PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 6, June 2015, Pages 2613–2622
S 0002-9939(2015)12468-9
Article electronically published on February 16, 2015

REMARKS ON SELF-SIMILAR SOLUTIONS FOR THE SURFACE QUASI-GEOSTROPHIC EQUATION AND ITS GENERALIZATION

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(Communicated by Joachim Krieger)

Abstract. We prove some nonexistence results of self-similar singular solutions for the surface quasi-geostrophic equation and its generalization by relying on the fundamental local L^p-inequality of the self-similar quantity.

1. Introduction

In this paper we focus on the following generalized surface quasi-geostrophic equation:

$$(1.1)\qquad \begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ u = \mathcal{R}_\gamma^\perp\theta = (-\mathcal{R}_{2,\gamma}\theta, \mathcal{R}_{1,\gamma}\theta), \\ \theta|_{t=0} = \theta_0, \end{cases}$$

where $\mathcal{R}_{i,\gamma} = \partial_{x_i}(-\Delta)^{-\frac{\gamma}{2}}$, $i=1,2$, $\gamma\in[1,2]$ are pseudo-differential operators which generalize the usual Riesz transform. When $\gamma=1$, (1.1) is just the surface quasi-geostrophic equation which arises from the geostrophic fluids and is viewed as a two-dimensional model of the 3D Euler system (cf. [7,12]). When $\gamma=2$, (1.1) corresponds to the classical 2D Euler equations in vorticity form. (1.1) in the case $1<\gamma<2$ is the intermediate toy model introduced by Constantin et al. [6].

It is well-known that the 2D Euler equations preserve the global regularity for the smooth data (e.g. [1, 15]) by using the L^∞-norm conservation of θ, while for the SQG equation and its generalization with $1\le\gamma<2$, although they are of very simple form and have been intensely studied, it still remains open whether the solutions blow up at finite time or not.

The equation (1.1) is invariant under the scaling transformations that for $\alpha>-1$,

$$\theta(t,x)\mapsto\theta_\lambda(t,x)=\lambda^{\alpha+\gamma-1}\theta(\lambda^{\alpha+1}t,\lambda x),\quad \forall\lambda>0.$$

We say a solution is self-similar if $\theta=\theta_\lambda$ for all $\lambda>0$. The self-similar blowup singularity is an important scenario that may occur in the evolution of θ and is the main concern in this note. More precisely, we assume there exists $(t_*,x_*)\in\mathbb{R}^+\times\mathbb{R}^2$ such that the solution θ develops a self-similar singularity at (t_*,x_*) of the form

$$(1.2)\qquad \theta(t,x)=\frac{1}{(t_*-t)^{\frac{\alpha+\gamma-1}{1+\alpha}}}\Theta\left(\frac{x-x_*}{(t_*-t)^{\frac{1}{1+\alpha}}}\right),\quad \alpha>-1$$

Received by the editors October 11, 2013 and, in revised form, February 5, 2014.
2010 *Mathematics Subject Classification.* Primary 76B03, 35Q31, 35Q35, 35Q86.
Key words and phrases. Self-similar solutions, surface quasi-geostrophic equation, nonexistence.

for all $x \in \mathbb{R}^2$ and $0 < t < t_*$. In terms of the profile function $\Theta = \theta|_{t=1}$, we formally get

$$\begin{cases} \frac{\alpha+\gamma-1}{\alpha+1}\Theta + \frac{1}{\alpha+1}\, y \cdot \nabla\Theta + U \cdot \nabla\Theta = 0, \\ U = \mathcal{R}_\gamma^\perp \Theta. \end{cases} \tag{1.3}$$

The finite time singularity of self-similar type has been studied for many nonlinear evolution equations, and one can see the recent review paper [11], especially; we refer to [3, 8, 14] and references therein for the nonexistence results of self-similar solutions for the 3D Euler equations. For the general transport equation (1.1), Chae in [3] proved that there exists no nontrivial self-similar solution (1.2) if $\Theta \in L^{p_1} \cap L^{p_2}(\mathbb{R}^2)$ with $p_1, p_2 \in]0, \infty]$, $p_1 < p_2$, and the particle trajectory generated by the velocity u is a C^1-diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ for all $t \in]0, t_*[$. For the SQG equation, i.e. (1.1) in the case $\gamma = 1$, Castro and Córdoba in [2] considered a particular type of solution $\theta(t, x) = x_2 f_{x_1}(t, x_1)$, which has infinite energy, and constructed self-similar blowup solutions to the reduced equation of $f(t, x_1)$ for any $\alpha > -1$.

In this short paper, being different from [3], we show the nonexistence results for self-similar solutions by relying only on the intrinsic local L^p-inequality of Θ, and also give some insight on the further study of the self-similar blowup problem. Our first result is as follows.

Theorem 1.1. *Let $\gamma \in [1, 2]$, $p \in]1, \frac{2}{\gamma-1}[$ (we adopt the convention $\frac{2}{\gamma-1} = \infty$ for $\gamma = 1$). Suppose that $\Theta \in C^1_{\mathrm{loc}} \cap L^q(\mathbb{R}^2)$ with $\frac{2p+2}{\gamma+1} \le q < \frac{2}{\gamma-1}$. Then for all $-1 < \alpha \le \frac{2}{q} - \gamma + 1$ or $\alpha > \frac{2}{p} - \gamma + 1$, we have $\Theta \equiv 0$, while for all $\frac{2}{q} - \gamma + 1 < \alpha \le \frac{2}{p} - \gamma + 1$, we have*

$$\int_{|y| \le M} |\Theta(y)|^p \mathrm{d}y \lesssim M^{2-p(\alpha+\gamma-1)}, \quad \forall M \gg 1. \tag{1.4}$$

Moreover, for $\frac{2}{q} - \gamma + 1 < \alpha < \frac{2}{p} - \gamma + 1$, if Θ additionally satisfies that

$$\int_{|y| \le M} |\Theta(y)|^p \mathrm{d}y = M^{2-p(\alpha+\gamma-1)} o_M(1), \quad \text{with} \quad \lim_{M \to \infty} o_M(1) = 0, \tag{1.5}$$

then we have $\Theta \equiv 0$.

As a corollary of the upper theorem, we have the following excluding result for all of the range of α.

Corollary 1.2. *Suppose that $\Theta \in C^1_{loc} \cap L^{\frac{2p+2}{\gamma+1}} \cap L^{\frac{2(2p+3+\gamma)}{(\gamma+1)^2}}(\mathbb{R}^2)$ with $p \in]1, \frac{2}{\gamma-1}[$. Then for all $\alpha > -1$, we have $\Theta \equiv 0$.*

If $\theta_0 \in L^\infty(\mathbb{R}^2)$, we a priori have $\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$ for all $t \in [0, t_*[$, and thus (1.2) necessarily implies that

$$|\Theta(y)| \lesssim |y|^{-(\alpha+\gamma-1)}, \qquad \forall |y| \gg 1. \tag{1.6}$$

Since $p < \frac{2p+2}{\gamma+1} < \frac{2(2p+3+\gamma)}{(\gamma+1)^2}$ and they all tend to $\frac{2}{\gamma-1}-$ as $p \to \frac{2}{\gamma-1}-$, we see that for $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\alpha > 0$ there exists some $p \in]1, \frac{2}{\gamma-1}[$ so that the assumption of Corollary 1.2 can be satisfied and it yields $\Theta \equiv 0$.

If $\theta_0 \in \dot{H}^{-\gamma/2}(\mathbb{R}^2)$, then due to the structure of nonlinearity, we a priori have $\theta(t) \in \dot{H}^{-\gamma/2}(\mathbb{R}^2)$ for all $t \in [0, t_*[$. Thus for every $\gamma \in]1, 2]$, by interpolation we find that

$$\|u(t)\|_{L^\infty} = \|\mathcal{R}_\gamma^\perp \theta(t)\|_{L^\infty} \lesssim \|\theta(t)\|_{\dot H^{-\gamma/2}\cap L^\infty} \lesssim \|\theta_0\|_{\dot H^{-\gamma/2}\cap L^\infty}. \tag{1.7}$$

Noting that

$$u(t,x) = \frac{1}{(t_* - t)^{\frac{\alpha}{1+\alpha}}} U\Big(\frac{x - x_*}{(t_* - t)^{\frac{1}{1+\alpha}}}\Big), \quad \alpha > -1,$$

and from (1.7), we necessarily have

$$|U(y)| \lesssim |y|^{-\alpha}, \qquad \forall |y| \gg 1. \tag{1.8}$$

The scenario (1.6) and (1.8) for $\alpha > 0$ can be excluded by Corollary 1.2, while for $1 < \gamma \leq 2$ and $1 - \gamma < \alpha \leq 0$, by using the local L^p-inequality with $p \in]\frac{2}{\gamma-1}, \infty[$, we have the following result.

Proposition 1.3. *Let $\gamma \in]1,2]$. Assume that $\Theta \in C^1_{loc}(\mathbb{R}^2)$ and for all $1 - \gamma < \alpha \leq 0$,*

$$|\Theta(y)| \lesssim \frac{1}{|y|^{\alpha+\gamma-1}} \quad \textit{and} \quad |U(y)| \lesssim |y|^{-\alpha}, \qquad \forall |y| \gg 1. \tag{1.9}$$

Then we have $\Theta \equiv U \equiv 0$.

As mentioned above, the starting point of the argument in showing Theorem 1.1 and Proposition 1.3 is the local L^p-inequality of Θ (2.2). By appropriately choosing the numbers m_1, m_2 in (2.2), and from the integrable and asymptotic assumptions of Θ and U, we can use the iteration method to show $\int_{|y|\leq M} |\Theta(y)|^p \mathrm{d}y \to 0$ as $M \to \infty$, which implies $\Theta \equiv 0$. We also notice that (2.2) is derived from the local L^p-equality of θ (2.1), which in turn can be ensured for the weak solution θ under the condition $\theta \in C^1_{\mathrm{loc}}((0,t_*)\times\mathbb{R}^2) \cap L^\infty_t(L^p_x \cap L^{2/(\gamma-1)}_x)$ (one can refer to [4] for a less regular assumption to guarantee (2.1)).

Remark 1.4. For the 2D Euler system, i.e. (1.1) in the case $\gamma = 2$, Proposition 1.3 guarantees that if $\Theta \in C^1_{\mathrm{loc}}(\mathbb{R}^2)$ and (1.9) is satisfied, then there are no nontrivial self-similar solutions for any $\alpha > -1$. This is consistent with Yudovich's classical result (cf. [15]) that the 2D Euler system has a global unique solution for the initial data with bounded vorticity and finite energy (equivalently, $\theta_0 \in L^\infty \cap \dot H^{-1}(\mathbb{R}^2)$). But for the equation (1.1) in the case $1 \leq \gamma < 2$, there is still a possibility to have nontrivial self-similar singular solutions for some $-1 < \alpha \leq 1 - \gamma$.

For the SQG equation, i.e. (1.1) in the case $\gamma = 1$, if $\Theta \in C^1_{\mathrm{loc}} \cap L^{p+1} \cap L^{p+2}(\mathbb{R}^2)$ with $p \in]1, \infty[$, we have $\Theta \equiv 0$ for all $\alpha > -1$. In particular, this excludes the scenario that Θ has the decaying asymptotics, but it remains open for the case that Θ has nondecaying asymptotics, for instance, the scenario from (1.6) that

$$1 \lesssim |\Theta(y)| \lesssim |y|^{-\alpha}, \qquad \text{for} \quad -1 < \alpha < 0, \ \forall |y| \gg 1. \tag{1.10}$$

A similar problem also holds for the 3D Euler system, and since what we can rely on is just the L^2-energy conservation of velocities, it leaves open more scenarios for the corresponding self-similar solutions, e.g. the scenario that the self-similar velocities for each $0 < \alpha < \frac{3}{2}$ having the decay asymptotics of $|y|^{-\alpha}$ is still not excluded (cf. [14]).

At last, we recall the related results for some 1D models of the SQG equation and 3D Euler equations. The typical examples are the Burgers equation $\partial_t\theta + \theta\theta_x = 0$ and the nonlocal CCF model $\partial_t\theta + H(\theta)\,\theta_x = 0$, where θ is a 1D scalar and H

is the Hilbert transform. It is known that the Burgers equation forms the finite-time shock singularity, while for the CCF equation, it was proved in [9] that some solutions develop the cusp singularity at finite time. The authors in [10,11] moreover considered the structures of such singularities for the Burgers and CCF equations to find that the singularities are of self-similar type with some indexes in the range $-1 < \alpha < 0$. Despite being in a different setting, this is compatible with our results above and suggests that the scenarios like (1.10) are potentially serious cases to be further considered.

The outline of this paper is the following: we prove Theorem 1.1 in Section 2, and we present the proofs for Corollary 1.2 and Proposition 1.3 in Section 3.

2. Proof of Theorem 1.1

2.1. **Local L^p-inequality.** The first basic assumption is that the solution θ is regular enough to satisfy the local L^p-equality ($1 < p < \infty$):

$$\begin{aligned}&\int_{\mathbb{R}^2} |\theta(t_2,x)|^p \chi(t_2,x)\mathrm{d}x - \int_{\mathbb{R}^2} |\theta(t_1,x)|^p \chi(t_1,x)\mathrm{d}x \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(t,x)|^p \partial_t \chi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(t,x)|^p (\mathcal{R}_\gamma^\perp \theta)\cdot \nabla\chi(t,x)\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.1}$$

where $\chi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and $0 < t_1 < t_2 < t_*$. This can be guaranteed by the locally regular solution $\theta \in C^1_{\mathrm{loc}}((0,t_*)\times\mathbb{R}^2) \cap L^\infty_t(L^p_x \cap L_x^{2/(\gamma-1)})$.

With no loss of generality, we assume that $x_* = 0$ and $t_* = 1$. Let $\phi(x) = \phi(|x|)$ be a radial smooth test function such that $0 \le \phi \le 1$, $\phi(x) \equiv 1$ for $0 \le |x| \le \frac{1}{2}$, and $\phi(x) \equiv 0$ for $|x| > 1$. Let $\chi = \phi$; then (2.1) becomes

$$\int_{\mathbb{R}^2} |\theta(t_2,x)|^p\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^2} |\theta(t_1,x)|^p\phi(x)\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(t,x)|^p(\mathcal{R}_\gamma^\perp\theta)\cdot\nabla\phi(x)\,\mathrm{d}x\mathrm{d}t.$$

According to (1.2), we see that

$$\begin{aligned}&(1-t_2)^{\frac{2-p(\alpha+\gamma-1)}{1+\alpha}} \int_{|y|\le (1-t_2)^{-\frac{1}{1+\alpha}}} |\Theta(y)|^p \phi(y(1-t_2)^{\frac{1}{1+\alpha}})\,\mathrm{d}y \\ &= (1-t_1)^{\frac{2-p(\alpha+\gamma-1)}{1+\alpha}} \int_{|y|\le (1-t_1)^{-\frac{1}{1+\alpha}}} |\Theta(y)|^p \phi(y(1-t_1)^{\frac{1}{1+\alpha}})\,\mathrm{d}y \\ &\quad + \int_{t_1}^{t_2}\int_{\frac{1}{2}(1-t)^{-\frac{1}{1+\alpha}}\le|y|\le(1-t)^{-\frac{1}{1+\alpha}}} (1-t)^{\frac{2-\alpha-p(\alpha+\gamma-1)}{1+\alpha}} |\Theta(y)|^p (\mathcal{R}_\gamma^\perp\Theta) \\ &\qquad\qquad \cdot \nabla\phi(y(1-t)^{\frac{1}{1+\alpha}})\,\mathrm{d}y\mathrm{d}t.\end{aligned}$$

By denoting $m_i = (1-t_i)^{-\frac{1}{1+\alpha}}$ and integrating on the t-variable in the last integral, we get that for all $0 < m_1 < m_2$,

$$\begin{aligned}&\Bigg| \frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\le m_2} |\Theta(y)|^p \phi(y m_2^{-1})\,\mathrm{d}y - \\ &\qquad - \frac{1}{m_1^{2-p(\alpha+\gamma-1)}} \int_{|y|\le m_1} |\Theta(y)|^p\phi(y m_1^{-1})\,\mathrm{d}y \Bigg| \\ &\le C \int_{\frac{m_1}{2}\le|y|\le m_2} \frac{|\Theta(y)|^p |\mathcal{R}_\gamma^\perp\Theta(y)|}{|y|^{3-p(\alpha+\gamma-1)}}\,\mathrm{d}y.\end{aligned} \tag{2.2}$$

2.2. **Proof of Theorem 1.1.** First we consider the case $-1 < \alpha \leq \frac{2}{q} - \gamma + 1$ and $\frac{2p+2}{\gamma+1} \leq q < \infty$. We claim that as $m_2 \to \infty$,

$$\frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq m_2} |\Theta(y)|^p \phi(y m_2^{-1}) \mathrm{d}y \to 0.$$

Indeed, for a large number $K \gg 1$ and $m_2 > K$, by the Hölder inequality we have

$$\begin{aligned} &\frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq m_2} |\Theta(y)|^p \phi(y m_2^{-1}) \mathrm{d}y \\ &\leq \frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq K} |\Theta(y)|^p \mathrm{d}y + \frac{C_0}{m_2^{2p/q-p(\alpha+\gamma-1)}} \Big(\int_{K\leq |y| \leq m_2} |\Theta(y)|^q \mathrm{d}y \Big)^{\frac{p}{q}} \\ &\leq \frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq K} |\Theta(y)|^p \mathrm{d}y + C_0 \Big(\int_{|y|\geq K} |\Theta(y)|^q \mathrm{d}y \Big)^{\frac{p}{q}}. \end{aligned}$$

Passing m_2 to ∞ and then K to ∞, the claim is followed. Now (2.2) reduces to the following form (by choosing $m_1/2 = M$):

$$\frac{1}{M^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq M} |\Theta(y)|^p \mathrm{d}y \leq C \int_{|y|\geq M} \frac{|\Theta(y)|^p |\mathcal{R}_\gamma^\perp \Theta(y)|}{|y|^{3-p(\alpha+\gamma-1)}} \mathrm{d}y. \tag{2.3}$$

From the Hölder inequality, Calderón-Zygmund inequality and Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned} \frac{1}{M^{2-p(\alpha+\gamma-1)}} \int_{|y|\leq M} |\Theta(y)|^p \mathrm{d}y &\leq C \frac{1}{M^{2-\gamma-p(\alpha+\gamma-1)+2(p+1)/q}} \| |\Theta|^p \|_{L^{\frac{q}{p}}} \| \mathcal{R}_\gamma^\perp \Theta \|_{L^r} \\ &\leq C \frac{1}{M^{2-\gamma-p(\alpha+\gamma-1)+2(p+1)/q}} \|\Theta\|_{L^q}^{p+1}, \end{aligned}$$

with r the index such that $\frac{2}{r} = \frac{2}{q} - (\gamma - 1)$. Hence

$$\int_{|y|\leq M} |\Theta(y)|^p \mathrm{d}y \leq C M^{a_q}, \qquad \text{with} \quad a_q = \gamma - \frac{2(p+1)}{q}. \tag{2.4}$$

Clearly, if $a_q < 0$, that is, $q < 2(p+1)/\gamma$, we conclude the proof by passing $M \to \infty$. Otherwise, by interpolation, we get

$$\int_{|y|\leq M} |\Theta(y)|^{\frac{2p+2}{\gamma+1}} \mathrm{d}y \leq C M^{a_q b_q}, \qquad \text{with} \quad b_q = \frac{q - 2(p+1)/(\gamma+1)}{q-p} < 1. \tag{2.5}$$

In order to control the growth of the nonlocal term $u = \mathcal{R}_\gamma^\perp \Theta$ ($\gamma \in [1,2]$), we have the following lemma.

Lemma 2.1. *Assume that* $\int_{|y|\leq M} |\Theta(y)|^p \mathrm{d}y \leq C M^a$ *for all* $1 < p < \frac{2}{\gamma-1}$, $a < 2 - p(\gamma-1)$ *and* $M \geq 1$. *Then we have*

$$\int_{|y|\leq M} |\mathcal{R}_\gamma^\perp \Theta(y)|^{\frac{2p}{2-p(\gamma-1)}} \mathrm{d}y \leq C M^{\frac{2a}{2-p(\gamma-1)}}. \tag{2.6}$$

In our application, from $a_q = \gamma - \frac{2p+2}{q} < \gamma - (p+1)(\gamma-1) < 2 - \frac{2p+2}{\gamma+1}(\gamma-1)$ (due to $q < \frac{2}{\gamma-1}$) and (2.5), we have

$$\int_{|y|\leq M} |\mathcal{R}_\gamma^\perp \Theta(y)|^{\frac{2p+2}{2-p(\gamma-1)}} \mathrm{d}y \leq C M^{\frac{(\gamma+1)a_q b_q}{2-p(\gamma-1)}}. \tag{2.7}$$

Proof of Lemma 2.1. From the expression

$$\mathcal{R}_\gamma^\perp\Theta(y) = p.v.\int_{\mathbb{R}^2} K_\gamma^\perp(y-z)\Theta(z)\mathrm{d}z, \quad \text{with} \quad K_\gamma^\perp(z) = C_0\frac{(-z_2,z_1)}{|z|^{4-\gamma}},$$

we have the splitting

$$\begin{aligned}\int_{|y|\leq M}|\mathcal{R}_\gamma^\perp\Theta(y)|^{\frac{2p}{2-p(\gamma-1)}}\mathrm{d}y =& \int_{|y|\leq M}\Big|\int_{|z|\leq 2M}K_\gamma^\perp(y-z)\Theta(z)\mathrm{d}z\Big|^{\frac{2p}{2-p(\gamma-1)}}\mathrm{d}y\\ &+\int_{|y|\leq M}\Big|\int_{|z|\geq 2M}K_\gamma^\perp(y-z)\Theta(z)\mathrm{d}z\Big|^{\frac{2p}{2-p(\gamma-1)}}\mathrm{d}y\\ =& I + II.\end{aligned}$$

For I, from the Calderón-Zygmund inequality and Hardy-Littlewood-Sobolev inequality, we get

$$I \leq C\Big(\int_{|z|\leq 2M}|\Theta(z)|^p\mathrm{d}z\Big)^{\frac{2}{2-p(\gamma-1)}} \leq CM^{\frac{2a}{2-p(\gamma-1)}}.$$

For II, by the dyadic decomposition, we find that

$$\begin{aligned}II &\leq C\int_{|y|\leq M}\Big(\sum_{k=1}^\infty\int_{2^kM\leq|z|\leq 2^{k+1}M}\frac{1}{|y-z|^{3-\gamma}}|\Theta(z)|\mathrm{d}z\Big)^{\frac{2p}{2-p(\gamma-1)}}\mathrm{d}y\\ &\leq C\int_{|y|\leq M}\Big(\sum_{k=1}^\infty\frac{1}{2^{(3-\gamma)k}M^{3-\gamma}}\int_{2^kM\leq|z|\leq 2^{k+1}M}|\Theta(z)|\mathrm{d}z\Big)^{\frac{2p}{2-p(\gamma-1)}}\mathrm{d}y\\ &\leq CM^2\Big(\sum_{k=1}^\infty\frac{1}{2^{k(3-\gamma)}M^{3-\gamma}}(2^kM)^{2-\frac{2}{p}}(2^{k+1}M)^{\frac{a}{p}}\Big)^{\frac{2p}{2-p(\gamma-1)}}\\ &\leq CM^{\frac{2a}{2-p(\gamma-1)}}\Big(\sum_{k=1}^\infty 2^{-k\frac{2-p(\gamma-1)-a}{p}}\Big)^{\frac{2p}{2-p(\gamma-1)}} \leq CM^{\frac{2a}{2-p(\gamma-1)}}.\end{aligned}$$

Gathering the upper estimates leads to the desired result. □

Now from (2.5) and (2.7), we treat (2.3) as follows:

$$\begin{aligned}&\frac{1}{M^{2-p(\alpha+\gamma-1)}}\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\\ &\leq\sum_{k=0}^\infty\frac{C}{(2^kM)^{3-p(\alpha+\gamma-1)}}\Big(\int_{|y|\leq 2^{k+1}M}|\Theta(y)|^{\frac{2p+2}{\gamma+1}}\mathrm{d}y\Big)^{\frac{(\gamma+1)p}{2p+2}}\\ &\qquad\times\Big(\int_{|y|\leq 2^{k+1}M}|\mathcal{R}_\gamma^\perp\Theta|^{\frac{2p+2}{2-(\gamma-1)p}}\mathrm{d}y\Big)^{\frac{2-(\gamma-1)p}{2p+2}}\\ &\leq\frac{C}{M^{3-p(\alpha+\gamma-1)-a_qb_q(\gamma+1)/2}}\sum_{k=0}^\infty\frac{1}{2^{k\left(3-p(\alpha+\gamma-1)-a_qb_q(\gamma+1)/2\right)}}\\ &\leq\frac{C}{M^{3-p(\alpha+\gamma-1)-a_qb_q(\gamma+1)/2}},\end{aligned}\tag{2.8}$$

where we have used the fact that $3-p(\alpha+\gamma-1)-a_qb_q(\gamma+1)/2 > 3-2p/q-a_q > 0$. Hence

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y \leq CM^{a_q\tilde{b}_q-1}, \quad \text{with} \quad \tilde{b}_q = \frac{b_q(\gamma+1)}{2} < 1.$$

If $a_q\tilde{b}_q - 1 < 0$, then the proof is over. Otherwise,

$$\int_{|y|\le M} |\Theta(y)|^{\frac{2p+2}{\gamma+1}} \mathrm{d}y \le CM^{(a_q\tilde{b}_q-1)b_q}.$$

Once again we have

$$\begin{aligned}\int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y &\le \frac{C}{M}\sum_{k=0}^{\infty} \frac{1}{2^{k(3-p(\alpha+\gamma-1))}} \int_{2^k M\le |y|\le 2^{k+1}M} |\Theta(y)|^p |\mathcal{R}_\gamma^\perp \Theta(y)| \mathrm{d}y \\ &\le CM^{a_q\tilde{b}_q^2-\tilde{b}_q-1}.\end{aligned}$$

By repeating the above steps, we deduce that

$$\int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y \le CM^{a_q\tilde{b}_q^N-\tilde{b}_q^{N-1}-\cdots-\tilde{b}_q-1}.$$

Since $\tilde{b}_q < 1$, by choosing N large enough, the power of M will be negative, which yields $\Theta \equiv 0$.

Then we consider the case $\alpha > \frac{2}{p} - \gamma + 1$. We also begin with the local L^p-inequality

$$\begin{aligned}\frac{1}{m_2^{2-p(\alpha+\gamma-1)}} \int_{|y|\le m_2/2} |\Theta(y)|^p \mathrm{d}y &\le \frac{1}{m_1^{2-p(\alpha+\gamma-1)}} \int_{|y|\le m_1} |\Theta|^p \mathrm{d}y \\ &\quad + C\int_{m_1/2\le |y|\le m_2} \frac{|\Theta(y)|^p |\mathcal{R}_\gamma^\perp \Theta|}{|y|^{3-p(\alpha+\gamma-1)}} \mathrm{d}y.\end{aligned}$$

By choosing $m_2 = 2M \gg 1$ and $m_1 = 2$, we have

$$\frac{1}{M^{2-p(\alpha+\gamma-1)}} \int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y \le C + C\int_{1\le |y|\le 2M} \frac{|\Theta(y)|^p |\mathcal{R}_\gamma^\perp \Theta(y)|}{|y|^{3-p(\alpha+\gamma-1)}} \mathrm{d}y. \tag{2.9}$$

The Hölder inequality leads to

$$\begin{aligned}&\frac{1}{M^{2-p(\alpha+\gamma-1)}} \int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y \\ &\quad\le C + C\Big(\int_{1\le |y|\le 2M} \frac{1}{|y|^{\frac{(3-p(\alpha+\gamma-1))q}{(\gamma+1)q/2-p-1}}} \mathrm{d}y\Big)^{\frac{(\gamma+1)q/2-p-1}{q}} \|\Theta\|_{L^q}^p \|\mathcal{R}_\gamma^\perp \Theta\|_{L^r},\end{aligned}$$

where r is the index such that $\frac{2}{r} = \frac{2}{q} - (\gamma - 1)$. The only case we need to treat is when $\frac{(3-p(\alpha+\gamma-1))q}{(\gamma+1)q/2-(p+1)} < 2$, and in this case we get

$$\int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y \le CM^{2-p(\alpha+\gamma-1)} + CM^{a_q}, \quad \text{with} \quad a_q = \gamma - \frac{2(p+1)}{q}.$$

If $a_q < 0$, i.e. $q < 2(p+1)/\gamma$, then the proof is finished. Otherwise, we get

$$\int_{|y|\le M} |\Theta(y)|^p \mathrm{d}y \le CM^{a_q} \quad \text{and} \quad \int_{|y|\le M} |\Theta(y)|^{\frac{2p+2}{\gamma+1}} \mathrm{d}y \le CM^{a_q b_q}. \tag{2.10}$$

Now we use a bootstrap argument as before, but with suitable modification. Inserting (2.10) into (2.9) we obtain

$$\begin{aligned}\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y &\leq CM^{2-p(\alpha+\gamma-1)}\\ &\quad+\frac{C}{M}\sum_{k=-1}^{[\log_2 M]}2^{k(3-p(\alpha+\gamma-1))}\int_{\frac{M}{2^{k+1}}\leq|y|\leq\frac{M}{2^k}}|\Theta(y)|^p|\mathcal{R}_\gamma^\perp\Theta|\mathrm{d}y\\ &\leq CM^{2-p(\alpha+\gamma-1)}+CM^{a_q\tilde{b}_q-1}\sum_{k=-1}^{[\log_2 M]}2^{k(3-p(\alpha+\gamma-1)-a_q\tilde{b}_q)},\end{aligned}$$

where $[\log_2 M]$ is the integer part of $\log_2 M$. If $3-p(\alpha+\gamma-1)-a_q\tilde{b}_q\geq 0$, then

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\leq CM^{2-p(\alpha+\gamma-1)}+CM^{2-p(\alpha+\gamma-1)}\log_2 M\to 0,\quad \text{as } M\to\infty.$$

Otherwise, we get

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\leq CM^{2-p(\alpha+\gamma-1)}+CM^{a_q\tilde{b}_q-1}.$$

If $a_q\tilde{b}_q-1<0$, then the proof is over. Otherwise, we can repeat the above process to get

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\leq CM^{2-p(\alpha+\gamma-1)}+CM^{a_q\tilde{b}_q^N-\tilde{b}_q^{N-1}-\cdots-\tilde{b}_q-1},$$

and for N large enough, we finish the proof.

Next we consider the case $\frac{2}{q}-\gamma+1<\alpha\leq\frac{2}{p}-\gamma+1$ to show the estimate (1.4). The proof is quite similar to the above, and we only need to consider the situation that

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\leq CM^{2-p(\alpha+\gamma-1)}\log_2 M.$$

In this case, for some $\epsilon\in]0,\min\{1,p(\frac{2}{q}-\gamma+1)\}[$, we roughly have

$$\int_{|y|\leq M}|\Theta(y)|^p\mathrm{d}y\leq CM^{2-p(\alpha+\gamma-1)+\epsilon}$$

and

$$\int_{|y|\leq M}|\Theta(y)|^{\frac{2p+2}{\gamma+1}}\mathrm{d}y\leq CM^{(2-p(\alpha+\gamma-1)+\epsilon)b_q}.$$

Since $2-p(\alpha+\gamma-1)+\epsilon<2-p(\gamma-1)$, we can apply Lemma 2.1 again to get

$$\begin{aligned}\int_{|y|\leq M}|\Theta|^p\mathrm{d}y &\leq CM^{2-p(\alpha+\gamma-1)}\\ &\quad+\frac{C}{M}\sum_{k=-1}^{[\log_2 M]}2^{k(3-p(\alpha+\gamma-1))}\int_{\frac{M}{2^{k+1}}\leq|y|\leq\frac{M}{2^k}}|\Theta(y)|^p|\mathcal{R}_\gamma^\perp\Theta(y)|\mathrm{d}y\\ &\leq CM^{2-p(\alpha+\gamma-1)}+CM^{(2-p(\alpha+\gamma-1)+\epsilon)\tilde{b}_q-1}\\ &\qquad\times\sum_{k=-1}^{[\log_2 M]}2^{k(3-p(\alpha+\gamma-1))}2^{-k(2-p(\alpha+\gamma-1)+\epsilon)\tilde{b}_q}\\ &\leq CM^{2-p(\alpha+\gamma-1)}.\end{aligned}$$

Finally, we treat the case $\frac{2}{q}-\gamma+1<\alpha<\frac{2}{p}-\gamma+1$ under the additional assumption (1.5). From (1.5), we can eliminate the m_2-integral in (2.2) by passing m_2 to infinity. Setting $m_1=2M>0$, we have

$$\frac{1}{M^{2-p(\alpha+\gamma-1)}}\int_{|y|\le M}|\Theta(y)|^p\mathrm{d}y\le C\int_{|y|\ge M}\frac{|\Theta(y)|^p|\mathcal{R}_\gamma^\perp\Theta(y)|}{|y|^{3-p(\alpha+\gamma-1)}}\mathrm{d}y.$$

Dyadic decomposition leads to

$$\int_{|y|\le M}|\Theta(y)|^p\mathrm{d}y\le\frac{C}{M}\sum_{k=0}^{\infty}\frac{1}{2^{k(3-p(\alpha+\gamma-1))}}\int_{2^kM\le|y|\le2^{k+1}M}|\Theta(y)|^p|\mathcal{R}_\gamma^\perp\Theta(y)|\mathrm{d}y.$$

From (1.4) and Lemma 2.1, similarly to obtaining (2.8), we get

$$\begin{aligned}\int_{|y|\le M}|\Theta(y)|^p\mathrm{d}y&\le CM^{(2-p(\alpha+\gamma-1))\tilde{b}_q-1}\sum_{k=0}^{\infty}\frac{1}{2^{k(3-p(\alpha+\gamma-1))}}2^{k(2-p(\alpha+\gamma-1))\tilde{b}_q}\\&\le CM^{(2-p(\alpha+\gamma-1))\tilde{b}_q-1}.\end{aligned}$$

If $(2-p(\alpha+\gamma-1))\tilde{b}_q-1<0$, then the proof is over. Otherwise, by iteration we obtain

$$\int_{|y|\le M}|\Theta(y)|^p\mathrm{d}y\le CM^{(2-p(\alpha+\gamma-1))\tilde{b}_q^N-\tilde{b}_q^{N-1}-\cdots-\tilde{b}_q-1}.$$

Since $\tilde{b}_q<1$, for N sufficiently large, the power will be negative and the proof is finished.

3. Proofs of Corollary 1.2 and Proposition 1.3

3.1. **Proof of Corollary 1.2.** We apply Theorem 1.1 based on the L^p and $L^{\frac{2p+2}{\gamma+1}}$ local inequality respectively: since $\Theta\in C^1_{\mathrm{loc}}\cap L^{\frac{2p+2}{\gamma+1}}$, we get $\Theta\equiv0$ for all $-1<\alpha\le\frac{\gamma+1}{p+1}-\gamma+1$; while since $\Theta\in C^1_{\mathrm{loc}}\cap L^{\frac{2(2p+\gamma+3)}{(\gamma+1)^2}}$, we get $\Theta\equiv0$ for all $\alpha>\frac{\gamma+1}{p+1}-\gamma+1$. Therefore $\Theta\equiv0$ for all $\alpha>-1$.

3.2. **Proof of Proposition 1.3.** We start with the local L^p-inequality (2.2). For some $p\in]\frac{2}{\alpha+\gamma-1},\infty[$, e.g. $p=\frac{4}{\alpha+\gamma-1}$, it directly satisfies $\alpha>\frac{2}{p}+1-\gamma$. Then by choosing $m_1=2M_0$ and $m_2=2M\gg M_0$ (M_0 is a constant such that (1.9) holds for $|y|\ge M_0$) in (2.2), we have

$$\begin{aligned}\int_{|y|\le M}|\Theta|^p\mathrm{d}y\le CM^{2-p(\alpha+\gamma-1)}&\int_{|y|\le2M_0}|\Theta|^p\mathrm{d}y+CM^{2-p(\alpha+\gamma-1)}\\&\times\int_{M_0\le|y|\le2M}\frac{|\Theta(y)|^p|U(y)|}{|y|^{3-p(\alpha+\gamma-1)}}\mathrm{d}y.\end{aligned}\tag{3.1}$$

Taking advantage of (1.9) and the fact that $\alpha+1>0$, we get

$$\begin{aligned}\int_{|y|\le M}|\Theta(y)|^p\mathrm{d}y&\le CM^{2-p(\alpha+\gamma-1)}+CM^{2-p(\alpha+\gamma-1)}\int_{M_0\le|y|\le2M}\frac{1}{|y|^{3+\alpha}}\mathrm{d}y\\&\le CM^{2-p(\alpha+\gamma-1)}.\end{aligned}$$

Passing M to infinity leads to $\Theta\equiv0$. From $U=\mathcal{R}_\gamma^\perp\Theta$, we have $U\equiv0$.

References

[1] C. Bardos, *Existence et unicité de la solution de l'équation d'Euler en dimension deux* (French), J. Math. Anal. Appl. **40** (1972), 769–790. MR0333488 (48 #11813)

[2] A. Castro and D. Córdoba, *Infinite energy solutions of the surface quasi-geostrophic equation*, Adv. Math. **225** (2010), no. 4, 1820–1829, DOI 10.1016/j.aim.2010.04.018. MR2680191 (2011g:35239)

[3] Dongho Chae, *Nonexistence of self-similar singularities for the 3D incompressible Euler equations*, Comm. Math. Phys. **273** (2007), no. 1, 203–215, DOI 10.1007/s00220-007-0249-8. MR2308755 (2008h:35280)

[4] Dongho Chae, *On the conserved quantities for the weak solutions of the Euler equations and the quasi-geostrophic equations*, Comm. Math. Phys. **266** (2006), no. 1, 197–210, DOI 10.1007/s00220-006-0018-0. MR2231970 (2007j:76019)

[5] Dongho Chae, Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos, *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Adv. Math. **194** (2005), no. 1, 203–223, DOI 10.1016/j.aim.2004.06.004. MR2141858 (2006a:76116)

[6] Peter Constantin, Gautam Iyer, and Jiahong Wu, *Global regularity for a modified critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J. **57** (2008), no. 6, 2681–2692, DOI 10.1512/iumj.2008.57.3629. MR2482996 (2009k:35120)

[7] Peter Constantin, Andrew J. Majda, and Esteban Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar*, Nonlinearity **7** (1994), no. 6, 1495–1533. MR1304437 (95i:76107)

[8] Dongho Chae and Roman Shvydkoy, *On formation of a locally self-similar collapse in the incompressible Euler equations*, Arch. Ration. Mech. Anal. **209** (2013), no. 3, 999–1017, DOI 10.1007/s00205-013-0630-z. MR3067830

[9] Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. (2) **162** (2005), no. 3, 1377–1389, DOI 10.4007/annals.2005.162.1377. MR2179734 (2007b:35011)

[10] F. de la Hoz and M. A. Fontelos, *The structure of singularities in nonlocal transport equations*, J. Phys. A **41** (2008), no. 18, 185204, 12, DOI 10.1088/1751-8113/41/18/185204. MR2453970 (2009j:82068)

[11] Jens Eggers and Marco A. Fontelos, *The role of self-similarity in singularities of partial differential equations*, Nonlinearity **22** (2009), no. 1, R1–R44, DOI 10.1088/0951-7715/22/1/001. MR2470260 (2010f:35033)

[12] Isaac M. Held, Raymond T. Pierrehumbert, Stephen T. Garner, and Kyle L. Swanson, *Surface quasi-geostrophic dynamics*, J. Fluid Mech. **282** (1995), 1–20, DOI 10.1017/S0022112095000012. MR1312238 (95i:76017)

[13] Roman Shvydkoy, *A study of energy concentration and drain in incompressible fluids*, Nonlinearity **26** (2013), no. 2, 425–436, DOI 10.1088/0951-7715/26/2/425. MR3007898

[14] L. Xue, Discretely self-similar singular solutions for the incompressible Euler equations. ArXiv:1408.6619v1[math.AP].

[15] V. I. Yudovič, *Non-stationary flows of an ideal incompressible fluid* (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066. MR0158189 (28 #1415)

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