ERRATUM: GLOBAL REGULARITY OF NON-DIFFUSIVE TEMPERATURE FRONTS FOR THE 2D VISCOUS BOUSSINESQ SYSTEM*

DONGHO CHAE†, QIANYUN MIAO‡, AND LIUTANG XUE§

Abstract. There is an error of misusing a commutator formula in the proof of Lemma 2.7 in [*SIAM J. Math. Anal.*, 54 (2022), 4043-4103] so that this lemma is not correct. However, by establishing a weaker version of this commutator lemma, the main results including Theorem 1.1 in [*SIAM J. Math. Anal.*, 54 (2022), 4043-4103] still hold true. In addition, we confirm the validity of Lemma 2.4-(3) and Lemma 2.5-(3) in [*SIAM J. Math. Anal.*, 54 (2022), 4043-4103] although similar error occurs in the proof.

Key words. Boussinesq system, temperature patch problem, commutator estimate, striated estimates.

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In [1], there is an error of misusing a commutator formula in the proof of Lemma 2.7 so that the commutator estimate (2.23) is not valid. But we instead can show a weaker result as follows, which is sufficient for our use.

LEMMA 0.1. *Let $s \in (0,1)$, $p \in [2,\infty]$, $r \in [1,\infty]$. Let $\mathcal{R}_{-1} := m(D)\Lambda^{-1}$, $\Lambda = (-\Delta)^{1/2}$ and $m(D)$ be a zero-order pseudo-differential operator with its symbol $m(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\})$. Assume that $u = (u_1, \cdots, u_d)$ is a smooth divergence-free vector field and ϕ is a smooth scalar function. Then we have*

$$\|[\mathcal{R}_{-1}, u\cdot\nabla]\phi\|_{B^s_{p,r}} \leq C\|\nabla u\|_{L^p}\big(\|\phi\|_{B^{s-1}_{\infty,r}} + \|\phi\|_{L^2}\big) + C\|u\|_{L^2}\|\phi\|_{L^2}, \tag{0.1}$$

where $C > 0$ is a constant depending only on s, p and d.

Proof. Bony's decomposition gives

$$\begin{aligned}
&[\mathcal{R}_{-1}, u\cdot\nabla]\phi \\
&= \sum_{q\in\mathbb{N}}[\mathcal{R}_{-1}, S_{q-1}u\cdot\nabla]\Delta_q\phi + \sum_{q\in\mathbb{N}}[\mathcal{R}_{-1}, \Delta_q u\cdot\nabla]S_{q-1}\phi + \sum_{q\geq -1}[\mathcal{R}_{-1}, \Delta_q u\cdot\nabla]\widetilde{\Delta}_q\phi \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where we have adopted the standard notations in the Littlewood-Paley theory (see [1, Section 2]). For I, there exists a bump function $\widetilde{\psi} \in C_c^\infty(\mathbb{R})$ supported on an annulus of $\mathbb{R}^d$ away from zero such that

$$\mathrm{I} = \sum_{q\in\mathbb{N}}[\mathcal{R}_{-1}\widetilde{\psi}(2^{-q}D), S_{q-1}u\cdot\nabla]\Delta_q\phi \tag{0.2}$$

with $\mathcal{R}_{-1}\widetilde{\psi}(2^{-q}D) = 2^{q(d-1)}\overline{h}(2^q\cdot)*$ and $\overline{h} \in \mathcal{S}(\mathbb{R}^d)$. Thus by using Minkowski's

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†Department of Mathematics, Chung-Ang University, Dongjak-gu, Heukseok-ro 84, Seoul 06974, Republic of Korea (dchae@cau.ac.kr).

‡School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China (qianyunm@bit.edu.cn).

§Laboratory of Mathematics and Complex Systems (MOE), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R. China (xuelt@bnu.edu.cn).

inequality we find that for every $j \geq -1$,

$$
\begin{aligned}
2^{js}\|\Delta_j \mathrm{I}\|_{L^p} &\leq C 2^{js} \sum_{q\in\mathbb{N},|q-j|\leq 4} \|[\mathcal{R}_{-1}\widetilde{\psi}(2^{-q}D), S_{q-1}u\cdot\nabla]\Delta_q\phi\|_{L^p} \\
&\leq C 2^{js} \sum_{|q-j|\leq 4} \int_{R^d} 2^{q(d-1)}|\overline{h}(2^q y)||y|\mathrm{d}y\, \|\nabla S_{q-1}u\|_{L^p}\|\nabla\Delta_q\phi\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^p} \sum_{|q-j|\leq 4} 2^{q(s-1)}\|\Delta_q\phi\|_{L^\infty},
\end{aligned}
$$

which ensures that

$$
\|\mathrm{I}\|_{B^s_{p,r}} \leq C\|\nabla u\|_{L^p}\|\phi\|_{B^{s-1}_{\infty,r}}.
$$

The error in [1, Lemma 2.7] appears in the treating of II, where it was thought a similar formula (0.2) holds for II, but indeed it is not correct. Instead, for II, noting that

$$
\mathrm{II} = \sum_{q\in\mathbb{N}} \Big(\mathcal{R}_{-1}\widetilde{\psi}(2^{-q}D)\big(\Delta_q u\cdot\nabla S_{q-1}\phi\big) - \Delta_q u\cdot\nabla\mathcal{R}_{-1}S_{q-1}\phi\Big),
$$

and using the fact $\mathcal{R}_{-1}\nabla\Delta_j$ $(j\in\mathbb{N})$ is bounded on $L^p(\mathbb{R}^d)$ for $p\in[2,\infty]$, we obtain

$$
\begin{aligned}
& 2^{js}\|\Delta_j \mathrm{II}\|_{L^p} \\
&\leq C 2^{js} \sum_{q\in\mathbb{N},|q-j|\leq 4} \Big(2^{-q}\|\big(\Delta_q u\cdot\nabla S_{q-1}\phi\big)\|_{L^p} + \|\Delta_q u\|_{L^p}\|\nabla\mathcal{R}_{-1}S_{q-1}\phi\|_{L^\infty}\Big) \\
&\leq C \sum_{q\in\mathbb{N},|q-j|\leq 4} 2^{qs}\|\Delta_q u\|_{L^p}\bigg(\|S_{q-1}\phi\|_{L^\infty} + \|\nabla\mathcal{R}_{-1}\Delta_{-1}\phi\|_{L^\infty} + \sum_{0\leq l\leq q-1}\|\Delta_l\phi\|_{L^\infty}\bigg) \\
&\leq C\|\nabla u\|_{L^p} \sum_{|q-j|\leq 4}\bigg(2^{q(s-1)}\|\Delta_{-1}\phi\|_{L^2} + \sum_{0\leq l\leq q-1} 2^{(q-l)(s-1)}2^{l(s-1)}\|\Delta_l\phi\|_{L^\infty}\bigg),
\end{aligned}
$$

which combined with the discrete Young's inequality leads to that for every $s<1$,

$$
\|\mathrm{II}\|_{B^s_{p,r}} \leq C\|\nabla u\|_{L^p}\big(\|\phi\|_{B^{s-1}_{\infty,r}} + \|\phi\|_{L^2}\big).
$$

For III, in light of the divergence-free property of u, we split it as the following

$$
\begin{aligned}
\mathrm{III} &= \sum_{q\geq 3}\mathcal{R}_{-1}\nabla\cdot\big(\Delta_q u\,\widetilde{\Delta}_q\phi\big) - \sum_{q\geq 3}\Delta_q u\cdot\nabla\mathcal{R}_{-1}\widetilde{\Delta}_q\phi + \sum_{-1\leq q\leq 2}[\mathcal{R}_{-1}\nabla\cdot, \Delta_q u]\widetilde{\Delta}_q\phi \\
&=: \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.
\end{aligned}
$$

For III_1, since $\mathcal{R}_{-1}\nabla$ is bounded on $L^p(\mathbb{R}^d)$ for $p\in[2,\infty)$, we deduce that for $j=-1$ (realizing $\frac{2p}{p+2}=2$ for $p=\infty$),

$$
\begin{aligned}
2^{-s}\|\Delta_{-1}\mathrm{III}_1\|_{L^\infty} &\leq C\sum_{q\geq 3}\|\Delta_q u\,\widetilde{\Delta}_q\phi\|_{L^{\frac{2p}{p+2}}} \\
&\leq C\sum_{q\geq 3}\|\Delta_q u\|_{L^p}\|\widetilde{\Delta}_q\phi\|_{L^2} \leq C\|\nabla u\|_{L^p}\|\phi\|_{L^2},
\end{aligned}
$$

and for every $j \in \mathbb{N}$ and $s > 0$,

$$\begin{aligned}
2^{js}\|\Delta_j \mathrm{III}_1\|_{L^p} &\le C 2^{js} \sum_{q\ge 3, q\ge j-3} \|\Delta_j \mathcal{R}_{-1}\nabla\cdot(\Delta_q u\, \widetilde{\Delta}_q \phi)\|_{L^p} \\
&\le C \sum_{q\ge 3, q\ge j-3} 2^{(j-q)s} 2^q \|\Delta_q u\|_{L^p} 2^{q(s-1)} \|\widetilde{\Delta}_q \phi\|_{L^\infty} \\
&\le C c_j \|\nabla u\|_{L^p} \|\phi\|_{B^{s-1}_{\infty,r}},
\end{aligned}$$

where $\{c_j\}_{j\in\mathbb{N}}$ is such that $\|c_j\|_{\ell^r} = 1$. The estimation of III_2 is similar as that of III_1, and we get

$$\|\mathrm{III}_1\|_{B^s_{p,r}} + \|\mathrm{III}_2\|_{B^s_{p,r}} \le C\|\nabla u\|_{L^p}\big(\|\phi\|_{B^{s-1}_{\infty,r}} + \|\phi\|_{L^2}\big).$$

For III_3, we directly have

$$\begin{aligned}
\|\mathrm{III}_3\|_{B^s_{p,r}} &\le C \sum_{-1\le j\le 6}\sum_{-1\le q\le 2} \Big(\|\Delta_j \mathcal{R}_{-1}\nabla\cdot(\Delta_q u\, \widetilde{\Delta}_q\phi)\|_{L^p} + \|\Delta_q u \cdot \nabla \mathcal{R}_{-1}\widetilde{\Delta}_q \phi\|_{L^p}\Big) \\
&\le C \sum_{-1\le q\le 2} \Big(\|\Delta_q u\, \widetilde{\Delta}_q \phi\|_{L^1} + \|\Delta_q u\cdot\nabla\mathcal{R}_{-1}\widetilde{\Delta}_q\phi\|_{L^1}\Big) \\
&\le C \sum_{-1\le q\le 2} \|\Delta_q u\|_{L^2}\|\widetilde{\Delta}_q\phi\|_{L^2} \le C\|u\|_{L^2}\|\phi\|_{L^2}.
\end{aligned}$$

Therefore, collecting the above estimates yields the wanted estimate (0.1). ∎

We can use Lemma 0.1 to replace [1, Lemma 2.7] in the application, which is mainly used in the section 3 and appendix B of [1]. Indeed, in (3.10) and (3.20) of [1], it needs to estimate $\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_T(L^2)}$ and $\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_T(L^p)}$ $(p>2)$, and they both are controlled by the following

$$\begin{aligned}
&\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_T(B^{1-\frac{2}{p}}_{2,1})} \\
&\le C\|\nabla u\|_{L^1_T(L^2)}\Big(\|\theta\|_{L^\infty_T(B^{-\frac{2}{p}}_{\infty,1})} + \|\theta\|_{L^\infty_T(L^2)}\Big) + C\|u\|_{L^1_T(L^2)}\|\theta\|_{L^\infty_T(L^2)} \\
&\le C\|u\|_{L^1_T(H^1)}\|\theta\|_{L^\infty_T(L^2\cap L^\infty)};
\end{aligned}$$

while in (3.45) and (B.7), it suffices to control $\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_t(B^{\gamma'}_{\infty,1})}$ and $\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_t(B^{\gamma'}_{r,1})}$ for every $0<\gamma'<1-\frac{2}{p}$ and $r\ge 2$, and from the Besov embedding they can be bounded as follows

$$\begin{aligned}
&\|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_t(B^{\gamma'+\frac{2}{p}}_{p,1})} + \|[\mathcal{R}_{-1}, u\cdot\nabla]\theta\|_{L^1_t(B^{\gamma'}_{2,1})} \\
&\le C\|\nabla u\|_{L^1_t(L^p\cap L^2)}\Big(\|\theta\|_{L^\infty_t(B^{\gamma'+\frac{2}{p}-1}_{\infty,1})} + \|\theta\|_{L^\infty_t(B^{\gamma'-1}_{\infty,1}\cap L^2)}\Big) + \|u\|_{L^1_t(L^2)}\|\theta\|_{L^\infty_t(L^2)} \\
&\le C\|\nabla u\|_{L^1_t(L^p\cap L^2)}\|\theta\|_{L^\infty_t(L^2\cap L^\infty)} + \|u\|_{L^1_t(L^2)}\|\theta\|_{L^\infty_t(L^2)}.
\end{aligned}$$

Hence, the main results including Theorem 1.1 in [1] still hold true.

Finally, we remark that the error of misusing commutator formula also appears in the proof of Lemma 2.5-(3) and Lemma 2.4-(3) in [1], but they can be easily fixed without using the commutator structure, so the validity of Lemma 2.5-(3) and Lemma

2.4-(3) is not affected. Indeed, in Lemma 2.4-(3), it suffices to estimate $\|\mathcal{II}\|_{B^{-\epsilon}_{p,r}}$ with $\epsilon \in (-1,1)$, $(p,r) \in [1,\infty]^2$ and

$$\mathcal{II} := \sum_{j\in\mathbb{N}} [m(D), \Delta_j u \cdot \nabla] S_{j-1}\phi = \mathcal{II}_1 - \mathcal{II}_2,$$

and

$$\mathcal{II}_1 := \sum_{j\in\mathbb{N}} m(D)\widetilde{\psi}(2^{-j}D)\big(\Delta_j u \cdot \nabla S_{j-1}\phi\big), \quad \mathcal{II}_2 := \sum_{j\in\mathbb{N}} \Delta_j u \cdot \nabla m(D) S_{j-1}\phi,$$

where $m(D)$ is a zero-order pseudo-differential operator with symbol $m(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $\widetilde{\psi}$ is defined as in (0.2), then we have that for $q \geq -1$,

$$\begin{aligned}
&2^{-q\epsilon}\|\Delta_q \mathcal{II}\|_{L^p} \\
&\leq C 2^{-q\epsilon} \sum_{j\in\mathbb{N}, |j-q|\leq 4} \Big(\|m(D)\widetilde{\psi}(2^{-j}D)(\Delta_j u \cdot \nabla S_{j-1}\phi)\|_{L^p} + \|\Delta_j u \, \nabla m(D) S_{j-1}\phi\|_{L^p} \Big) \\
&\leq C 2^{-q\epsilon} \sum_{j\in\mathbb{N}, |j-q|\leq 4} \|\Delta_j u\|_{L^\infty} \Big(\|\nabla S_{j-1}\phi\|_{L^p} + \|\nabla m(D) S_{j-1}\phi\|_{L^p} \Big) \\
&\leq C \|\nabla u\|_{L^\infty} \sum_{j\in\mathbb{N}, |j-q|\leq 4} 2^{-j(\epsilon+1)} \Big(\sum_{-1\leq j'\leq j-1} 2^{j'} \|\Delta_{j'}\phi\|_{L^p} \Big) \\
&\leq C \|\nabla u\|_{L^\infty} \sum_{|j-q|\leq 4} \sum_{-1\leq j'\leq j-1} 2^{(j'-j)(1+\epsilon)} 2^{-j'\epsilon} \|\Delta_{j'}\phi\|_{L^p},
\end{aligned}$$

which leads to the desired inequality

$$\|\mathcal{II}\|_{B^{-\epsilon}_{p,r}} \leq C \|\nabla u\|_{L^\infty} \|\phi\|_{B^{-\epsilon}_{p,r}}.$$

While for Lemma 2.4-(3), it suffices to show (5.29) in [1], that is, for every $\epsilon \in (0,1)$ and $(p,r) \in [1,\infty]^2$,

$$\|\mathcal{II}_1\|_{\widetilde{\mathcal{B}}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}} + \|\mathcal{II}_2\|_{\widetilde{\mathcal{B}}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}} \lesssim \|\nabla u\|_{\mathcal{B}^{0,\ell+1}_{\mathcal{W}}} \|\phi\|_{\mathcal{B}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}}. \tag{0.3}$$

In fact, using Lemmas 5.1 and 5.2 in [1], we find that for every $q \geq -1$ and $\lambda \in \{0, 1, \cdots, \ell+1\}$,

$$\begin{aligned}
&2^{-q\epsilon}\|\Delta_q (T_{\mathcal{W}\cdot\nabla})^\lambda \mathcal{II}_1\|_{L^p} \\
&\lesssim 2^{-q\epsilon} \sum_{j\in\mathbb{N}, j\sim q} \|(T_{\mathcal{W}\cdot\nabla})^\lambda m(D)\widetilde{\psi}(2^{-j}D)(\Delta_j u \cdot \nabla S_{j-1}\phi)\|_{L^p} \\
&\lesssim 2^{-q\epsilon} \sum_{j\in\mathbb{N}, j\sim q} \sum_{\mu=0}^{\lambda} \|(T_{\mathcal{W}\cdot\nabla})^\mu (\Delta_j u \cdot \nabla S_{j-1}\phi)\|_{L^p} \\
&\lesssim 2^{-q\epsilon} \sum_{j\in\mathbb{N}, j\sim q} \sum_{\mu_1+\mu_2\leq\lambda} \|(T_{\mathcal{W}\cdot\nabla})^{\mu_1} \Delta_j u\|_{L^\infty} \|(T_{\mathcal{W}\cdot\nabla})^{\mu_2} \nabla S_{j-1}\phi\|_{L^p} \\
&\lesssim \sum_{j\in\mathbb{N}, j\sim q} \sum_{\mu_1+\mu_2\leq\lambda} 2^{-j(1+\epsilon)} \Big(\sum_{j_1\sim j} \sum_{\mu_3=0}^{\mu_1} \|(T_{\mathcal{W}\cdot\nabla})^{\mu_3} \Delta_{j_1} \nabla u\|_{L^\infty} \Big) \Big(\sum_{j'\leq j-1} \|(T_{\mathcal{W}\cdot\nabla})^{\mu_2} \nabla \Delta_{j'}\phi\|_{L^p} \Big) \\
&\lesssim c_q \sum_{\mu_1+\mu_2\leq\lambda} \Big(\sum_{\mu_3=0}^{\mu_1} \|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\mu_3}_{\mathcal{W}}} \Big) \Big\| \Big(\sum_{j'\leq j-1} 2^{(j'-j)(1+\epsilon)} 2^{-j'(1+\epsilon)} \|(T_{\mathcal{W}\cdot\nabla})^{\mu_2} \Delta_{j'} \nabla\phi\|_{L^p} \Big)_{j\in\mathbb{N}} \Big\|_{\ell^r} \\
&\lesssim c_q \|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\ell+1}_{\mathcal{W}}} \sum_{\mu_2=0}^{\lambda} \|\nabla\phi\|_{\widetilde{\mathcal{B}}^{-1-\epsilon,\mu_2}_{p,r,\mathcal{W}}} \\
&\lesssim c_q \|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\ell+1}_{\mathcal{W}}} \|\phi\|_{\widetilde{\mathcal{B}}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}} \lesssim c_q \|\nabla u\|_{\mathcal{B}^{0,\ell+1}_{\mathcal{W}}} \|\phi\|_{\mathcal{B}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}},
\end{aligned}$$

with $\{c_q\}_{q\geq -1}$ satisfying $\|c_q\|_{\ell^r}=1$, and similarly,

$$
\begin{aligned}
&2^{-q\epsilon}\|\Delta_q(T_{\mathcal{W}}\cdot\nabla)^{\lambda}\mathcal{I}\mathcal{I}_2\|_{L^p}\\
&\lesssim 2^{-q\epsilon}\sum_{j\in\mathbb{N},j\sim q}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda}\big(\Delta_j u\cdot\nabla m(D)S_{j-1}\phi\big)\|_{L^p}\\
&\lesssim 2^{-q\epsilon}\sum_{j\in\mathbb{N},j\sim q}\sum_{\lambda_1+\lambda_2\leq\lambda}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_1}\Delta_j u\|_{L^\infty}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_2}\nabla m(D)S_{j-1}\phi\|_{L^p}\\
&\lesssim \sum_{j\sim q}\sum_{\lambda_1+\lambda_2\leq\lambda}2^{-j(1+\epsilon)}\Big(\sum_{j_1\sim j}\sum_{\lambda_3=0}^{\lambda_1}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_3}\Delta_{j_1}\nabla u\|_{L^\infty}\Big)\Big(\sum_{j'=-1}^{j-1}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_2}\nabla m(D)\Delta_{j'}\phi\|_{L^p}\Big)\\
&\lesssim \sum_{j\sim q}\sum_{\lambda_1+\lambda_2\leq\lambda}2^{-j(1+\epsilon)}\Big(\sum_{\lambda_3=0}^{\lambda_1}\|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\lambda_3}_{\mathcal{W}}}\Big)\times\\
&\quad\times\bigg(\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_2}\nabla m(D)\Delta_{-1}\phi\|_{L^p}+\sum_{j'=0}^{j-1}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_2}\nabla m(D)\Delta_{j'}\phi\|_{L^p}\bigg)\\
&\lesssim \|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\ell+1}_{\mathcal{W}}}\sum_{j\sim q}\sum_{\lambda_2=0}^{\lambda}2^{-j(1+\epsilon)}\bigg(\|\nabla m(D)\Delta_{-1}\phi\|_{L^p}+\sum_{j'=0}^{j-1}\sum_{\lambda_4=0}^{\lambda_2}2^{j'}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_4}\Delta_{j'}\phi\|_{L^p}\bigg)\\
&\lesssim c_q\|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\ell+1}_{\mathcal{W}}}\bigg(\|\Delta_{-1}\phi\|_{L^p}+\sum_{\lambda_4=0}^{\lambda}\bigg\|\bigg(\sum_{0\leq j'\leq j-1}2^{(j'-j)(1+\epsilon)}2^{-j'\epsilon}\|(T_{\mathcal{W}}\cdot\nabla)^{\lambda_4}\Delta_{j'}\phi\|_{L^p}\bigg)_{j\in\mathbb{N}}\bigg\|_{\ell^r}\bigg)\\
&\lesssim c_q\|\nabla u\|_{\widetilde{\mathcal{B}}^{0,\ell+1}_{\mathcal{W}}}\|\phi\|_{\widetilde{\mathcal{B}}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}}\lesssim c_q\|\nabla u\|_{\mathcal{B}^{0,\ell+1}_{\mathcal{W}}}\|\phi\|_{\mathcal{B}^{-\epsilon,\ell+1}_{p,r,\mathcal{W}}},
\end{aligned}
$$

then the desired inequality (0.3) follows immediately.

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