by proposition 3.1 and the Banach fixed point theorem.

Then we derive the uniform-in-n estimates for approximate solutions by induction. Indeed, by arguing as proposition 4.2, under the smallness condition (1.9), we prove that, for any $n \in \mathbb{Z}_+$ and $k \leqslant n$, there hold that for any $t > 0$,

$$E_2(w^k) := \|w^k\|_{L_t^\infty(L^2)} + \|\Lambda^\alpha w^k\|_{L_t^2(L^2)} \leqslant C\|a_0\|_{L^2\cap L^\infty} e^{C\|u_0\|_{H^1}^2}, \tag{5.4}$$

and

$$E_p(w^k) := \|w^k\|_{L_t^\infty(\dot{B}_{p,2}^\alpha)} + \|(\partial_\tau w^k, \Lambda^{2\alpha} w^k, \nabla p^k)\|_{L_t^2(L^p)} \leqslant \widetilde{C}\|a_0\|_{L^2\cap L^\infty} \exp\left\{\widetilde{C}\|u_0\|_{H^1\cap\dot{B}_{p,2}^\alpha}^2\right\}, \tag{5.5}$$

with $C, \widetilde{C} > 0$ depending only on α, p. In addition, since $\nabla a_0 = \nabla \rho_0 \in L^{\frac{2}{\alpha}}$, one can easily show that for every $n \in \mathbb{N}$,

$$\|\nabla a^{n+1}\|_{L_t^\infty(L^{\frac{2}{\alpha}})} \leqslant \|\nabla a_0\|_{L^{\frac{2}{\alpha}}} \exp\left\{\int_0^t \|\nabla u^n\|_{L^\infty} \mathrm{d}\tau\right\} \leqslant \|\nabla a_0\|_{L^{\frac{2}{\alpha}}} e^{C(u_0)\sqrt{t}}. \tag{5.6}$$

Based on the uniform-in-n estimates (5.4)–(5.6), we can show that $\{(a^n, w^n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the L^2-energy space on a small interval $[0, t_*]$ with $t_* > 0$. Thus, there exists a limit function (a, w) such that

$$(a_n, w_n) \to (a, w) \text{ in } L^\infty(0, t_*; L^2(\mathbb{R}^2)) \times \big(L^\infty(0, t_*; L^2(\mathbb{R}^2)) \cap L^2(0, t_*; \dot{H}^\alpha(\mathbb{R}^2))\big).$$

By virtue of (5.4), (5.5) and interpolation, we can pass to the limit $n \to \infty$ in the system (1.7) to deduce that (a, w) solves the perturbed system (1.7) in the distributional sense. By Fatou's lemma, we see that (a, w) satisfies $E_2(w) + E_p(w) < \infty$ on $[0, t_*]$ and $a \in L^\infty(0, t_*, W^{1,\frac{2}{\alpha}})$.

Since $a \in L^\infty(0, T; L^\infty \cap W^{1,\frac{2}{\alpha}})$ and w is regular enough, we can show the uniqueness in the standard L^2-topology in the Eulerian framework.

Finally, let $T_* > 0$ be the maximal existence time of the above constructed strong solution (a, w) solving (1.7), then under the smallness condition (1.9), we can use a bootstrapping argument to show $T_* = \infty$. Combined with the estimates of $\bar{u}$, the proof of proposition 5.1 is complete. □

5.2. *Global existence of solution for the 2D fINS system with rough density*

Owing to the low regularity of $a_0 = \rho_0 - 1$ (now we do not assume $\nabla a_0 \in L^{\frac{2}{\alpha}}$ any more), we cannot prove the convergence of the approximate sequences in the L^2-topology as in the previous subsection. Thus we shall use the compactness arguments instead. For completeness, we outline the proof as follows.

For every $\varepsilon > 0$, let $\chi_\epsilon(\cdot) = \epsilon^{-2}\chi(\frac{\cdot}{\epsilon})$ and $\chi \in C_c^\infty(\mathbb{R}^2)$ be a standard mollifier. Let $\rho_0^\epsilon = \chi_\epsilon * \rho_0$, then it satisfies $\nabla \rho_0^\epsilon \in L^{\frac{2}{\alpha}}(\mathbb{R}^2)$. According to proposition 5.1, the perturbed system (1.1) with initial data (ρ_0^ϵ, u_0) admits a unique global-in-time strong solution $(\rho^\epsilon, u^\epsilon)$ satisfying the uniform-in-ε bounds (1.10) and (1.11). Thus we are allowed to pick a subsequence ε_k ($\epsilon_k \to 0$ as $k \to \infty$) such that

$$\rho^{\epsilon_k} - 1 \rightharpoonup^* \rho - 1 \quad \text{in } L^\infty(\mathbb{R}_+; L^2 \cap L^\infty(\mathbb{R}^2)), \tag{5.7}$$

and

$$\begin{aligned} &u^{\epsilon_k} - \bar{u} \rightharpoonup^* u - \bar{u} \quad \text{in } L^\infty(\mathbb{R}_+; L^2 \cap \dot{B}_{p.2}^\alpha(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; \dot{W}^{2\alpha,p}(\mathbb{R}^2)), \\ &\partial_t u^{\epsilon_k} - \partial_t \bar{u} \rightharpoonup \partial_t u - \partial_t \bar{u} \quad \text{in } L^2(\mathbb{R}_+; L^p(\mathbb{R}^2)). \end{aligned} \tag{5.8}$$

In addition, utilizing the diagonal argument together with the Rellich-type theorems applied for the compact (space-time) subsets of $\mathbb{R}^2\times\mathbb{R}_+$, we conclude

$$u^{\epsilon_k}\to u \quad \text{a.e. in } \mathbb{R}_+\times\mathbb{R}^2. \tag{5.9}$$

In view of

$$\rho^{\epsilon_k}\partial_t u^{\epsilon_k}+\rho^{\epsilon_k}(u^{\epsilon_k}\cdot\nabla u^{\epsilon_k})=\partial_t(\rho^{\epsilon_k}u^{\epsilon_k})+\mathrm{div}(\rho^{\epsilon_k}u^{\epsilon_k}\otimes u^{\epsilon_k}),$$

the above convergence is sufficient to pass to the limit in (1.1) in the distributional sense and hence (ρ,u) is indeed a distributional solution to the system (1.1). By Fatou's lemma, the solution (ρ,u) is also regular enough and satisfies estimates (1.10) and (1.11). Therefore, the existence part of theorem 1.1 is proved.

6. Proof of theorem 1.1: the uniqueness part

This section is devoted to proving the uniqueness of constructed solutions in theorem 1.1.

Because of the hyperbolic nature of the coupled system (1.1) and the low-regularity of density, the Eulerian framework used in the uniqueness proof of proposition 5.1 seems not effective, and we shall employ the Lagrangian approach as in [18, 19, 48] to tackle with the uniqueness issue. Inspired by [19], we intend to show the uniqueness by establishing the $L^\infty_T(\dot H^\alpha)\cap L^2_T(\dot H^{2\alpha})$-estimates of the difference δv of two velocity fields in Lagrangian coordinates (it seems almost impossible to prove the uniqueness in the usual $L^\infty_T(L^2)\cap L^2_T(\dot H^\alpha)$ framework due to the fact that one cannot control the term $\nabla\delta v$ on the right-hand side). We write the system of δv as the twisted fractional Stokes system (6.18) and we derive the crucial $L^2_T(L^2)$ maximal regularity estimate (6.23) on a small time interval. Meanwhile, some right-hand terms in (6.23) arising from the nonlocal dissipation seem hard to be controlled using the (natural) quantity $\|v_i\|_{L^2_T(\dot W^{2\alpha,p})}$, instead we have to adopt $\|v_i\|_{L^2_T(\dot B^{2\alpha}_{p,1})}$ as the bound, which in turn need the stronger regularity $u_i\in L^2_T(\dot B^{2\alpha}_{p,1})$ obtained in proposition 4.4.

In order to derive the 2D fINS system (1.1) in the Lagrangian coordinates, we firstly introduce some basic results. The particle trajectory $X_t(\cdot)$ associated with the velocity u is defined by the ordinary differential equation

$$\frac{\mathrm{d}X_t(y)}{\mathrm{d}t}=u(t,X_t(y)),\quad X_t(y)|_{t=0}=y, \tag{6.1}$$

that is,

$$X_t(y)=y+\int_0^t u(\tau,X_\tau(y))\,\mathrm{d}\tau,\quad y\in\mathbb{R}^2, \tag{6.2}$$

which maps the Lagrangian coordinate y to the Eulerian coordinate $x=X_t(y)$. According to (4.28), we know that $u\in L^1(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^2))$, and equation (6.1) admits a unique solution $X_t(\cdot):\mathbb{R}^2\to\mathbb{R}^2$ for every $t\in[0,\infty)$ which is a measure-preserving bi-Lipschitzian homeomorphism satisfying $X_t^{\pm1}\in L^\infty(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^2))$. Note that the inverse mapping $X_t^{-1}(\cdot):\mathbb{R}^2\to\mathbb{R}^2$ solves that

$$X_t^{-1}(x)=x-\int_0^t u(\tau,X_\tau\circ X_t^{-1}(x))\mathrm{d}\tau. \tag{6.3}$$

By letting $0<T_1<1$ be small enough, we can assume

$$\int_0^{T_1}\|\nabla_x u\|_{L^\infty}\mathrm{d}t\leqslant\frac{1}{2}. \tag{6.4}$$

Denote by

$$\eta(t,y) := \rho(t,X_t(y)), \quad v(t,y) := u(t,X_t(y)), \quad \Pi(t,y) := \pi(t,X_t(y)). \tag{6.5}$$

In the above notation, system (1.1) can be expressed as follows

$$\begin{cases} \partial_t \eta = 0, \\ \eta \partial_t v + \Lambda_v^{2\alpha} v + \nabla_v \Pi = 0, \\ \mathrm{div}_v\, v = 0, \\ (\eta, v)|_{t=0}(y) = (\rho_0(y), u_0(y)), \end{cases} \tag{6.6}$$

where $\Lambda_v^{2\alpha} v(t,y) := \Lambda^{2\alpha} u(t,x) = (\Lambda^{2\alpha} u)(t,X_t(y))$, $\nabla_v \Pi(t,y) := \nabla_x \pi(t,x) = (\nabla_x \pi)(t,X_t(y))$, $\mathrm{div}_v\, v(t,y) := \mathrm{div}_x\, u(t,x) = (\mathrm{div}_x\, u)(t,X_t(y))$. We set

$$A(t,y) := (D_y X_t)^{-1}(y) = \big(D_x X_t^{-1}\big) \circ X_t(y), \quad \text{with } (D_y X_t)_{ij} := \partial_{y_j} X_t^i, \tag{6.7}$$

and set A^{T} the transpose matrix of A, then by the chain rule, some elementary calculation gives that (e.g. see [18, appendix] or [19, equation (35)])

$$\nabla_v \Pi = A^{\mathrm{T}} \nabla_y \Pi, \qquad \mathrm{div}_v\, v = \mathrm{div}_y(Av) = A^{\mathrm{T}} : \nabla v. \tag{6.8}$$

Since $X_t(\cdot)$ is a measure-preserving mapping, according to lemma 2.9, we find that

$$\begin{aligned} (\Lambda^{2\alpha} u)(t,X_t(y)) &= -\frac{\nabla\cdot}{\Lambda^{2-2\alpha}} \nabla u(t,X_t(y)) \\ &= -c_\alpha \int_{\mathbb{R}^2} \frac{(X_t(y)-z)\cdot(\nabla u(t,X_t(y)) - \nabla u(t,z)))}{|X_t(y)-z|^{2+2\alpha}} \mathrm{d}z \\ &= -c_\alpha\, \mathrm{p.v.} \int_{\mathbb{R}^2} \frac{(X_t(y)-X_t(\tilde z))\cdot(\nabla u(t,X_t(y)) - \nabla u(t,X_t(\tilde z))}{|X_t(y)-X_t(\tilde z)|^{2+2\alpha}} \mathrm{d}\tilde z \\ &= -c_\alpha\, \mathrm{p.v.} \int_{\mathbb{R}^2} \frac{(X_t(y)-X_t(z))\cdot\big(A^{\mathrm{T}}(t,y)\nabla v(t,y) - A^{\mathrm{T}}(t,z)\nabla v(t,z)\big)}{|X_t(y)-X_t(z)|^{2+2\alpha}} \mathrm{d}z \\ &=: \Lambda_v^{2\alpha} v(t,y), \end{aligned} \tag{6.9}$$

where $c_\alpha = \frac{\alpha 4^\alpha \Gamma(\alpha)}{2\pi \Gamma(1-\alpha)}$ and $\Gamma(s)$ is the Gamma function.

Now let $(\rho_i, u_i, \pi_i), i = 1,2$ be two solutions of the 2D fINS system (1.1) with the same initial data (ρ_0, u_0). Define

$$\eta_i(t,y) := \rho_i(t,X_{i,t}(y)), \quad v_i(t,y) := u_i(t,X_{i,t}(y)), \quad \Pi_i(t,y) := \pi_i(t,X_{i,t}(y)), \tag{6.10}$$

where $X_{i,t}(y)$ is the particle trajectory generated by velocity u_i:

$$X_{i,t}(y) = y + \int_0^t u_i(\tau, X_{i,\tau}(y))\, \mathrm{d}\tau = y + \int_0^t v_i(\tau,y) \mathrm{d}\tau, \quad i = 1,2. \tag{6.11}$$

Thanks to propositions 4.3 and 4.4, we have the following estimates for the solutions in Lagrangian coordinates.

Proposition 6.1. *Let* $\frac{1}{2}<\alpha<1$, $p>\frac{2}{2\alpha-1}$, $u_0\in H^1\cap\dot{B}^{\alpha+s}_{p,2}(\mathbb{R}^2)$, $s\in(0,1)$, *and* $\rho_0-1\in L^2\cap L^\infty\cap\mathcal{M}(\dot{B}^s_{p,2})\cap\mathcal{M}(\dot{B}^{\frac{2}{p}+1-2\alpha}_{p,1})$ *satisfying conditions (1.9) and (1.12). Then for* $i=1,2$ *we have*

$$\|v_i\|_{L^\infty(\mathbb{R}_+;L^2\cap\dot{B}^\alpha_{p,2})}+\|(\partial_t v_i,\nabla\Pi_i)\|_{L^2(\mathbb{R}_+;L^p)}+\|\Lambda^\alpha v_i\|_{L^2(\mathbb{R}_+;L^2)}\leqslant C, \tag{6.12}$$

and there exists a constant $T_1\in(0,1]$ *small enough such that*

$$\int_0^{T_1}\|\nabla u_i(t)\|_{L^\infty}\mathrm{d}t\leqslant\frac{1}{2},\quad \textit{and}\quad \int_0^{T_1}\|\nabla v_i(t)\|_{L^\infty}\mathrm{d}t\leqslant\frac{1}{2}, \tag{6.13}$$

and

$$\|v_i\|_{L^2(0,T_1;\dot{B}^{2\alpha}_{p,1})}\leqslant C. \tag{6.14}$$

Proof of proposition 6.1. From (4.28) and lemma 2.6, we immediately obtain the estimates of $\|v_i\|_{L^\infty(\mathbb{R}_+;\dot{B}^\alpha_{p,2})}$ and $\|\Lambda^\alpha v_i\|_{L^2(\mathbb{R}_+;L^2)}$ as in (6.12). By virtue of (6.11), we get that

$$\|\nabla X_{i,t}\|_{L^\infty(\mathbb{R}_+;L^\infty)}\leqslant e^{\|\nabla u_i\|_{L^1(\mathbb{R}_+;L^\infty)}}\leqslant C.$$

Then proposition 4.3 and a direct calculation yield

$$\begin{aligned}
&\|\partial_t v_i\|_{L^2(\mathbb{R}_+;L^p)}\leqslant\|\partial_t u_i\|_{L^2(\mathbb{R}_+;L^p)}+\|\nabla u_i\|_{L^2(\mathbb{R}_+;L^p)}\|u_i\|_{L^\infty(\mathbb{R}_+;L^\infty)}\leqslant C,\\
&\|\nabla\Pi_i\|_{L^2(\mathbb{R}_+;L^p)}\leqslant\|\nabla\pi_i\|_{L^2(\mathbb{R}_+;L^p)}\|\nabla X_{i,t}\|_{L^\infty(\mathbb{R}_+;L^\infty)}\leqslant C,\\
&\|\nabla v_i\|_{L^1(\mathbb{R}_+;L^\infty)}\leqslant\|\nabla u_i\|_{L^1(\mathbb{R}_+;L^\infty)}\|\nabla X_{i,t}\|_{L^\infty(\mathbb{R}_+;L^\infty)}\leqslant C.
\end{aligned}\tag{6.15}$$

By letting $T_1>0$ be small enough, the estimate (6.13) follows from (4.28) and (6.15).

Next, we prove the estimate (6.14) from (4.29). Noticing that Next, we prove the estimat

$$\nabla v_i(t,y)=(\nabla X^{\mathrm{T}}_{i,t}(y))\nabla u(t,X_{i,t}(y))=(\nabla X^{\mathrm{T}}_{i,t}(y)-\mathrm{Id})\nabla u_i(t,X_{i,t}(y))+\nabla u_i(t,X_{i,t}(y)),$$

and $\nabla X_{i,t}(y)=\mathrm{Id}+\int_0^t\nabla v_i(\tau,y)\mathrm{d}\tau$, by virtue of (2.1) and proposition 4.3, we find that

$$\begin{aligned}
\|\nabla v_i\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}&\leqslant\|(\nabla X^{\mathrm{T}}_{i,t}-\mathrm{Id})\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}+\|\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\\
&\leqslant C\|\nabla X_{i,t}-\mathrm{Id}\|_{L^\infty_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\|\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(L^\infty)}\\
&\quad+C\big(\|\nabla X_{i,t}-\mathrm{Id}\|_{L^\infty_{T_1}(L^\infty)}+1\big)\|\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\\
&\leqslant C\sqrt{T_1}\|\nabla u_i\|_{L^2_{T_1}(L^\infty)}\|\nabla v_i\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\\
&\quad+C\big(\|\nabla v_i\|_{L^1_{T_1}(L^\infty)}+1\big)\|\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}.
\end{aligned}$$

Letting $T_1>0$ small enough, and utilizing lemma 2.6 and (4.29), we obtain

$$\|\nabla v_i\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\leqslant C\|\nabla u_i\circ X_{i,t}\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\leqslant C\|\nabla u_i\|_{L^2_{T_1}(\dot{B}^{2\alpha-1}_{p,1})}\leqslant C,$$

which immediately leads to (6.14). □

In view of (6.10), the system (6.6) holds with (η_i,v_i,Π_i) in place of (η,v,Π) and with the same initial data (ρ_0,u_0). The equation $(6.6)_1$ gives that

$$\eta_i(t,y)\equiv\rho_0(y),\quad \text{for } i=1,2. \tag{6.16}$$

Set

$$\delta v:=v_1-v_2,\quad \delta\Pi:=\Pi_1-\Pi_2,\quad \delta A:=A_1-A_2, \tag{6.17}$$

with $A_i(t,y) := (\nabla_y X_{i,t})^{-1}(y)$. Then, we arrive at

$$\begin{cases} \partial_t \delta v + \Lambda^{2\alpha}_{v_1}\delta v + \nabla \delta\Pi = (1-\rho_0)\partial_t\delta v + \delta f_1 + \delta f_2, \\ \operatorname{div}\delta v = \operatorname{div}\delta g, \\ \delta v|_{t=0} = 0, \end{cases} \tag{6.18}$$

where

$$\Lambda^{2\alpha}_{v_1}\delta v(t,y) := -c_\alpha\,\text{p.v.}\int_{\mathbb{R}^2} \frac{(X_{1,t}(y)-X_{1,t}(z))\cdot(A_1^{\mathrm{T}}(t,y)\nabla\delta v(t,y) - A_1^{\mathrm{T}}(t,z)\nabla\delta v(t,z))}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}}\mathrm{d}z, \tag{6.19}$$

$$\delta f_1 := (\nabla - \nabla_{v_1})\Pi_1 - (\nabla-\nabla_{v_2})\Pi_2 = (\mathrm{Id} - A_1^{\mathrm{T}})\nabla\delta\Pi - (\delta A^{\mathrm{T}})\nabla\Pi_2, \tag{6.20}$$

$$\begin{aligned}\delta f_2 := c_\alpha\,\text{p.v.}\int_{\mathbb{R}^2}\Bigg(&\frac{(X_{1,t}(y)-X_{1,t}(z))\cdot\big(A_1^{\mathrm{T}}(t,y)\nabla v_2(t,y) - A_1^{\mathrm{T}}(t,z)\nabla v_2(t,z)\big)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}} \\ &- \frac{(X_{2,t}(y)-X_{2,t}(z))\cdot\big(A_2^{\mathrm{T}}(t,y)\nabla v_2(t,y) - A_2^{\mathrm{T}}(t,z)\nabla v_2(t,z)\big)}{|X_{2,t}(y)-X_{2,t}(z)|^{2+2\alpha}}\Bigg)\mathrm{d}z,\end{aligned} \tag{6.21}$$

and

$$\delta g := (\mathrm{Id}-A_1)v_1 - (\mathrm{Id}-A_2)v_2 = (\mathrm{Id}-A_1)\delta v - \delta A\, v_2. \tag{6.22}$$

Concerning the twisted fractional Stokes system (6.18), we have the following $L^2_{T_1}(L^2)$ maximal regularity result on a short time interval.

Proposition 6.2. *Let $(\delta v, \delta\Pi)$ be the solution to the system (6.18). There exists a sufficiently small constant $T_1 > 0$ depending on α, p, s and $\|u_0\|_{H^1\cap\dot B^{\alpha+s}_{p,2}}$ such that*

$$\begin{aligned}\delta E(T_1) &:= \|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)} + \|(\partial_t\delta v, \Lambda^{2\alpha}\delta v, \nabla\delta\Pi)\|_{L^2_{T_1}(L^2)} \\ &\leqslant C\|\operatorname{div}\delta g\|_{L^2_{T_1}(\dot H^{2\alpha-1})} + C\|(\delta f_1,\delta f_2,\partial_t\delta g)\|_{L^2_{T_1}(L^2)},\end{aligned} \tag{6.23}$$

where C *depends only on α and the upper bounds in propositions 4.4 and 6.1.*

Remark 6.3. Following the ideas of [19, 48], it is convenient to write the system (6.18) as

$$\begin{cases} \partial_t\delta v + \Lambda^{2\alpha}\delta v + \nabla\delta\Pi = \delta h_1 + \delta F, \\ \operatorname{div}\delta v = \operatorname{div}\delta g, \quad \delta v|_{t=0} = 0, \end{cases} \tag{6.24}$$

where

$$\delta h_1 := c_\alpha\,\text{p.v.}\int_{\mathbb{R}^2}\left(\frac{X_{1,t}(y)-X_{1,t}(z)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}} - \frac{y-z}{|y-z|^{2+2\alpha}}\right)\cdot\big(\nabla\delta v(t,y) - \nabla\delta v(t,z)\big)\mathrm{d}z,$$

and δF denotes the remaining terms. According to proposition 3.1, one can easily build the following estimate that for $T_1 > 0$ sufficiently small,

$$\delta E(T_1) \leqslant C\|\operatorname{div}\delta g\|_{L^2_{T_1}(\dot H^{2\alpha-1})} + C\|\delta h_1\|_{L^2_{T_1}(L^2)} + C\|(\delta F, \partial_t\delta g)\|_{L^2_{T_1}(L^2)}.$$

However, by letting $T_1 > 0$ small enough and using lemma 2.3, one has the following estimate

$$\begin{aligned}\|\delta h_1\|_{L^2_{T_1}(L^2)} &\leqslant \varepsilon\Big\|\int_{\mathbb{R}^2}\frac{|\nabla\delta v(t,y) - \nabla\delta v(t,z)|}{|y-z|^{1+2\alpha}}\mathrm{d}z\Big\|_{L^2_{T_1}(L^2_y)} \\ &\leqslant \varepsilon\int_{\mathbb{R}^2}\frac{\|\nabla\delta v(t,y) - \nabla\delta v(t,y+z)\|_{L^2_y}}{|z|^{1+2\alpha}}\mathrm{d}z \leqslant \varepsilon\|\nabla\delta v\|_{L^2_{T_1}(\dot B^{2\alpha-1}_{2,1})}\end{aligned}$$

with $\varepsilon>0$ sufficiently small, but it seems difficult to control $\|\delta h_1\|_{L^2_{T_1}(L^2)}$ with the upper bound $\varepsilon\|\nabla\delta v\|_{L^2_{T_1}(\dot{H}^{2\alpha})}$. Hence, we instead treat the system (6.18) directly to derive the key estimate (6.23), so the proof is more complicated than that in the 2D INS system (1.4).

Proof of proposition 6.2. Taking the inner product of $(6.18)_1$ with $\Lambda^{2\alpha}_{v_1}\delta v(t,y)$, we find

$$\begin{aligned}&\int_{\mathbb{R}^2}\partial_t\delta v(t,y)\,\Lambda^{2\alpha}_{v_1}\delta v(t,y)\mathrm{d}y+\int_{\mathbb{R}^2}|\Lambda^{2\alpha}_{v_1}\delta v(t,y)|^2\mathrm{d}y\\&\quad=\int_{\mathbb{R}^2}\big(-\nabla\delta\Pi+(1-\rho_0)\partial_t\delta v+\delta f_1+\delta f_2\big)\,\Lambda^{2\alpha}_{v_1}\delta v(t,y)\mathrm{d}y\\&\quad\leqslant\varepsilon\|\Lambda^{2\alpha}_{v_1}\delta v\|^2_{L^2}+\frac{3}{4\varepsilon}\Big(\|\nabla\delta\Pi\|^2_{L^2}+\|a_0\|^2_{L^\infty}\|\partial_t\delta v\|^2_{L^2}+\|(\delta f_1,\delta f_2)\|^2_{L^2}\Big),\end{aligned}\tag{6.25}$$

where $\varepsilon>0$ is a small constant chosen later. Denoting by $\mathcal{P}:=\nabla\Delta^{-1}\operatorname{div}$, we see that

$$\nabla\delta\Pi=-\mathcal{P}\partial_t\delta v-\mathcal{P}\Lambda^{2\alpha}_{v_1}\delta v+\mathcal{P}(-a_0\,\partial_t\delta v)+\mathcal{P}(\delta f_1)+\mathcal{P}(\delta f_2),$$

which leads to that

$$\begin{aligned}\|\nabla\delta\Pi\|^2_{L^2}\leqslant&\big(\|\mathcal{P}\partial_t\delta v\|_{L^2}+\|\mathcal{P}\Lambda^{2\alpha}_{v_1}\delta v\|_{L^2}+\|\mathcal{P}(a_0\partial_t\delta v)\|_{L^2}+\|\mathcal{P}(\delta f_1,\delta f_2)\|_{L^2}\big)^2\\\leqslant&4\Big(\|\partial_t\delta g\|^2_{L^2}+\|\mathcal{P}(\Lambda^{2\alpha}_{v_1}\delta v)\|^2_{L^2}+\|a_0\|^2_{L^\infty}\|\partial_t\delta v\|^2_{L^2}+\|(\delta f_1,\delta f_2)\|^2_{L^2}\Big).\end{aligned}\tag{6.26}$$

Utilizing the equations $(6.18)_1$ and (6.26) gives

$$\begin{aligned}&\|\partial_t\delta v\|^2_{L^2}+\frac14\|\nabla\delta\Pi\|^2_{L^2}\\&\quad\leqslant\|\Lambda^{2\alpha}_{v_1}\delta v\|^2_{L^2}+\|a_0\|^2_{L^\infty}\|\partial_t\delta v\|^2_{L^2}+\frac54\|\nabla\delta\Pi\|^2_{L^2}+\|(\delta f_1,\delta f_2)\|^2_{L^2}\\&\quad\leqslant\|\Lambda^{2\alpha}_{v_1}\delta v\|^2_{L^2}+5\|\partial_t\delta g\|^2_{L^2}+5\|\mathcal{P}(\Lambda^{2\alpha}_{v_1}\delta v)\|^2_{L^2}+6\|a_0\|^2_{L^\infty}\|\partial_t\delta v\|^2_{L^2}+6\|(\delta f_1,\delta f_2)\|^2_{L^2},\end{aligned}$$

then by assuming $\|a_0\|_{L^\infty}\leqslant\frac14$ without loss of generality, we infer that

$$\|\partial_t\delta v\|^2_{L^2}+\frac{\|\nabla\delta\Pi\|^2_{L^2}}{2}\leqslant2\|\Lambda^{2\alpha}_{v_1}\delta v\|^2_{L^2}+10\|\partial_t\delta g\|^2_{L^2}+10\|\mathcal{P}(\Lambda^{2\alpha}_{v_1}\delta v)\|^2_{L^2}+12\|(\delta f_1,\delta f_2)\|^2_{L^2}.\tag{6.27}$$

Letting $\varepsilon_1>0$ be a small constant chosen later, we insert (6.26) into (6.25) and then combine it with $\varepsilon_1\times$ (6.27) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\partial_t\delta v\,\Lambda^{2\alpha}_{v_1}\delta v(t,y)\mathrm{d}y+\big(1-(\varepsilon+2\varepsilon_1)\big)\|\Lambda^{2\alpha}_{v_1}\delta v\|^2_{L^2}+\Big(\varepsilon_1-\frac{15\|a_0\|^2_{L^\infty}}{4\varepsilon}\Big)\|\partial_t\delta v\|^2_{L^2}+\frac{\varepsilon_1}{2}\|\delta\Pi\|^2_{L^2}\\&\quad\leqslant\Big(\frac{3}{\varepsilon}+10\varepsilon_1\Big)\big(\|\partial_t\delta g\|^2_{L^2}+\|\mathcal{P}(\Lambda^{2\alpha}_{v_1}\delta v)\|^2_{L^2}\big)+\Big(\frac{15}{4\varepsilon}+12\varepsilon_1\Big)\|(\delta f_1,\delta f_2)\|^2_{L^2}.\end{aligned}$$

Hence by setting $\varepsilon=\varepsilon_1=\frac14$, and assuming $\|a_0\|_{L^\infty}\leqslant\frac19$ without loss of generality, it leads to

$$\begin{aligned}&\int_{\mathbb{R}^2}\partial_t\delta v(t,y)\,\Lambda^{2\alpha}_{v_1}\delta v(t,y)\mathrm{d}y+\frac14\|\Lambda^{2\alpha}_{v_1}\delta v(t)\|^2_{L^2}+\frac1{16}\|\partial_t\delta v(t)\|^2_{L^2}+\frac18\|\nabla\delta\Pi(t)\|^2_{L^2}\\&\quad\leqslant15\|\partial_t\delta g(t)\|^2_{L^2}+15\|\mathcal{P}(\Lambda^{2\alpha}_{v_1}\delta v)(t)\|^2_{L^2}+18\|(\delta f_1,\delta f_2)(t)\|^2_{L^2}.\end{aligned}\tag{6.28}$$

Integrating in the time variable shows that for every $t\in[0,T_1]$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\partial_\tau\delta v\,\Lambda_{v_1}^{2\alpha}\delta v(\tau,y)\mathrm{d}y\mathrm{d}\tau+\frac{1}{4}\|\Lambda_{v_1}^{2\alpha}\delta v\|_{L_t^2(L^2)}^2+\frac{1}{16}\|\partial_\tau\delta v\|_{L_t^2(L^2)}^2+\frac{1}{8}\|\nabla\delta\Pi\|_{L_t^2(L^2)}^2\\&\quad\leqslant 15\|\partial_t\delta g\|_{L_t^2(L^2)}^2+15\|\mathcal{P}(\Lambda_{v_1}^{2\alpha}\delta v)\|_{L_t^2(L^2)}^2+18\|(\delta f_1,\delta f_2)\|_{L_t^2(L^2)}^2.\end{aligned}\tag{6.29}$$

Next, observing that by (6.9), (6.7) and the change of variables

$$\begin{aligned}&\|\Lambda_{v_1}^{2\alpha}\delta v(t,y)\|_{L_y^2}^2\\&\quad=c_\alpha^2\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\frac{(X_{1,t}(y)-X_{1,t}(z))\cdot\big(A_1^{\mathrm{T}}(t,y)\nabla\delta v(t,y)-A_1^{\mathrm{T}}(t,z)\nabla\delta v(t,z)\big)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}}\mathrm{d}z\Big|^2\mathrm{d}y\\&\quad=c_\alpha^2\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\frac{(x-z)\cdot\Big(\nabla\big(\delta v(t,X_{1,t}^{-1}(x))\big)-\nabla\big(\delta v(t,X_{1,t}^{-1}(z))\big)\Big)}{|x-z|^{2+2\alpha}}\mathrm{d}z\Big|^2\mathrm{d}x\\&\quad=\|\Lambda^{2\alpha-2}\nabla\cdot\nabla\big(\delta v(t,X_{1,t}^{-1}(x))\big)\|_{L_x^2}^2=\|\Lambda^{2\alpha}\big(\delta v(t,X_{1,t}^{-1}(x))\big)\|_{L_x^2}^2,\end{aligned}\tag{6.30}$$

we get

$$\begin{aligned}\|\Lambda_{v_1}^{2\alpha}\delta v(t,y)\|_{L_y^2}^2&\geqslant\|\Lambda^{2\alpha-1}\nabla_x\big(\delta v(t,X_{1,t}^{-1}(x))\big)\|_{L_x^2}^2\\&=\|\Lambda^{2\alpha-1}\big(\nabla X_{1,t}^{-1}(x)\,(\nabla\delta v)(t,X_{1,t}^{-1}(x))\big)\|_{L_x^2}^2\\&\geqslant\frac{1}{2}\|\nabla\delta v(t,X_{1,t}^{-1}(x))\|_{\dot H_x^{2\alpha-1}}^2-\|(\mathrm{Id}-\nabla X_{1,t}^{-1})\,\nabla\delta v(t,X_{1,t}^{-1}(x))\|_{\dot H_x^{2\alpha-1}}^2\\&=:N_1+N_2,\end{aligned}\tag{6.31}$$

where in the third line we have used the simple inequality $(a-b)^2\geqslant\frac{1}{2}a^2-b^2$ for $a,b>0$. Notice that (owing to lemma 2.3)

$$\begin{aligned}N_1&\geqslant\frac{1}{2C_1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\nabla\delta v(t,X_{1,t}^{-1}(x))-\nabla\delta v(t,X_{1,t}^{-1}(z))|^2}{|x-z|^{4\alpha}}\mathrm{d}x\mathrm{d}z\\&=\frac{1}{2C_1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\nabla\delta v(t,y)-\nabla\delta v(t,z)|^2}{|X_{1,t}(y)-X_{1,t}(z)|^{4\alpha}}\mathrm{d}y\mathrm{d}z.\end{aligned}\tag{6.32}$$

Making use of (6.2) and the mean value theorem yields that for $i=1,2$,

$$\begin{aligned}|X_{i,t}(y)-X_{i,t}(z)|&\leqslant|y-z|+\int_0^t\big|u_i(\tau,X_{i,\tau}(y))-u_i(\tau,X_{i,\tau}(z))\big|\mathrm{d}\tau\\&\leqslant|y-z|+\int_0^t\|\nabla u_i\|_{L^\infty}\big|X_{i,\tau}(y)-X_{i,\tau}(z)\big|\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}|y-z|&\leqslant\big|X_{i,t}(y)-X_{i,t}(z)\big|+\int_0^t\big|v_i(\tau,y)-v_i(\tau,z)\big|\mathrm{d}\tau\\&\leqslant\big|X_{i,t}(y)-X_{i,t}(z)\big|+\int_0^t\|\nabla v_i\|_{L^\infty}|y-z|\mathrm{d}\tau.\end{aligned}$$

Thus Gronwall's inequality guarantees that

$$|y-z|e^{-\int_0^t\|\nabla v_i\|_{L^\infty}\mathrm{d}\tau}\leqslant|X_{i,t}(y)-X_{i,t}(z)|\leqslant|y-z|e^{\int_0^t\|\nabla u_i\|_{L^\infty}\mathrm{d}\tau}.$$

Hence, by taking $T_1 > 0$ small enough, we have that for any $t \leqslant T_1$,

$$\frac{3}{4} \leqslant \frac{|y-z|}{|X_{i,t}(y) - X_{i,t}(z)|} \leqslant \frac{4}{3}, \quad \forall y \neq z \in \mathbb{R}^2, \tag{6.33}$$

or equivalently,

$$\frac{3}{4} \leqslant \frac{|X_{i,t}^{-1}(y) - X_{i,t}^{-1}(z)|}{|y-z|} \leqslant \frac{4}{3}, \quad \forall y \neq z \in \mathbb{R}^2. \tag{6.34}$$

Thus, it follows from (6.32) and (6.33) that

$$\begin{aligned} N_1 &\geqslant \frac{1}{2(4/3)^{4\alpha} C_1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\nabla \delta v(t,y) - \nabla \delta v(t,z)|^2}{|y-z|^{4\alpha}} \mathrm{d}y\mathrm{d}z \\ &\geqslant \frac{1}{8C_1} \|\Lambda^{2\alpha-1} \nabla \delta v\|_{L^2}^2 \geqslant \frac{1}{8C_1} \|\Lambda^{2\alpha} \delta v\|_{L^2}^2, \end{aligned}$$

and then for every $t \in [0, T_1]$, we can lower bound

$$\|N_1\|_{L^1([0,t])} \geqslant \frac{1}{8C_1} \|\Lambda^{2\alpha} \delta v\|_{L_t^2(L^2)}^2. \tag{6.35}$$

For the term N_2 given by (6.31), noticing that $\nabla X_{1,t}^{-1}(x) = A_1^{\mathrm{T}}(t,y) = A_1^{\mathrm{T}}(t, X_{1,t}^{-1}(x))$ (from (6.7)), we can apply lemmas 2.6 and 2.7 to find that

$$\begin{aligned} \|N_2\|_{L^1([0,T_1])} &\leqslant C \|\nabla \delta v(t, X_{1,t}^{-1}(x))\|_{L_{T_1}^2(\dot{H}^{2\alpha-1})}^2 \|\mathrm{Id} - A_1(t, X_{1,t}^{-1}(x))\|_{L_{T_1}^\infty(L^\infty)}^2 \\ &\quad + C \|\nabla \delta v(t, X_{1,t}^{-1}(x))\|_{L_{T_1}^2(L^{\frac{2p}{p-2}})}^2 \|\mathrm{Id} - A_1(t, X_{1,t}^{-1}(x))\|_{L_{T_1}^\infty(\dot{W}^{2\alpha-1,p})}^2 \\ &\leqslant C \|\nabla \delta v\|_{L_{T_1}^2(\dot{H}^{2\alpha-1})}^2 \|\mathrm{Id} - A_1\|_{L_{T_1}^\infty(L^\infty)}^2 + C \|\nabla \delta v\|_{L_{T_1}^2(L^{\frac{2p}{p-2}})}^2 \|\mathrm{Id} - A_1\|_{L_{T_1}^\infty(\dot{B}_{p,1}^{2\alpha-1})}^2, \end{aligned} \tag{6.36}$$

where in the last line we also have used the embedding $\dot{B}_{p,1}^{2\alpha-1} \hookrightarrow \dot{W}^{2\alpha-1,p}$. Recalling that under the condition (6.13),

$$A_i(t,y) = (DX_{i,t}(y))^{-1} = (\mathrm{Id} + B_i(t,y))^{-1} = \mathrm{Id} + \sum_{k=1}^{\infty} (-B_i(t,y))^k, \quad i = 1,2, \tag{6.37}$$

with $B_i(t,y) := DX_{i,t}(y) - \mathrm{Id} = \int_0^t Dv_i(\tau,y)\,\mathrm{d}\tau$, it is easy to see that

$$\|\mathrm{Id} - A_i\|_{L_{T_1}^\infty(L^\infty)} \leqslant \sum_{k=1}^{\infty} \|B_i\|_{L_{T_1}^\infty(L^\infty)}^k \leqslant \sum_{k=1}^{\infty} \|\nabla v_i\|_{L_{T_1}^1(L^\infty)}^k \leqslant 2\|\nabla v_i\|_{L_{T_1}^1(L^\infty)} \leqslant 2CT_1^{\frac{1}{2}}. \tag{6.38}$$

For the term $\|\mathrm{Id} - A_i\|_{L_T^\infty(\dot{B}_{p,1}^{2\alpha-1})}$, due to $2\alpha - 1 - \frac{2}{p} > 0$, the nonhomogeneous space $B_{p,1}^{2\alpha-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ is a Banach algebra, thus choosing $T_1 > 0$ small enough, proposition 6.1 yields that for $i = 1,2$,

$$\big\|\mathrm{Id}-A_i\big\|_{L^\infty_{T_1}(\dot B^{2\alpha-1}_{p,1})}\leqslant\sum_{k=1}^{\infty}\|B_i(t,y)\|^k_{L^\infty_{T_1}(B^{2\alpha-1}_{p,1})}$$
$$\leqslant\sum_{k=1}^{\infty}\Big(CT_1^{\frac12}\|\nabla v_i\|_{L^2_{T_1}(B^{2\alpha-1}_{p,1})}\Big)^k$$
$$\leqslant\sum_{k=1}^{\infty}\Big(CT_1^{\frac12}\|v_i\|_{L^2_{T_1}(\dot B^{2\alpha}_{p,1}\cap\dot H^\alpha)}\Big)^k$$
$$\leqslant C\sqrt{T_1}\|v_i\|_{L^2_{T_1}(\dot B^{2\alpha}_{p,1}\cap\dot H^\alpha)}\leqslant C\sqrt{T_1}. \tag{6.39}$$

Using the Sobolev embedding $\dot H^\alpha\cap\dot H^{2\alpha}(\mathbb{R}^2)\hookrightarrow\dot W^{1,\frac{2p}{p-2}}(\mathbb{R}^2)$ (due to $p>\frac{2}{2\alpha-1}$), we also get

$$\big\|\nabla\delta v\big\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})}\leqslant C\|\delta v\|^{\frac{(2\alpha-1)p-2}{\alpha p}}_{L^2_{T_1}(\dot H^\alpha)}\|\delta v\|^{\frac{(1-\alpha)p+2}{\alpha p}}_{L^2_{T_1}(\dot H^{2\alpha})}\leqslant C\|\delta v\|_{L^2_{T_1}(\dot H^\alpha\cap\dot H^{2\alpha})}. \tag{6.40}$$

Collecting estimates (6.36) and (6.38)–(6.40) yields

$$\|N_2\|_{L^1([0,T_1])}\leqslant CT_1\|\delta v\|^2_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|^2_{L^2_{T_1}(\dot H^\alpha)}. \tag{6.41}$$

We then consider the first term on the left-hand side of (6.28). In light of (6.19) and (6.30), we see that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\partial_t\delta v(t,y)\,\Lambda^{2\alpha}_{v_1}\delta v(t,y)\mathrm{d}y\\
&=c_\alpha\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_t\delta v(t,y)\,\frac{(X_{1,t}(y)-X_{1,t}(z))\cdot(A_1^{\mathrm T}(t,y)\nabla\delta v(t,y)-A_1^{\mathrm T}(t,z)\nabla\delta v(t,z))}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}}\mathrm{d}z\mathrm{d}y\\
&=\int_{\mathbb{R}^2}\partial_t\delta v\big(t,X^{-1}_{1,t}(x)\big)\,\Lambda^{2\alpha}_x\big(\delta v(t,X^{-1}_{1,t}(x))\big)\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\partial_t\big(\delta v(t,X^{-1}_{1,t}(x))\big)\,\Lambda^{2\alpha}_x\big(\delta v(t,X^{-1}_{1,t}(x))\big)\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^2}\partial_tX^{-1}_{1,t}(x)\cdot\nabla\delta v(t,X^{-1}_{1,t}(x))\,\Lambda^{2\alpha}_x\big(\delta v(t,X^{-1}_{1,t}(x))\big)\mathrm{d}x\\
&=:\Psi_1+\Psi_2.
\end{aligned}$$

For Ψ_1, notice that

$$\begin{aligned}
\Psi_1&=\frac12\frac{\mathrm d}{\mathrm dt}\|\big(\delta v(t,X^{-1}_{1,t}(x))\big)\|^2_{\dot H^\alpha_x}\\
&=\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\delta v(t,X^{-1}_{1,t}(x))-\delta v(t,X^{-1}_{1,t}(z))|^2}{|x-z|^{2+2\alpha}}\mathrm{d}x\mathrm{d}z\\
&=\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\delta v(t,y)-\delta v(t,\tilde z)|^2}{|X_{1,t}(y)-X_{1,t}(\tilde z)|^{2+2\alpha}}\mathrm{d}y\mathrm{d}\tilde z.
\end{aligned}$$

Thus, by letting $T_1>0$ be small enough so that (6.33) holds, we integrate in the time variable and then use (6.33) to deduce that for every $t\in[0,T_1]$,

$$\int_0^t \Psi_1 \mathrm{d}\tau \geqslant \frac{1}{2\cdot(4/3)^{2+2\alpha}}\|\delta v(t,\cdot)\|_{\dot H^\alpha}^2 \geqslant \frac{1}{8}\|\delta v(t,\cdot)\|_{\dot H^\alpha}^2. \tag{6.42}$$

Noting that (from (6.3))

$$\partial_t X_{i,t}^{-1}(x) = -u_i(t,x) - \int_0^t \nabla u_i\big(\tau, X_{i,\tau}\circ X_{i,t}^{-1}(x)\big)\,\nabla X_{i,\tau}\circ X_t^{-1}(x)\mathrm{d}\tau\,\partial_t X_{i,t}^{-1}(x),$$

and

$$\partial_t X_{i,t}^{-1}(x) = -\Big(\mathrm{Id} + \int_0^t \nabla u_i\big(\tau, X_{i,\tau}\circ X_{i,t}^{-1}(x)\big)\,\nabla X_{i,\tau}\circ X_t^{-1}(x)\mathrm{d}\tau\Big)^{-1} u_i(t,x),$$

we apply (6.13) and the estimate $\|\nabla X_{i,t}\|_{L^\infty} \leqslant e^{\|\nabla u_i\|_{L^1_t(L^\infty)}} \leqslant \sqrt{e}$ to deduce that for every $t\in[0,T_1]$,

$$\|\partial_t X_{i,t}^{-1}\|_{L^p} \leqslant \Big(1 - \int_0^t \|\nabla u_i(\tau)\|_{L^\infty}\mathrm{d}\tau \|\nabla X_{i,\tau}\|_{L^\infty_t(L^\infty)}\Big)^{-1}\|u_i(t)\|_{L^p} \leqslant C.$$

Using the above inequality, (6.30) and interpolation inequality (6.40), the term Ψ_2 can be estimated as

$$\begin{aligned}
\int_0^t |\Psi_2|\mathrm{d}\tau &\leqslant \int_0^t \|\partial_\tau X_{1,\tau}^{-1}(x)\|_{L^p_x}\|\nabla\delta v(\tau, X_{1,\tau}^{-1}(x))\|_{L_x^{\frac{2p}{p-2}}}\|\delta v(\tau, X_{1,\tau}^{-1}(x))\|_{\dot H_x^{2\alpha}}\mathrm{d}\tau \\
&\leqslant C\|\partial_\tau X_{1,\tau}^{-1}\|_{L^\infty_t(L^p)}\|\nabla\delta v\|_{L^2_t(L^{\frac{2p}{p-2}})}\|\delta v(\tau, X_{1,\tau}^{-1}(x))\|_{L^2_t(\dot H^{2\alpha})} \\
&\leqslant C\|\delta v\|_{L^2_t(\dot H^\alpha)}^{\frac{2(2\alpha-1)p-4}{\alpha p}}\|\delta v\|_{L^2_t(\dot H^{2\alpha})}^{\frac{2(1-\alpha)p+4}{\alpha p}} + \frac{1}{16}\|\Lambda_{\nu_1}^{2\alpha}\delta v(\tau,y)\|_{L^2_t(L^2)}^2 \\
&\leqslant C\|\delta v\|_{L^2_t(\dot H^\alpha)}^2 + \frac{\|\delta v\|_{L^2_t(\dot H^{2\alpha})}^2}{128C_1} + \frac{1}{16}\|\Lambda_{\nu_1}^{2\alpha}\delta v\|_{L^2_t(L^2)}^2 \\
&\leqslant Ct\|\delta v\|_{L^\infty_t(\dot H^\alpha)}^2 + \frac{1}{128C_1}\|\delta v\|_{L^2_t(\dot H^{2\alpha})}^2 + \frac{1}{16}\|\Lambda_{\nu_1}^{2\alpha}\delta v\|_{L^2_t(L^2)}^2.
\end{aligned} \tag{6.43}$$

Integrating in the time interval $[0,t]$ yields that for every $t\in[0,T_1]$,

$$\int_0^t\int_{\mathbb{R}^2}\partial_\tau\delta v\,\Lambda_{\nu_1}^{2\alpha}\delta v\mathrm{d}y\mathrm{d}\tau \geqslant \frac{\|\delta v(t)\|_{\dot H^\alpha}^2}{8} - Ct\|\delta v\|_{L^\infty_t(\dot H^\alpha)}^2 - \frac{\|\delta v\|_{L^2_t(\dot H^{2\alpha})}^2}{128C_1} - \frac{\|\Lambda_{\nu_1}^{2\alpha}\delta v\|_{L^2_t(L^2)}^2}{16}. \tag{6.44}$$

Now we consider $\|\mathcal{P}(\Lambda_{\nu_1}^{2\alpha}\delta v)\|_{L^2}$ (recall that $\mathcal{P} := \nabla\Delta^{-1}\mathrm{div}$). By arguing as in the proof of (6.30), and using (6.8), (6.19) and the change of variables, we find that

$$\begin{aligned}
&\|\mathcal{P}(\Lambda_{\nu_1}^{2\alpha}\delta v)\|_{L^2}^2 \\
&\quad\leqslant \|\mathrm{div}(\Lambda_{\nu_1}^{2\alpha}\delta v)\|_{\dot H^{-1}}^2 \\
&\quad\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\frac{\mathrm{div}(\Lambda_{\nu_1}^{2\alpha}\delta v)(y)}{|x-y|}\mathrm{d}y\Big|^2\mathrm{d}x \\
&\quad\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\frac{1}{|x-y|}\,\mathrm{div}_y\Big(\int_{\mathbb{R}^2}\frac{(X_{1,t}(y)-X_{1,t}(z))\cdot(A_1^{\mathrm{T}}(t,y)\nabla\delta v(t,y)-A_1^{\mathrm{T}}(t,z)\nabla\delta v(t,z))}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}}\mathrm{d}z\Big)\mathrm{d}y\Big|^2\mathrm{d}x \\
&\quad\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\frac{1}{|X_{1,t}^{-1}(\tilde x)-X_{1,t}^{-1}(\tilde y)|}\,\mathrm{div}_{\tilde y}\Big(\big(DX_{1,t}^{-1}(\tilde y)\big)^{-1}\int_{\mathbb{R}^2}\frac{(\tilde y-\tilde z)\cdot\big(\nabla[\delta v(t,X_{1,t}^{-1}(\tilde y))]-\nabla[\delta v(t,X_{1,t}^{-1}(\tilde z))]\big)}{|\tilde y-\tilde z|^{2+2\alpha}}\mathrm{d}\tilde z\Big)\mathrm{d}\tilde y\Big|^2\mathrm{d}\tilde x
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\,\mathrm{div}_{\tilde{y}}\Big(\big(DX_{1,t}^{-1}(\tilde{y})\big)^{-1}\Lambda^{2\alpha}\big(\delta v(t,X_{1,t}^{-1}(\tilde{y}))\big)\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\,\mathrm{div}_{\tilde{y}}\Big(\Lambda^{2\alpha}\big(\delta v(t,X_{1,t}^{-1}(\tilde{y}))\big)\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\quad+\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\,\mathrm{div}_{\tilde{y}}\Big(\big(\mathrm{Id}-\big(DX_{1,t}^{-1}(\tilde{y})\big)^{-1}\big)\Lambda^{2\alpha}\big(\delta v(t,X_{1,t}^{-1}(\tilde{y}))\big)\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&=:\Upsilon_1+\Upsilon_2.
\end{aligned}
$$

For Υ_1, by using the relation (6.18)$_2$ and the change of variables again, it follows that

$$
\begin{aligned}
\Upsilon_1 &\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\Big(\Lambda^{2\alpha}\,\mathrm{div}\,\big(\delta v(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\Big(\Lambda^{2\alpha}\big(\nabla X_{1,t}^{-1}(\cdot):\nabla\delta v(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\Big(\Lambda^{2\alpha}\big(\mathrm{div}\,\delta v(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\quad+\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\Big(\Lambda^{2\alpha}\big((\mathrm{Id}-\nabla X_{1,t}^{-1}(\cdot)):\nabla\delta v(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\,\mathrm{div}_{\tilde{y}}\Big(\nabla\Lambda^{2\alpha-2}\big(\mathrm{div}\,\delta g(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\quad+\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|X_{1,t}^{-1}(\tilde{x})-X_{1,t}^{-1}(\tilde{y})|}\,\mathrm{div}_{\tilde{y}}\Big(\nabla\Lambda^{2\alpha-2}\big((\mathrm{Id}-\nabla X_{1,t}^{-1}(\cdot)):\nabla\delta v(t,X_{1,t}^{-1}(\cdot))\big)(\tilde{y})\Big)\mathrm{d}\tilde{y}\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|x-y|}\,\mathrm{div}_y\Big(A_1(t,y)\big(\big[\nabla\Lambda^{2\alpha-2}\big(\mathrm{div}\,\delta g(t,X_{1,t}^{-1})\big)\big]\circ X_{1,t}(y)\big)\Big)\mathrm{d}y\Big|^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\tfrac{1}{|x-y|}\,\mathrm{div}_y\Big(A_1(t,y)\big(\big[\nabla\Lambda^{2\alpha-2}\big((\mathrm{Id}-\nabla X_{1,t}^{-1}):\nabla\delta v(t,X_{1,t}^{-1})\big)\big]\circ X_{1,t}(y)\big)\Big)\mathrm{d}y\Big|^2\mathrm{d}\tilde{x}\\
&\lesssim \Big\|\Lambda^{-1}\,\mathrm{div}\Big(A_1(t,\cdot)\big[\nabla\Lambda^{2\alpha-2}\big(\mathrm{div}\,\delta g(t,X_{1,t}^{-1})\big)\big]\circ X_{1,t}(\cdot)\Big)\Big\|_{L^2}^2\\
&\quad+\Big\|\Lambda^{-1}\,\mathrm{div}\Big(A_1(t,\cdot)\big[\nabla\Lambda^{2\alpha-2}\big((\mathrm{Id}-\nabla X_{1,t}^{-1}):\nabla\delta v(t,X_{1,t}^{-1})\big)\big]\circ X_{1,t}(\cdot)\Big)\Big\|_{L^2}^2\\
&\leqslant C\|A_1(t,y)\|_{L_y^\infty}^2\|\nabla\Lambda^{2\alpha-2}\big(\mathrm{div}\,\delta g(t,X_{1,t}^{-1}(\cdot))\big)(x)\|_{L_x^2}^2\\
&\quad+C\|A_1(t,y)\|_{L_y^\infty}^2\|\nabla\Lambda^{2\alpha-2}\big((\mathrm{Id}-\nabla X_{1,t}^{-1}(\cdot)):\nabla\delta v(t,X_{1,t}^{-1}(\cdot))\big)(x)\|_{L_x^2}^2.
\end{aligned}
$$

Similarly to the proof of (6.36), by applying lemma 2.6, proposition 6.1 and letting $T_1>0$ be small enough, we infer that

$$
\begin{aligned}
\|\Upsilon_1\|_{L^1_{T_1}} &\lesssim \|\,\mathrm{div}\,\delta g(t,X_{1,t}^{-1}(\cdot))\|_{L^2_{T_1}(\dot H^{2\alpha-1})}^2+\|\big(\mathrm{Id}-A_1(t,X_{1,t}^{-1}(x))\big):\nabla\delta v(t,X_{1,t}^{-1}(x))\|_{L^2_{T_1}(\dot H^{2\alpha-1})}^2\\
&\leqslant C\|\,\mathrm{div}\,\delta g\|_{L^2_{T_1}(\dot H^{2\alpha-1})}^2+C\|\mathrm{Id}-A_1\|_{L^\infty_{T_1}(L^\infty)}^2\|\nabla\delta v\|_{L^2_{T_1}(\dot H^{2\alpha-1})}^2\\
&\quad+C\|\mathrm{Id}-A_1\|_{L^\infty_{T_1}(\dot B^{2\alpha-1}_{p,1})}^2\|\nabla\delta v\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})}^2\\
&\leqslant C\|\,\mathrm{div}\,\delta g\|_{L^2_{T_1}(\dot H^{2\alpha-1})}^2+CT_1\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}^2+CT_1\|\delta v\|_{L^2_{T_1}(\dot H^{\alpha})}^2.
\end{aligned}
\tag{6.45}
$$

For Υ_2, observing that $(DX_{1,t}^{-1}(\tilde{y}))^{-1}=DX_{1,t}(y)=(DX_{1,t})\circ X_{1,t}^{-1}(\tilde{y})$, and by letting $T_1>0$ small enough (so that (6.34) and the last inequality in (6.46) hold true), we obtain that for every $t\in[0,T_1]$,

$$\begin{aligned}
\|\Upsilon_2\|_{L^1_t} &\lesssim \Big\| \int_{\mathbb{R}^2} \frac{1}{|\tilde{x}-\tilde{y}|} \mathrm{div}_{\tilde{y}} \Big(\big(\mathrm{Id} - (DX_{1,\tau}) \circ X^{-1}_{1,\tau}(\tilde{y})\big) \left[\Lambda^{2\alpha}(\delta v(\tau, X^{-1}_{1,\tau}))\right] \circ X_{1,\tau}(\tilde{y}) \Big) \mathrm{d}\tilde{y} \Big\|^2_{L^2_t(L^2_{\tilde{x}})} \\
&\lesssim \Big\| \Lambda^{-1} \mathrm{div} \Big(\big(\mathrm{Id} - (DX_{1,\tau}) \circ X^{-1}_{1,\tau}(\cdot)\big) \left[\Lambda^{2\alpha}(\delta v(\tau, X^{-1}_{1,\tau}))\right] \circ X_{1,\tau}(\cdot) \Big) \Big\|^2_{L^2_t(L^2_{\tilde{x}})} \\
&\lesssim \|\mathrm{Id} - DX_{1,\tau}\|^2_{L^\infty_t(L^\infty)} \big\| \Lambda^{2\alpha} \big(\delta v(\tau, X^{-1}_{1,\tau}(\cdot)) \big) \big\|^2_{L^2_t(L^2)} \\
&\leqslant C \|\nabla v_1\|^2_{L^1_t(L^\infty)} \|\Lambda^{2\alpha}_{v_1} \delta v(\tau, x)\|^2_{L^2_t(L^2_x)} \\
&\leqslant C T_1 \|\Lambda^{2\alpha}_{v_1} \delta v(\tau, x)\|^2_{L^2_t(L^2)} \leqslant \frac{1}{16} \|\Lambda^{2\alpha}_{v_1} \delta v(\tau, x)\|^2_{L^2_t(L^2)},
\end{aligned} \tag{6.46}$$

where in the last line we have used (6.30).

Gathering (6.29) and (6.30), (6.35), (6.41), (6.44)–(6.46), we conclude that for every $t \in [0, T_1]$,

$$\begin{aligned}
&\frac{\|\delta v(t)\|^2_{\dot{H}^\alpha}}{8} + \frac{1}{128 C_1} \|\Lambda^{2\alpha} \delta v\|^2_{L^2_t(L^2)} + \frac{1}{16} \|\partial_\tau \delta v\|^2_{L^2_t(L^2)} + \frac{1}{8} \|\nabla \delta \Pi\|^2_{L^2_t(L^2)} \\
&\leqslant C T_1 \big(\|\delta v\|^2_{L^\infty_{T_1}(\dot{H}^\alpha)} + \|\delta v\|^2_{L^2_{T_1}(\dot{H}^{2\alpha})} \big) + C \|\mathrm{div}\, \delta g\|^2_{L^2_{T_1}(\dot{H}^{2\alpha-1})} + C \|(\delta f_1, \delta f_2, \partial_t g)\|^2_{L^2_{T_1}(L^2)}.
\end{aligned}$$

Hence, by taking the supremum over $[0, T_1]$ and then by letting $T_1 > 0$ be small enough, we conclude the desired estimate (6.23).

□

Now in order to get the uniqueness result in theorem 1.1, we need to check the terms in the right-hand side of (6.23). For the term $\|\delta f_1\|_{L^2_{T_1}(L^2)}$ with $\delta f_1 = (\mathrm{Id} - A_1^{\mathrm{T}}) \nabla \delta \Pi - (\delta A^{\mathrm{T}}) \nabla \Pi_2$, recalling (6.37) and (6.38) and noting that

$$\begin{aligned}
\delta A(t, y) &= \sum_{k=1}^{\infty} \Big((-B_1(t, y))^k - (-B_2(t, y))^k \Big) \\
&= \Big(\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} (-1)^k B_1^j B_2^{k-1-j} \Big) \int_0^t D\delta v(\tau, y)\, \mathrm{d}\tau,
\end{aligned} \tag{6.47}$$

with $B_i(t, y) = \int_0^t D v_i(\tau, y) \mathrm{d}\tau$, by letting $T_1 > 0$ be small enough, it can be controlled as follows

$$\begin{aligned}
\|\delta f_1\|_{L^2_{T_1}(L^2)} &\leqslant \|\mathrm{Id} - A_1\|_{L^\infty_{T_1}(L^\infty)} \|\nabla \delta \Pi\|_{L^2_{T_1}(L^2)} + \|\delta A\|_{L^\infty_{T_1}\left(L^{\frac{2p}{p-2}}\right)} \|\nabla \Pi_2\|_{L^2_{T_1}(L^p)} \\
&\leqslant C T_1^{\frac{1}{2}} \|\nabla \delta \Pi\|_{L^2_{T_1}(L^2)} + C T_1^{\frac{1}{2}} \|\nabla \delta v\|_{L^2_{T_1}\left(L^{\frac{2p}{p-2}}\right)} \\
&\leqslant C T_1^{\frac{1}{2}} \|\nabla \delta \Pi\|_{L^2_{T_1}(L^2)} + C T_1^{\frac{1}{2}} \|\delta v\|_{L^2_{T_1}(\dot{H}^{2\alpha})} + C T_1 \|\delta v\|_{L^\infty_{T_1}(\dot{H}^\alpha)},
\end{aligned} \tag{6.48}$$

where in the last line we also have used (6.40).

Next, let us treat the term δf_2 given by (6.21). Observe that

$$\delta f_2 = c_\alpha \int_{\mathbb{R}^2} \frac{\big(X_{1,t}(y) - X_{1,t}(z)\big) \cdot \big(A_1^{\mathrm{T}}(t,y) - A_2^{\mathrm{T}}(t,y) - (A_1^{\mathrm{T}}(t,z) - A_2^{\mathrm{T}}(t,z))\big)\nabla v_2(t,y)}{|X_{1,t}(y) - X_{1,t}(z)|^{2+2\alpha}} \mathrm{d}z$$
$$+ c_\alpha \int_{\mathbb{R}^2} \frac{\big(X_{1,t}(y) - X_{1,t}(z)\big) \cdot \big(A_1^{\mathrm{T}}(t,z) - A_2^{\mathrm{T}}(t,z)\big)\big(\nabla v_2(t,y) - \nabla v_2(t,z)\big)}{|X_{1,t}(y) - X_{1,t}(z)|^{2+2\alpha}} \mathrm{d}z$$
$$+ c_\alpha \int_{\mathbb{R}^2} \left[\frac{X_{1,t}(y)-X_{1,t}(z)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}} - \frac{X_{2,t}(y)-X_{2,t}(z)}{|X_{2,t}(y)-X_{2,t}(z)|^{2+\alpha}}\right] \cdot \big(A_2^{\mathrm{T}}(t,y) - A_2^{\mathrm{T}}(t,z)\big)\nabla v_2(t,y)\mathrm{d}z$$
$$+ c_\alpha \int_{\mathbb{R}^2} \left[\frac{X_{1,t}(y)-X_{1,t}(z)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}} - \frac{X_{2,t}(y)-X_{2,t}(z)}{|X_{2,t}(y)-X_{2,t}(z)|^{2+\alpha}}\right] \cdot A_2^{\mathrm{T}}(t,z)(\nabla v_2(t,y) - \nabla v_2(t,z))\mathrm{d}z$$
$$:= \delta f_2^1 + \delta f_2^2 + \delta f_2^3 + \delta f_2^4.$$

For δf_2^1, by changing variables and using lemma 2.6, we get

$$\|\delta f_2^1\|_{L^2_y}^2 = c_\alpha^2 \int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \frac{\big(X_{1,t}(y) - X_{1,t}(z)\big) \cdot \big(\delta A^{\mathrm{T}}(t,y) - \delta A^{\mathrm{T}}(t,z)\big)}{|X_{1,t}(y) - X_{1,t}(z)|^{2+2\alpha}} \mathrm{d}z \, \nabla v_2(t,y)\Big|^2 \mathrm{d}y$$
$$= c_\alpha^2 \int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \frac{\big(x - \tilde z\big) \cdot \big(\delta A^{\mathrm{T}}(t,X_{1,t}^{-1}(x)) - \delta A^{\mathrm{T}}(t,X_{1,t}^{-1}(\tilde z))\big)}{|x - \tilde z|^{2+2\alpha}} \mathrm{d}\tilde z \, \nabla v_2(t,X_{1,t}^{-1}(x))\Big|^2 \mathrm{d}x$$
$$= C \int_{\mathbb{R}^2} \Big| \big[\Lambda^{2\alpha-2} \nabla \cdot (\delta A^{\mathrm{T}}(t,X_{1,t}^{-1}(x)))\big] \nabla v_2(t,X_{1,t}^{-1}(x))\Big|^2 \mathrm{d}x$$
$$\leqslant C \|\delta A(t,X_{1,t}^{-1}(x))\|_{\dot H^{2\alpha-1}}^2 \|\nabla v_2(t,X_{1,t}^{-1}(x))\|_{L^\infty}^2$$
$$\leqslant C \|\delta A(t,y)\|_{\dot H_y^{2\alpha-1}}^2 \|\nabla v_2(t)\|_{L^\infty}^2.$$

In view of (6.47) and lemma 2.7, and using estimates (6.38)–(6.40), we have

$$\begin{aligned}
\big\|\delta A\big\|_{L^\infty_{T_1}(\dot H^{2\alpha-1})} &\leqslant C \big\|\nabla \delta v\big\|_{L^1_{T_1}(\dot H^{2\alpha-1})} \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \|B_1^j B_2^{k-1-j}\|_{L^\infty_{T_1}(L^\infty)} \\
&\quad + C \big\|\nabla \delta v\big\|_{L^1_{T_1}(L^{\frac{2p}{p-2}})} \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \|B_1^j B_2^{k-1-j}\|_{L^\infty_{T_1}(W^{2\alpha-1,p})} \\
&\leqslant C T^{\frac12} \|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})} \sum_{k=1}^{\infty} k \Big(T_1^{\frac12} \|(\nabla v_1, \nabla v_2)\|_{L^2_{T_1}(L^\infty)}\Big)^k \\
&\quad + C T_1^{\frac12} \|\nabla \delta v\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})} \sum_{k=1}^{\infty} k \Big(C T_1^{\frac12} \|(v_1, v_2)\|_{L^2_{T_1}(\dot B^{2\alpha}_{p,1} \cap \dot H^\alpha)}\Big)^k \\
&\leqslant C T^{\frac12} \|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})} + T_1 \|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)},
\end{aligned} \tag{6.49}$$

where $T_1 > 0$ is small enough so that $CT^{\frac12} \|(v_1, v_2)\|_{L^2_{T_1}(\dot B^{2\alpha}_{p,1} \cap \dot H^\alpha)} \leqslant \frac12$. Thus, combining the above two estimates yields that

$$\|\delta f_2^1\|_{L^2_{T_1}(L^2)} \leqslant \|\delta A\|_{L^\infty_{T_1}(\dot H^{2\alpha-1})} \|\nabla v_2\|_{L^2_{T_1}(L^\infty)} \leqslant C T_1^{\frac12} \|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})} + C T_1 \|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}. \tag{6.50}$$

Let us check the estimation of term δf_2^4. Defining

$$h(t,y,z,\theta) := \theta\big(X_{1,t}(y) - X_{1,t}(z)\big) + (1-\theta)\big(X_{2,t}(y) - X_{2,t}(z)\big),$$

and using the fundamental theorem of calculus, we find that

$$\begin{aligned}
&\frac{X_{1,t}(y)-X_{1,t}(z)}{|X_{1,t}(y)-X_{1,t}(z)|^{2+2\alpha}}-\frac{X_{2,t}(y)-X_{2,t}(z)}{|X_{2,t}(y)-X_{2,t}(z)|^{2+2\alpha}}\\
&\quad=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{h(t,x,y,\theta)}{|h(t,x,y,\theta)|^{2+2\alpha}}\mathrm{d}\theta\\
&\quad=-(1+2\alpha)\int_0^1\frac{1}{|h(x,y,t,\theta)|^{2+2\alpha}}\mathrm{d}\theta\cdot\Big(X_{1,t}(y)-X_{2,t}(y)-\big(X_{1,t}(z)-X_{2,t}(z)\big)\Big)\\
&\quad=-(1+2\alpha)\int_0^1\frac{1}{|h(x,y,t,\theta)|^{2+2\alpha}}\mathrm{d}\theta\cdot\int_0^t\big(\delta v(\tau,y)-\delta v(\tau,z)\big)\mathrm{d}\tau.
\end{aligned}$$

This gives that

$$\begin{aligned}
\delta f_2^4=&-c_\alpha(1+2\alpha)\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{\big(\delta v(\tau,y)-\delta v(\tau,z)\big)\cdot A_2^{\mathrm{T}}(t,z)(\nabla v_2(t,y)-\nabla v_2(t,z))}{|h(t,x,y,\theta)|^{2+2\alpha}}\mathrm{d}z\mathrm{d}\tau\mathrm{d}\theta\\
=&-c_\alpha(1+2\alpha)\int_0^1\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{(y-z)\cdot\nabla\delta v(\tau,\tilde\theta y+(1-\tilde\theta)z)}{|h(t,x,y,\theta)|^{2+2\alpha}}\cdot\\
&\qquad\cdot A_2^{\mathrm{T}}(t,z)(\nabla v_2(t,y)-\nabla v_2(t,z))\mathrm{d}z\mathrm{d}\tau\mathrm{d}\theta\mathrm{d}\tilde\theta.
\end{aligned}$$

Note that

$$\begin{aligned}
|h(t,y,z,\theta)-(y-z)|&\leqslant\big(\theta\|\nabla v_1\|_{L^1_t(L^\infty)}+(1-\theta)\|\nabla v_2\|_{L^1_t(L^\infty)}\big)|y-z|\\
&\leqslant T^{\frac12}\|(\nabla v_1,\nabla v_2)\|_{L^2_t(L^\infty)}|y-z|,
\end{aligned}$$

so by choosing $T_1>0$ small enough, we get $h(t,y,z,\theta)\approx|y-z|$ for every $y\neq z$ and $t\leqslant T_1$. Then taking advantage of Minkowski's inequality, lemma 2.3 and the estimates (6.14), (6.40), we infer that

$$\begin{aligned}
&\|\delta f_2^4\|_{L^2_{T_1}(L^2)}\\
&\quad\lesssim\Big\|\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla\delta v(\tau,\tilde\theta y+(1-\tilde\theta)z)||\nabla v_2(t,y)-\nabla v_2(t,z)|}{|y-z|^{1+2\alpha}}|A_2(t,z)|\mathrm{d}z\mathrm{d}\tau\mathrm{d}\tilde\theta\Big\|_{L^2_{T_1}(L^2_y)}\\
&\quad\lesssim\Big\|\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla\delta v(\tau,y+(1-\tilde\theta)z)||\nabla v_2(t,y)-\nabla v_2(t,y+z)|}{|z|^{1+2\alpha}}\mathrm{d}z\mathrm{d}\tau\mathrm{d}\tilde\theta\Big\|_{L^2_{T_1}(L^2_y)}\|A_2\|_{L^\infty_{T_1}(L^\infty)}\\
&\quad\lesssim\Big\|\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{\big\||\nabla\delta v(\tau,y+(1-\tilde\theta)z)\,|\nabla v_2(t,y)-\nabla v_2(t,y+z)|\big\|_{L^2_y}}{|z|^{1+2\alpha}}\mathrm{d}z\mathrm{d}\tau\mathrm{d}\tilde\theta\Big\|_{L^2_{T_1}}\\
&\quad\lesssim T_1^{\frac12}\Big\|\int_{\mathbb{R}^2}\frac{\|\nabla v_2(t,y)-\nabla v_2(t,y+z)\|_{L^p_y}}{|z|^{1+2\alpha}}\mathrm{d}z\Big\|_{L^2_{T_1}}\|\nabla\delta v\|_{L^2_{T_1}\left(L^{\frac{2p}{p-2}}\right)}\\
&\quad\leqslant CT_1^{\frac12}\|\nabla v_2\|_{L^2_{T_1}(\dot B^{2\alpha-1}_{p,1})}\|\nabla\delta v\|_{L^2_{T_1}\left(L^{\frac{2p}{p-2}}\right)}\\
&\quad\leqslant CT_1^{\frac12}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}.
\end{aligned}\tag{6.51}$$

The terms δf_2^2 and δf_2^3 can be estimated similarly as above: due to (6.33) and (6.48), we have the following bounds

$$\begin{aligned}
\|\delta f_2^2\|_{L^2_{T_1}(L^2)} &\leqslant C\Big\|\int_{\mathbb{R}^2}\frac{|\delta A^{\mathrm{T}}(t,y+z)|\,|\nabla v_2(t,y)-\nabla v_2(t,y+z)|}{|z|^{1+2\alpha}}\mathrm{d}z\Big\|_{L^2_{T_1}(L^2_y)}\\
&\leqslant C\Big\|\int_{\mathbb{R}^2}\frac{\|\nabla v_2(t,y)-\nabla v_2(t,y+z)\|_{L^p_y}}{|z|^{1+2\alpha}}\mathrm{d}z\Big\|_{L^2_{T_1}}\|\delta A\|_{L^\infty_{T_1}(L^{\frac{2p}{p-2}})}\\
&\leqslant CT_1^{\frac12}\|\nabla v_2\|_{L^2_{T_1}(\dot B^{2\alpha-1}_{p,1})}\|\nabla\delta v\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})}\\
&\leqslant CT_1^{\frac12}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)},
\end{aligned}\tag{6.52}$$

and

$$\begin{aligned}
&\|\delta f_2^3\|_{L^2_{T_1}(L^2)}\\
&\leqslant C\Big\|\int_0^1\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla\delta v(\tau,y+(1-\tilde\theta)z)|\,|A_2(t,y)-A_2(t,y+z)|}{|z|^{1+2\alpha}}|\nabla v_2(t,y)|\mathrm{d}z\mathrm{d}\tau\mathrm{d}\tilde\theta\Big\|_{L^2_{T_1}(L^2)}\\
&\leqslant CT_1^{\frac12}\|\nabla\delta v\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})}\Big\|\int_{\mathbb{R}^2}\frac{\|(\mathrm{Id}-A_2(t,y))-(\mathrm{Id}-A_2(t,y+z))\|_{L^p_y}}{|z|^{1+2\alpha}}\mathrm{d}z\Big\|_{L^\infty_{T_1}}\|\nabla v_2\|_{L^2_{T_1}(L^\infty)}\\
&\leqslant CT_1^{\frac12}\|\nabla\delta v\|_{L^2_{T_1}(L^{\frac{2p}{p-2}})}\|\mathrm{Id}-A_2\|_{L^\infty_{T_1}(\dot B^{2\alpha-1}_{p,1})}\|\nabla v_2\|_{L^2_{T_1}(L^\infty)}\\
&\leqslant C\big(T_1^{\frac12}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+T_1\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}\big)\|\mathrm{Id}-A_2\|_{L^\infty_{T_1}(\dot B^{2\alpha-1}_{p,1})}.
\end{aligned}$$

By (2.1) and the embedding $B^{2\alpha-1}_{p,1}(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, similarly to the proof of (6.39), we see that for $i=1,2$,

$$\|\mathrm{Id}-A_i\|_{L^\infty_T(\dot B^{2\alpha-1}_{p,1})}\leqslant\sum_{k=1}^\infty\Big(CT_1^{\frac12}\|\nabla v_i\|_{L^2_{T_1}(B^{2\alpha-1}_{p,1})}\Big)^k\leqslant\sum_{k=1}^\infty\Big(CT_1^{\frac12}\|v_i\|_{L^2_{T_1}(\dot B^{2\alpha}_{p,1}\cap\dot H^\alpha)}\Big)^k\leqslant CT^{\frac12},$$

so this implies

$$\|\delta f_2^3\|_{L^2_{T_1}(L^2)}\leqslant CT_1^{\frac12}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}.\tag{6.53}$$

Collecting the above estimates on f_2^1–f_2^4 yields that for $T_1>0$ small enough,

$$\|\delta f_2\|_{L^2_{T_1}(L^2)}\leqslant CT_1^{\frac12}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}.\tag{6.54}$$

Next we consider the estimation related to δg given by (6.22). The algebraic relation (6.8) gives

$$\mathrm{div}\,\delta g=(\mathrm{Id}-A_1^{\mathrm{T}}):\nabla\delta v-(\delta A^{\mathrm{T}}):\nabla v_2,$$

thus by using lemma 2.7, along with (6.38)–(6.40) and (6.48)–(6.49), we deduce that

$$\begin{aligned}\|\operatorname{div}\delta g\|_{L^2_{T_1}(\dot H^{2\alpha-1})} &\leqslant \|(\mathrm{Id}-A_1^{\mathrm T}):\nabla\delta v\|_{L^2_{T_1}(\dot H^{2\alpha-1})}+\|\delta A^{\mathrm T}:\nabla v_2\|_{L^2_{T_1}(\dot H^{2\alpha-1})}\\ &\leqslant C\|\mathrm{Id}-A_1\|_{L^\infty_{T_1}(\dot W^{2\alpha-1,p})}\|\nabla\delta v\|_{L^2_{T_1}\left(L^{\frac{2p}{p-2}}\right)}+C\|\mathrm{Id}-A_1\|_{L^\infty_{T_1}(L^\infty)}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}\\ &\quad +C\|\delta A\|_{L^\infty_{T_1}(\dot H^{2\alpha-1})}\|\nabla v_2\|_{L^2_{T_1}(L^\infty)}+C\|\delta A\|_{L^\infty_{T_1}\left(L^{\frac{2p}{p-2}}\right)}\|\nabla v_2\|_{L^2_{T_1}(\dot W^{2\alpha-1,p})}\\ &\leqslant CT_1^{1/2}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+T_1\|\delta v\|_{L^\infty_{T_1}(\dot H^{\alpha})}. \end{aligned}\tag{6.55}$$

We split $\partial_t\delta g$ into the following four terms:

$$\begin{aligned}\partial_t\delta g&=\partial_t[(\mathrm{Id}-A_1)\delta v-(\delta A)\,v_2]\\ &=-(\partial_tA_1)\,\delta v+(\mathrm{Id}-A_1)\,\partial_t\delta v-(\partial_t\delta A)\,v_2-(\delta A)\,\partial_tv_2.\end{aligned}$$

Noting that for $i=1,2$ (from (6.37))

$$\partial_tA_i(t,y)=\sum_{k=1}^{\infty}(-1)^kk\big(B_i(t,y)\big)^{k-1}Dv_i(t,y),\quad B_i(t,y)=\int_0^tDv_i(\tau,y)\mathrm d\tau,\tag{6.56}$$

and

$$\|\delta v(t,y)\|_{L^2_y}=\|\delta v(t,y)-\delta v(0,y)\|_{L^2_y}\leqslant\int_0^t\|\partial_\tau\delta v(\tau,y)\|_{L^2_y}\mathrm d\tau\leqslant t^{\frac12}\|\partial_\tau\delta v\|_{L^2_t(L^2)},\tag{6.57}$$

so by choosing $T_1>0$ small enough, we find that

$$\begin{aligned}\|\partial_tA_1\,\delta v\|_{L^2_{T_1}(L^2)}&\leqslant\|\partial_tA_1\|_{L^2_{T_1}(L^\infty)}\|\delta v\|_{L^\infty_{T_1}(L^2)}\\ &\leqslant CT_1^{1/2}\|\nabla v_1\|_{L^2_{T_1}(L^\infty)}\|\partial_t\delta v\|_{L^2_{T_1}(L^2)}\leqslant CT_1^{1/2}\|\partial_t\delta v\|_{L^2_{T_1}(L^2)}.\end{aligned}\tag{6.58}$$

Thanks to (6.38) and (6.47), we immediately get

$$\|(\mathrm{Id}-A_1)\,\partial_t\delta v\|_{L^2_{T_1}(L^2)}\leqslant\|\mathrm{Id}-A_1\|_{L^\infty_{T_1}(L^\infty)}\|\partial_t\delta v\|_{L^2_{T_1}(L^2)}\leqslant CT_1^{1/2}\|\partial_t\delta v\|_{L^2_{T_1}(L^2)},\tag{6.59}$$

and

$$\begin{aligned}\|(\delta A)\,\partial_tv_2\|_{L^2_{T_1}(L^2)}&\leqslant\|\delta A\|_{L^\infty_{T_1}(L^{\frac{2p}{p-2}})}\|\partial_tv_2\|_{L^2_{T_1}(L^p)}\\ &\leqslant CT_1^{1/2}\|\nabla\delta v\|_{L^2_T\left(L^{\frac{2p}{p-2}}\right)}\leqslant CT_1^{1/2}\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}+CT_1\|\delta v\|_{L^\infty_{T_1}(\dot H^{\alpha})}.\end{aligned}\tag{6.60}$$

In view of the following formula (from (6.56))

$$\begin{aligned}\partial_t\delta A(t,y)=&-D\delta v(t,y)+\sum_{k=2}^{\infty}(-1)^kkB_2^{k-1}D\delta v(t,y)\\ &+\sum_{k=2}^{\infty}\sum_{j=0}^{k-2}(-1)^kkB_1^{j-1}B_2^{k-1-j}\Big(\int_0^tD\delta v(\tau,y)\,\mathrm d\tau\Big)Dv_1(t,y).\end{aligned}$$

Using the Gagliardo–Nirenberg inequality $\|\nabla\delta v\|_{L^2}\leqslant C\|\delta v\|_{\dot H^\alpha}^{2-\frac1\alpha}\|\delta v\|_{\dot H^{2\alpha}}^{\frac1\alpha-1}$, we infer that

$$\begin{aligned}\|(\partial_t\delta A)\,v_2\|_{L^2_{T_1}(L^2)}&\leqslant\|\partial_t\delta A\|_{L^2_{T_1}(L^2)}\|v_2\|_{L^\infty_{T_1}(L^\infty)}\\ &\leqslant C\|v_2\|_{L^\infty_{T_1}(L^\infty)}\big(\|\nabla\delta v\|_{L^2_{T_1}(L^2)}+\|\nabla\delta v\|_{L^1_{T_1}(L^2)}\|\nabla v_1\|_{L^2_{T_1}(L^\infty)}\big)\\ &\leqslant CT_1^{1-\frac1{2\alpha}}\big(\|\delta v\|_{L^\infty_{T_1}(\dot H^\alpha)}+\|\delta v\|_{L^2_{T_1}(\dot H^{2\alpha})}\big).\end{aligned}\tag{6.61}$$

Noticing that $T_1 \leqslant 1$ and $\alpha \in (\frac{1}{2}, 1)$, we collect estimates (6.58)–(6.61) to obtain

$$\|\partial_t \delta g\|_{L^2_{T_1}(L^2)} \leqslant C T_1^{\frac{1}{2}} \|\partial_t \delta v\|_{L^2_{T_1}(L^2)} + C T_1^{1-\frac{1}{2\alpha}} \left(\|\delta v\|_{L^2_{T_1}(\dot{H}^{2\alpha})} + \|\delta v\|_{L^\infty_{T_1}(\dot{H}^{\alpha})} \right). \tag{6.62}$$

Therefore, plugging inequalities (6.48), (6.54), (6.55), and (6.62) into (6.23), we find that for $T_1 \in (0,1]$ small enough,

$$\begin{aligned} \delta E(T_1) &\leqslant C T_1^{\frac{1}{2}} \|(\partial_t \delta v, \nabla \delta \Pi)\|_{L^2_{T_1}(L^2)} + C T_1^{1-\frac{1}{2\alpha}} \left(\|\delta v\|_{L^2_{T_1}(\dot{H}^{2\alpha})} + \|\delta v\|_{L^\infty_{T_1}(\dot{H}^{\alpha})} \right) \\ &\leqslant C T_1^{1-\frac{1}{2\alpha}} \delta E(T_1). \end{aligned}$$

By letting $T_1 > 0$ be a even smaller constant (if necessary) so that $C T_1^{1-\frac{1}{2\alpha}} \leqslant \frac{1}{2}$, we conclude that $\delta E(t) \equiv 0$ on $[0, T_1]$. The Sobolev inequality $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$ or estimate (6.57) further implies that $\delta v \equiv 0$ for a.e. $(y,t) \in \mathbb{R}^2 \times [0, T_1]$. By using (6.11) and coming back to the Eulerian coordinates, we also get $X_{1,t}(y) \equiv X_{2,t}(y)$ and $u_1(t,x) \equiv u_2(t,x)$ for a.e. $(x,t) \in \mathbb{R}^2 \times [0, T_1]$.

Repeating the above procedure and arguing as the corresponding part in [47], we can further prove $u_1 = u_2$ on $\mathbb{R}^2 \times [T_1, 2T_1], \mathbb{R}^2 \times [2T_1, 3T_1], \ldots$, where $T_1 > 0$ is a small constant depending only on α, p, s, and the norms of (u_i, π_i) in propositions 4.3 and 4.4. Hence the uniqueness part of theorem 1.1 is proved, which completes the proof of theorem 1.1.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

The authors would like to express their deep gratitude to the anonymous referees for the valuable suggestions and comments. LX is partly supported by National Key Research and Development Program of China (No. 2020YFA0712900) and NSFC (Nos. 12271045, 11771043). YL is partly supported by NSFC (No. 12071043) and Educational Commission Science Programm of Jiangxi Province (Grant No. GJJ2200521). QM is partly supported by NSFC (Nos. 12001041, 12171031).

ORCID iD

Yatao Li https://orcid.org/0000-0001-6734-2927

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