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Emergence of time periodic solutions for the generalized surface quasi-geostrophic equation in the disc ☆

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A B S T R A C T

In this paper we address the existence of time periodic solutions for the generalized inviscid SQG equation in the unit disc with homogeneous Dirichlet boundary condition when $\alpha \in (0,1)$. We show the existence of a countable family of bifurcating curves from the radial patches. In contrast with the preceding studies in active scalar equations, the Green function is no longer explicit and we circumvent this issue by a suitable splitting into a singular explicit part (which coincides with the planar one) and a smooth implicit one induced by the boundary of the domain. Another problem is connected to the analysis of the linear frequencies which admit a complicated form through a discrete sum involving Bessel functions and their zeros. We overcome this difficulty by using Sneddon's

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Bifurcation theory
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formula leading to a suitable integral representation of the frequencies.

Contents

1. Introduction

In this paper we investigate some special structures of the vortical motions for the inviscid generalized surface quasi-geostrophic (abbr. gSQG) equation in the unit disc $\mathbb{D} \subset \mathbb{R}^2$. This model describes the evolution of the potential temperature ω governed by the transport equation,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & (t, x) \in [0, \infty) \times \mathbb{D}, \\ u = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \omega, \\ \omega_{|t=0}(x) = \omega_0(x). \end{cases} \tag{1}$$

Here $u = (u_1, u_2)$ refers to the velocity field, $\nabla^\perp = (-\partial_2, \partial_1)$, $\alpha \in [0, 1)$ is a real parameter. The fractional Laplacian operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is defined via the eigenfunction expansion of the Laplacian in $\mathbb{D}$ with homogeneous Dirichlet boundary condition (see (9)-(12) below). This model was introduced in [16] for the flat case $\mathbb{R}^2$ as an interpolation between 2D Euler equation and the surface quasi-geostrophic (abbr. SQG) model, corresponding to $\alpha = 0$ and $\alpha = 1$ in (1), respectively. Notice that the SQG equation was used as a simplified model to track the atmospheric circulation near the tropopause [43,36] and the ocean dynamics in the upper layers [49]. A strong analogy with the vorticity formulation of the 3D incompressible Euler equations was discussed in [13].

These aforementioned active scalar equations have attracted a lot of attention in the past decades and important progress has been settled in various directions. As to the local well-posedness of classical solutions in the whole space, it was performed in various function spaces. For instance, we refer to [8] where the solutions are constructed in the framework of Sobolev spaces. The global well-posedness issue is still open except for the Euler case $\alpha = 0$. However, L^2-weak solutions in the whole space are known to be global, see [57,53,50]. The nonuniqueness of weak solutions to SQG equation has been explored recently in [3,42]. Another class of solutions widely discussed in the literature is described by the patches where the initial data takes the form of the characteristic function of a smooth bounded domain D, that is, $\omega_0 = \mathbf{1}_D$. In this case, the patch structure is preserved for a short time and the boundary evolves according to a suitable contour dynamics equation, see [8,58,25]. Similar studies have been achieved for a half plane [45,46,26] and for any smooth bounded domain [47]. The global in time persistence of the boundary regularity is only known for the case $\alpha = 0$ according to Chemin's result [9], see also Bertozzi and Constantin [2] for another proof. Notice that some numerical experiments show strong evidence for the singularity formation in finite time, see for instance [16,59,60]. For the patch problem associated to gSQG equation in the half plane, a finite-time singularity result with multi-signed patches has been established by Kiselev et al. [45] for the case $0 < \alpha < \frac{1}{12}$ and Gancedo et al. [26] for $0 < \alpha < \frac{1}{3}$.

The analysis of SQG type equations in bounded smooth domains is much involved than the flat case due in part to the Green function which is not explicit. This study was initiated by Constantin and Ignatova [10,11]. They considered the SQG equation with critical dissipation and obtained the global existence of L^2-weak solutions with a global Lipschitz a priori interior estimates. We also refer to the papers [12,41,62] for more results and discussions. Concerning the inviscid model (1) in smooth bounded domains, the L^2 global weak solutions was constructed by Constantin and Nguyen [15] for the SQG case $\alpha = 1$, and later generalized in [56] to the case $\alpha \in (1, 2)$ with more singular constitutive law in the velocity. The local well-posedness issue in the framework of classical solutions for the inviscid SQG equation (in bounded smooth domains) was performed in [14]. We point out that with some slight modification, the results of [14,15,56] can be extended to the gSQG equation with $\alpha \in (0, 1)$.

The aim of this work is to construct time periodic solutions in the patch form for the gSQG model in bounded smooth domains. We will in particular focus on the class of *V-states* or *rotating patches*, whose dynamics is described by a rigid body transformation. In this setting, the problem reduces to finding some domains D subject to uniform rotation around their centers of mass. Observe that during the motion the support D_t of the patch solution does not change the shapes and is determined by $D_t = \mathbf{R}_{x_0,\Omega t}\, D$, where $\mathbf{R}_{x_0,\Omega t}$ denotes the planar rotation with center x_0 and angle Ωt. The parameter $\Omega \in \mathbb{R}$ is called the angular velocity of the rotating domain.

The V-states problem has a long history and it is still the subject of an intensive research, and many important contributions have been achieved in the last few decades at the analytical and numerical levels. The first example of rotating patches for the 2D Euler equation in the plane was discovered by Kirchhoff [44] who proved that an ellipse of semi-axes a and b rotates perpetually with the uniform angular velocity $\Omega = \frac{ab}{(a+b)^2}$, we also refer to [48, p. 232] and [52, p. 304]. One century later, Deem and Zabusky [20] gave numerical evidence of the existence of implicit V-states with m-fold symmetry. Afterwards, Burbea [4] provided an analytical proof of this fact using local bifurcation theory and conformal parametrization. In particular, he proved that for each symmetry $m \geqslant 2$ a curve of non trivial V-states bifurcates from Rankine vorticity (the radial shape) at the angular velocities $\Omega = \frac{m-1}{2m}$. See also Hmidi, Mateu and Verdera [40], where the C^∞ boundary regularity and the convexity of these bifurcated V-states are established. Real analyticity of the boundary was further obtained by Castro, Córdoba and Gómez-Serrano [7]. Hassainia, Masmoudi and Wheeler in [35] studied through some global bifurcation arguments the analyticity of the V-states along the whole bifurcating branches. Besides the preceding results, several families of V-states with different topological structures were recently explored. For instance, it was shown in [7,37] that a second family of countable branches bifurcate from Kirchhoff's ellipses. Rotating patches with only one hole exist near the annulus as proved in [22,38], concentrated multi vortices centered at regular n-gons or distributed according to suitable periodic spatial patterns are analyzed in [39,32,27,28]. The study of V-states in radial domains was performed for Euler equation in [21]. In particular, De la Hoz, Hassainia, Hmidi and Mateu [21] proved the existence of m-fold symmetric V-states for the Euler equation in the unit disc, which bifurcates from the trivial solution $\mathbf{1}_{b\mathbb{D}}$ at the angular velocity $\Omega = \frac{m-1+b^{2m}}{2m}$ for any $m \geqslant 1$ and $b \in (0,1)$.

The existence of the V-states for the gSQG equation (1) starts with the work of Hassainia and Hmidi [33] where they showed similar results to Burbea curves in the whole plane and for $\alpha \in (0,1)$. Later, Castro, Córdoba and Gómez-Serrano [6] proved the existence of V-states for the range $\alpha \in [1,2)$ and obtained the C^∞-regularity of their boundary for all $\alpha \in (0,2)$, see also [7] for the real analyticity of the patch boundary. For other connected topics we refer to the papers [1,5,6,23,21,29,30,39] and the references therein.

In this paper, we shall focus on the existence of the V-states for the gSQG equation (1) with $\alpha \in (0,1)$ in the unit disc $\mathbb{D}$. More precisely, we want to construct rigid periodic solutions around radial stationary patches $\mathbf{1}_{b\mathbb{D}}$, $b \in (0,1)$ using bifurcation tools. We remind that the case $\alpha = 0$ was discussed in [21]. As a by-product we construct an infinite family of non-stationary global solutions for the gSQG equation in the bounded domain $\mathbb{D}$, although the global well-posedness/blow up issue is not well understood and remains an open problem. The situation in bounded domains turns out to be more tricky due to the non explicit form of Green function associated to the spectral fractional Laplacian. This has an impact in the study of the regularity of the functional that will describe the V-states. Later, in Section 3 we shall explore how to recover the boundary equation of the V-states close to the patch $\mathbf{1}_{b\mathbb{D}}$ in terms of polar coordinates $\theta \in \mathbb{R} \mapsto \sqrt{b^2 + 2r(\theta)}e^{i\theta}$. In this regard, the deformation radius r solves a nonlinear integro-differential equation of the following type,

$$F(\Omega, r(\theta)) \triangleq \Omega\, r'(\theta) + \partial_\theta \left(\int_0^{2\pi} \int_0^{R(\eta)} K^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big)\rho\, \mathrm{d}\rho \mathrm{d}\eta \right) = 0, \quad R(\eta) \triangleq \sqrt{b^2 + 2r(\eta)}, \tag{2}$$

where K^α stands for the Green function of the spectral fractional Laplacian $(-\Delta)^{-1+\frac{\alpha}{2}}$ on the unit disc $\mathbb{D}$, defined via the relation

$$(-\Delta)^{-1+\frac{\alpha}{2}} f(x) = \int_{\mathbb{D}} K^\alpha(x, y) f(y) \mathrm{d}y.$$

One may easily verify that $F(\Omega, 0) = 0$ for every $\Omega \in \mathbb{R}$ and therefore the next step is to check that the bifurcation tools such as the Crandall-Rabinowitz theorem [18] applies in this framework. To state our main result, we shall introduce the following angular velocities,

$$\Omega_{m,b}^\alpha \triangleq 2 \sum_{k \geqslant 1} x_{0,k}^{\alpha-2} \frac{J_1^2(x_{0,k} b)}{J_1^2(x_{0,k})} - 2 \sum_{k \geqslant 1} x_{m,k}^{\alpha-2} \frac{J_m^2(x_{m,k} b)}{J_{m+1}^2(x_{m,k})}, \tag{3}$$

where J_m denotes the Bessel functions of order m and $x_{m,k}$ denotes the k-th positive root of $J_m(x) = 0$, see Section 2.3 for more details. Now, we are in a position to give our main result.

Theorem 1.1. *Let (α, b, m) satisfy one of the following conditions*

$$\alpha \in (0,1),\ b \in \left(0, \left(\tfrac{1-\alpha}{2-\alpha/2}\right)^{\frac{1}{2}}\right],\ m \geqslant 1; \tag{4}$$

$$\alpha \in (0,1),\ b \in (0,1),\ m \geqslant m^*; \tag{5}$$

$$\alpha \in (0, \alpha^*),\ b \in (0,1),\ m \geqslant 1; \tag{6}$$

with $m^* = m^*(\alpha, b) \in \mathbb{N}$ (a rough bound is $m^* \leqslant \frac{1}{\log b}\big(\log \frac{1-\alpha}{1-\alpha/2-(e\log b)^{-1}}\big)$) and $\alpha^* = \alpha^*(b) > 0$ a small number. Then there exists a family of m-fold symmetric V-states $(V_m)_{m\geqslant 1}$ for the gSQG equation (1) bifurcating from the trivial solution $\omega_0 = \mathbf{1}_{b\mathbb{D}}$ at the angular velocity $\Omega_{m,b}^\alpha$ given by (3). In addition, the boundary of the V-states belongs to the Hölder class $C^{2-\alpha}$.

More precisely, there exist a constant $a > 0$ and two continuous functions $\Omega : (-a, a) \to \mathbb{R}$, $r : (-a, a) \to C^{2-\alpha}(\mathbb{T})$ satisfying $\Omega(0) = \Omega_{m,b}^\alpha$, $r(0) = 0$, such that $(r_s)_{-a<s<a}$ is a one-parameter non-trivial solution of the equation (2) describing V-states. Moreover, r_s admits the expansion

$$\forall \theta \in \mathbb{R}, \quad r_s(\theta) = s\cos(m\theta) + s\sum_{n\geqslant 1} b_{nm}(s)\cos(nm\theta), \quad b_{nm} = O(s),$$

and the mapping $\theta \mapsto \sqrt{b^2 + 2r_s(\theta)}\, e^{i\theta}$ maps the torus $\mathbb{T}$ to the boundary of an m-fold rotating patch with the angular velocity $\Omega(s)$.

Before giving the ideas of the proof, we shall make some comments.

Remark 1.1. Theorem 1.1 shows the existence of global solutions near Rankine vortices. This issue is open for general initial data.

Remark 1.2. When the domain of the fluid is the ball $B(0,R)$, with $R>1$, then by a scaling argument and applying the preceding theorem, the bifurcation from the unit disc $\mathbb{D}$ occurs at the angular velocities (see [33, Proposition 3])

$$\widetilde{\Omega}_{m,R}^{\alpha} \triangleq R^{-\alpha}\Omega_{m,R^{-1}}^{\alpha}.$$

According to (84), (87) and (88), and using (95) and (108) to control the remainder terms, we obtain

$$\lim_{R\to\infty}\widetilde{\Omega}_{m,R}^{\alpha}=\frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\Big(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\Big), \tag{7}$$

which corresponds to Hassainia-Hmidi's result in [33].

On the other hand, when $\alpha\to 0$, as indicated in Lemma 5.1-(i), we recover the result of De la Hoz, Hassainia, Hmidi and Mateu [21] in the limit.

Remark 1.3. For the SQG equation corresponding to the case $\alpha=1$ the situation is more delicate. By reformulating the boundary equation of V-states to relax the violent singularity of the kernel, we can give a full description of the linearized operator $\partial_r F(\Omega,0)$ and the associated dispersion relation. It turns out that the spectrum $\Omega_{m,b}^1$ coincides with the limit of $\Omega_{m,b}^{\alpha}$ when $\alpha\to 1$ as (85) shows. However, the function spaces used here are not well-adapted due to a logarithmic loss in frequency when $\alpha=1$. We believe that the L^2 weighted spaces introduced in [6] could be used in order to generate the V-states in the critical case $\alpha=1$. Notice that the same techniques could give that the boundary is analytic.

In the proof of Theorem 1.1, the first difficulty that one should face is related to the kernel function K^α which has no explicit form as in the whole space for the gSQG equation [33] or Euler equation in the disc [21]. This makes the regularity problem of the functional F introduced in (2) more complicated and to circumvent this issue we establish in Lemma 2.3 the following decomposition in $\mathbb{D}\times\mathbb{D}$

$$K^\alpha(x,y)=\frac{c_\alpha}{|x-y|^\alpha}+K_1^\alpha(x,y)$$

where K_1^α is a smooth function in $\mathbb{D}\times\mathbb{D}$. We emphasize that this splitting is valid for any smooth bounded domain and extends a classical result for 2D Euler equation [21,24]. Remark that for the specific case of the unit disc one can recover this kernel from the eigenfunctions of the Laplacian which are explicitly described through Bessel functions, see (2.5) below. However, it is not at all clear how to deal with this series and deduce the splitting mentioned above. Then by virtue of this decomposition together with Lemma 2.6 dealing with singular kernel integrals on the torus, we can prove the desired regularity properties of F needed in the Crandall-Rabinowitz theorem (see Theorem 6.1). The spectral problem is carefully studied in Section 5 and one important delicate point lies on the analysis of angular velocity sequence $\{\Omega_{m,b}^{\alpha}, m \geqslant 1\}$. In order to get a one dimensional kernel we need to check the monotonicity of these frequencies. The formula (3) seems to be out of use due to the complexity of the sum. Surprisingly, this complicated form admits a nice integral representation by virtue of Sneddon's formula [61], from which we conduct a careful analysis and manage to show the key monotonicity property of the sequence $\{\Omega_{m,b}^{\alpha}, m \geqslant 1\}$ under some constraints on α, b and the symmetry. For this discussion, we refer to Lemma 2.7 and the proof of Lemma 5.1. Notice that the rest of the conditions of Crandall-Rabinowitz theorem are satisfied allowing to get an affirmative answer for the existence of the V-states for the gSQG equation (1) in the disc $\mathbb{D}$ when $\alpha \in (0,1)$.

The reminder of the paper is organized as follows. In the next section, we shall present some technical results related to the spectral fractional Laplacian and the associated Green function in bounded smooth domain, and then we shall introduce some estimates on singular kernel integrals and also recall Sneddon's formula. In Section 3, we shall write down the boundary equation of V-states in the unit disc. The Sections 4 and 5 are devoted to the proof of Theorem 1.1. In Section 4, we study the linearization and regularity of the nonlinear functionals in the boundary equation. In Section 5, we conduct the spectral study of the linearized operator around zero and under suitable assumptions we obtain a Fredholm operator of zero index. Finally, in the last section we recall the Crandall-Rabinowitz theorem and give the proof of some auxiliary lemmas used in the paper.

Notation. Throughout this space we shall use the following convention and notation.

- C denotes a positive constant that may change its value from line to line.
- The set $\mathbb{N} = \{0, 1, 2, \cdots\}$ is the set of nonnegative numbers, and $\mathbb{N}^+ = \{1, 2, ..\}$ denotes the set of all positive integers.
- Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology.
- We denote by $\mathbf{D}$ a bounded open domain with smooth boundary of the Euclidean space $\mathbb{R}^d$, while we use the notation $\mathbb{D}$ to denote the unit disc of the Euclidean space $\mathbb{R}^2$.

2. Tools

This section is dedicated to some technical results related to the structure of the heat semi-flow and the Green function of fractional Laplacian in bounded smooth domains.

We shall also discuss some estimates on integrals with singular periodic kernels and recall Sneddon's formula which plays a central role on the spectral study.

2.1. Spectral fractional Laplacian and Green function

The main goal of this subsection is to explore the structure of Green function associated to Dirichlet fractional Laplacian.

We shall first recall how to define the spectral fractional Laplacian in bounded domains. Let $\mathbf{D} \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded open domain with smooth boundary. The $L^2(\mathbf{D})$-normalized eigenfunctions of the operator $-\Delta$ supplemented with Dirichlet boundary condition are denoted by ϕ_j, and the associated eigenvalues counted with their multiplicities are positive real numbers λ_j such that

$$\text{for} \quad j \geqslant 1, \quad -\Delta\phi_j = \lambda_j\phi_j, \quad \phi_j|_{\partial\mathbf{D}} = 0, \quad \int_{\mathbf{D}} \phi_j^2(x)\mathrm{d}x = 1. \tag{8}$$

It is a classical fact that

$$0 < \lambda_1 < \lambda_2 \leqslant ... \leqslant \lambda_j \to \infty.$$

Now, according to the functional calculus, the spectral fractional Laplacian $(-\Delta)^{-1+\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$ is defined through

$$\begin{aligned}(-\Delta)^{-1+\frac{\alpha}{2}} f(x) &= \frac{1}{\Gamma(1-\frac{\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}} e^{t\Delta} f(x)\mathrm{d}t \\ &= \int_{\mathbf{D}} K^\alpha(x,y) f(y)\mathrm{d}y,\end{aligned} \tag{9}$$

with Γ being the classical Gamma function and K^α the Green function. Let $H_{\mathbf{D}}(t,\cdot,\cdot)$ denote the kernel of the heat semigroup $e^{t\Delta}$ on the domain $\mathbf{D}$ with Dirichlet boundary condition, then

$$e^{t\Delta} f(x) = \int_{\mathbf{D}} H_{\mathbf{D}}(t,x,y) f(y)\mathrm{d}y.$$

It is a classical fact that this kernel can be reconstructed from the eigenfunctions (8) as follows

$$\forall\ t \geqslant 0, x, y \in \mathbf{D}, \quad H_{\mathbf{D}}(t,x,y) = \sum_{j\geqslant 1} e^{-\lambda_j t}\phi_j(x)\phi_j(y). \tag{10}$$

Consequently, the Green function K^α admits different representations

$$K^\alpha(x,y) = \frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^\infty t^{-\frac{\alpha}{2}} H_{\mathbf{D}}(t,x,y)\mathrm{d}t \tag{11}$$

$$= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^\infty t^{-\frac{\alpha}{2}} \sum_{j\geqslant 1} e^{-\lambda_j t}\phi_j(x)\phi_j(y)\mathrm{d}t$$

$$= \sum_{j\geqslant 1} \lambda_j^{\frac{\alpha}{2}-1}\phi_j(x)\phi_j(y). \tag{12}$$

There is an abundant literature dealing with the analytic properties of the heat kernels, see for instance [10,19,64]. Here, we shall restrict the discussion to some of them that will be needed later. In view of the points (31)-(32) in [10] or [19,64], there exists a time $T_0 > 0$ and positive constant C depending only on the domain $\mathbf{D}$ such that for all $0 \leqslant t \leqslant T_0$ and $x, y \in \mathbf{D}$,

$$0 < H_{\mathbf{D}}(t,x,y) \leqslant C \min\left\{\tfrac{\phi_1(x)}{|x-y|}, 1\right\} \min\left\{\tfrac{\phi_1(y)}{|x-y|}, 1\right\} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{C\,t}} \tag{13}$$

and

$$\tfrac{|\nabla_x H_{\mathbf{D}}(t,x,y)|}{H_{\mathbf{D}}(t,x,y)} \leqslant \begin{cases} \frac{C}{d(x)}, & \text{if } \sqrt{t} \geqslant d(x), \\ \frac{C}{\sqrt{t}}\Big(1 + \frac{|x-y|}{\sqrt{t}}\Big), & \text{if } \sqrt{t} \leqslant d(x), \end{cases} \tag{14}$$

with $d(x) \triangleq d(x, \partial\mathbf{D})$ being the Euclidean distance between x and the boundary $\partial\mathbf{D}$. The function ϕ_1 is the first eigenfunction of $-\Delta$ as in (8).

Before stating the main result of this section, see Lemma 2.3, on the Dirichlet Green function K^α, we give two auxiliary results on the Dirichlet heat kernel. The first one is on the higher differentiability of $H_{\mathbf{D}}(t)$ inside the domain $\mathbf{D}$ for a short time whose proof is classical and will be postponed later in the Appendix. Its statement reads as follows.

Lemma 2.1. *We have that for every $(x, y) \in \mathbf{D} \times \mathbf{D}$, $n \in \mathbb{N}$ and for every $0 < t \leqslant \min\{d(x)^2, T_0\}$,*

$$|\nabla_x^n H_{\mathbf{D}}(t,x,y)| \leqslant C t^{-\frac{n+d}{2}} e^{-\frac{|x-y|^2}{Ct}},$$

with C depending on d, n and T_0.

Notice that the latter estimate in the preceding lemma degenerates on the diagonal $x = y$ for small time t in a similar way to the Gauss kernel of the heat semigroup $e^{t\Delta}$ on the whole space $\mathbb{R}^d$ explicitly given by $(x, y) \mapsto G_t(x-y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$. The second auxiliary result deals with the description of this defect and shows that $H_{\mathbf{D}}(t)$ differs from G_t by a smooth contribution uniformly for small time. The proof will be provided in the Appendix.

Lemma 2.2. *Let $(x,y) \in \mathbf{D}\times\mathbf{D}$, $k,l\in\mathbb{N}$ and fix $0<t\leqslant \min\{d(x)^2, d(y)^2, T_0\}$. Then, we have*

$$\left|\nabla_x^k\nabla_y^l\big(H_{\mathbf{D}}(t,x,y)-G_t(x-y)\big)\right|\leqslant C,$$

where $C>0$ depends on $d,k,l,T_0,d(x)$ and $d(y)$.

Now, we are ready to state the main result of this section dealing with a natural decomposition of Dirichlet Green function K^α. It can be split into a singular one that coincides with the whole-space Green function and a smooth term with bounded derivatives inside the domain $\mathbf{D}\times\mathbf{D}$.

Lemma 2.3. *Let $\mathbf{D}\subset\mathbb{R}^d$ $(d\geq 2)$ be a bounded open domain with smooth boundary. Let $\alpha\in(0,2)$ and K^α be the kernel function given by (12). Then we have that for every $x\neq y\in\mathbf{D}\times\mathbf{D}$,*

$$K^\alpha(x,y)=\frac{c_\alpha}{|x-y|^{\alpha+d-2}}+K_1^\alpha(x,y), \tag{15}$$

with $c_\alpha=\frac{4^{\alpha/2-1}\Gamma(\frac{\alpha}{2}+\frac{d}{2}-1)}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$ and $K_1^\alpha\in C^\infty(\mathbf{D}\times\mathbf{D})$.

Remark 2.1. This result is compatible with the classical one known for $\alpha=0$. For instance when $\alpha=0$ and $d=2$, we have

$$K^0(x,y)=-\frac{1}{2\pi}\log|x-y|+K_1^0(x,y),$$

where K_1^0 is a smooth function in the open domain $\mathbf{D}\times\mathbf{D}$ and partially harmonic in x and y ([24]).

Proof of Lemma 2.3. For $x,y\in\mathbf{D}$ fixed, there exists an open domain $\mathbf{D}_0\subset\overline{\mathbf{D}_0}\subset\mathbf{D}$ such that $x,y\in\mathbf{D}_0$. Denote $d_0\triangleq\min\{\sqrt{T_0}, d(\overline{\mathbf{D}_0},\partial\mathbf{D})\}>0$. Then, we get from (11) the decomposition

$$\begin{aligned}
K^\alpha(x,y)&=\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2}t^{-\frac{\alpha}{2}}H_{\mathbf{D}}(t,x,y)\mathrm{d}t+\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_{d_0^2}^{\infty}t^{-\frac{\alpha}{2}}H_{\mathbf{D}}(t,x,y)\mathrm{d}t\\
&=\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2}t^{-\frac{\alpha}{2}}G_t(x-y)\mathrm{d}t+\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2}t^{-\frac{\alpha}{2}}\big(H_{\mathbf{D}}(t,x,y)-G_t(x-y)\big)\mathrm{d}t\\
&\quad+\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_{d_0^2}^{\infty}t^{-\frac{\alpha}{2}}H_{\mathbf{D}}(t,x,y)\mathrm{d}t
\end{aligned}$$

$$\triangleq \frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2} t^{-\frac{\alpha}{2}} G_t(x-y)\mathrm{d}t + K_{11}^\alpha(x,y) + K_{12}^\alpha(x,y).$$

To deal with the first term, we use a change of variables allowing to get

$$\begin{aligned}\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2} t^{-\frac{\alpha}{2}} G_t(x-y)\mathrm{d}t &= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^{d_0^2} t^{-\frac{\alpha}{2}} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}\mathrm{d}t\\ &= \frac{4^{\alpha/2-1}}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x-y|^{\alpha+d-2}}\int_{\frac{|x-y|^2}{4d_0^2}}^{\infty} \tau^{\frac{\alpha}{2}+\frac{d}{2}-2} e^{-\tau}\mathrm{d}\tau\\ &= \frac{4^{\alpha/2-1}\Gamma(\frac{\alpha}{2}+\frac{d}{2}-1)}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x-y|^{\alpha+d-2}} - K_{13}^\alpha(x,y),\end{aligned}$$

with

$$K_{13}^\alpha(x,y) \triangleq \frac{4^{\alpha/2-1}}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x-y|^{\alpha+d-2}}\int_0^{\frac{|x-y|^2}{4d_0^2}} \tau^{\frac{\alpha}{2}+\frac{d}{2}-2} e^{-\tau}\mathrm{d}\tau. \tag{16}$$

Therefore, we may decompose K^α as in (15) with

$$K_1^\alpha(x,y) = K_{11}^\alpha(x,y) + K_{12}^\alpha(x,y) - K_{13}^\alpha(x,y).$$

In what follows we intend to show that all these functions are C^∞-smooth in $\mathbf{D}\times\mathbf{D}$. For K_{11}^α, by virtue of Lemma 2.2, we infer that for every $k,l \geq 0$ and $\alpha \in (0,2)$,

$$\begin{aligned}|\nabla_x^k \nabla_y^l K_{11}^\alpha(x,y)| &\leqslant C\int_0^{d_0^2} t^{-\frac{\alpha}{2}}\mathrm{d}t\\ &\leqslant C.\end{aligned}$$

The smoothness of K_{12}^α is a direct consequence of Lemma 6.1. Indeed, using (10) we infer

$$\begin{aligned}|\nabla_x^k \nabla_y^l K_{12}^\alpha(x,y)| &= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\Big|\int_{d_0^2}^{\infty} t^{-\frac{\alpha}{2}}\Big(\sum_{j\geqslant 1} e^{-\lambda_j t}\nabla_x^k\phi_j(x)\nabla_y^l\phi_j(y)\Big)\mathrm{d}t\Big|\\ &\leqslant C\sum_{j\geqslant 1}\int_{d_0^2}^{\infty} t^{-\frac{\alpha}{2}} e^{-\lambda_j t}\|\nabla^k\phi_j\|_{L^\infty}\|\nabla^l\phi_j\|_{L^\infty}\mathrm{d}t.\end{aligned}$$

Applying the estimates of Lemma 6.1 with Sobolev embeddings we get in view of Weyl's law stating that $\lambda_j \sim j$ as $j \to \infty$,

$$|\nabla_x^k \nabla_y^l K_{12}^\alpha(x,y)| \leqslant C\Big(\sum_{j\geqslant 1} e^{-\frac{d_0^2}{2}\lambda_j}(1+\lambda_j)^{\frac{k+l}{2}+d+1}\Big)\int_{d_0^2}^{\infty} t^{-\frac{\alpha}{2}} e^{-\frac{\lambda_1}{2}t}\mathrm{d}t$$
$$\leqslant C.$$

Now we shall move to K_{13}^α defined in (16) and show that it is smooth. According to Taylor expansion of $e^{-\tau}$, we find

$$K_{13}^\alpha(x,y) = \frac{4^{\alpha/2-1}}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x-y|^{\alpha+d-2}}\int_0^{\frac{|x-y|^2}{4d_0^2}} \tau^{\frac{\alpha}{2}+\frac{d}{2}-2}\Big(\sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}\Big)\mathrm{d}\tau$$
$$= \frac{1}{2^{2n+d}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})d_0^{\alpha+d-2}}\sum_{n=0}^{\infty}\frac{(-1)^n|x-y|^{2n}}{n!(n+\frac{\alpha}{2}+\frac{d}{2}-1)d_0^{2n}},$$

which is an absolutely convergent power series and is C^∞-smooth in x and y. Hence, gathering the above estimates leads to the desired splitting (15) with $K_1^\alpha \in C^\infty(\mathbf{D}\times\mathbf{D})$. $\square$

The next goal is to explore how the symmetry of the domain can be reflected on the Green function, in the sense that if the domain $\mathbf{D}$ is invariant under suitable planar transformations then the kernel functions $H_{\mathbf{D}}$ and K^α will enjoy an adequate symmetry property. More precisely, we shall establish the following result.

Lemma 2.4. *Let $\mathbf{D} \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary. Let $H_{\mathbf{D}}(t,\cdot,\cdot)$ be the kernel of the heat semigroup $e^{t\Delta}$ on the domain $\mathbf{D}$ and $K^\alpha(\cdot,\cdot)$ be the Green function defined via (9). Then the following statements hold true.*

(i) *Let $\bar{x} = (x_1,-x_2)$ be the reflection point of $x=(x_1,x_2)$. If $\mathbf{D}$ is invariant by reflection with respect to the real axis, then*

$$\forall\, x,y\in\mathbf{D},\quad H_{\mathbf{D}}(t,\bar{x},\bar{y}) = H_{\mathbf{D}}(t,x,y)\quad \textit{and}\quad K^\alpha(\bar{x},\bar{y}) = K^\alpha(x,y).$$

(ii) *If $e^{i\theta}\mathbf{D}=\mathbf{D}$ for some $\theta\in\mathbb{R}$, then*

$$\forall\, x,y\in\mathbf{D},\quad H_{\mathbf{D}}(t,e^{i\theta}x,e^{i\theta}y) = H_{\mathbf{D}}(t,x,y)\quad \textit{and}\quad K^\alpha(e^{i\theta}x,e^{i\theta}y) = K^\alpha(x,y).$$

Proof of Lemma 2.4. **(i)** Observe that for each $y \in \mathbf{D}$,

$$\begin{cases} \partial_t H_{\mathbf{D}}(t,x,y) - \Delta_x H_{\mathbf{D}}(t,x,y) = 0, & x \in \mathbf{D}, x \neq y, \\ H_{\mathbf{D}}(t,x,y) = 0, & x \in \partial\mathbf{D}, \\ H_{\mathbf{D}}(0,x,y) = \delta(x-y), \end{cases} \tag{17}$$

where $\delta(\cdot)$ is the Dirac δ-function centered at the origin. Now we view $H_{\mathbf{D}}(t,\bar{x},\bar{y})$ as a function of x and y as a parameter. Then, we can check that

$$H_{\mathbf{D}}(0,\bar{x},\bar{y}) = \delta(\bar{x}-\bar{y}) = \delta(x-y) = H_{\mathbf{D}}(0,x,y)$$

and $\Delta_x\big(H_{\mathbf{D}}(t,\bar{x},\bar{y})\big) = \big(\Delta_x H_{\mathbf{D}}\big)(t,\bar{x},\bar{y})$. In addition, using the fact that $\bar{x} \in \partial\mathbf{D} \Leftrightarrow x \in \partial\mathbf{D}$, we have $H_{\mathbf{D}}(t,\bar{x},\bar{y}) = 0$ for any $x \in \partial\mathbf{D}$. Hence, for each $y \in \mathbf{D}$, the mapping $x \in \mathbf{D} \mapsto H_{\mathbf{D}}(t,\bar{x},\bar{y})$ solves (17). Owing to the uniqueness of the initial-boundary value problem (17), we conclude that $H_{\mathbf{D}}(t,x,y) = H_{\mathbf{D}}(t,\bar{x},\bar{y})$. In view of (11), the desired equality $K^\alpha(\bar{x},\bar{y}) = K^\alpha(x,y)$ directly follows.

(ii) The proof of statement (ii) is quite similar as above, and thus we omit the details. □

Hereafter we shall give a precise description of the Green function K^α when the domain $\mathbf{D}$ is the planar unit disc $\mathbb{D}$. Actually, in this radial case the eigenvalues and eigenfunctions of the spectral Laplacian $-\Delta$ on $\mathbb{D}$ have precise expression formula through Bessel functions and thus the Dirichlet Green function K^α might be explicitly calculated. We actually have the following result.

Lemma 2.5. *Let $\mathbf{D} = \mathbb{D}$ be the unit disc of $\mathbb{R}^2$ and let K^α given by (12). Then the eigenvalues and the eigenfunctions solving the spectral problem (8) are described by double index families $\{\lambda_{n,k}\}_{n\in\mathbb{N},k\geqslant 1}$ and $\{(\phi^{(1)}_{n,k},\phi^{(2)}_{n,k})\}_{n\in\mathbb{N},k\geqslant 1}$ such that*

$$\lambda_{n,k} = x_{n,k}^2, \quad \phi^{(1)}_{n,k}(x) = J_n(x_{n,k}|x|)A_{n,k}\cos(n\theta), \quad \phi^{(2)}_{n,k}(x) = J_n(x_{n,k}|x|)A_{n,k}\sin(n\theta),$$

where

$$\pi A_{0,k}^2 = \frac{1}{J_1^2(x_{0,k})} \quad \text{and} \quad \pi A_{n,k}^2 = \frac{2}{J_{n+1}^2(x_{n,k})}, \quad \forall n \geqslant 1, \tag{18}$$

and J_n denotes the first kind Bessel function of order n and $\{x_{n,k}, k \geqslant 1\}$ are its zeroes. Furthermore, we have

$$K^\alpha(x,y) = \sum_{n\in\mathbb{N},k\geqslant 1} x_{n,k}^{\alpha-2}\Big(\phi^{(1)}_{n,k}(x)\phi^{(1)}_{n,k}(y) + \phi^{(2)}_{n,k}(x)\phi^{(2)}_{n,k}(y)\Big). \tag{19}$$

Proof of Lemma 2.5. The explicit formula of the eigenvalues and the normalized eigenfunctions of $-\Delta$ on the disc $\mathbb{D}$ are well-known, and for instance one can refer to Section

5.5 of Chapter V in [17] for the proof. As a result, the formula (19) is an immediate consequence of the identity (12). □

Remark 2.2. Let $x = \rho_1 e^{i\theta}, y = \rho_2 e^{i\eta} \in \mathbb{D}$, then by setting $K^\alpha(x,y) = G(\rho_1,\theta,\rho_2,\eta)$, we get from the expression (19),

$$\begin{aligned} K^\alpha(x,y) = G(\rho_1,\theta,\rho_2,\eta) &= \sum_{\substack{n\in\mathbb{N},k\geqslant 1\\ 1\leqslant j\leqslant 2}} x_{n,k}^{\alpha-2}\phi_{n,k}^{(j)}\big(\rho_1 e^{i\theta}\big)\phi_{n,k}^{(j)}\big(\rho_2 e^{i\eta}\big) \\ &= \sum_{\substack{n\in\mathbb{N}\\ k\geqslant 1}} x_{n,k}^{\alpha-2}A_{n,k}^2 J_n(x_{n,k}\rho_1)J_n(x_{n,k}\rho_2)\big(\cos(n\theta)\cos(n\eta)+\sin(n\theta)\sin(n\eta)\big) \\ &= \sum_{\substack{n\in\mathbb{N}\\ k\geqslant 1}} x_{n,k}^{\alpha-2}A_{n,k}^2 J_n(x_{n,k}\rho_1)J_n(x_{n,k}\rho_2)\cos\big(n(\theta-\eta)\big). \end{aligned} \tag{20}$$

Straightforward computations yield

$$\nabla_x K^\alpha(x,y) = \begin{pmatrix} \partial_{\rho_1}G\cos\theta - \partial_\theta G\frac{\sin\theta}{\rho_1} \\ \partial_{\rho_1}G\sin\theta + \partial_\theta G\frac{\cos\theta}{\rho_1} \end{pmatrix}, \quad \nabla_y K^\alpha(x,y) = \begin{pmatrix} \partial_{\rho_2}G\cos\eta - \partial_\eta G\frac{\sin\eta}{\rho_2} \\ \partial_{\rho_2}G\sin\eta + \partial_\eta G\frac{\cos\eta}{\rho_2} \end{pmatrix}. \tag{21}$$

These identities will be useful later in the explicit computation of the linearized operator at the equilibrium state, see the proof of Proposition 5.1.

2.2. Singular kernel integrals on the torus

In this subsection, we intend to deal with integrals with singular kernels of the following type

$$\mathcal{T}(f)(\theta) \triangleq \int_{\mathbb{T}} K(\theta,\eta)\, f(\eta)\mathrm{d}\eta, \tag{22}$$

where $\mathbb{T}$ is the periodic torus (identified with $[0,2\pi)$), $K:\mathbb{T}\times\mathbb{T}\to\mathbb{C}$ is a suitable singular kernel, and $f:\mathbb{T}\to\mathbb{C}$ is a 2π-periodic function. This structure will appear later when we will explore the regularity of the nonlinear operator in the rotating patches formalism, see Section 4. The result that we shall present below is more or less classical and is analogous to [33,55], and we shall provide a complete proof for the self-containing of the paper.

Lemma 2.6. *Let $0 \leqslant \alpha < 1$ and assume that there exists $C_0 > 0$ such that $K:\mathbb{T}\times\mathbb{T}\to\mathbb{C}$ satisfies the following properties.*

(i) *K is measurable on $\mathbb{T}\times\mathbb{T}\backslash\{(\theta,\theta),\,\theta\in\mathbb{T}\}$ and*

$$|K(\theta,\eta)|\leqslant\frac{C_0}{|\sin\frac{\theta-\eta}{2}|^{\alpha}},\quad\forall\eta\neq\theta\in\mathbb{T}.\tag{23}$$

(ii) *For each $\eta\in\mathbb{T}$, the mapping $\theta\mapsto K(\theta,\eta)$ is differentiable in $\mathbb{T}\backslash\{\eta\}$ and*

$$|\partial_\theta K(\theta,\eta)|\leqslant\frac{C_0}{|\sin\frac{\theta-\eta}{2}|^{1+\alpha}},\quad\forall\theta\neq\eta\in\mathbb{T}.\tag{24}$$

Then the linear operator $\mathcal{T}$ given by (22) is continuous from $L^\infty(\mathbb{T})$ to $C^{1-\alpha}(\mathbb{T})$. More precisely, there exists a constant $C_\alpha>0$ depending only on α such that

$$\|\mathcal{T}(f)\|_{C^{1-\alpha}}\leqslant C_\alpha C_0\|f\|_{L^\infty}.\tag{25}$$

Proof of Lemma 2.6. We first prove that $\mathcal{T}(f)$ is bounded on $\mathbb{T}$. For every $\theta\in\mathbb{T}$, thanks to (23), we see that

$$|\mathcal{T}(f)(\theta)|\leq C_0\|f\|_{L^\infty}\Big|\int_{\mathbb{T}}\frac{d\eta}{|\sin(\frac{\theta-\eta}{2})|^{\alpha}}\Big|=C_0\|f\|_{L^\infty}\int_{-\pi}^{\pi}\frac{d\eta}{|\sin\frac{\eta}{2}|^{\alpha}}\leqslant C_\alpha C_0\|f\|_{L^\infty}.$$

Next, for every $\theta_1,\theta_2\in\mathbb{T}$ such that $0<|\theta_1-\theta_2|\leqslant\pi$, it is obvious that

$$\tfrac{2}{\pi}|\theta_1-\theta_2|\leqslant|e^{i\theta_1}-e^{i\theta_2}|=2\Big|\sin\tfrac{\theta_1-\theta_2}{2}\Big|\leqslant|\theta_1-\theta_2|.$$

Set $r\triangleq|e^{i\theta_1}-e^{i\theta_2}|$ and define $B_r(\theta)\triangleq\big\{\eta\in\mathbb{T}\,:\,|2\sin(\frac{\eta-\theta}{2})|=|e^{i\theta}-e^{i\eta}|\leq r\big\}$. With no loss of generality assume that $r\leqslant\frac14$, then we have

$$\begin{aligned}\Big|\mathcal{T}(f)(\theta_1)-\mathcal{T}(f)(\theta_2)\Big|\leqslant&\Big|\int_{B_{3r}(\theta_1)}|f(\eta)||K(\theta_1,\eta)|d\eta\Big|+\Big|\int_{B_{3r}(\theta_1)}|f(\eta)||K(\theta_2,\eta)|d\eta\Big|\\&+\Big|\int_{B_{2r}^c(\theta_1)}|f(\eta)||K(\theta_1,\eta)-K(\theta_2,\eta)|d\tau\Big|\\\triangleq&J_1+J_2+J_3.\end{aligned}$$

Applying (23) together with a change of variables allow to get the estimate

$$\begin{aligned}J_1+J_2\leqslant&C_0\|f\|_{L^\infty}\Bigg(\Big|\int_{B_{3r}(\theta_1)}\frac{d\eta}{|\sin(\frac{\eta-\theta_1}{2})|^{\alpha}}\Big|+\Big|\int_{B_{3r}(\theta_2)}\frac{d\eta}{|\sin(\frac{\eta-\theta_2}{2})|^{\alpha}}\Big|\Bigg)\\\leqslant&C_\alpha C_0\|f\|_{L^\infty}\Big|\int_0^{\frac32 r}\frac{dw}{|w|^{\alpha}\sqrt{1-w^2}}\Big|\leqslant C_\alpha C_0\|f\|_{L^\infty}|\theta_1-\theta_2|^{1-\alpha}.\end{aligned}$$

To estimate the third term J_3, noting that for every $\eta\in B^c_{2r}(\theta_1)$ and for any $\kappa\in[0,1]$,

$$\begin{aligned}2\Big|\sin\tfrac{\theta_1-\eta+(1-\kappa)(\theta_2-\theta_1)}{2}\Big| &= |e^{i(\theta_1-\eta)}-e^{i(1-\kappa)(\theta_1-\theta_2)}|\\ &\geqslant |e^{i(\theta_1-\eta)}-1|-|e^{i(1-\kappa)(\theta_1-\theta_2)}-1|\\ &\geqslant |e^{i(\theta_1-\eta)}-1|-|e^{i(\theta_1-\theta_2)}-1| = |e^{i\eta}-e^{i\theta_1}|-|e^{i\theta_1}-e^{i\theta_2}|\\ &\geqslant \tfrac{1}{2}|e^{i\eta}-e^{i\theta_1}| = \Big|\sin\tfrac{\eta-\theta_1}{2}\Big|.\end{aligned}$$

Therefore, applying the mean value theorem, (24) and the preceding estimate we find

$$|K(\theta_1,\eta)-K(\theta_2,\eta)|\leqslant C_\alpha C_0\frac{|\theta_1-\theta_2|}{|\sin\frac{\eta-\theta_1}{2}|^{1+\alpha}},\quad \forall\,\eta\in B^c_{2r}(\theta_1).$$

Consequently, we obtain

$$\begin{aligned}J_3 &\leqslant C_\alpha C_0\|f\|_{L^\infty}\int\limits_{B^c_{2r}(\theta_1)}\frac{|\theta_1-\theta_2|}{|\sin\frac{\eta-\theta_1}{2}|^{1+\alpha}}\mathrm{d}\eta\\ &\leqslant C_\alpha C_0\|f\|_{L^\infty}\int\limits_r^1\frac{|\theta_1-\theta_2|}{|w|^{1+\alpha}\sqrt{1-w^2}}\mathrm{d}w\\ &\leqslant C_\alpha C_0\|f\|_{L^\infty}|\theta_1-\theta_2|^{1-\alpha}.\end{aligned}$$

Hence gathering the above estimates concludes the proof of (25). □

2.3. Special functions

The main task of this subsection is to recall Sneddon's formula which is very crucial in the spectral problem associated to the linearization of the vortex patch equations around radial solutions that will be explored in Section 5. It allows in our context to derive a suitable representation of the angular velocities from which periodic solutions bifurcate from Rankine vortices. Before stating this formula, we need to remind some special functions and notations (e.g. see [31]). First, the Gamma function $\Gamma:\mathbb{C}\setminus(-\mathbb{N})\to\mathbb{C}$ is the analytic continuation to the negative half plane of the usual Gamma function defined on $\{\mathrm{Re}\,z>0\}$ by the integral formula

$$\Gamma(z)=\int\limits_0^\infty t^{z-1}e^{-t}\mathrm{d}t.$$

It satisfies the classical relation

$$\Gamma(z+1)=z\Gamma(z),\quad \forall z\in\mathbb{C}\setminus(-\mathbb{N}).\tag{26}$$

On the other hand, for every $z \in \mathbb{C}$ we denote $(z)_n$ the Pokhhammer's symbol defined by

$$(z)_n \triangleq \begin{cases} z(z+1)\cdots(z+n-1), & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0. \end{cases}$$

The following relations are straightforward

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad (z)_n = (-1)^n \frac{\Gamma(1-z)}{\Gamma(1-z-n)}, \tag{27}$$

provided that all the right-hand quantities are well-defined. In what follows we intend to recall Bessel functions and some of their variations. The Bessel functions of order $\nu \in \mathbb{C}$ is defined by

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)| < \pi.$$

Next, we shall introduce Bessel functions of imaginary argument also called modified Bessel functions of first and second kind, (e.g. see p.66 of [51])

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)| < \pi \tag{28}$$

and

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}, \quad |\arg(z)| < \pi.$$

However, when $j \in \mathbb{Z}$, K_j is defined through the formula $K_j(z) = \lim_{\nu \to j} K_\nu(z)$.
It is known that for $\nu > 0$ the zeros of the Bessel function J_ν are given by a countable family $\{x_{\nu,k}, k \in \mathbb{N}\}$ of positive increasing numbers with the following asymptotics

$$x_{\nu,k} = (k + \tfrac{\nu}{2} - \tfrac{1}{4})\pi + O(k^{-1}), \quad k \to \infty.$$

We also recall that for any real numbers $c_1, c_2 \in \mathbb{R}$, $c_3 \in \mathbb{R} \setminus (-\mathbb{N})$ the hypergeometric function $z \mapsto F(c_1, c_2; c_3; z)$ is defined on the open unit disc $\mathbb{D}$ by the power series

$$F(c_1, c_2; c_3; z) = \sum_{n=0}^{\infty} \frac{(c_1)_n (c_2)_n}{(c_3)_n} \frac{z^n}{n!}, \quad \forall z \in \mathbb{D},$$

where $(x)_n$ is Pokhhammer's symbol. The hypergeometric series converges in the unit disc $\mathbb{D}$. Assume that $\mathrm{Re}(c_3) > \mathrm{Re}(c_2) > 0$, then the hypergeometric function has an integral representation as follows

$$F(c_1,c_2;c_3;z)=\frac{\Gamma(c_3)}{\Gamma(c_2)\Gamma(c_3-c_2)}\int_0^1 x^{c_2-1}(1-x)^{c_3-c_2-1}(1-zx)^{-c_1}\mathrm{d}x,\quad |z|<1.$$

When $\mathrm{Re}(c_1+c_2-c_3)<0$, the hypergeometric series is absolutely convergent on the closed unit disc and one has the following expression

$$F(c_1,c_2;c_3;1)=\frac{\Gamma(c_3)\Gamma(c_3-c_1-c_2)}{\Gamma(c_3-c_1)\Gamma(c_3-c_2)}.$$

Now, we are ready to state Sneddon's formula that can be found for instance in (2.2.9) in [61], or (24)-(25) in [54].

Lemma 2.7. *Let $0<a,b\leqslant 1$, $n,\beta,\gamma\in\mathbb{N}$ and $1<q<\beta+\gamma-2n+2$. Then we have*

$$\sum_{k=1}^{\infty}\frac{J_\beta(ax_{n,k})J_\gamma(bx_{n,k})}{x_{n,k}^{q}J_{n+1}^2(x_{n,k})}=\mathbb{J}+\frac{1}{\pi}\sin\left(\tfrac{\pi}{2}(\beta+\gamma-2n-q)\right)\int_0^\infty \rho^{1-q}I_\beta(a\rho)I_\gamma(b\rho)\frac{K_n(\rho)}{I_n(\rho)}\mathrm{d}\rho, \tag{29}$$

where

$$\begin{aligned}\mathbb{J}&\triangleq\frac{1}{\pi}\sin\left(\tfrac{\pi}{2}(\gamma-\beta+q)\right)\int_0^\infty\rho^{1-q}I_\beta(a\rho)K_\gamma(b\rho)\mathrm{d}\rho\\&=\frac{a^\beta\Gamma\left(1+\frac{\beta+\gamma-q}{2}\right)}{2^qb^{2+\beta-q}\Gamma(\beta+1)\Gamma(\frac{\gamma-\beta+q}{2})}F\left(1+\tfrac{\beta+\gamma-q}{2},1+\tfrac{\beta-\gamma-q}{2};\beta+1;\tfrac{a^2}{b^2}\right).\end{aligned}\tag{30}$$

In particular, if $a=b$, it holds that

$$\mathbb{J}|_{a=b}=\frac{\Gamma(1+\frac{\beta+\gamma-q}{2})\Gamma(q-1)}{2^qa^{2-q}\Gamma(\frac{\beta+\gamma+q}{2})\Gamma(\frac{\gamma-\alpha+q}{2})\Gamma(\frac{\beta-\gamma+q}{2})}.\tag{31}$$

Remark 2.3. For the explicit formula of $\mathbb{J}$ stated in the formula (30)-(31), it is enough to apply the following identities. First, for every $b>a>0$, $n,\beta,\gamma\in\mathbb{N}$, $q<2+\beta-\gamma$, (see for instance 6.576 of [31])

$$\begin{aligned}&\int_0^\infty\rho^{1-q}I_\beta(a\rho)K_\gamma(b\rho)\mathrm{d}\rho\\&\quad=\frac{a^\beta\Gamma(1+\frac{\beta+\gamma-q}{2})\Gamma(1-\frac{\gamma-\beta+q}{2})}{2^qb^{2+\beta-q}\Gamma(\beta+1)}F\left(1+\tfrac{\beta+\gamma-q}{2},1+\tfrac{\beta-\gamma-q}{2};\beta+1;\tfrac{a^2}{b^2}\right).\end{aligned}$$

Second, it gives that in the particular case of $a=b$, $n,\beta,\gamma\in\mathbb{N}$, $1<q<2+\beta-\gamma$, (see for instance 9.122 of [31])

$$\int_0^\infty \rho^{1-q} I_\beta(a\rho)K_\gamma(a\rho)\mathrm{d}\rho = \frac{\Gamma(1-\frac{\gamma-\beta+q}{2})\Gamma(1+\frac{\beta+\gamma-q}{2})\Gamma(q-1)}{2^q a^{2-q}\Gamma(\frac{\beta+\gamma+q}{2})\Gamma(\frac{\beta-\gamma+q}{2})}. \tag{32}$$

3. Boundary equation of rigid periodic patches

This section focuses on the vortex patch motion to the gSQG equation (1). In this setting, the solution takes at least for a short time the form $\omega(t) = \mathbf{1}_{D_t}$, where $D_t \subset \mathbb{D}$ is smooth and will be chosen close to the small disc $b\mathbb{D}$ $(0 < b < 1)$. Then identifying the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, one might use the polar coordinates as follows, see for instance [34],

$$\begin{aligned} z(t):\ \mathbb{T} &\mapsto \partial D_t \\ \theta &\mapsto R(\theta)e^{i\theta} \triangleq \sqrt{b^2+2r(t,\theta)}e^{i\theta}. \end{aligned} \tag{33}$$

We denote by $\mathbf{n}(t,z(t,\theta)) = i\partial_\theta z(t,\theta)$ an inward normal vector to the boundary ∂D_t of the patch at the point $z(t,\theta)$. According to [40, p. 174], the vortex patch equation writes

$$\begin{aligned} \partial_t z(t,\theta)\cdot\mathbf{n} &= u(t,z(t,\theta))\cdot\mathbf{n} \\ &= -\partial_\theta[\psi(t,z(t,\theta))], \end{aligned}$$

where $\psi(t,x) = (-\Delta)^{-1+\frac{\alpha}{2}}\theta(t,x)$ is the stream function. Notice that

$$\psi(t,z(t,\theta)) = \int_{D_t} K^\alpha(z(t,\theta),y)\mathrm{d}y.$$

Now we shall write the patch equation in the particular case of rotating domains $D_t = e^{it\Omega}D$ with some $\Omega \in \mathbb{R}$, that is,

$$z(t,\theta) = e^{it\Omega}z(\theta) = e^{it\Omega}\big(b^2+2r(\theta)\big)^{\frac{1}{2}}e^{i\theta}. \tag{34}$$

Then making a change of variables we deduce from Lemma 2.4 that

$$\begin{aligned} \psi(t,z(t,\theta)) &= \int_D K^\alpha(e^{it\Omega}z(\theta),e^{it\Omega}y)\mathrm{d}y \\ &= \int_D K^\alpha(z(\theta),y)\mathrm{d}y. \end{aligned}$$

In addition

$$\partial_t z(t,\theta) = i\Omega z(t,\theta) = i\,\Omega\,e^{it\Omega}\sqrt{b^2+2r(\theta)}e^{i\theta},$$

and

$$\partial_t z(t,\theta)\cdot \mathbf{n}(t,z(t,\theta)) = \mathrm{Im}\left(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\right) \overset{(34)}{=} \Omega r'(\theta).$$

Therefore we find the equation

$$\begin{aligned}\Omega r'(\theta) &= -\partial_\theta[\Psi(z(t,\theta))] \\ &= -\partial_\theta \int_D K^\alpha(z(\theta),y)\mathrm{d}y.\end{aligned} \tag{35}$$

Using the polar coordinates yields

$$\int_D K^\alpha(z(\theta),y)\mathrm{d}y = \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha\left(R(\theta)e^{i\theta},\rho e^{i\eta}\right)\rho\mathrm{d}\rho\mathrm{d}\eta, \quad R(\eta)=\sqrt{b^2+2r(\eta)}. \tag{36}$$

According to Lemma 2.3, we have the splitting

$$K^\alpha(x,y) = K_0^\alpha(x,y)+K_1^\alpha(x,y) = K_0^\alpha(x-y)+K_1^\alpha(x,y), \tag{37}$$

where $(x,y)\in\mathbb{D}^2\mapsto K_1^\alpha(x,y)$ is smooth and we make an abuse of notation

$$K_0^\alpha(x,y) = K_0^\alpha(x-y) \triangleq \frac{4^{\alpha/2-1}\Gamma(\frac{\alpha}{2})}{\pi\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x-y|^\alpha} = \frac{c_\alpha}{|x-y|^\alpha}\cdot \tag{38}$$

Therefore, we get due to the fact $\nabla_x K_0^\alpha(x,y) = -\nabla_y K_0^\alpha(x,y)$,

$$\begin{aligned}\partial_\theta\Psi(r(\theta)) &= \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K^\alpha\left(R(\theta)e^{i\theta},\rho e^{i\eta}\right)\cdot\partial_\theta(R(\theta)e^{i\theta})\rho\mathrm{d}\rho\mathrm{d}\eta \\ &= \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K_1^\alpha\left(R(\theta)e^{i\theta},\rho e^{i\eta}\right)\cdot\partial_\theta(R(\theta)e^{i\theta})\rho\mathrm{d}\rho\mathrm{d}\eta \\ &\quad - \int_0^{2\pi}\int_0^{R(\eta)} \nabla_y K_0^\alpha\left(R(\theta)e^{i\theta},\rho e^{i\eta}\right)\cdot\partial_\theta(R(\theta)e^{i\theta})\rho\mathrm{d}\rho\mathrm{d}\eta.\end{aligned} \tag{39}$$

To deal with the last integral term we apply Gauss-Green theorem,

$$\int_0^{2\pi}\int_0^{R(\eta)} \nabla_y K_0^\alpha\left(R(\theta)e^{i\theta},\rho e^{i\eta}\right)\cdot\partial_\theta\left(R(\theta)e^{i\theta}\right)\rho\,\mathrm{d}\rho\mathrm{d}\eta$$

$$= \iint_D \nabla_y K_0^\alpha \left(R(\theta)e^{i\theta}, y\right) \cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)\,\mathrm{d}y$$

$$= \int_0^{2\pi} K_0^\alpha \left(R(\theta)e^{i\theta}, R(\eta)e^{i\eta}\right)\big(-i\partial_\eta(R(\eta)e^{i\eta})\big) \cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)\mathrm{d}\eta.$$

Consequently, we find

$$\begin{aligned} \partial_\theta \Psi(r(\theta)) =& \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K_1^\alpha \left(R(\theta)e^{i\theta}, \rho e^{i\eta}\right) \cdot \partial_\theta(R(\theta)e^{i\theta})\rho \mathrm{d}\rho \mathrm{d}\eta \\ &- \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\big(-i\partial_\eta(R(\eta)e^{i\eta})\big) \cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)\,\mathrm{d}\eta. \end{aligned} \tag{40}$$

Hence the V-states equation reads as follows

$$\begin{aligned} F(\Omega, r(\theta)) &\triangleq \Omega r'(\theta) + \partial_\theta\Psi(r(\theta)) \\ &= F_1(\Omega, r(\theta)) + F_2(r(\theta)) = 0, \end{aligned} \tag{41}$$

with

$$\begin{aligned} F_1(\Omega, r(\theta)) &\triangleq \Omega r'(\theta) - \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\big(-i\partial_\eta(R(\eta)e^{i\eta})\big) \cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)\,\mathrm{d}\eta \\ &= \Omega r'(\theta) - \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\,\mathrm{Im}\big(\partial_\eta(R(\eta)e^{i\eta})\overline{\partial_\theta(R(\theta)e^{i\theta})}\big)\,\mathrm{d}\eta \end{aligned} \tag{42}$$

and

$$F_2(r(\theta)) \triangleq \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K_1^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big) \cdot \partial_\theta(R(\theta)e^{i\theta})\,\rho \mathrm{d}\rho \mathrm{d}\eta. \tag{43}$$

In the previous decomposition we recognize two terms: the first one F_1 is the same functional as in the flat space $\mathbb{R}^2$ describing the induced patch effect, see (15) in [33], and the second one F_2 describes the rigid boundary effect on the patch.

4. Linearization and regularity of the functional F

In order to apply the Crandall-Rabinowitz theorem stated in Theorem 6.1, we need first to fix the function spaces and check the regularity of the functional F introduced in (41) with respect to these spaces. We should look for Banach spaces X and Y such that $F:\mathbb{R}\times X\to Y$ is well-defined and satisfies the assumptions of Theorem 6.1. Let $\alpha\in(0,1)$, $m\in\mathbb{N}^+$ and consider the m-fold Banach spaces

$$X=X_m\triangleq\Big\{f\in C^{2-\alpha}(\mathbb{T}):\ f(\theta)=\sum_{n\geqslant 1}b_n\cos(nm\theta),\ b_n\in\mathbb{R},\ \theta\in\mathbb{T}\Big\}$$

and

$$Y=Y_m\triangleq\Big\{f\in C^{1-\alpha}(\mathbb{T}):\ f(\theta)=\sum_{n\geqslant 1}b_n\sin(nm\theta),\ b_n\in\mathbb{R},\ \theta\in\mathbb{T}\Big\}$$

equipped with their usual norms. For $\epsilon_0\ll 1$, we denote by B_{ϵ_0} the open ball of X_m with center 0 and radius ϵ_0,

$$B_{\epsilon_0}\triangleq\big\{r\in X_m:\|r\|_{X_m}<\epsilon_0\big\}.$$

Recall from (39) and (41) that

$$\begin{aligned}F(\Omega,r)&=\Omega\,r'(\theta)+\int_0^{2\pi}\int_0^{R(\eta)}\nabla_x K^\alpha\big(R(\theta)e^{i\theta},\rho e^{i\eta}\big)\cdot\Big(\frac{r'(\theta)}{R(\theta)}e^{i\theta}+R(\theta)ie^{i\theta}\Big)\rho\mathrm{d}\rho\mathrm{d}\eta\\&=F_1(\Omega,r)+F_2(r)=0,\end{aligned}\tag{44}$$

where $\Omega\in\mathbb{R}$, $r\in X_m$ and the functionals F_1,F_2 are defined by (42)-(43). Notice that Rankine vortices are stationary solutions and therefore

$$F(\Omega,0)\equiv 0,\quad\forall\,\Omega\in\mathbb{R}.\tag{45}$$

This can be analytically checked using the fact that the stream function is radial, which follows from the rotation invariance of the Green function stated in Lemma 2.4 via the following identity

$$\nabla_x K^\alpha\big(be^{i\theta},\rho e^{i\eta}\big)\cdot(ie^{i\theta})=b^{-1}\partial_\theta G(b,\theta,\rho,\eta).\tag{46}$$

4.1. Linearization

The next goal is to linearize the nonlinear equation (44) around an arbitrary small state r. The computations below are formal that can be implemented from the Gateaux derivative

$$\partial_r F(\Omega,r)h(\theta)=\frac{\mathrm{d}}{\mathrm{d}s}F(\Omega,r+sh)\Big|_{s=0}.$$

They can be rigorously checked in the Fréchet sense. Denoting by $B(r)(\theta) \triangleq \partial_\theta(R(\theta)e^{i\theta})$, and via straightforward computations based on (39) and (41)-(43) we get that for any $h \in X_m$,

$$\begin{aligned}\partial_r F(\Omega, r)h(\theta) = \Omega h'(\theta) + \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big) \cdot \frac{\mathrm{d}}{\mathrm{d}s}(B(r+sh)(\theta))\Big|_{s=0} \rho \,\mathrm{d}\rho\mathrm{d}\eta \\
+ \frac{h(\theta)}{R(\theta)}\int_0^{2\pi}\int_0^{R(\eta)} \big(\nabla_x^2 K_1^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big)\cdot e^{i\theta}\big)\cdot B(r)(\theta)\rho\,\mathrm{d}\rho\mathrm{d}\eta \\
+ \int_0^{2\pi} \nabla_x K_1^\alpha\big(R(\theta)e^{i\theta}, R(\eta)e^{i\eta}\big)\cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)h(\eta)\mathrm{d}\eta \\
- \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\Big(-i\frac{\mathrm{d}}{\mathrm{d}s}B(r+sh)(\eta)\Big|_{s=0}\Big)\cdot B(r)(\theta)\mathrm{d}\eta \\
- \int_0^{2\pi} \nabla_x K_0^\alpha\left(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\right) \\
\cdot\Big(e^{i\theta}\frac{h(\theta)}{R(\theta)} - e^{i\eta}\frac{h(\eta)}{R(\eta)}\Big)\big(-iB(r)(\eta)\big)\cdot B(r)(\theta)\,\mathrm{d}\eta.\end{aligned}$$

Noting that

$$\frac{\mathrm{d}}{\mathrm{d}s}B(r+sh)(\theta)\Big|_{s=0} = \partial_\theta\left(\frac{h(\theta)e^{i\theta}}{R(\theta)}\right) = \frac{h'(\theta)}{R(\theta)}e^{i\theta} + \partial_\theta\left(\frac{e^{i\theta}}{R(\theta)}\right)h(\theta),$$

and

$$\partial_\eta\big(-if(\eta)e^{i\eta}\big)\cdot\partial_\theta\big(g(\theta)e^{i\theta}\big) = \partial_\eta\partial_\theta\big(f(\eta)g(\theta)\sin(\eta-\theta)\big), \quad \forall f,g\in C^1, \tag{47}$$

we can rewrite the above equation as

$$\partial_r F(\Omega, r)h(\theta) = \big[\Omega + V_1(r)(\theta)\big]h'(\theta) + V_2(r)(\theta)h(\theta) + V_3(r,h)(\theta) + V_4(r,h)(\theta) \tag{48}$$

with

$$V_1(r)(\theta) \triangleq R^{-1}(\theta)\int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big)\cdot e^{i\theta}\rho\,\mathrm{d}\rho\mathrm{d}\eta, \tag{49}$$

and

$$\begin{aligned} V_2(r)(\theta) \triangleq & R^{-1}(\theta) \int_0^{2\pi}\int_0^{R(\eta)} \left(\nabla_x^2 K_1^\alpha(R(\theta)e^{i\theta}, \rho e^{i\eta}) \cdot e^{i\theta}\right) \cdot \partial_\theta\big(R(\theta)e^{i\theta}\big)\, \rho\, \mathrm{d}\rho \mathrm{d}\eta \\ & + \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K^\alpha\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big) \cdot \partial_\theta \left(\frac{e^{i\theta}}{R(\theta)}\right) \rho\, \mathrm{d}\rho \mathrm{d}\eta, \end{aligned} \tag{50}$$

and

$$\begin{aligned} V_3(r,h)(\theta) \triangleq & \int_0^{2\pi} \nabla_x K_1^\alpha\big(R(\theta)e^{i\theta}, R(\eta)e^{i\eta}\big) \cdot \partial_\theta(R(\theta)e^{i\theta}) h(\eta) \mathrm{d}\eta \\ & - \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big) L_A(h) \mathrm{d}\eta, \end{aligned} \tag{51}$$

$$\begin{aligned} V_4(r,h)(\theta) \triangleq & -\int_0^{2\pi} \nabla_x K_0^\alpha\left(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\right) \cdot \left(e^{i\theta}\frac{h(\theta)}{R(\theta)} - e^{i\eta}\frac{h(\eta)}{R(\eta)}\right) \\ & \times \partial_\eta \partial_\theta \Big(R(\eta)R(\theta)\sin(\eta-\theta)\Big)\, \mathrm{d}\eta, \end{aligned} \tag{52}$$

where

$$L_A(h) \triangleq \partial_\eta\partial_\theta\Big(\frac{h(\eta)R(\theta)\sin(\eta-\theta)}{R(\eta)}\Big). \tag{53}$$

4.2. Strong regularity

This subsection is devoted to the regularity of the functional F described by the formula (44). One gets the following result.

Proposition 4.1. *Let $\alpha \in (0,1)$, there exists $\epsilon_0 > 0$ sufficiently small such that the following statements hold true for any $m \in \mathbb{N}^+$.*

(i) *$F : \mathbb{R} \times B_{\epsilon_0} \to Y_m$ is well-defined.*
(ii) *$F : \mathbb{R} \times B_{\epsilon_0} \to Y_m$ is of class C^1.*
(iii) *The partial derivative $\partial_\Omega \partial_r F : \mathbb{R} \times B_{\epsilon_0} \to \mathcal{L}(X_m, Y_m)$ exists and is continuous.*

Proof of Proposition 4.1. (i) We shall use the expression of F detailed in (41), (42) and (43). First, notice that the regularity of $\theta \in \mathbb{T} \mapsto r'(\theta)$ is obvious since $r' \in Y_m$ whenever $r \in X_m$. Second, since $\|r\|_{C^{2-\alpha}} \leq \epsilon_0 \ll 1$, then $x = R(\theta)e^{i\theta} = \sqrt{b^2 + 2r(\theta)}e^{i\theta}$ is in the compact set $B(0, \sqrt{b^2 + 2\epsilon_0}) \subset \mathbb{D}$. Next we shall check the regularity of the second part of F_1 given by (42). Set

$$F_{1,1}(r(\theta)) \triangleq \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\,\big(-i\partial_\eta(R(\eta)e^{i\eta})\big)\cdot\partial_\theta\big(R(\theta)e^{i\theta}\big)\,\mathrm{d}\eta$$
$$= \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\big(-i\partial_\eta(R(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\partial_\theta\big(R(\theta)e^{i\theta}\big). \tag{54}$$

Since $r \in C^{1-\alpha}(\mathbb{T})$ then using the law of product we deduce that

$$\theta \in \mathbb{T} \mapsto \partial_\theta(R(\theta)e^{i\theta}) = \Big(\tfrac{r'(\theta)}{R(\theta)}e^{i\theta} + i\,R(\theta)e^{i\theta}\Big) \in C^{1-\alpha}(\mathbb{T}). \tag{55}$$

Now, to analyze the regularity of the first term in the right-hand side of (54) it suffices to combine Lemma 2.6 with the estimates below

$$\forall\theta \neq \eta \in \mathbb{T}, \quad |K_0^\alpha(R(\theta)e^{i\theta}, R(\eta)e^{i\eta})| \leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|^{-\alpha}, \tag{56}$$

and

$$|\partial_\theta K_0^\alpha(R(\theta)e^{i\theta}, R(\eta)e^{i\eta})| \leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|^{-(1+\alpha)}, \tag{57}$$

in order to get

$$\theta \in \mathbb{T} \mapsto \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta}, R(\eta)e^{i\eta}\big)\big(-i\partial_\eta(R(\eta)e^{i\eta})\big)\mathrm{d}\eta \in C^{1-\alpha}(\mathbb{T}). \tag{58}$$

Thus the classical law of product gives in view of (54), $F_{1,1}(r) \in C^{1-\alpha}(\mathbb{T})$ and then $F_1(r) \in C^{1-\alpha}(\mathbb{T})$. Now, let us check how to get (56) and (57). The first step is to show the following

$$\forall\,\theta,\eta \in \mathbb{R}, \quad C_1\left|\sin\tfrac{\theta-\eta}{2}\right| \leqslant |R(\theta)e^{i\theta} - R(\eta)e^{i\eta}| \leqslant C_2\left|\sin\tfrac{\theta-\eta}{2}\right|, \tag{59}$$

with some constants $0 < C_1 \leqslant C_2$. To do so, we write

$$R(\theta)e^{i\theta} - R(\eta)e^{i\eta} = be^{i\eta}\big(e^{i(\theta-\eta)} - 1\big) + \Big(\big(R(\theta)e^{i\theta} - be^{i\theta}\big) - \big(R(\eta)e^{i\eta} - be^{i\eta}\big)\Big),$$
$$|\partial_\theta\big(R(\theta)e^{i\theta} - be^{i\theta}\big)| \leqslant \frac{|r| + |r'(\theta)|}{\sqrt{b^2 - 2\epsilon_0}} \leqslant \frac{2\epsilon_0}{\sqrt{b^2 - 2\epsilon_0}},$$

and

$$\tfrac{2|\theta|}{\pi} \leqslant |e^{i\theta} - 1| = |2\sin\tfrac{\theta}{2}| \leqslant |\theta|, \qquad \text{for} \ \ |\theta| \leqslant \pi, \tag{60}$$

that we combine with Taylor's formula in order to get

$$C_1\left|\sin\tfrac{\theta-\eta}{2}\right|\leqslant|R(\theta)e^{i\theta}-R(\eta)e^{i\eta}|\leqslant C_2\left|\sin\tfrac{\theta-\eta}{2}\right|,\quad\forall\,|\theta-\eta|\leqslant\pi.\tag{61}$$

For the general case $\theta,\eta\in\mathbb{R}$ there exists a k_0 such that $|\theta+2k_0\pi-\eta|\leqslant\pi$ and by periodicity we infer $R(\theta+2k_0\pi)e^{i(\theta+2k_0\pi)}=R(\theta)e^{i\theta}$, and then (61) applies, leading to (59). Therefore using (59) and the explicit formula of K_0^α given in (38), we find (56) and by differentiation

$$\begin{aligned}|\partial_\theta K_0^\alpha(R(\theta)e^{i\theta},R(\eta)e^{i\eta})|&\leqslant|\nabla_x K_0^\alpha(R(\theta)e^{i\theta},R(\eta)e^{i\eta})||\partial_\theta(R(\theta)e^{i\theta})|\\&\leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|^{-(1+\alpha)},\end{aligned}$$

achieving (57). The next task is to show that the functional F_2 given by (43) belongs to $C^{1-\alpha}(\mathbb{T})$. Recall that (55) holds true, then to get the suitable regularity for F_2 it is enough to show that

$$\theta\in\mathbb{T}\mapsto F_{2,1}(r(\theta))\triangleq\int_0^{2\pi}\int_0^{R(\eta)}\nabla_x K_1^\alpha(R(\theta)e^{i\theta},\rho e^{i\eta})\rho\,\mathrm{d}\rho\mathrm{d}\eta\in C^{1-\alpha}(\mathbb{T}).\tag{62}$$

Since K_1^α is smooth inside the domain $\mathbb{D}^2$ as in Lemma 2.3 and the patch is far away the boundary $\partial\mathbb{D}$ then it is plain that $F_{2,1}(r)\in L^\infty(\mathbb{T})$. To establish the Hölder regularity, we consider two points $x_1=R(\theta_1)e^{i\theta_1}$ and $x_2=R(\theta_2)e^{i\theta_2}$, and write by the mean value theorem and the boundedness of $\nabla_x^2K_1^\alpha$ in any compact set of $\mathbb{D}^2$,

$$\begin{aligned}|F_{2,1}(r(\theta_1))-F_{2,1}(r(\theta_2))|\leqslant&\int_0^{2\pi}\int_0^{R(\eta)}|\nabla_x K_1^\alpha(x_1,\rho e^{i\eta})-\nabla_x K_1^\alpha(x_2,\rho e^{i\eta})|\rho\mathrm{d}\rho\mathrm{d}\eta\\\leqslant&|x_1-x_2|\int_0^1\iint_{B(0,\sqrt{b^2+2\epsilon_0})}|\nabla_x^2K_1^\alpha(\kappa x_1+(1-\kappa)x_2,y)|\mathrm{d}y\mathrm{d}\kappa\\\leqslant&C|R(\theta_1)e^{i\theta_1}-R(\theta_2)e^{i\theta_2}|,\end{aligned}\tag{63}$$

where $C>0$ is independent of x_1,x_2. Therefore, applying (59) gives that $F_{2,1}(r)$ is Lipschitz and consequently it belongs to the space $C^{1-\alpha}$ achieving the proof of the claim (62). To sum, we have proven that the functions $\theta\in\mathbb{T}\mapsto\partial_\theta\Psi(r(\theta)),F(\Omega,r(\theta))$ belong to $C^{1-\alpha}(\mathbb{T})$. It remains to check the symmetry for these functions in order to be in the space Y_m. According to the first line of (41), it suffices to show the symmetry for the function $\theta\in\mathbb{T}\mapsto\partial_\theta\Psi(r(\theta))$.

Now we want to show that $\partial_\theta\Psi(r(\theta))$ and $F(\Omega,r(\theta))$ has the series expansion as in Y_m. In light of (36), statement (i) of Lemma 2.4 and the fact that $R(-\eta)=R(\eta),\forall\eta\in\mathbb{R}$, one obtains

$$\begin{aligned}\Psi(r(-\theta)) &= \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha(R(-\theta)e^{-i\theta}, R(\eta)e^{i\eta})\rho\,\mathrm{d}\rho\mathrm{d}\eta\\ &= \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha(R(\theta)e^{-i\theta}, R(\eta)e^{-i\eta})\rho\,\mathrm{d}\rho\mathrm{d}\eta\\ &= \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha(R(\theta)e^{i\theta}, R(\eta)e^{i\eta})\rho\,\mathrm{d}\rho\mathrm{d}\eta\\ &= \Psi(r(\theta)),\end{aligned}$$

which implies that

$$\Psi(r(\theta)) = \sum_{n=0}^{\infty} c_n \cos n\theta, \quad \text{with } \ c_n \in \mathbb{R}.$$

Thanks to the statement (ii) of Lemma 2.4, the fact that $R(\eta) = R(\eta + \frac{2\pi}{m})$ for every $\eta \in \mathbb{R}$ and the change of variable $\eta \mapsto \eta + \frac{2\pi}{m}$ we deduce by elementary operations

$$\begin{aligned}\Psi\Big(r\big(\theta + \tfrac{2\pi}{m}\big)\Big) &= \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha\Big(R\big(\theta + \tfrac{2\pi}{m}\big)e^{i(\theta+\frac{2\pi}{m})}, R(\eta)e^{i\eta}\Big)\rho\,\mathrm{d}\rho\mathrm{d}\eta\\ &= \int_0^{2\pi}\int_0^{R(\eta)} K^\alpha\big(R(\theta)e^{i(\theta+\frac{2\pi}{m})}, R(\eta)e^{i(\eta+\frac{2\pi}{m})}\big)\rho\,\mathrm{d}\rho\mathrm{d}\eta\\ &= \Psi(r(\theta)).\end{aligned}$$

Thus

$$\begin{aligned}\Psi(r(\theta)) = \Psi\Big(r\big(\theta + \tfrac{2\pi}{m}\big)\Big) &= \sum_{k=0}^{+\infty} c_n \cos\Big(n\theta + \tfrac{2n\pi}{m}\Big)\\ &= \sum_{k=0}^{+\infty} c_n\Big(\cos(n\theta)\,\cos\big(\tfrac{2n\pi}{m}\big) - \sin(n\theta)\,\sin\big(\tfrac{2n\pi}{m}\big)\Big).\end{aligned}$$

By the uniqueness of Fourier coefficients, we infer that $\cos\big(\frac{2n\pi}{m}\big) = 1$ and $\sin\big(\frac{2\pi n}{m}\big) = 0$. Hence

$$\Psi(r(\theta)) = \sum_{n=0}^{+\infty} c_{nm}\cos(nm\theta), \quad \text{with } \ c_{nm} \in \mathbb{R}.$$

Consequently $\partial_\theta \Psi(r) \in Y_m$ and therefore $F : \mathbb{R} \times B_{\epsilon_0} \to Y_m$ is well-defined.

(ii) It amounts to showing that the partial derivatives $\partial_\Omega F$ and $\partial_r F$ exist in the Frechet sense and they are both continuous. For $\partial_\Omega F$, it is obvious to see that

$$\partial_\Omega F(\Omega, r)(\theta) = r'(\theta),$$

and thus $\partial_\Omega F$ is a linear bounded operator from X_m to Y_m and is independent of Ω. Then we only need to check that $\partial_r F(\Omega, r) \in \mathcal{L}(X_m, Y_m)$ and it is continuous. In light of (48)-(52), we have the decomposition

$$\partial_r F(\Omega, r)h(\theta) = \big[\Omega + V_1(r)(\theta)\big]h'(\theta) + V_2(r)(\theta)h(\theta) + V_3(r,h)(\theta) + V_4(r,h)(\theta). \quad (64)$$

For the first term on the right-hand side of (64), we may argue as for the estimate of $\partial_\theta \Psi(r(\theta))$ developed in the preceding point **(i)**, leading to

$$\theta \mapsto \int_0^{2\pi}\int_0^{R(\eta)} (\nabla_x K^\alpha)\left(R(\theta)e^{i\theta}, \rho e^{i\eta}\right) \cdot e^{i\theta} \rho \, \mathrm{d}\rho \mathrm{d}\eta \in C^{1-\alpha}(\mathbb{T}),$$

and thus combined with the fact that $R^{-1}(\theta) \in C^{2-\alpha}$, we deduce from the product law that

$$\begin{aligned} \|(\Omega + V_1(r))h'\|_{C^{1-\alpha}} &\lesssim \|h'\|_{C^{1-\alpha}} + \|V_1(r)\|_{C^{1-\alpha}}\|h'\|_{C^{1-\alpha}} \\ &\lesssim \|h\|_{C^{2-\alpha}}. \end{aligned} \quad (65)$$

For the second term $V_2(r)h$, described in (50), one easily obtains

$$\|V_2(r)h\|_{C^{1-\alpha}} \leqslant C\|V_2(r)\|_{C^{1-\alpha}}\|h\|_{C^{1-\alpha}}.$$

To check that $V_2(r)$ belongs to $C^{1-\alpha}(\mathbb{T})$, it is enough to show that

$$\theta \mapsto \int_0^{2\pi}\int_0^{R(\eta)} \left(\nabla_x^2 K_1^\alpha\left(R(\theta)e^{i\theta}, \rho e^{i\eta}\right) \cdot e^{i\theta}\right) \cdot \partial_\theta\left(R(\theta)e^{i\theta}\right)\rho \, \mathrm{d}\rho \mathrm{d}\eta \in C^{1-\alpha}(\mathbb{T}),$$

since the treatment of the remaining term in (50) is quite similar to $V_1(r)$. This latter term is easily estimated from the product law since K_1^α is highly smooth inside the domain $\mathbb{D}^2$. For the third term $V_3(r,h)$ given by (51), one can easily estimate as before the first term connected to K_1^α:

$$\left\|\theta \mapsto \int_0^{2\pi} \nabla_x K_1^\alpha\left(R(\theta)e^{i\theta}, R(\eta)e^{i\eta}\right) \cdot \partial_\theta\left(R(\theta)e^{i\theta}\right)h(\eta)\, \mathrm{d}\eta\right\|_{C^{1-\alpha}} \leqslant C\|h\|_{L^\infty}.$$

As to the remaining term in $V_3(r,h)$, noting that

$$L_A(h)=\operatorname{Im}\Big(\partial_\eta\big(\tfrac{h(\eta)}{R(\eta)}e^{i\eta}\big)\,\overline{\partial_\theta(R(\theta)e^{i\theta})}\Big),$$

one finds by using (55), (56), (57) and Lemma 2.6,

$$\begin{aligned}&\left\|\theta\mapsto\int_0^{2\pi}K_0^\alpha\big(R(\theta)e^{i\theta}-R(\eta)e^{i\eta}\big)L_A(h)\mathrm{d}\eta\right\|_{C^{1-\alpha}}\\ &\lesssim\left\|\int_0^{2\pi}K_0^\alpha\big(R(\theta)e^{i\theta}-R(\eta)e^{i\eta}\big)\partial_\eta\left(\tfrac{h(\eta)e^{i\eta}}{R(\eta)}\right)\mathrm{d}\eta\right\|_{C^{1-\alpha}}\|\partial_\theta\big(R(\theta)e^{-i\theta}\big)\|_{C^{1-\alpha}}\\ &\lesssim\|h\|_{C^1}.\end{aligned}$$

Combining with the above two estimates leads to

$$\|V_3(r,h)\|_{C^{1-\alpha}}\leqslant C\|h\|_{C^1}. \tag{66}$$

Now we focus on the last term $V_4(r,h)$ given by (52). In view of (47), we write it as

$$V_4(r,h)=\int_0^{2\pi}\widetilde{H}(\theta,\eta)\big(i\partial_\eta(R(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\big(\partial_\theta\big(R(\theta)e^{i\theta}\big)\big),$$

with

$$\widetilde{H}(\theta,\eta)\triangleq\nabla_xK_0^\alpha\big(R(\theta)e^{i\theta}-R(\eta)e^{i\eta}\big)\cdot\left(e^{i\theta}\tfrac{h(\theta)}{R(\theta)}-e^{i\eta}\tfrac{h(\eta)}{R(\eta)}\right). \tag{67}$$

From the mean value theorem we infer

$$\left|e^{i\theta}\tfrac{h(\theta)}{R(\theta)}-e^{i\eta}\tfrac{h(\eta)}{R(\eta)}\right|\leqslant C|\theta-\eta|\|h\|_{C^1},$$

and since h and R are 2π-periodic functions then we can argue as (59) in order to get

$$\left|e^{i\theta}\tfrac{h(\theta)}{R(\theta)}-e^{i\eta}\tfrac{h(\eta)}{R(\eta)}\right|\leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|\|h\|_{C^1}. \tag{68}$$

Combining this with (38) and (56)-(57) allows to get

$$|\widetilde{H}(\theta,\eta)|\leqslant C\frac{\|h\|_{C^1}}{\big|\sin\frac{\theta-\eta}{2}\big|^\alpha},\quad|\partial_\theta\widetilde{H}(\theta,\eta)|\leqslant C\frac{\|h\|_{C^1}}{\big|\sin\frac{\theta-\eta}{2}\big|^{1+\alpha}},\quad\forall\eta\neq\theta\in\mathbb{T}.$$

Applying Lemma 2.6 and (55), we derive that

$$\|V_4(r,h)\|_{C^{1-\alpha}} \leqslant C\|h\|_{C^1}. \tag{69}$$

In conclusion, gathering the above estimates, we prove that

$$\|\partial_r F(\Omega,r)h\|_{C^{1-\alpha}} \leqslant C\|h\|_{C^{2-\alpha}},$$

which implies that $\partial_r F(\Omega,r) \in \mathcal{L}(X_m, Y_m)$.

The next step is to prove that for given $\Omega \in \mathbb{R}$, $\partial_r F(\Omega,r)$ is a continuous mapping taking values in the space of bounded linear operators from X_m to Y_m. In other words, we will show that, for every r_1, $r_2 \in B_{\epsilon_0} \subset X_m$,

$$\sup_{\|h\|_{C^{2-\alpha}}\leqslant 1} \|\partial_r F(\Omega,r_1)h - \partial_r F(\Omega,r_2)h\|_{C^{1-\alpha}} \to 0, \quad \text{as } \|r_1 - r_2\|_{C^{2-\alpha}} \to 0.$$

Thanks to (64) and the algebra structure of Hölder spaces, this can be guaranteed by the following continuous result that as $\|r_1 - r_2\|_{C^{2-\alpha}} \to 0$,

$$\sum_{j=1}^{2} \|V_j(r_1) - V_j(r_2)\|_{C^{1-\alpha}} + \sup_{\|h\|_{C^{2-\alpha}}\leqslant 1} \sum_{j=3}^{4} \|V_j(r_1,h) - V_j(r_2,h)\|_{C^{1-\alpha}} \to 0. \tag{70}$$

To prove the continuity result (70) regarding V_1, given by (49), we proceed first in a similar way as in the derivation (40), by writing

$$\begin{aligned} V_1(r)(\theta) =& R^{-1}(\theta) \int_0^{2\pi}\int_0^{R(\eta)} \big(-\nabla_y K_0^\alpha + \nabla_x K_1^\alpha\big)\big(R(\theta)e^{i\theta}, \rho e^{i\eta}\big)\cdot e^{i\theta}\rho\, \mathrm{d}\rho \mathrm{d}\eta \\ =& \frac{1}{R(\theta)} \int_0^{2\pi}\int_0^{R(\eta)} \nabla_x K_1^\alpha(R(\theta)e^{i\theta}, \rho e^{i\eta})\cdot e^{i\theta}\rho\, \mathrm{d}\rho \mathrm{d}\eta \\ &+ \frac{1}{R(\theta)} \int_0^{2\pi} K_0^\alpha\big(R(\theta)e^{i\theta} - R(\eta)e^{i\eta}\big)\big(i\partial_\eta(R(\eta)e^{i\eta})\big)\cdot e^{i\theta}\, \mathrm{d}\eta \\ \triangleq& \frac{V_{11}(r)(\theta) + V_{12}(r)(\theta)}{R(\theta)}, \end{aligned} \tag{71}$$

with $R(\theta) \triangleq \sqrt{b^2 + 2r(\theta)}$. The continuity of the mapping $r \mapsto R$ is immediate. Then it remains to explore the continuity of the mappings $r \mapsto V_{11}(r), V_{12}(r)$. With the notation $R_j(\theta) \triangleq \sqrt{b^2 + 2r_j(\theta)}$ one gets

$$V_{11}(r_1) - V_{11}(r_2) = \int_0^{2\pi}\int_{R_2(\eta)}^{R_1(\eta)} \nabla_x K_1^\alpha\big(R_1(\theta)e^{i\theta}, \rho e^{i\eta}\big)\cdot e^{i\theta}\rho\, \mathrm{d}\rho \mathrm{d}\eta$$

$$+\int_0^{2\pi}\int_0^{R_2(\eta)}\Big(\nabla_x K_1^\alpha\big(R_1(\theta)e^{i\theta},\rho e^{i\eta}\big)-\nabla_x K_1^\alpha\big(R_2(\theta)e^{i\theta},\rho e^{i\eta}\big)\Big)\cdot e^{i\theta}\rho\,\mathrm{d}\rho\mathrm{d}\eta.$$

In view of the fact that $|R_1(\theta)-R_2(\theta)|\leqslant\frac{|r_1(\theta)-r_2(\theta)|}{\sqrt{b^2-2\epsilon_0}}$ and as K_1^α is smooth in $\mathbb{D}^2$, we can use the mean value theorem in order to get

$$\|V_{11}(r_1)-V_{11}(r_2)\|_{L^\infty}\leqslant C\|r_1-r_2\|_{L^\infty}.$$

Similarly, by $|R_1'(\theta)-R_2'(\theta)|\leqslant C\frac{\|r_1-r_2\|_{C^1}}{\sqrt{b^2-2\epsilon_0}}$, we obtain

$$\|V_{11}(r_1)-V_{11}(r_2)\|_{C^1}\leqslant C\|r_1-r_2\|_{C^1}.$$

Hence we find by Sobolev embedding,

$$\|V_{11}(r_1)-V_{11}(r_2)\|_{C^{1-\alpha}}\leqslant C\|r_1-r_2\|_{C^{2-\alpha}}.$$

For V_{12} defined by (71), we write

$$\begin{aligned}V_{12}(r_1)-V_{12}(r_2)&=\int_0^{2\pi}H(\theta,\eta)\,\partial_\eta\big(iR_1(\eta)e^{i\eta}\big)\cdot e^{i\theta}\mathrm{d}\eta\\&\quad+\int_0^{2\pi}K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\,\partial_\eta\big(iR_1(\eta)e^{i\eta}-iR_2(\eta)e^{i\eta}\big)\cdot e^{i\theta}\,\mathrm{d}\eta\\&\triangleq J_1+J_2,\end{aligned}$$

where

$$H(\theta,\eta)\triangleq K_0^\alpha\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)-K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big).\tag{72}$$

To estimate J_2 we proceed as in the estimate of (54), leading to

$$\begin{aligned}\|J_2\|_{C^{1-\alpha}}&\leqslant C\|\partial_\eta\big(R_1(\eta)e^{i\eta}-R_2(\eta)e^{i\eta}\big)\|_{L^\infty}\\&\leqslant C\|r_1-r_2\|_{C^1}.\end{aligned}$$

As to the first term J_1, we will apply Lemma 2.6 to control the above right-hand term. Note that

$$\begin{aligned}\big|\big(R_1(\theta)-R_2(\theta)\big)e^{i\theta}-\big(R_1(\eta)-R_2(\eta)\big)e^{i\eta}\big|&\leqslant\big\|\partial_\theta\big((R_1(\theta)-R_2(\theta))e^{i\theta}\big)\big\|_{L^\infty}|\theta-\eta|\\&\leqslant C\|r_1-r_2\|_{C^1}|\theta-\eta|,\end{aligned}$$

which implies that (owing to (61) and the 2π-periodicity of R_i)

$$\begin{aligned}
&\big||R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}|-|R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}|\big|\\
&\quad\leqslant\big|(R_1(\theta)-R_2(\theta))e^{i\theta}-(R_1(\eta)-R_2(\eta))e^{i\eta}\big|\\
&\quad\leqslant C\|r_1-r_2\|_{C^1}\left|\sin\tfrac{\theta-\eta}{2}\right|. \qquad (73)
\end{aligned}$$

Using (38), (59), (73) and the following inequality

$$|A^\alpha-B^\alpha|\leqslant C_\alpha\big(A^{\alpha-1}+B^{\alpha-1}\big)|A-B|,\quad \forall A,B>0,$$

applied with $A=|R_1(\theta)-R_1(\eta)|^{-1}$ and $B=|R_2(\theta)-R_2(\eta)|^{-1}$ we obtain

$$|H(\theta,\eta)|\leqslant C\frac{\|r_1-r_2\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^\alpha}.$$

On the other hand, straightforward computations yield the splitting

$$\begin{aligned}
\partial_\theta H(\theta,\eta)&=\nabla_x K_0^\alpha\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)\cdot\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\\
&\quad-\nabla_x K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\cdot\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\\
&=H_1(\theta,\eta)+H_2(\theta,\eta)+H_3(\theta,\eta)
\end{aligned}$$

where

$$\begin{aligned}
H_1(\theta,\eta)&\triangleq-\alpha c_\alpha\frac{(R_1(\theta)-R_2(\theta))e^{i\theta}-(R_1(\eta)-R_2(\eta))e^{i\eta}}{|R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}|^{\alpha+2}}\cdot\partial_\theta\big(R_1(\theta)e^{i\theta}\big),\\
H_2(\theta,\eta)&\triangleq-\alpha c_\alpha\frac{R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}}{|R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}|^{\alpha+2}}\cdot\partial_\theta\big(R_1(\theta)e^{i\theta}-R_2(\theta)e^{i\theta}\big),\\
H_3(\theta,\eta)&\triangleq\alpha c_\alpha\,\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\cdot(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta})\\
&\quad\times\frac{|R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}|^{\alpha+2}-|R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}|^{\alpha+2}}{|R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}|^{\alpha+2}|R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}|^{\alpha+2}}.
\end{aligned}$$

For H_1 and H_2, using (59) and (73), we readily infer that

$$|H_1(\theta,\eta)|+|H_2(\theta,\eta)|\leqslant C\frac{\|r_1-r_2\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{1+\alpha}},\quad\forall\theta\neq\eta\in\mathbb{T}. \qquad (74)$$

Concerning the last term H_3, by virtue of the following inequality

$$|A^{k+\alpha}-B^{k+\alpha}|\leqslant C_{k,\alpha}|A-B|(A^{k+\alpha-1}+B^{k+\alpha-1}),\quad A,B\geqslant0,k\in\mathbb{N},\,0<\alpha<1, \qquad (75)$$

and using (59) and (73), we find

$$|H_3(\theta,\eta)| \leqslant C\frac{\|r_1-r_2\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{1+\alpha}}, \quad \forall\theta\neq\eta\in\mathbb{T}. \tag{76}$$

Putting together (74) and (76) leads to the wanted estimate of $\partial_\theta H(\theta,\eta)$. Hence, the kernel H satisfies the required assumptions in Lemma 2.6, and consequently,

$$\|J_1\|_{C^{1-\alpha}} \leqslant C\|r_1-r_2\|_{C^1}\|\partial_\eta\big(R_1(\eta)e^{i\eta}\big)\|_{L^\infty} \leqslant C\|r_1-r_2\|_{C^1}.$$

Hence, we conclude that $r\in X_m \mapsto (h\mapsto V_1(r)h)$ as a mapping from X_m to $\mathcal{L}(X_m,Y_m)$ is continuous.

Concerning the continuity of $r\mapsto V_2(r)$ introduced in (50) it can be checked in a similar way to V_1 discussed before. Therefore we will skip it.

Concerning the continuity of $r\mapsto V_3(r,h)$ given by (51), we start with the first term

$$\begin{aligned} J_4 \triangleq &\int_0^{2\pi} \nabla_x K_1^\alpha\left(R_1(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\right)\cdot\partial_\theta\big(R_1(\theta)e^{i\theta}\big)h(\eta)\mathrm{d}\eta \\ &-\int_0^{2\pi} \nabla_x K_1^\alpha\left(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta}\right)\cdot\partial_\theta\big(R_2(\theta)e^{i\theta}\big)h(\eta)\mathrm{d}\eta, \end{aligned}$$

with $R_j(\theta)=\sqrt{b^2+2r_j(\theta)}$, $j=1,2$. It is easy to see

$$\begin{aligned} J_4 = &\int_0^{2\pi} \nabla_x K_1^\alpha\left(R_1(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\right)\cdot\Big(\partial_\theta\big(R_1(\theta)e^{i\theta}\big)-\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\Big)h(\eta)\mathrm{d}\eta \\ &+\int_0^{2\pi}\Big(\nabla_x K_1^\alpha\big(R_1(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\big)-\nabla_x K_1^\alpha\big(R_2(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\big)\Big) \\ &\cdot\partial_\theta\big(R_2(\theta)e^{i\theta}\big)h(\eta)\mathrm{d}\eta \\ &+\int_0^{2\pi}\Big(\nabla_x K_1^\alpha\big(R_2(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\big)-\nabla_x K_1^\alpha\big(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta}\big)\Big) \\ &\cdot\partial_\theta\big(R_2(\theta)e^{i\theta}\big)h(\eta)\mathrm{d}\eta \\ \triangleq\, & J_{4,1}+J_{4,2}+J_{4,3}. \end{aligned}$$

For $J_{4,1}$, we use the following estimate

$$\begin{aligned} \|\partial_\theta\big(R_1(\theta)e^{i\theta}\big)-\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\|_{C^{1-\alpha}} &\leqslant \|R_1-R_2\|_{C^{1-\alpha}}+C\|R_1^{-1}\|_{C^{1-\alpha}}\|r_1'-r_2'\|_{C^{1-\alpha}} \\ &\quad + C\|R_1^{-1}-R_2^{-1}\|_{C^{1-\alpha}}\|r_2'\|_{C^{1-\alpha}} \\ &\leqslant C\|r_1-r_2\|_{C^{2-\alpha}}, \end{aligned} \tag{77}$$

and argue as (54) in order to get

$$\|J_{4,1}\|_{C^{1-\alpha}} \leq C\|h\|_{L^\infty}\|r_1 - r_2\|_{C^{2-\alpha}}.$$

For $J_{4,2}$, we apply the mean value theorem and the smoothness of K_1^α inside $\mathbb{D}^2$

$$J_{4,2} = \int_0^1\int_0^{2\pi} \partial_\theta\big(R_2 e^{i\theta}\big)\cdot\Big(\nabla_x^2 K_1^\alpha\big(\kappa R_1 e^{i\theta} + (1-\kappa)R_2\big)e^{i\theta}, R_1 e^{i\eta}\big)\Big) \cdot \big(R_1 e^{i\theta} - R_2 e^{i\theta}\big)h(\eta)\,\mathrm{d}\eta\mathrm{d}\kappa,$$

allowing to get

$$\begin{aligned}\|J_{4,2}\|_{C^{1-\alpha}} &\leqslant C\|h\|_{L^\infty}\|R_1(\theta)e^{i\theta} - R_2(\theta)e^{i\theta}\|_{C^{1-\alpha}}\\ &\leqslant C\|h\|_{L^\infty}\|r_1 - r_2\|_{C^1}.\end{aligned}$$

The term $J_{4,3}$ can be analogously treated as above to obtain that

$$\begin{aligned}\|J_{4,3}\|_{C^{1-\alpha}} &\leqslant C\|h\|_{L^\infty}\|R_1(\eta)e^{i\eta} - R_2(\eta)e^{i\eta}\|_{L^\infty}\\ &\leqslant C\|h\|_{L^\infty}\|r_1 - r_2\|_{L^\infty}.\end{aligned}$$

Hence from the preceding estimates we obtain the continuity for J_4. For the second term in $V_3(r,h)$ given by (51), we consider

$$J_5 \triangleq \int_0^{2\pi} K_0^\alpha(R_1(\theta)e^{i\theta}, R_1(\eta)e^{i\eta})L_{A_1}(h)\mathrm{d}\eta - \int_0^{2\pi} K_0^\alpha(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta})L_{A_2}(h)\mathrm{d}\eta,$$

where for $j = 1, 2$,

$$L_{A_j}(h) \triangleq \partial_\eta\partial_\theta\Big(\tfrac{h(\eta)R_j(\theta)\sin(\eta-\theta)}{R_j(\eta)}\Big).$$

Notice that

$$\begin{aligned}J_5 &= \int_0^{2\pi} H(\theta,\eta)\, L_{A_1}(h)\mathrm{d}\eta + \int_0^{2\pi} K_0^\alpha\big(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta}\big)\big(L_{A_1}(h) - L_{A_2}(h)\big)\mathrm{d}\eta\\ &\triangleq J_{5,1} + J_{5,2},\end{aligned}$$

where H is the kernel function defined by (72). For $J_{5,1}$, it can be estimated following the same lines as J_2, leading to

$$\begin{aligned}\|J_{5,1}\|_{C^{1-\alpha}} &\leqslant \Big\| \int_0^{2\pi} H(\theta,\eta)\partial_\eta\left(\tfrac{h(\eta)e^{i\eta}}{R_1(\eta)}\right)\mathrm{d}\eta\Big\|_{C^{1-\alpha}} \|\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\|_{C^{1-\alpha}}\\ &\leqslant C\|h\|_{C^1}\|r_1-r_2\|_{C^1}.\end{aligned}$$

For the term $J_{5,2}$, by using (47), (77) and Lemma 2.6, we similarly get

$$\begin{aligned}\|J_{5,2}\|_{C^{1-\alpha}} \lesssim& \Big\|\int_0^{2\pi} K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\partial_\eta\bigg(\tfrac{h(\eta)e^{i\eta}}{R_1(\eta)}\bigg)\mathrm{d}\eta\Big\|_{C^{1-\alpha}}\|\partial_\theta\big(R_1e^{i\theta}-R_2e^{i\theta}\big)\|_{C^{1-\alpha}}\\ &+C\Big\|\int_0^{2\pi} K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\partial_\eta\left[\left(\tfrac{h(\eta)e^{i\eta}}{R_1(\eta)}\right)-\left(\tfrac{h(\eta)e^{i\eta}}{R_2(\eta)}\right)\right]\mathrm{d}\eta\Big\|_{C^{1-\alpha}}\\ \leqslant& C\|h\|_{C^1}\|r_1-r_2\|_{C^{2-\alpha}}.\end{aligned}$$

Hence, based the preceding results, we claim that $V_3(r,h)$ as a mapping from X_m to $\mathcal{L}(X_m,Y_m)$ is continuous.
Now we shall investigate the continuity estimate of the last term $V_4(r,h)$. In view of (47) and (52), we need to consider

$$\begin{aligned}J_6 \triangleq& \int_0^{2\pi}\widetilde{H}_1(\theta,\eta)\big(i\partial_\eta(R_1(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\Big(\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\Big)\\ &-\int_0^{2\pi}\widetilde{H}_2(\theta,\eta)\big(i\partial_\eta(R_2(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\Big(\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\Big),\end{aligned}$$

with $R_j(\theta)=\sqrt{b^2+2r_j(\theta)}$, $j=1,2$ and

$$\widetilde{H}_j(\theta,\eta)\triangleq\nabla_x K_0^\alpha\big(R_j(\theta)e^{i\theta}-R_j(\eta)e^{i\eta}\big)\cdot\left(e^{i\theta}\tfrac{h(\theta)}{R_j(\theta)}-e^{i\eta}\tfrac{h(\eta)}{R_j(\eta)}\right).$$

Observe that

$$\begin{aligned}J_6 =& \int_0^{2\pi}\big(\widetilde{H}_1(\theta,\eta)-\widetilde{H}_2(\theta,\eta)\big)\big(i\partial_\eta(R_1(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\Big(\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\Big)\\ &+\int_0^{2\pi}\widetilde{H}_2(\theta,\eta)\big(i\partial_\eta(R_1(\eta)e^{i\eta})-i\partial_\eta(R_2(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\Big(\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\Big)\\ &+\int_0^{2\pi}\widetilde{H}_2(\theta,\eta)\big(i\partial_\eta(R_1(\eta)e^{i\eta})\big)\mathrm{d}\eta\cdot\Big(\partial_\theta\big(R_1(\theta)e^{i\theta}\big)-\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\Big)\\ \triangleq& J_{6,1}+J_{6,2}+J_{6,3}.\end{aligned}\tag{78}$$

Concerning the terms $J_{6,2}$ and $J_{6,3}$, we may proceed in a similar way to $J_{5,1}$ and $J_{5,2}$, and one gets

$$\|J_{6,2}\|_{C^{1-\alpha}} + \|J_{6,3}\|_{C^{1-\alpha}} \leqslant C\|h\|_{C^1}\|r_1 - r_2\|_{C^{2-\alpha}}.$$

For $J_{6,1}$, we first notice by the product law

$$\|J_{6,1}\|_{C^{1-\alpha}} \leqslant C\Big\| \int_0^{2\pi} \big(\widetilde{H}_1(\theta,\eta) - \widetilde{H}_2(\theta,\eta)\big)\big(i\partial_\eta(R_1(\eta)e^{i\eta})\big)\mathrm{d}\eta\Big\|_{C^{1-\alpha}},$$

and we shall use Lemma 2.6 in order to tackle the above right-hand term. Direct computations yield the following decomposition

$$\begin{aligned}
&\widetilde{H}_1(\theta,\eta) - \widetilde{H}_2(\theta,\eta)\\
&= \big(\nabla_x K_0^\alpha\big(R_1(\theta)e^{i\theta}, R_1(\eta)e^{i\eta}\big) - \nabla_x K_0^\alpha\big(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta}\big)\big)\cdot\Big(e^{i\theta}\tfrac{h(\theta)}{R_1(\theta)} - e^{i\eta}\tfrac{h(\eta)}{R_1(\eta)}\Big)\\
&\quad + \nabla_x K_0^\alpha\big(R_2(\theta)e^{i\theta}, R_2(\eta)e^{i\eta}\big)\cdot\Big(e^{i\theta}\tfrac{h(\theta)}{R_1(\theta)} - e^{i\eta}\tfrac{h(\eta)}{R_1(\eta)} - e^{i\theta}\tfrac{h(\theta)}{R_2(\theta)} + e^{i\eta}\tfrac{h(\eta)}{R_2(\eta)}\Big)\\
&\triangleq S_1(\theta,\eta) + S_2(\theta,\eta).
\end{aligned}$$

Noting that

$$\begin{aligned}
\nabla_x K_0^\alpha(x_1 - y_1) - \nabla_x K_0^\alpha(x_2 - y_2) &= -\alpha c_\alpha \frac{x_1 - y_1}{|x_1 - y_1|^{\alpha+2}} + \alpha c_\alpha \frac{x_2 - y_2}{|x_2 - y_2|^{\alpha+2}}\\
&= \alpha c_\alpha (x_1 - y_1)\left(\frac{|x_1 - y_1|^{\alpha+2} - |x_2 - y_2|^{\alpha+2}}{|x_1 - y_1|^{\alpha+2}|x_2 - y_2|^{\alpha+2}}\right)\\
&\quad + \alpha c_\alpha \frac{(x_2 - y_2) - (x_1 - y_1)}{|x_2 - y_2|^{\alpha+2}},
\end{aligned}$$

and using (59), (68), (73) and (75), we have that

$$|S_1(\theta,\eta)| \leqslant C\frac{\|h\|_{C^1}\|r_1 - r_2\|_{C^1}}{\big|\sin\frac{\theta-\eta}{2}\big|^\alpha}, \quad \forall \theta \neq \eta \in \mathbb{T}.$$

Using mean value theorem and the 2π-periodicity of R_j and h, we find

$$\left|\frac{e^{i\theta}h(\theta)}{R_1(\theta)} - \frac{e^{i\eta}h(\eta)}{R_1(\eta)} - \frac{e^{i\theta}h(\theta)}{R_2(\theta)} + \frac{e^{i\eta}h(\eta)}{R_2(\eta)}\right| \leqslant C\|h\|_{C^1}\|r_1 - r_2\|_{C^1}\left|\sin\frac{\theta-\eta}{2}\right|. \tag{79}$$

According to (57) and (79), we obtain

$$|S_2(\theta,\eta)| \leqslant C\frac{\|r_1 - r_2\|_{C^1}\|h\|_{C^1}}{\big|\sin\frac{\theta-\eta}{2}\big|^\alpha}, \quad \forall \theta \neq \eta \in \mathbb{T}.$$

Consequently,

$$|\widetilde{H}_1(\theta,\eta)-\widetilde{H}_2(\theta,\eta)|\leqslant C\frac{\|r_1-r_2\|_{C^1}\|h\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{\alpha}},\quad \forall\theta\neq\eta\in\mathbb{T}. \tag{80}$$

Now we intend to estimate $\partial_\theta S_1(\theta,\eta)$ and $\partial_\theta S_2(r,\theta)$. We see that

$$\begin{aligned}\partial_\theta S_1(\theta,\eta)=&\Big(\nabla_x^2 K_0^\alpha\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)\cdot\partial_\theta\big(R_1(\theta)e^{i\theta}\big)\\&-\nabla_x^2 K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\cdot\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\Big)\cdot\Big(e^{i\theta}\tfrac{h(\theta)}{R_1(\theta)}-e^{i\eta}\tfrac{h(\eta)}{R_1(\eta)}\Big)\\&+\big(\nabla_x K_0^\alpha\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)-\nabla_x K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\big)\\&\cdot\partial_\theta\Big(e^{i\theta}\tfrac{h(\theta)}{R_1(\theta)}\Big).\end{aligned}$$

Note that for every $x=(x^1,x^2)$ and $y=(y^1,y^2)$ satisfying $x\neq y$,

$$\begin{aligned}\nabla_x^2 K_0^\alpha(x-y)=&\frac{-\alpha c_\alpha}{|x-y|^{\alpha+2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}\\&+\frac{\alpha(\alpha+2)c_\alpha}{|x-y|^{\alpha+4}}\begin{pmatrix}(x^1-y^1)^2&(x^1-y^1)(x^2-y^2)\\(x^1-y^1)(x^2-y^2)&(x^2-y^2)^2\end{pmatrix},\end{aligned}$$

and

$$|\nabla_x^3 K_0^\alpha(x-y)|\leqslant\frac{C}{|x-y|^{\alpha+3}},$$

then using (59) (and its suitable variation), we know that

$$|\nabla_x^2 K_0^\alpha(R(\theta)e^{i\theta}-R(\eta)e^{i\eta})|\leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|^{-(\alpha+2)}, \tag{81}$$

and for every $\kappa\in[0,1]$,

$$\Big|\nabla_x^3 K_0^\alpha\Big(\kappa\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)+(1-\kappa)\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\Big)\Big|\leqslant C\left|\sin\tfrac{\theta-\eta}{2}\right|^{-(\alpha+3)}.$$

Thanks to the above estimates, together with (59), (68) and (73), we conclude that

$$\big|\nabla_x^2 K_0^\alpha\big(R_1(\theta)e^{i\theta}-R_1(\eta)e^{i\eta}\big)-\nabla_x^2 K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\big|\leqslant C\frac{\|r_1-r_2\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{\alpha+2}},$$

and

$$|\partial_\theta S_1(\theta,\eta)|\leqslant C\frac{\|r_1-r_2\|_{C^1}\|h\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{1+\alpha}},\quad\forall\theta\neq\eta\in\mathbb{T}.$$

On the other hand, by direct computations we get

$$\partial_\theta S_2(\theta,\eta)=\partial_\theta\big(R_2(\theta)e^{i\theta}\big)\cdot\nabla_x^2K_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\\ \cdot\left(\tfrac{e^{i\theta}h(\theta)}{R_1(\theta)}-\tfrac{e^{i\eta}h(\eta)}{R_1(\eta)}-\tfrac{e^{i\theta}h(\theta)}{R_2(\theta)}+\tfrac{e^{i\eta}h(\eta)}{R_2(\eta)}\right)\\ +\nabla_xK_0^\alpha\big(R_2(\theta)e^{i\theta}-R_2(\eta)e^{i\eta}\big)\cdot\partial_\theta\left(\tfrac{e^{i\theta}h(\theta)}{R_1(\theta)}-\tfrac{e^{i\theta}h(\theta)}{R_2(\theta)}\right),$$

and

$$\left|\partial_\theta\left(\tfrac{e^{i\theta}h(\theta)}{R_1(\theta)}-\tfrac{e^{i\theta}h(\theta)}{R_2(\theta)}\right)\right|\leqslant C\|h\|_{C^1}\|r_1-r_2\|_{C^1}. \tag{82}$$

Together with (79) and (81), we infer

$$|\partial_\theta S_2(\theta,\eta)|\leqslant C_\alpha\frac{\|r_1-r_2\|_{C^1}\|h\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{1+\alpha}}.$$

It follows that

$$|\partial_\theta\widetilde{H}_1(\theta,\eta)-\partial_\theta\widetilde{H}_2(\theta,\eta)|\leqslant C_\alpha\frac{\|r_1-r_2\|_{C^1}\|h\|_{C^1}}{\left|\sin\frac{\theta-\eta}{2}\right|^{1+\alpha}}. \tag{83}$$

At this stage we can use (80), (83) and Lemma 2.6 to show that

$$\|J_{6,1}\|_{C^{1-\alpha}}\leqslant C\|h\|_{C^1}\|r_1-r_2\|_{C^1}.$$

Hence from (78) and the above estimates, we have

$$\|J_6\|_{C^{1-\alpha}}\leqslant C\|h\|_{C^1}\|r_1-r_2\|_{C^{2-\alpha}},$$

which ensures that $r\mapsto(h\mapsto V_4(r,h))$ is a continuous mapping from X_m to $\mathcal{L}(X_m,Y_m)$. In conclusion, we have established that $r\mapsto\partial_rF(\Omega,r)$ is a continuous mapping from the small ball B_{ϵ_0} of X_m to $\mathcal{L}(X_m,Y_m)$.

(iii) We shall compute $\partial_\Omega\partial_rF(\Omega,r)$ and prove the continuity of this function. Let $r\in B_{\epsilon_0}$ and $h\in X_m$, then in view of (48) one has

$$\partial_\Omega\partial_rF(\Omega,r)h(\theta)=h'(\theta),$$

which is independent of r and Ω. Hence, the continuity of $\partial_\Omega\partial_rF(\Omega,r)$ is obvious. This concludes the proof of the Proposition 4.1. □

5. Spectral study

In this section we focus on the spectral study of the linearized operator of $F(\Omega, r)$ around zero, which is denoted by $\partial_r F(\Omega, 0)$. It turns out that only some discrete values of Ω are allowed to generate a non-trivial kernel. In addition, we will see that all the spectral properties required by Theorem 6.1 are satisfied at least for large symmetry m. The main result of this section reads as follows.

Proposition 5.1. *Let (α, b, m) satisfy one of the cases (4)-(5)-(6). Then the following statements hold true.*

(i) *The kernel of $\partial_r F(\Omega, 0)$ in X_m is non-trivial if and only if $\Omega = \Omega^\alpha_{\ell m,b}$, for some $\ell \in \mathbb{N}^+$ with*

$$\begin{aligned}\Omega^\alpha_{m,b} &= -V_1(0) - \alpha_{m,b}\\ &\triangleq 2\sum_{k\geqslant 1} x_{0,k}^{\alpha-2}\frac{J_1^2(x_{0,k}b)}{J_1^2(x_{0,k})} - 2\sum_{k\geqslant 1} x_{m,k}^{\alpha-2}\frac{J_m^2(x_{m,k}b)}{J_{m+1}^2(x_{m,k})},\end{aligned} \tag{84}$$

and in this case, it is a one-dimensional vector space in X_m generated by $\theta \mapsto \cos(\ell m\theta)$.

(ii) *The range of $\partial_r F(\Omega^\alpha_{\ell m,b}, 0)$ is closed in Y_m and is of co-dimension one. It is given by*

$$R\left(\partial_r F(\Omega^\alpha_{\ell m,b}, 0)\right) = \left\{ r \in C^{1-\alpha}(\mathbb{T}) : r(\theta) = \sum_{\substack{n\geqslant 1\\ n\neq \ell}} a_n \sin(nm\theta),\ a_n \in \mathbb{R} \right\}.$$

(iii) *Transversality assumption:*

$$\partial_\Omega \partial_r F(\Omega^\alpha_{\ell m,b}, 0)\big(\cos(\ell m\theta)\big) \notin R(\partial_r F(\Omega^\alpha_{\ell m,b}, 0)).$$

As one can easily observe, the proof of Theorem 1.1 is a direct consequence of Theorem 6.1, Proposition 5.1 and Proposition 4.1.

The proof of Proposition 5.1 will be done in several steps through the subsections below.

5.1. Analysis of the linear frequencies

Before proceeding forward, we collect some properties on the asymptotic behavior of the sequence $\{\Omega^\alpha_{m,b}\}_{m\geq 1}$ with respect to m and α.

Lemma 5.1. *We have the following results.*

(i) *Let* (α, b, m) *satisfy one of cases* (4)-(5)-(6) *with* $m^* = m^*(\alpha, b) \in \mathbb{N}^+$ *(a rough bound is* $m^* \leqslant \frac{1}{\log b}\Big(\log \frac{1-\alpha}{1-\frac{\alpha}{2}-(e\log b)^{-1}}\Big)$*) and* $\alpha^* = \alpha^*(b) > 0$ *a small number. Then the map* $m \mapsto \Omega^{\alpha}_{m,b}$ *is strictly increasing. In addition, for any* $m \geqslant 1$ *and* $b \in (0,1)$,

$$\lim_{\alpha\to 0} \Omega^{\alpha}_{m,b} = \frac{m-1+b^{2m}}{2m},$$

and

$$\lim_{\alpha\to 1} \Omega^{\alpha}_{m,b} = \frac{2}{\pi b}\sum_{k=1}^{m-1}\frac{1}{2k+1} + \frac{2}{\pi}\int_0^{\infty}\left(\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)} + \frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\right)\mathrm{d}\rho. \tag{85}$$

(ii) *For* $\alpha, b \in (0,1)$ *fixed and* $m \in \mathbb{N}$ *large enough,*

$$\alpha_{m,b} = \frac{2^{\alpha-1}\Gamma(1-\alpha)}{b^{\alpha}\Gamma^2(1-\frac{\alpha}{2})}m^{\alpha-1} + O\left(\tfrac{1}{m^{3-\alpha}}\right). \tag{86}$$

Proof of Lemma 5.1. (i) By using (84) combined with Sneddon's formula (29) and (31), we have

$$-V_1(0) = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{\alpha}{2})}{b^{\alpha}2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})} + \frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})\int_0^{\infty}\rho^{\alpha-1}\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)}\mathrm{d}\rho, \tag{87}$$

and

$$\begin{aligned}\alpha_{m,b} &= \frac{2}{\pi}\sin\left(\tfrac{\alpha\pi}{2}\right)\int_0^{\infty}\rho^{\alpha-1}I_m(b\rho)K_m(b\rho)\mathrm{d}\rho - \frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})\int_0^{\infty}\rho^{\alpha-1}\frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\mathrm{d}\rho \\ &= \frac{\Gamma(1-\alpha)\Gamma(m+\frac{\alpha}{2})}{b^{\alpha}2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})\Gamma(m+1-\frac{\alpha}{2})} - \frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})\int_0^{\infty}\rho^{\alpha-1}\frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\mathrm{d}\rho \\ &\triangleq \alpha^{(1)}_{m,b} - \alpha^{(2)}_{m,b},\end{aligned} \tag{88}$$

where I_m and K_m are the modified Bessel functions introduced in Section 2.3.
In order to show that $\Omega^{\alpha}_{m,b}$ is strictly increasing in m, we shall analyze the monotonicity of the sequence $\{\alpha_{m,b}\}_{m\geq 1}$. Consider the Wallis quotient defined by

$$\mathtt{W}_{\alpha}(m) \triangleq \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})},$$

then we easily see that

$$\alpha_{m,b}^{(1)} = \frac{2^{\alpha-1}\Gamma(1-\alpha)}{b^\alpha\Gamma^2(1-\frac{\alpha}{2})}\mathtt{W}_\alpha(m). \tag{89}$$

Straightforward computation based on the identity $\Gamma(1+x) = x\Gamma(x)$ allows us to get

$$\mathtt{W}_\alpha(m+1) - \mathtt{W}_\alpha(m) = -\frac{1-\alpha}{1+m-\frac{\alpha}{2}}\mathtt{W}_\alpha(m). \tag{90}$$

In particular this implies that $\{\mathtt{W}_\alpha(m)\}_{m\geqslant 1}$ and $\{\alpha_{m,b}^{(1)}\}_{m\geqslant 1}$ are strictly decreasing. Now let us move to the analysis of $\alpha_{m,b}^{(2)}$. Recalling from (28) that

$$I_m(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{m+2n}}{n!(m+n)!}, \tag{91}$$

and (e.g. see 6.22 (5) of [63])

$$K_m(x) = \int_0^{+\infty} e^{-x\cosh t}\cosh(mt)\mathrm{d}t > 0, \quad \forall x \in \mathbb{R},$$

we see that

$$I_m(bx) \leqslant b^m I_m(x), \qquad \forall\, x \geqslant 0, 0 < b < 1, \tag{92}$$

and

$$I_m(x)K_m(x) \geqslant 0, \quad \forall\, x \geqslant 0.$$

A refined version of (92) is that for all $x > 0$ and $0 < b < 1$,

$$I_m(bx) = \big(b^m - r_{m,b}(x)\big)I_m(x), \quad \text{with } \ 0 \leqslant r_{m,b}(x) \leqslant b^m \min\big\{1, \tfrac{x^2}{4m}\big\}, \tag{93}$$

which can be easily seen from the following formula

$$\begin{aligned} I_m(bx) - b^m I_m(x) &= -\sum_{n=1}^{\infty}\big(b^m - b^{m+2n}\big)\frac{(\frac{1}{2}x)^{m+2n}}{n!(m+n)!} \\ &= -\frac{x^2}{4}\sum_{n=0}^{\infty}\big(b^m - b^{m+2n+2}\big)\frac{(\frac{1}{2}x)^{m+2n}}{(n+1)!(m+n+1)!}. \end{aligned}$$

Hence, we infer that

$$0 \leqslant \alpha_{m,b}^{(2)} = \frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})b^m \int_0^{\infty} \rho^{\alpha-1} I_m(b\rho)K_m(\rho)\mathrm{d}\rho$$

$$\begin{aligned}&-\frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})\int_0^\infty \rho^{\alpha-1}r_{m,b}(\rho)I_m(b\rho)K_m(\rho)\mathrm{d}\rho\\
&=\frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})b^{2m}\int_0^\infty \rho^{\alpha-1}I_m(\rho)K_m(\rho)\mathrm{d}\rho\\
&\quad-\frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})\int_0^\infty \rho^{\alpha-1}r_{m,b}(\rho)\big(I_m(b\rho)+b^mI_m(\rho)\big)K_m(\rho)\mathrm{d}\rho\\
&\triangleq \alpha_{m,b}^{(21)}-\alpha_{m,b}^{(22)}. \qquad (94)\end{aligned}$$

In view of the formula (32) and the fact that (see e.g. 8.334 of [31])

$$\Gamma(\tfrac{\alpha}{2})\Gamma(1-\tfrac{\alpha}{2})=\frac{\pi}{\sin(\pi\frac{\alpha}{2})},$$

we get

$$\begin{aligned}\alpha_{m,b}^{(2)}\leqslant\alpha_{m,b}^{(21)}&=\frac{2}{\pi}\sin(\tfrac{\alpha\pi}{2})b^{2m}\frac{\Gamma(\frac{\alpha}{2})\Gamma(1-\alpha)}{2^{2-\alpha}\Gamma(1-\frac{\alpha}{2})}\frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})} \qquad (95)\\
&=\frac{1}{\pi}\sin(\tfrac{\alpha\pi}{2})b^{2m}\frac{\Gamma(\frac{\alpha}{2})\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}\mathtt{W}_\alpha(m)\\
&=\frac{2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}b^{2m}\mathtt{W}_\alpha(m).\end{aligned}$$

For the remainder term $\alpha_{m,b}^{(22)}$, we can show from the second inequality of (93) that

$$\forall\delta\in(0,1),\quad r_{m,b}(\rho)\leqslant b^m(\tfrac{1}{4m})^{\delta/2}\rho^\delta,$$

combined with (32) it yields that for every $\delta\in(0,1-\alpha)$,

$$\begin{aligned}0\leqslant\alpha_{m,b}^{(22)}&\leqslant\frac{4}{\pi}\frac{b^{2m}}{(4m)^{\delta/2}}\sin(\tfrac{\alpha\pi}{2})\int_0^\infty\rho^{\alpha-1+\delta}I_m(\rho)K_m(\rho)\mathrm{d}\rho\\
&\leqslant\frac{4}{\pi}\frac{b^{2m}}{(4m)^{\delta/2}}\sin(\tfrac{\alpha\pi}{2})\frac{\Gamma(\frac{\alpha+\delta}{2})\Gamma(m+\frac{\alpha+\delta}{2})\Gamma(1-\alpha-\delta)}{2^{2-\alpha-\delta}\Gamma(m+1-\frac{\alpha+\delta}{2})\Gamma(1-\frac{\alpha+\delta}{2})}, \qquad (96)\end{aligned}$$

which will be useful in the sequel. Notice that $\{\alpha_{m,b}^{(2)}\}$ is positive and $\{\mathtt{W}_\alpha(m)\}$ is positive and decreasing, then

$$\alpha_{m+1,b}^{(2)}-\alpha_{m,b}^{(2)}\geqslant-\alpha_{m,b}^{(2)}\geqslant-\frac{b^{2m}2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\mathtt{W}_\alpha(m).$$

Thus in combination with (88), (89) and (90) we obtain

$$\begin{aligned}\alpha_{m+1,b}-\alpha_{m,b} &\leqslant -\frac{2^{\alpha-1}\Gamma(1-\alpha)}{b^{\alpha}\Gamma^2(1-\frac{\alpha}{2})}\frac{1-\alpha}{1+m-\frac{\alpha}{2}}\mathtt{W}_\alpha(m)+\frac{b^{2m}2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\mathtt{W}_\alpha(m)\\ &\leqslant \frac{2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{b^{-\alpha}}{1+m-\frac{\alpha}{2}}\mathtt{W}_\alpha(m)\Big(-(1-\alpha)+(1+m-\tfrac{\alpha}{2})b^{2m+\alpha}\Big).\end{aligned}$$

Now, we intend to check the monotonicity of $\{\alpha_{m,b}\}_{m\geq 1}$ for fixed α, b and for m large enough.
Since for fixed $\alpha, b\in(0,1)$, $\lim\limits_{m\to\infty}(m+1)b^{2m}=0$, there exists a constant $m^*=m(\alpha,b)\in\mathbb{N}$ such that $\alpha-1+(1+m-\frac{\alpha}{2})b^{2m+\alpha}<0$ for every $m\geq m^*$, thus we have

$$\forall\, m\geqslant m^*,\quad \alpha_{m+1,b}-\alpha_{m,b}<0.$$

On the other hand by using the inequality $mb^{2m}\leqslant(-\frac{1}{e\log b})b^m$, $\forall m\geqslant 1$, we infer

$$-(1-\alpha)+(1+m-\tfrac{\alpha}{2})b^{2m+\alpha}<-(1-\alpha)+\big(1-\tfrac{\alpha}{2}-\tfrac{1}{e\log b}\big)b^m,$$

then we may choose $m^*\leqslant\frac{1}{\log b}\big(\log\frac{1-\alpha}{1-\frac{\alpha}{2}-(e\log b)^{-1}}\big)$. This gives the condition (5).
Next, we shall check the condition (4) dealing with the monotonicity of $\{\alpha_{m,b}\}_{m\geqslant 1}$ for fixed $\alpha\in(0,1)$ but small b. For this aim we introduce the function

$$\varphi(m)\triangleq(1+m-\tfrac{\alpha}{2})b^{2m+\alpha},\quad m\in[1,\infty).$$

Differentiating in m gives

$$\varphi'(m)=b^{2m+\alpha}\Big(1+2(\log b)(1+m-\tfrac{\alpha}{2})\Big).$$

We can observe that if $\varphi'(1)\leqslant 0$ then $\varphi'(m)\leqslant 0$ for any $m\geqslant 1$. Let $b^*\in(0,1)$ be such that

$$1+2(\log b^*)(2-\tfrac{\alpha}{2})\leqslant 0. \tag{97}$$

Then for any $b\in[0,b^*]$ and $m\geqslant 1$,

$$\varphi'(m)\leqslant 0.$$

Consequently, for any $b\in[0,b^*]$

$$\begin{aligned}\forall\, m\geqslant 1,\quad \alpha-1+\varphi(m)\leqslant \alpha-1+\varphi(1)&=\alpha-1+(2-\tfrac{\alpha}{2})b^{2+\alpha}\\ &\leqslant \alpha-1+(2-\tfrac{\alpha}{2})b^2.\end{aligned}$$

Fix $b^*\triangleq\sqrt{\frac{1-\alpha}{2-\frac{\alpha}{2}}}$, then the condition (97) becomes

$$1+\log(\tfrac{1-\alpha}{2-\frac{\alpha}{2}})(2-\tfrac{\alpha}{2})\leqslant 0.$$

This inequality holds true for any $\alpha\in(0,1)$. Finally we get the following

$$\forall \alpha\in(0,1), b\in(0,b^*), m\geqslant 1,\quad \alpha_{m+1,b}-\alpha_{m,b}<0.$$

The last point to discuss is (6) related to the monotonicity of $\{\alpha_{m,b}\}_{m\geqslant 1}$ for fixed $b\in(0,1)$ and small α. In view of (89), (94) and (95), we have

$$\alpha_{m,b}=\frac{2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\Big(\frac{1}{b^\alpha}-b^{2m}\Big)\mathtt{W}_\alpha(m)+\alpha_{m,b}^{(22)}. \tag{98}$$

By choosing $\delta=\frac12$ in (96), we infer that (under the additional constraint $\alpha\in(0,\frac12)$)

$$|\alpha_{m,b}^{(22)}|\leqslant \frac{4}{\pi}\frac{b^{2m}}{(4m)^{1/4}}\sin(\tfrac{\alpha\pi}{2})\frac{\Gamma(\frac{\alpha}{2}+\frac14)\Gamma(m+\frac{\alpha}{2}+\frac14)\Gamma(\frac12-\alpha)}{2^{3/2-\alpha}\Gamma(m+\frac34-\frac{\alpha}{2})\Gamma(\frac34-\frac{\alpha}{2})}. \tag{99}$$

Applying Gautschi's inequality

$$\forall x>0, \forall s\in(0,1),\quad x^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<(x+1)^{1-s},$$

we deduce that for any $m\geqslant 1$, $a,b\in(0,1)$,

$$\frac{(1+m)^{a-1}}{m^{b-1}}\leqslant\frac{\Gamma(m+a)}{\Gamma(m+b)}\leqslant\frac{m^{a-1}}{(m+1)^{b-1}},$$

leading to

$$\tfrac12 m^{a-b}\leqslant\frac{\Gamma(m+a)}{\Gamma(m+b)}\leqslant 2m^{a-b}. \tag{100}$$

It follows from the right-hand side inequality of (100) that

$$\forall m\geqslant 1,\forall\alpha\in(0,1),\quad \frac{\Gamma(m+\frac{\alpha}{2}+\frac14)}{\Gamma(m+\frac34-\frac{\alpha}{2})}\leqslant 2m^{\alpha-\frac12}. \tag{101}$$

Inserting (101) into (99) yields for any $m\geqslant 1$, $\alpha\in(0,\frac13)$,

$$\begin{aligned}|\alpha_{m,b}^{(22)}|&\leqslant \tfrac{4\sqrt{2}}{\pi}\frac{b^{2m}}{m^{\frac34-\alpha}}\sin(\tfrac{\alpha\pi}{2})\frac{\Gamma(\frac{\alpha}{2}+\frac14)\Gamma(\frac12-\alpha)}{2^{3/2-\alpha}\Gamma(\frac34-\frac{\alpha}{2})}\\ &\leqslant \tfrac{C_0 b^{2m}}{m^{3/4-\alpha}}\alpha,\end{aligned} \tag{102}$$

with C_0 an absolute constant. On the other hand, by applying the right-hand side inequality in (100) we infer

$$\forall m \geqslant 1, \forall \alpha \in (0,1), \quad \mathtt{W}_\alpha(m) \leqslant 2m^{\alpha-1}. \tag{103}$$

Combining this inequality with (90) we deduce that

$$\begin{aligned} \forall m \geqslant 1, \forall \alpha \in (0,1), \quad \mathtt{W}_\alpha(m+1) - \mathtt{W}_\alpha(m) &\leqslant -\frac{1-\alpha}{2(1+m-\frac{\alpha}{2})m^{1-\alpha}} \\ &\leqslant -\frac{1}{8m^{2-\alpha}}\cdot \end{aligned} \tag{104}$$

Putting together (98) with (102), (103) and (104), allows us to get for any $m \geqslant 1, \alpha \in (0, \frac{1}{3})$,

$$\alpha_{m+1,b} - \alpha_{m,b} \leqslant -C_1 m^{\alpha-2} + C_2 b^{2m},$$

for some absolute constants $C_1, C_2 > 0$. Therefore, for fixed $b \in (0,1)$ we can find $\overline{m} = \overline{m}(b) \in \mathbb{N}$ such that

$$\forall\, \alpha \in (0, \tfrac{1}{3}),\ \forall m \geqslant \overline{m}, \quad \alpha_{m+1,b} - \alpha_{m,b} < 0. \tag{105}$$

By virtue of (98) we may write

$$\alpha_{m,b} = \mathtt{V}_m(\alpha) + \alpha_{m,b}^{(22)}, \qquad \mathtt{V}_m(\alpha) \triangleq \frac{2^{\alpha-1}\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\Big(\frac{1}{b^\alpha} - b^{2m}\Big)\mathtt{W}_\alpha(m),$$

which can be decomposed as follows

$$\begin{aligned} \alpha_{m,b} &= \mathtt{V}_m(0) + \big(\mathtt{V}_m(\alpha) - \mathtt{V}_m(0)\big) + \alpha_{m,b}^{(22)} \\ &\triangleq \mathtt{V}_m(0) + \varrho_m(\alpha). \end{aligned} \tag{106}$$

Using the mean value theorem together with (102) yield

$$\forall \alpha \in [0, \tfrac{1}{3}],\ \forall m \in [1, \overline{m}+1], \quad |\varrho_m(\alpha)| \leqslant C_3 \alpha, \tag{107}$$

for some constant $C_3 = C_3(b)$ depending only in b. It is obvious that

$$\mathtt{V}_m(0) = \frac{1-b^{2m}}{2m}.$$

According to [21, Proposition 15], the sequence $\{\mathtt{V}_m(0)\}_{m\geqslant 1}$ is strictly decreasing in m. Then we can find $\varepsilon = \varepsilon(b) > 0$ such that

$$\forall m \in [1, \overline{m}], \quad \mathtt{V}_{m+1}(0) - \mathtt{V}_m(0) < -\varepsilon.$$

Combining this inequality with (106) and (107) implies

$$\forall m \in [1, \overline{m}], \quad \alpha_{m+1,b} - \alpha_{m,b} < -\varepsilon + 2C_3\alpha.$$

By taking $\alpha^* \triangleq \frac{\varepsilon}{2C_3}$, which depends only on b, we get

$$\forall \alpha \in (0, \alpha^*), \forall m \in [1, \overline{m}], \quad \alpha_{m+1,b} - \alpha_{m,b} < 0.$$

It follows from (105) that

$$\forall \alpha \in (0, \alpha^*), \forall m \geqslant 1, \quad \alpha_{m+1,b} - \alpha_{m,b} < 0.$$

This concludes the proof of the monotonicity.

Let us move to the computation of $\lim\limits_{\alpha\to 0} \Omega^\alpha_{m,b}$. Putting together (98), (102) yields

$$\lim_{\alpha\to 0} \alpha_{m,b} = \tfrac{1}{2}\Big(1 - b^{2m}\Big)\mathtt{W}_0(m) = \frac{1-b^{2m}}{2m}\cdot$$

For $-V_1(0)$ given by (84), by virtue of (87), one can expect that

$$\lim_{\alpha\to 0}\big(-V_1(0)\big) = \frac{1}{2} + \frac{2}{\pi}\lim_{\alpha\to 0}\left(\sin(\tfrac{\alpha\pi}{2})\int_0^\infty \rho^{\alpha-1}\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)}\mathrm{d}\rho\right).$$

In light of (91), we observe that

$$0 \leqslant \frac{I_1(bx)}{I_0(x)} \leqslant \tfrac{1}{2}bx, \quad \forall x \geqslant 0.$$

Then we use Remark 2.3 and (26) to get

$$\begin{aligned}\Big|\int_0^\infty \rho^{\alpha-1}\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)}\mathrm{d}\rho\Big| &\leqslant \frac{b}{2}\int_0^\infty \rho^\alpha I_1(b\rho)K_0(\rho)\mathrm{d}\rho \\ &= \frac{b^2\Gamma^2(1+\frac{\alpha}{2})}{2^{2-\alpha}}F\big(1+\tfrac{\alpha}{2}, 1+\tfrac{\alpha}{2}; 2; b^2\big).\end{aligned} \tag{108}$$

Note that the hypergeometric function $F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 2; b^2)$ is a convergent series for every $\alpha \in (0,1)$ and $b \in (0,1)$, and it converges to $F(1,1;2;b^2) = -\frac{\log(1-b^2)}{b^2}$ as $\alpha \to 0$ (e.g. see 9.12 of [31]). Thus we find

$$\lim_{\alpha\to 0}\big(-V_1(0)\big) = \tfrac{1}{2}.$$

Hence, we have $\lim\limits_{\alpha\to 0}\Omega^\alpha_{m,b} = \lim\limits_{\alpha\to 0}\big(-V_1(0) - \alpha_{m,b}\big) = \frac{m-1+b^{2m}}{2m}$ as desired.

Finally, we consider $\lim\limits_{\alpha\to 1}\Omega^\alpha_{m,b}$. In view of (87) and (88), and using the monotone convergence theorem, we have

$$\lim_{\alpha\to 1}\Omega_{m,b}^{\alpha} = \frac{1}{\pi b}\lim_{\alpha\to 1}\Gamma(1-\alpha)\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right) + \frac{2}{\pi}\int_0^{\infty}\left(\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)} + \frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\right)\mathrm{d}\rho.$$

As proved by Lemma 3-(1) in [33], we obtain

$$\frac{1}{\pi b}\lim_{\alpha\to 1}\Gamma(1-\alpha)\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right) = \frac{2}{\pi b}\sum_{k=1}^{m-1}\frac{1}{2k+1}.$$

In addition, using Remark 2.3, (92) and (108), we infer that

$$\begin{aligned}\int_0^{\infty}\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)}\mathrm{d}\rho &\leqslant \int_0^{\infty}\frac{b}{2}\rho I_1(b\rho)K_0(\rho)\mathrm{d}\rho\\ &= \frac{b^2\pi}{8}F\left(\tfrac{3}{2},\tfrac{3}{2};2;b^2\right) < +\infty,\end{aligned}$$

and

$$\begin{aligned}\int_0^{\infty}\frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\mathrm{d}\rho &\leqslant \int_0^{\infty} b^m I_m(b\rho)K_m(\rho)\mathrm{d}\rho\\ &= b^{2m}\frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}F(m+\tfrac{1}{2},\tfrac{1}{2};m+1;b^2)\\ &\leqslant \tfrac{\sqrt{\pi}}{2}F(1,\tfrac{1}{2};1;b^2)b^{2m} < +\infty.\end{aligned}$$

Hence, it follows that

$$\lim_{\alpha\to 1}\Omega_{m,b}^{\alpha} = \frac{2}{\pi b}\sum_{k=1}^{m-1}\frac{1}{2k+1} + \frac{2}{\pi}\int_0^{\infty}\left(\frac{I_1^2(b\rho)K_0(\rho)}{I_0(\rho)} + \frac{I_m^2(b\rho)K_m(\rho)}{I_m(\rho)}\right)\mathrm{d}\rho.$$

This ends the proof the first point **(i)**.

(ii) By virtue of Lemma A.1 in [34] we get

$$\mathtt{W}_m(\alpha) = \tfrac{1}{m^{1-\alpha}} + O\left(\tfrac{1}{m^{3-\alpha}}\right).$$

Then we deduce from (88), (89) and (95) that for any $\alpha, b \in (0,1)$,

$$\alpha_{m,b} = \frac{2^{\alpha-1}\Gamma(1-\alpha)}{b^{\alpha}\Gamma^2(1-\frac{\alpha}{2})}m^{\alpha-1} + O\left(\tfrac{1}{m^{3-\alpha}}\right) + O(b^{2m}).$$

This concludes the proof of (86). □

5.2. Proof of Proposition 5.1

(i) Consider

$$\theta \in \mathbb{R} \mapsto h(\theta) = \sum_{n=1}^{\infty} a_n \cos(nm\theta) \in X_m, \quad \text{with} \quad a_n \in \mathbb{R}, \tag{109}$$

and let us check that

$$\partial_r F(\Omega, 0) h(\theta) = -\sum_{n=1}^{\infty} a_n \big(\Omega - \Omega_{nm,b}^{\alpha}\big) nm \sin(nm\theta). \tag{110}$$

Then the result of statement **(i)** about the kernel structure follows immediately from this description. To proceed with, we apply first (48) with $r = 0$ in (48), leading to

$$\partial_r F(0) h(\theta) = \big[\Omega + V_1(0)(\theta)\big] h'(\theta) + V_2(0)(\theta) h(\theta) + V_3(0, h)(\theta) + V_4(0, h)(\theta). \tag{111}$$

Below we shall analyze the right-hand side terms of (111) one by one. From (49), we have

$$V_1(0)(\theta) = b^{-1} \int_0^{2\pi} \int_0^b \nabla_x K^\alpha \big(be^{i\theta}, \rho e^{i\eta}\big) \cdot e^{i\theta} \rho \, d\rho d\eta. \tag{112}$$

Noting that (owing to (21))

$$\nabla_x K^\alpha \big(be^{i\theta}, \rho_2 e^{i\eta}\big) \cdot e^{i\theta} = \partial_{\rho_1} G\big(b, \theta, \rho_2, \eta\big),$$

which yields in view of (20) and Fubini's theorem

$$\begin{aligned} V_1(0)(\theta) &= b^{-1} \int_0^{2\pi} \int_0^b (\partial_{\rho_1} G)\big(b, \theta, \rho, \eta\big) \rho \, d\rho d\eta \\ &= 2\pi b^{-1} \sum_{k \geqslant 1} x_{0,k}^{\alpha-1} A_{0,k}^2 J_0'\big(bx_{0,k}\big) \int_0^b J_0\big(x_{0,k}\rho\big) \rho d\rho \\ &= -2\pi b^{-1} \sum_{k \geqslant 1} x_{0,k}^{\alpha-1} A_{0,k}^2 J_1\big(bx_{0,k}\big) \int_0^b J_0\big(x_{0,k}\rho\big) \rho d\rho. \end{aligned}$$

In addition, thanks to the identity $\displaystyle\int_0^a t J_0(t) dt = a J_1(a)$ (see e.g. 6.561 of [31]), it follows that

$$\int_0^b J_0\big(x_{0,k}\rho\big)\rho\,\mathrm{d}\rho = \frac{b}{x_{0,k}} J_1\big(x_{0,k}b\big).$$

Hence we find that $V_1(0)(\theta)$ is independent of θ and by virtue of (18),

$$V_1(0) = -2\sum_{k\geqslant 1} x_{0,k}^{\alpha-2}\frac{J_1^2\big(x_{0,k}b\big)}{J_1^2\big(x_{0,k}\big)}\cdot \tag{113}$$

For $V_2(r)(\theta)$ given by (50), we use $\partial_\theta(R(\theta)e^{i\theta})|_{r=0} = ibe^{i\theta}$, together with (45)-(46) and (20) to find

$$\begin{aligned} V_2(0)(\theta) = & b^{-1}\int_0^{2\pi}\int_0^b \big(\nabla_x^2 K_1^\alpha(be^{i\theta},\rho e^{i\eta})\cdot e^{i\theta}\big)\cdot(ibe^{i\theta})\rho\,\mathrm{d}\rho\mathrm{d}\eta \\ & + b^{-1}\int_0^{2\pi}\int_0^b \nabla_x K^\alpha\left(be^{i\theta},\rho e^{i\eta}\right)\cdot(e^{i\theta}i)\rho\,\mathrm{d}\rho\mathrm{d}\eta \\ = & \int_0^{2\pi}\int_0^b \big(\nabla_x^2 K_1^\alpha(be^{i\theta},\rho e^{i\eta})\cdot e^{i\theta}\big)\cdot(ie^{i\theta})\rho\,\mathrm{d}\rho\mathrm{d}\eta + b^{-2}\int_0^{2\pi}\int_0^b \partial_\theta G(b,\theta,\rho,\eta)\rho\,\mathrm{d}\rho\mathrm{d}\eta \\ = & \int_0^{2\pi}\int_0^b \big(\nabla_x^2 K_1^\alpha(be^{i\theta},\rho e^{i\eta})\cdot e^{i\theta}\big)\cdot(ie^{i\theta})\rho\,\mathrm{d}\rho\mathrm{d}\eta. \end{aligned}$$

By virtue of the splitting (37) we may write

$$\nabla_x^2 K_1^\alpha = \nabla_x(\nabla_x K^\alpha - \nabla_x K_0^\alpha) = \nabla_x^2 K^\alpha + \nabla_x\nabla_y K_0^\alpha.$$

Applying Gauss-Green theorem and using straightforward computations we deduce that

$$\begin{aligned} & \int_0^{2\pi}\int_0^b \big(\nabla_y\nabla_x K_0^\alpha(be^{i\theta},\rho e^{i\eta})\cdot(ie^{i\theta})\big)\cdot e^{i\theta}\rho\mathrm{d}\rho\mathrm{d}\eta \\ & = \iint_{b\mathbb{D}} \big(\nabla_y\nabla_x K_0^\alpha(be^{i\theta},y)\cdot(ie^{i\theta})\big)\cdot e^{i\theta}\mathrm{d}y \\ & = \int_0^{2\pi} \big(\nabla_x K_0^\alpha(be^{i\theta},be^{i\eta})(be^{i\eta})\cdot(ie^{i\theta})\big)\cdot e^{i\theta}\mathrm{d}\eta \\ & = b\int_0^{2\pi} \nabla_x K_0^\alpha(be^{i\theta}-be^{i\eta})\cdot e^{i\theta}\sin(\eta-\theta)\mathrm{d}\eta \end{aligned}$$

$$= (-\alpha) c_\alpha b^{-\alpha} \int_0^{2\pi} \frac{1-\cos(\eta-\theta)}{|1-e^{i(\eta-\theta)}|^{\alpha+2}} \sin(\eta-\theta) \mathrm{d}\eta = 0.$$

Next, we intend to compute the following integral

$$\int_0^{2\pi} \int_0^b \left(\nabla_x^2 K^\alpha\big(be^{i\theta}, \rho e^{i\eta}\big) \cdot e^{i\theta}\right) \cdot (ie^{i\theta}) \rho \, \mathrm{d}\rho \mathrm{d}\eta.$$

Direct computations based on (21) give that

$$\begin{aligned}
\nabla_y K^\alpha\big(be^{i\theta}, \rho e^{i\eta}\big) \cdot (ie^{i\theta}) &= \sin(\eta-\theta)\partial_{\rho_2} G + \rho^{-1}\cos(\eta-\theta)\partial_\eta G, \\
\nabla_x K^\alpha\big(be^{i\theta}, \rho e^{i\eta}\big) \cdot e^{i\eta} &= \cos(\eta-\theta)\partial_{\rho_1} G + \sin(\eta-\theta) b^{-1}\partial_\theta G, \\
\nabla_y K^\alpha\big(be^{i\theta}, \rho e^{i\eta}\big) \cdot e^{i\eta} &= \partial_{\rho_2} G(b, \theta, \rho, \eta),
\end{aligned}$$

and

$$\Big(\nabla_x^2 K^\alpha (be^{i\theta}, \rho e^{i\eta}) \cdot e^{i\theta}\Big) \cdot (ie^{i\theta}) = b^{-1}\partial_{\rho_1}\partial_\theta G(b,\theta,\rho,\eta) - b^{-2}\partial_\theta G(b,\theta,\rho,\eta).$$

Thus, we infer from (20) and Fubini's theorem

$$\begin{aligned}
V_2(0)(\theta) &= \int_0^{2\pi}\int_0^b \left(\nabla_x^2 K^\alpha (be^{i\theta}, \rho e^{i\eta}) \cdot e^{i\theta}\right) \cdot (ie^{i\theta}) \rho \mathrm{d}\rho \mathrm{d}\eta \\
&= \int_0^{2\pi}\int_0^b \Big(\rho b^{-1}\partial_{\rho_1}\partial_\theta G(b,\theta,\rho,\eta) - \rho b^{-2}\partial_\theta G(b,\theta,\rho,\eta)\Big) \mathrm{d}\rho \mathrm{d}\eta \\
&= \int_0^{2\pi}\int_0^b \Bigg(\sum_{\substack{n\in\mathbb{N} \\ k\geqslant 1}} x_{n,k}^{\alpha-2} A_{n,k}^2 \rho J_n(x_{n,k}\rho) \Big(b^{-1} x_{n,k} J_n'(x_{n,k} b) \\
&\quad - b^{-2} J_n(x_{n,k} b)\Big) n \sin n(\eta-\theta)\Bigg) \mathrm{d}\rho \mathrm{d}\eta \\
&= 0. \qquad (114)
\end{aligned}$$

The next task is to evaluate the contributions induced by the nonlocal operators V_3 and V_4. The computation will be done in a formal way and can be justified by standard approximation arguments that we will omit here. For $V_3(r,h)$ given by (51), it is easy to check

$$L_A(h)|_{r=0} = -h'(\eta)\cos(\eta-\theta) + h(\eta)\sin(\eta-\theta) = -\partial_\eta\big(h(\eta)\cos(\eta-\theta)\big),$$

leading to

$$V_3(0,h)(\theta)=\int_0^{2\pi}\nabla_x K_1^\alpha(be^{i\theta},be^{i\eta})\cdot(ibe^{i\theta})h(\eta)\mathrm{d}\eta-\int_0^{2\pi}K_0^\alpha(be^{i\theta}-be^{i\eta})L_A(h)|_{r=0}(\eta)\mathrm{d}\eta.$$

Due to that

$$K_0^\alpha(be^{i\theta}-be^{i\eta})=K_0^\alpha(b-be^{i(\eta-\theta)}),$$

and $|1-e^{i\eta}|=\left|2\sin\frac{\eta}{2}\right|$, we use the integration by parts to infer that

$$\begin{aligned}
&\int_0^{2\pi}K_0^\alpha(be^{i\theta}-be^{i\eta})L_A(h)|_{r=0}(\eta)\mathrm{d}\eta\\
&=-\sum_{n\geqslant 1}a_n\int_0^{2\pi}K_0^\alpha(b-be^{i(\eta-\theta)})\partial_\eta\big(\cos(nm\eta)\cos(\eta-\theta)\big)\mathrm{d}\eta\\
&=-\sum_{n\geqslant 1}a_n\int_0^{2\pi}K_0^\alpha(b-be^{i\eta})\partial_\eta\big(\cos(nm\eta+nm\theta)\cos\eta\big)\mathrm{d}\eta\\
&=\sum_{n\geqslant 1}a_n\int_0^{2\pi}\partial_\eta\Big(\frac{c_\alpha}{b^\alpha(2\sin\frac{\eta}{2})^\alpha}\Big)\cos(nm\eta+nm\theta)\cos\eta\,\mathrm{d}\eta\\
&=-\sum_{n\geqslant 1}a_n\frac{\alpha c_\alpha}{b^\alpha}\int_0^{2\pi}\frac{\cos(nm\theta)\cos(nm\eta)-\sin(nm\theta)\sin(nm\eta)}{|2\sin\frac{\eta}{2}|^{\alpha+2}}\cos\eta\sin\eta\,\mathrm{d}\eta\\
&=\sum_{n\geqslant 1}a_n\frac{\alpha c_\alpha}{b^\alpha}\sin(nm\theta)\int_0^{2\pi}\frac{\sin(nm\eta)\cos\eta\sin\eta}{(2\sin\frac{\eta}{2})^{\alpha+2}}\mathrm{d}\eta.
\end{aligned}$$

Note that $\nabla_x K_0^\alpha(x-y)=-\alpha c_\alpha\frac{x-y}{|x-y|^{\alpha+2}}$, and for $e_{nm}(\theta)\triangleq e^{inm\theta}$, we have

$$\begin{aligned}
&\int_0^{2\pi}\nabla_x K_0^\alpha(be^{i\theta}-be^{i\eta})\cdot(ibe^{i\theta})h(\eta)\mathrm{d}\eta\\
&=\sum_{n\geqslant 1}a_n\,\mathrm{Re}\left(\int_0^{2\pi}\nabla_x K_0^\alpha(be^{i\theta}-be^{i\eta})\cdot(ibe^{i\theta})e_{nm}(\eta)\mathrm{d}\eta\right)\\
&=\sum_{n\geqslant 1}a_n\,\mathrm{Re}\left(\frac{\alpha c_\alpha}{b^\alpha}\int_0^{2\pi}\frac{(e^{i\eta}-e^{i\theta})\cdot(ie^{i\theta})}{|1-e^{i(\eta-\theta)}|^{\alpha+2}}e_{nm}(\eta)\mathrm{d}\eta\right)
\end{aligned}$$

$$= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\frac{\alpha c_\alpha}{b^\alpha} \int_0^{2\pi} \frac{\sin(\eta-\theta)}{|1-e^{i(\eta-\theta)}|^{\alpha+2}} e_{nm}(\eta) \mathrm{d}\eta \right).$$

By a change of variables and applying the orthogonality of trigonometric functions, we further deduce that

$$\begin{aligned}
&\int_0^{2\pi} \nabla_x K_0^\alpha (be^{i\theta} - be^{i\eta}) \cdot (ibe^{i\theta}) h(\eta) \mathrm{d}\eta \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\frac{\alpha c_\alpha}{b^\alpha} e_{nm}(\theta) \int_0^{2\pi} \frac{\sin \eta}{|1-e^{i\eta}|^{\alpha+2}} e_{nm}(\eta) \mathrm{d}\eta \right) \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(i \frac{\alpha c_\alpha}{b^\alpha} e_{nm}(\theta) \int_0^{2\pi} \frac{\sin \eta}{|1-e^{i\eta}|^{\alpha+2}} \sin(nm\eta) \mathrm{d}\eta \right) \\
&= -\sum_{n\geqslant 1} a_n \frac{\alpha c_\alpha}{b^\alpha} \sin(nm\theta) \int_0^{2\pi} \frac{\sin \eta \, \sin(nm\eta)}{|1-e^{i\eta}|^{\alpha+2}} \mathrm{d}\eta.
\end{aligned}$$

Using (18), (20) and (46), one can see that

$$\begin{aligned}
&\int_0^{2\pi} \nabla_x K^\alpha (be^{i\theta}, be^{i\eta}) \cdot \big(ibe^{i\theta}\big) h(\eta) \mathrm{d}\eta \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\int_0^{2\pi} \nabla_x K^\alpha (be^{i\theta}, be^{i\eta}) \cdot \big(ibe^{i\theta}\big) e_{nm}(\eta) \mathrm{d}\eta \right) \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\int_0^{2\pi} \partial_\theta G(b,\theta,b,\eta) e_{nm}(\eta) \mathrm{d}\eta \right) \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\Bigg(\sum_{\substack{\ell\in\mathbb{N}\\ k\geqslant 1}} x_{\ell,k}^{\alpha-2} A_{\ell,k}^2 J_\ell^2(x_{\ell,k} b) \int_0^{2\pi} \ell \sin \ell(\eta-\theta) e^{inm\eta} \mathrm{d}\eta \Bigg).
\end{aligned}$$

Consequently, we find

$$\begin{aligned}
&\int_0^{2\pi} \nabla_x K^\alpha (be^{i\theta}, be^{i\eta}) \cdot \big(ibe^{i\theta}\big) h(\eta) \mathrm{d}\eta \\
&= \sum_{n\geqslant 1} a_n \operatorname{Re}\Bigg(e_{nm}(\theta) \sum_{\substack{\ell\in\mathbb{N}\\ k\geqslant 1}} x_{\ell,k}^{\alpha-2} A_{\ell,k}^2 J_\ell^2(x_{\ell,k} b) \int_0^{2\pi} \ell \sin(\ell\eta) e^{inm\eta} \mathrm{d}\eta \Bigg)
\end{aligned}$$

$$= -\sum_{n\geqslant 1} a_n \, nm \, \alpha_{nm,b} \, \sin(nm\theta),$$

with

$$\alpha_{m,b} \triangleq 2\sum_{k\geqslant 1} x_{m,k}^{\alpha-2} \frac{J_m^2(x_{m,k}b)}{J_{m+1}^2(x_{m,k})}. \tag{115}$$

Hence, gathering the preceding identities yields

$$V_3(0,h)(\theta) = -\sum_{n\geqslant 1} a_n \left(nm \, \alpha_{nm,b} - \frac{\alpha c_\alpha}{b^\alpha} \int_0^{2\pi} \frac{\sin(nm\eta)(1-\cos\eta)\sin\eta}{|1-e^{i\eta}|^{\alpha+2}} \mathrm{d}\eta \right) \sin(nm\theta). \tag{116}$$

For $V_4(r,h)$ given by (52), when $r = 0$, one has

$$\nabla_x K_0^\alpha(be^{i\theta} - be^{i\eta}) \cdot e^{i\theta} = -\frac{c_\alpha \alpha}{b^{\alpha+1}} \frac{1-\cos(\eta-\theta)}{|1-e^{i(\eta-\theta)}|^{\alpha+2}} = -\nabla_x K_0^\alpha(be^{i\theta} - be^{i\eta}) \cdot e^{i\eta},$$

leading to

$$\begin{aligned} V_4(0,h) &= -\, b \int_0^{2\pi} \big(\nabla_x K_0^\alpha(be^{i\theta}, be^{i\eta}) \cdot \big(e^{i\theta} h(\theta) - e^{i\eta} h(\eta)\big)\big) \sin(\eta-\theta) \mathrm{d}\eta \\ &= \frac{c_\alpha \alpha}{b^\alpha} \int_0^{2\pi} \left(\frac{1-\cos(\eta-\theta)}{|1-e^{i(\eta-\theta)}|^{\alpha+2}} \big(h(\theta)+h(\eta)\big) \right) \sin(\eta-\theta) \mathrm{d}\eta. \end{aligned}$$

Recalling the expression of h in (109), we infer that

$$\begin{aligned} V_4(0,h) &= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\frac{c_\alpha \alpha}{b^\alpha} \int_0^{2\pi} \left(\frac{1-\cos(\eta-\theta)}{|1-e^{i(\eta-\theta)}|^{\alpha+2}} (e_{nm}(\theta) + e_{nm}(\eta)) \right) \sin(\eta-\theta) \mathrm{d}\eta \right) \\ &= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(\frac{c_\alpha \alpha}{b^\alpha} e_{nm}(\theta) \int_0^{2\pi} \left(\frac{1-\cos\eta}{|1-e^{i\eta}|^{\alpha+2}} (1 + e_{nm}(\eta)) \right) \sin\eta \, \mathrm{d}\eta \right) \\ &= \sum_{n\geqslant 1} a_n \operatorname{Re}\left(i \frac{c_\alpha \alpha}{b^\alpha} e_{nm}(\theta) \int_0^{2\pi} \frac{1-\cos\eta}{|1-e^{i\eta}|^{\alpha+2}} \sin(nm\eta) \sin\eta \, \mathrm{d}\eta \right) \\ &= -\sum_{n\geqslant 1} a_n \frac{\alpha c_\alpha}{b^\alpha} \sin(nm\theta) \int_0^{2\pi} \frac{1-\cos\eta}{|1-e^{i\eta}|^{\alpha+2}} \sin(nm\eta) \sin\eta \, \mathrm{d}\eta. \end{aligned} \tag{117}$$

Putting together (116) and (117) allows to get

$$V_3(0,h)+V_4(0,h)=-\sum_{n\geqslant 1}a_n\,\alpha_{nm,b}\,nm\sin(nm\theta). \tag{118}$$

Consequently, collecting equalities (111), (113), (114), (118), we obtain

$$\partial_r F(\Omega,0)h(\theta)=-\sum_{n=1}^{\infty}a_n\big(\Omega+V_1(0)+\alpha_{nm,b}\big)nm\sin(nm\theta). \tag{119}$$

In light of Lemma 5.1-(i), the map $m\mapsto\Omega_{m,b}^{\alpha}=-V_1(0)-\alpha_{m,b}$ is strictly increasing in the considered cases. Hence, the kernel of $\partial_r F(\Omega,0)$ is nontrivial if and only if there exists $\ell\in\mathbb{N}^+$ such that

$$\Omega=-V_1(0)-\alpha_{\ell m,b}=\Omega_{\ell m,b}^{\alpha}.$$

Moreover, the kernel of $\partial_r F(\Omega_{\ell m,b}^{\alpha},0)$ is one-dimensional vector space generated by the function $\theta\mapsto\cos(\ell m\theta)$, as desired.

(ii) Now we intend to show that for any $m,\ell\geqslant 1$ the range $R(\partial_r F(\Omega_{\ell m,b}^{\alpha},0))$ coincides with the subspace

$$Z_{\ell m}\triangleq\Big\{f\in C^{1-\alpha}(\mathbb{T}):\ f(\theta)=\sum_{\substack{n\geqslant 1\\ n\neq\ell}}b_n\sin(nm\theta),\ b_n\in\mathbb{R},\ \theta\in\mathbb{T}\Big\}. \tag{120}$$

Note that this sub-space is closed and of co-dimension one in the ambient space Y_m. In addition, one can easily deduce from (110) the trivial inclusion $R(\partial_r F(\Omega_{\ell m,b}^{\alpha},0))\subset Z_{\ell m}$, and therefore it remains to show the converse. For this purpose, let $f\in Z_{\ell m}$, we shall try to find a pre-image $h\in X_m$ satisfying $\partial_r F(\Omega_{\ell m,b}^{\alpha},0)(h)=f$. From the relation (119), it reduces to

$$a_n\big(\Omega_{\ell m,b}^{\alpha}-\Omega_{nm,b}^{\alpha}\big)nm=b_n,\quad \forall n\geqslant 1, n\neq\ell.$$

This uniquely determines the sequence $\{a_n\}_{n\geqslant 1,n\neq\ell}$ with

$$a_n=\frac{b_n}{nm(\Omega_{\ell m,b}^{\alpha}-\Omega_{nm,b}^{\alpha})},\quad \forall n\geqslant 1, n\neq\ell.$$

However, the coefficient a_ℓ is free and we can take it to be zero. Then, for $\theta\mapsto f(\theta)=\sum\limits_{\substack{n=1\\ n\neq\ell}}^{\infty}b_n\sin(nm\theta)\in Y_m$, in order to show $h\in X_m$, we only need to prove that

$$\theta\mapsto\sum_{\substack{n=1\\ n\neq\ell}}^{\infty}\frac{b_n}{nm(\alpha_{\ell m,b}-\alpha_{nm,b})}\cos(nm\theta)\in C^{2-\alpha}(\mathbb{T}),$$

or equivalently

$$\theta \mapsto \sum_{n\geqslant \ell+1}^{\infty} \frac{b_n}{n(\alpha_{\ell m,b}-\alpha_{nm,b})}\cos(nm\theta) \in C^{2-\alpha}(\mathbb{T}).$$

We shall skip the proof of this point because it is quite similar to that of Proposition 8-(2) in [33]. We use essentially the same arguments together with the asymptotic structure (86).

(iii) According to the continuity property of the second derivative $\partial_\Omega\partial_r F$, the transversality assumption reduces to

$$\partial_\Omega\partial_r F(\Omega,0)(h)\Big|_{\Omega=\Omega^\alpha_{\ell m,b},h=\cos(\ell m\theta)} \notin R(\partial_r F(\Omega^\alpha_{\ell m,b},0)).$$

This is indeed obvious by virtue of (120), due to that

$$\partial_\Omega\partial_r F(\Omega^\alpha_{\ell m,b},0)\cos(\ell m\theta) = -\ell m\sin(\ell m\theta) \notin R(\partial_r F(\Omega^\alpha_{\ell m,b},0)) = Z_{\ell m}.$$

This achieves the proof of Proposition 5.1.

6. Appendix

We intend to recall some tools used in the paper and discuss the proofs of some results established before. The first result concerns the classical Crandall-Rabinowitz theorem on the bifurcation from simple eigenvalues which can be stated as follows, see [18].

Theorem 6.1 *(Crandall-Rabinowitz theorem). Let X and Y be two Banach spaces, V a neighborhood of 0 in X and let $F:\mathbb{R}\times V\to Y$ be with the following properties:*

(i) $F(\lambda,0)=0$ *for any* $\lambda\in\mathbb{R}$.
(ii) *The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous.*
(iii) $N(\mathcal{L}_0)$ *and* $Y/R(\mathcal{L}_0)$ *are one-dimensional.*
(iv) *Transversality assumption:* $F_{tx}(0,0)x_0 \notin R(\mathcal{L}_0)$, *where*

$$N(\mathcal{L}_0) = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z is any complement of $N(\mathcal{L}_0)$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R}\times X$, an interval $(-a,a)$, and continuous functions $\varphi:(-a,a)\to\mathbb{R}$, $\psi:(-a,a)\to Z$ such that $\varphi(0)=0$, $\psi(0)=0$ and

$$F^{-1}(0)\cap U = \Big\{\big(\varphi(\xi),\xi x_0+\xi\psi(\xi)\big)\;:\;|\xi|<a\Big\}\cup\Big\{(\lambda,0)\;:\;(\lambda,0)\in U\Big\}.$$

The next result deals with the asymptotic growth of the normalized eigenfunctions to the spectral Laplacian in bounded smooth domains. The proof can be deduced from the standard elliptic estimates (see for instance Section 6.3 of [24]) and we here omit the details.

Lemma 6.1. *Let $\mathbf{D} \subset \mathbb{R}^d$ be a bounded smooth domain and $\{\phi_j, j \geqslant 1\}$ be the orthonormal basis of $L^2(\mathbf{D})$ of eigenfunctions of $-\Delta_{\mathbf{D}}$ satisfying the constraints (8). Then we have*

$$\|\phi_j\|_{H^{2n}(\mathbf{D})} \leqslant C_n(1+\lambda_j)^n, \quad \forall n \in \mathbb{N},$$

where $C_n > 0$ is a constant independent of j but may depend on n.

The current purpose is to prove Lemma 2.1 following the ideas developed in estimating (36) of Constantin and Ignatova [10].

Proof of Lemma 2.1. We take two points $(\bar{x}, y) \in \mathbf{D} \times \mathbf{D}$, and we consider $x \in B(\bar{x}, \frac{\delta}{2})$, where $\delta > 0$ is defined as

$$\delta := \begin{cases} \frac{d(\bar{x})}{8}, & \text{if } |\bar{x} - y| > \frac{d(\bar{x})}{4}, \\ \frac{d(\bar{x})}{2}, & \text{if } |\bar{x} - y| \leqslant \frac{d(\bar{x})}{4}. \end{cases}$$

Fix $(\bar{x}, y)$ and take the function $(t, z) \mapsto h(t, z) = H_{\mathbf{D}}(t, z, y)$, with $H_{\mathbf{D}}$ the heat kernel defined by (10). Now, we apply Green's formula on the domain $U = B(\bar{x}, \delta) \times (0, t)$ to obtain (denote $\mathbf{n}$ as the outer normal vector of $\partial B(\bar{x}, \delta)$)

$$\begin{aligned} 0 &= \int_U \Big[((\partial_s - \Delta_z) h(s,z))\, G_{t-s}(x-z) + h(s,z)(\partial_s + \Delta_z) G_{t-s}(x-z) \Big] \mathrm{d}z \mathrm{d}s \\ &= h(t,x) - G_t(x-y) \\ &\quad + \int_0^t \int_{\partial B(\bar{x},\delta)} \left[\frac{\partial G_{t-s}(x-z)}{\partial \mathbf{n}} h(s,z) - \frac{\partial h(s,z)}{\partial \mathbf{n}} G_{t-s}(x-z) \right] \mathrm{d}\sigma(z) \mathrm{d}s, \end{aligned}$$

which leads to

$$\begin{aligned} H_{\mathbf{D}}(t,x,y) &= G_t(x-y) \\ &\quad - \int_0^t \int_{\partial B(\bar{x},\delta)} \left[\frac{\partial G_{t-s}(x-z)}{\partial \mathbf{n}} h(s,z) - \frac{\partial h(s,z)}{\partial \mathbf{n}} G_{t-s}(x-z) \right] \mathrm{d}\sigma(z) \mathrm{d}s. \end{aligned}$$

By differentiating n-times in x in the above formula, and using the upper bounds (13)-(14), we have that for every $0 < t \leqslant T_0$,

$$\begin{aligned} &|\nabla_x^n H_{\mathbf{D}}(t,x,y) - \nabla_x^n G_t(x-y)| \\ &\leqslant C \int_0^t \int_{\partial B(\bar{x},\delta)} (t-s)^{-\frac{d+n+1}{2}} p_{n+1}\Big(\tfrac{|x-z|}{\sqrt{t-s}}\Big) e^{-\frac{|x-z|^2}{4(t-s)}} s^{-\frac{d}{2}} e^{-\frac{|y-z|^2}{Cs}} \mathrm{d}z \mathrm{d}s \end{aligned}$$

$$+C\int_0^{\min\{t,d(y)^2\}}\int_{\partial B(\bar{x},\delta)}(t-s)^{-\frac{d+n}{2}}p_n\Big(\tfrac{|x-z|}{\sqrt{t-s}}\Big)e^{-\frac{|x-y|^2}{4(t-s)}}s^{-\frac{d+1}{2}}p_1\Big(\tfrac{|y-z|}{\sqrt{s}}\Big)e^{-\frac{|y-z|^2}{Cs}}\mathrm{d}z\mathrm{d}s$$

$$+C\int_{\min\{t,d(y)^2\}}^{t}\int_{\partial B(\bar{x},\delta)}(t-s)^{-\frac{d+n}{2}}p_n\Big(\tfrac{|x-z|}{\sqrt{t-s}}\Big)e^{-\frac{|x-y|^2}{4(t-s)}}s^{-\frac{d}{2}}\tfrac{1}{d(y)}\tfrac{\phi_1(y)}{|y-z|}e^{-\frac{|y-z|^2}{Cs}}\mathrm{d}z\mathrm{d}s,$$

where $p_k(\xi)$ are polynomials of degree k. These integrals are all nonsingular. Note that since in the first integral of the right hand side one has the restriction $|x-z|\geqslant\frac{\delta}{2}$, then by elementary arguments we deduce that

$$\begin{aligned}(t-s)^{-\frac{k}{2}}p_l\big(\tfrac{|x-z|}{\sqrt{t-s}}\big)&\leqslant C|x-z|^{-k}e^{\frac{|x-z|^2}{8(t-s)}}\\&\leqslant C\delta^{-k}e^{\frac{|x-z|^2}{8(t-s)}}.\end{aligned}$$

Similarly, by noting that

$$|y-z|\geqslant\begin{cases}|\bar{x}-y|-\delta\geqslant\delta, & \text{if } |\bar{x}-y|>\frac{d(\bar{x})}{4},\\ |\bar{x}-z|-|\bar{x}-y|\geqslant\frac{\delta}{2}, & \text{if } |\bar{x}-y|\leqslant\frac{d(\bar{x})}{4},\end{cases}$$

we obtain

$$\begin{aligned}s^{-\frac{k}{2}}p_l\big(\tfrac{|y-z|}{\sqrt{s}}\big)&\leqslant C|y-z|^{-k}e^{\frac{|y-z|^2}{2Cs}}\\&\leqslant C\delta^{-k}e^{\frac{|y-z|^2}{2Cs}}.\end{aligned}$$

On the other hand, it is known that the first eigenfunction satisfies

$$0\leqslant\phi_1(y)\leqslant C_0d(y),$$

with $C_0>0$ a constant depending on $\mathbf{D}$, and it gives

$$\tfrac{\phi_1(y)}{|y-z|d(y)}\leqslant C\delta^{-1}.$$

From the inequality

$$\tfrac{|x-y|^2}{t}\leqslant 2\big(\tfrac{|x-z|^2}{t-s}+\tfrac{|y-z|^2}{s}\big),$$

we see that

$$e^{-\frac{|x-z|^2}{8(t-s)}-\frac{|y-z|^2}{2Cs}}\leqslant e^{-\frac{|x-y|^2}{\widetilde{C}t}},\quad \widetilde{C}\triangleq 16+4C.$$

Putting together the above estimates yields for every $0<t\leqslant\min\{T_0,d(x)^2\}$,

$$|\nabla_x^n H_{\mathbf{D}}(x,y,t) - \nabla_x^n G_t(x-y)| \leqslant Ce^{-\frac{|x-y|^2}{\widetilde{C}t}} t\delta^{-d-n-2} \leqslant Ct^{-\frac{d+n}{2}} e^{-\frac{|x-y|^2}{\widetilde{C}t}},$$

where in the last inequality we have used the fact that $\delta \approx d(\bar{x}) \approx d(x)$. Thus the desired estimate follows from the estimate of the heat kernel in the full plane G_t, which is easy to check. □

The next goal is to establish the proof of Lemma 2.2.

Proof of Lemma 2.2. We borrow the idea from the treatment of (171) in [10]. For x and y fixed, there exists an open domain $\mathbf{D}_0 \subset \overline{\mathbf{D}_0} \subset \mathbf{D}$ such that $x, y \in \mathbf{D}_0$. Denote by $d_0 \triangleq \min\{\sqrt{T_0}, d(\mathbf{D}_0, \partial\mathbf{D})\} > 0$. Let $\chi \in C^\infty$ be a cutoff function such that $\chi \equiv 1$ on $\{z | d(z, \mathbf{D}_0) \leqslant \frac{1}{3}d_0\}$ and $\chi \equiv 0$ on $\{z | d(z, \mathbf{D}_0) \geqslant \frac{1}{2}d_0\}$. Observe that $\widetilde{h}(t,z) = \chi(z)G_t(z-y)$ solves the following equations

$$\begin{aligned}
(\partial_t - \Delta)\widetilde{h}(t,z) &= -\big[(\Delta\chi(z))G_t(z-y) + 2(\nabla\chi(z))\cdot\nabla G_t(z-y)\big] \triangleq F(t,z,y),\\
\widetilde{h}(0,z) &= \chi(z)\delta(z-y),\\
\widetilde{h}(t,z)|_{\partial\mathbf{D}} &= 0,
\end{aligned}$$

with $\delta(\cdot)$ the Dirac mass centered at 0. Thus Duhamel's formula gives

$$\widetilde{h}(t,z) = e^{t\Delta}\widetilde{h}(0,z) + \int_0^t e^{(t-s)\Delta} F(s,z,y)\mathrm{d}s,$$

which, in combination with $(e^{t\Delta}f)(z) = \int_{\mathbf{D}} H_{\mathbf{D}}(t,z,w)f(w)\mathrm{d}w$ yields

$$\chi(z)G_t(z-y) = \chi(y)H_{\mathbf{D}}(t,z,y) + \int_0^t \int_{\mathbf{D}} H_{\mathbf{D}}(t-s,z,w)F(s,w,y)\mathrm{d}w\mathrm{d}s$$

for all $z \in \mathbf{D}$. Noting that $\chi(x) = \chi(y) = 1$, and setting $z = x$, we obtain

$$\begin{aligned}
H_{\mathbf{D}}(t,x,y) - G_t(x-y) = \int_0^t \int_{\mathbf{D}} & H_{\mathbf{D}}(t-s,x,w)\big[\Delta\chi(w)G_s(w-y) \\
&+ 2\nabla\chi(w)\cdot\nabla G_s(w-y)\big]\mathrm{d}w\mathrm{d}s. \qquad (121)
\end{aligned}$$

The integral on the right-hand side is not singular and indeed C^∞-smooth. In fact, notice that the fact $\operatorname{supp}\nabla\chi \subset \{w | \frac{1}{3}d_0 \leqslant d(w, \mathbf{D}_0) < \frac{1}{2}d_0\}$ ensures $\frac{1}{3}d_0 \leqslant |x-w|, |y-w| \leqslant C$, and by using the Gaussian upper bounds combined with Lemma 2.1, we have that for every $0 \leqslant t \leqslant d_0^2$,

$$\left|\nabla_x^k \nabla_y^l \Big(\int_0^t \int_{\mathbf{D}} H_{\mathbf{D}}(t-s,x,w) F(s,w,y)\, \mathrm{d}w \mathrm{d}s \Big)\right|$$

$$\leqslant C \int_0^t \int_{\frac{d_0}{3} \leqslant d(w,\mathbf{D}_0) < \frac{d_0}{2}} (t-s)^{-\frac{k+d}{2}} e^{-\frac{|x-w|^2}{C(t-s)}} \Big(|\nabla_y^l G_s(w-y)| + |\nabla_y^{l+1} G_s(w-y)| \Big) \mathrm{d}w \mathrm{d}s \leqslant C,$$

as desired. □

Data availability

No data was used for the research described in the article.

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