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Global regularity for the supercritical dissipative quasi-geostrophic equation with large dispersive forcing

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Abstract

We consider the 2-dimensional quasi-geostrophic equation with supercritical dissipation and dispersive forcing in the whole space. When the dispersive amplitude parameter is large enough, we prove the global well-posedness of strong solution to the equation with large initial data. We also show the strong convergence result as the amplitude parameter goes to ∞. Both results rely on the Strichartz-type estimates for the corresponding linear equation.

1. *Introduction*

In this paper, we consider the following 2-dimensional whole-space supercritical dissipative quasi-geostrophic (QG) equation with a dispersive forcing term

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|^\alpha\theta + Au_2 = 0, \\ u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \\ \theta|_{t=0}(x) = \theta_0(x), \quad x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\alpha\in]0,1[$, $\nu>0$, $A>0$, $\mathcal{R}_i = -\partial_i|D|^{-1}$, where $i=1,2$ is the usual Riesz transform, and the fractional differential operator $|D|^\alpha$ is defined via the Fourier transform

$$\widehat{|D|^\alpha f}(\xi) = |\xi|^\alpha\hat{f}(\xi).$$

Here θ is a real-valued scalar function, which can be interpreted as a buoyancy field, $u=(u_1,u_2)$ the velocity field and A the amplitude parameter. Equation (1.1) is a simplified model from the geostrophic fluid dynamics and describes the evolution of a surface buoyancy in the presence of an environmental horizontal buoyancy gradient (cf. [**15**]). From the physical viewpoint, the background buoyancy gradient generates dispersive waves, and thus equation (1.1) provides a model for the interaction between waves and turbulent motions in the 2-dimensional framework.

When $A=0$, equation (1.1) reduces to the known 2-dimensional dissipative QG equation, which also arises from the geostrophic fluid dynamics (cf. [**8**, **15**]) and recently has attracted intense attentions of many mathematicians (cf. [**4**, **7**–**12**, **17**, **20**, **22**] and references therein). According to the scaling transformation and the L^∞-maximum principle (cf. [**10**]), the cases $\alpha>1$, $\alpha=1$ and $\alpha<1$ are referred to as subcritical, critical and supercritical cases, respectively. Up to now, the study of subcritical and critical cases have been in a satisfactory state. For the delicate critical case, the issue of global regularity was independently solved by Kiselev, Nazarov and Volberg [**20**] and Caffarelli and Vasseur [**4**]. Kiselev *et al.* [**20**] proved the global well-posedness for the periodic smooth data by developing a new method called the non-local maximum principle method. Almost at the same time and from a totally different direction, Caffarelli and Vasseur [**4**] established the global regularity of weak solutions by deeply exploiting the DeGiorgi's iteration method. However, in the supercritical case whether

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solutions remain globally regular or not is a remarkable open problem. There are only some partial results, for instance: local well-posedness for large initial data and global well-posedness for small initial data concerning strong solutions (for example, [**7**, **12**, **16**, **17**]), the eventual regularity of the global weak solutions (cf. [**11**, **18**]).

Equation (1.1) is analogous to the 3-dimensional Navier–Stokes equation with Coriolis forcing, which is a basic model of oceanography and meteorology dealing with large-scale phenomena (cf. [**6**]),

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla P + \dfrac{1}{\epsilon} e_3 \times u = 0, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{1.2}$$

where $e_3 = (0,0,1)$, ϵ denotes the Rossby number and $u = (u_1, u_2, u_3)$ is the unknown velocity field. So far, it is known that global well-posedness of strong solutions to the 3-dimensional Navier–Stokes equations only holds for small initial data. But in the case of the 3-dimensional Navier–Stokes–Coriolis system (1.2), when ϵ is small enough, the presence of fast rotating term produces stabilization effect and ensures the global well-posedness of strong solution with large initial data. This result was shown by Babin *et al.* [**1**, **2**] and also by Gallagher [**13**] in the case of non-resonant periodic domains. For the case of whole space, it was proved by Chemin *et al.* [**5**] through establishing the Strichartz-type estimates of the corresponding linear system. By applying the method of Chemin, Desjardins, Gallagher and Grenier[**5**], Ngo [**21**], moreover, studied the case of small viscosity, that is, $\nu = \epsilon^\beta$ with $\beta \in]0, \beta_0]$ and some $\beta_0 > 0$, and proved the global existence of strong solution as ϵ small enough. The asymptotic behavior of weak solutions in the weak or strong sense as ϵ goes to 0 was also considered by Chemin, Gallagher and their collaborators (cf. [**6**, **14**]), and they showed that the limiting equation in the whole space (in general) is the 2-dimensional Navier–Stokes equations with the velocity field $\bar u$ three components

$$\begin{cases} \partial_t \bar u + \bar u_h \cdot \nabla^h \bar u - \nu\Delta_h \bar u + (\nabla^h P, 0) = 0, \\ \operatorname{div} \bar u_h = 0, \quad \bar u|_{t=0} = \bar u_0, \end{cases} \tag{1.3}$$

where $\bar u_h \triangleq (\bar u_1, \bar u_2)$, $\Delta_h \triangleq \partial_1^2 + \partial_2^2$ and $\nabla^h \triangleq (\partial_1, \partial_2)$.

For the dispersive dissipative QG equation (1.1), Kiselev and Nazarov [**19**] considered the critical case of $A > 0$ and $\alpha = 1$, and by applying the method of non-local maximum principle they proved the global regularity for the smooth solutions. Note that in their proof, the dispersive term always plays a negative role.

In this paper, we mainly focus on the dispersive dissipative QG equation (1.1) in the case of the supercritical regime $\alpha < 1$ and A large enough. Motivated by the results of the 3-dimensional Navier–Stokes–Coriolis equations, we shall develop the Strichartz-type estimate of the corresponding 2-dimensional linear equation to prove the global well-posedness of strong solution to (1.1) with large initial data. We shall also show the strong convergence result.

Before stating our main results, we first give some classical uniform existence results.

Proposition 1.1. *Let $\theta_0 \in L^2(\mathbb{R}^2)$ be a* 2-dimensional *real-valued scalar function. Then there exists a global weak solution θ (in the sense of distributions) to the dispersive dissipative QG equation* (1.1), *which also satisfies the following energy estimate, uniformly in A,*

$$\|\theta(t)\|_{L^2}^2 + 2\nu\int_0^t \||D|^{\alpha/2}\theta(\tau)\|_{L^2}^2\,d\tau \leqslant \|\theta_0\|_{L^2}^2 \quad \forall t > 0.$$

Moreover, if $\theta_0 \in H^{2-\alpha}(\mathbb{R}^2)$, then there is a time $T > 0$ independent of A such that

$$\theta \in \mathcal{C}([0,T]; H^{2-\alpha}) \cap L^2([0,T]; H^{2-\alpha/2})$$

with the norm independent of A, and all solutions to (1.1) coincide with θ on $[0,T]$. In particular, an absolute constant $c>0$ can be chosen such that if $\|\theta_0\|_{H^{2-\alpha}} \leqslant c\nu$, then the solution becomes global in time.

REMARK 1.2. Since for every $s\in\mathbb{R}$, $\overline{|D|^s\theta} = |D|^s\theta$ and $|\hat{\theta}(\xi)|^2 = \hat{\theta}(\xi)\hat{\theta}(-\xi)$, we know that

$$\int_{\mathbb{R}^2} |D|^s\mathcal{R}_1\theta(x)|D|^s\theta(x)\,dx = \langle |D|^s\mathcal{R}_1\theta, |D|^s\theta\rangle_{L^2} = -\int_{\mathbb{R}^2} i\xi_1|\xi|^{2s-1}|\hat{\theta}(\xi)|^2\,d\xi = 0, \tag{1.4}$$

thus the dispersive term does not contribute to the energy-type estimates. Therefore, the proof of Proposition 1.1 is almost identical to the corresponding classical proof for the supercritical dissipative QG equation, and we here omit it (cf. [**7**, **12**, **17**, **22**]).

Now we consider the asymptotic behaviour of equation (1.1) as A tends to infinity. This is reasonable since all bounds in the above statement are independent of A. In what follows, we shall also denote by θ^A the solutions in Proposition 1.1 to emphasize the dependence of A. The convergence result is as follows.

THEOREM 1.3. *Let $\theta_0(x) = \bar{\theta}_0(x_2) + \tilde{\theta}_0(x)$, with $\bar{\theta}_0 \in H^{3/2-\alpha}(\mathbb{R})$ a 1-dimensional real-valued scalar function and $\tilde{\theta}_0 \in L^2(\mathbb{R}^2)$ a 2-dimensional real-valued scalar function. Assume that $\bar{\theta}(t,x_2)$ is the unique solution of the following linear equation*:

$$\partial_t\bar{\theta} + \nu|D_2|^\alpha\bar{\theta} = 0, \quad \bar{\theta}(0,x_2) = \bar{\theta}_0(x_2). \tag{1.5}$$

Then there exists a global weak solution θ^A to the dispersive dissipative QG equation (1.1). Furthermore, for every $\sigma\in]2,4/(2-\alpha)[$ and $T>0$, we have

$$\lim_{A\to\infty}\int_0^T \|\theta^A(t) - \bar{\theta}(t)\|_{L^\sigma}^2\,dt = 0. \tag{1.6}$$

Next we consider the strong solutions, and we prove the following global result.

THEOREM 1.4. *Let $\theta_0(x)\in H^{2-\alpha}(\mathbb{R}^2)$ be a 2-dimensional real-valued scalar function, then there exists a positive number A_0 such that for every $A\geqslant A_0$, the dispersive dissipative QG equation (1.1) has a unique global solution θ^A satisfying $\theta^A\in\mathcal{C}(\mathbb{R}^+;H^{2-\alpha}(\mathbb{R}^2))\cap L^2(\mathbb{R}^+;\dot{H}^{2-\alpha/2}(\mathbb{R}^2))$. Moreover, if we denote by $\tilde{\theta}^A$ the solution of the following linear dispersive dissipative equation*:

$$\partial_t\tilde{\theta}^A + \nu|D|^\alpha\tilde{\theta}^A + A\mathcal{R}_1\tilde{\theta}^A = 0, \quad \tilde{\theta}^A|_{t=0} = \theta_0, \tag{1.7}$$

then as A goes to infinity,

$$\theta^A - \tilde{\theta}^A \longrightarrow 0 \quad \text{in } L^\infty(\mathbb{R}^+;H^{2-\alpha}(\mathbb{R}^2))\cap L^2(\mathbb{R}^+;\dot{H}^{2-\alpha/2}(\mathbb{R}^2)). \tag{1.8}$$

The proofs of both Theorems 1.3 and 1.4 are strongly based on the Strichartz-type estimate for the corresponding linear equation (1.7), which is the target of the whole of Section 3. The Fourier localization method and the para-differential calculus are also heavily used in the proof of Theorem 1.4, and for clarity, we place some needed commutator estimates and product estimates in the appendix section. The proofs of Theorems 1.3 and 1.4 are settled in Sections 4 and 5, respectively.

REMARK 1.5. The Strichartz-type estimate for the corresponding linear equation depends on the basic dispersive estimate, which is stated in Lemma 3.3. Compared with the dispersive estimate in the case of the 3-dimensional Navier–Stokes–Coriolis equations (cf. [**6**, Lemma 5.2]), Lemma 3.3 is much more delicate and the value (precisely, the argument) of z is more involved in the proof. The main reason is that the equation considered here is two dimensional, and the lower dimension makes it harder to develop the expected dispersive estimate. This can further be justified if we try to derive the dispersive estimate of the 'anisotropic' kernel function, which is as follows:

$$H(t,\mu,z_2,\xi_1) \triangleq \int_{\mathbb{R}} \Psi(\xi)\, e^{i\mu(\xi_1/|\xi|)+iz_2\xi_2-\nu t|\xi|^\alpha}\, d\xi_2, \tag{1.9}$$

with $z_2 \in \mathbb{R}$, $\mu > 0$ and Ψ defined by (3.5), and we find that it is rather difficult to obtain the needed dispersive estimate. Note that the suitable dispersive estimate for (1.9) will essentially be used if one treats the general data $\theta_0(x) = \bar{\theta}_0(x_2) + \tilde{\theta}_0(x)$ in Theorem 1.4.

REMARK 1.6. It is interesting to note that the limiting equation (1.5) is analogous to the 2-dimensional Navier–Stokes equation (1.3), and one can expect that the equation will play a similar role in other situations.

2. *Preliminaries*

In this preparatory section, we introduce some notation and present the definitions and some related results of the Sobolev and Besov spaces.

Some notation used in this paper are listed as follows.

(a) Throughout this paper, C stands for a constant which may be different from line to line. We sometimes use $A \lesssim B$ instead of $A \leqslant CB$, and use $A \lesssim_{\beta,\gamma,\dots} B$ instead of $A \leqslant C(\beta,\gamma,\dots)B$, with $C(\beta,\gamma,\dots)$ a constant depending on parameters $\beta,\gamma,\dots$.

(b) Denote by $\mathcal{D}(\mathbb{R}^n)$ the space of test functions which are smooth functions with compact support, $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing smooth functions, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ the quotient space of tempered distributions up to polynomials.

(c) $\mathcal{F}f$ or $\hat{f}$ denotes the Fourier transform, that is, $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$, while $\mathcal{F}^{-1}f$ the inverse Fourier transform, namely, $\mathcal{F}^{-1}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi$ (if there is no ambiguity, we sometimes omit $(2\pi)^{-n}$ for brevity).

(d) Denote by $\langle f,g\rangle_{L^2} \triangleq \int_{\mathbb{R}^n} f(x)\bar{g}(x)\, dx$ the inner product of the Hilbert space $L^2(\mathbb{R}^n)$.

(e) Denote by $B(x,r)$ the ball in $\mathbb{R}^n$ centred at x with radius r.

Now we give the definition of (L^2-based) Sobolev space. For $s \in \mathbb{R}$, the inhomogeneous Sobolev space

$$H^s \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{H^s}^2 \triangleq \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi < \infty \right\}.$$

Also one can define the corresponding homogeneous space:

$$\dot{H}^s \triangleq \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n); \|f\|_{\dot{H}^s}^2 \triangleq \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi < \infty \right\}.$$

In order to define the Besov spaces, we need the following dyadic partition of unity (cf. [**3**]). Choose two non-negative radial functions ζ, $\psi \in \mathcal{D}(\mathbb{R}^n)$ be supported, respectively, in the ball $\{\xi \in \mathbb{R}^n : |\xi| \leqslant \frac{4}{3}\}$ and the shell $\{\xi \in \mathbb{R}^n : \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\zeta(\xi) + \sum_{j\geqslant 0} \psi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n; \qquad \sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1 \quad \forall \xi \neq 0.$$

For all $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the non-homogeneous Littlewood–Paley operators

$$\Delta_{-1} f \triangleq \zeta(D) f; \quad \Delta_j f \triangleq \psi(2^{-j} D) f, \quad S_j f \triangleq \sum_{-1 \leqslant k \leqslant j-1} \Delta_k f \quad \forall j \in \mathbb{N},$$

And the homogeneous Littlewood–Paley operators can be defined as follows:

$$\dot{\Delta}_j f \triangleq \psi(2^{-j} D) f; \quad \dot{S}_j f \triangleq \sum_{k \in \mathbb{Z}, k \leqslant j-1} \dot{\Delta}_k f \quad \forall j \in \mathbb{Z}.$$

Then we introduce the definition of Besov spaces . Let $(p, r) \in [1, \infty]^2$, $s \in \mathbb{R}$, the non-homogeneous Besov space

$$B^s_{p,r} \triangleq \{ f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} \triangleq \|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j \geqslant -1}\|_{\ell^r} < \infty \}$$

and the homogeneous space

$$\dot{B}^s_{p,r} \triangleq \{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n); \|f\|_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js} \|\dot{\Delta}_j f\|_{L^p}\}_{j \in \mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty \}.$$

We point out that for all $s \in \mathbb{R}$, $B^s_{2,2} = H^s$ and $\dot{B}^s_{2,2} = \dot{H}^s$.

Next we introduce two kinds of space–time Besov spaces. The first one is the classical space–time Besov space $L^\rho([0,T], B^s_{p,r})$, abbreviated by $L^\rho_T B^s_{p,r}$, which is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{L^\rho_T B^s_{p,r}} \triangleq \|\|\{2^{js} \|\Delta_j f\|_{L^p}\}_{j \geqslant -1}\|_{\ell^r}\|_{L^\rho([0,T])} < \infty.$$

The second one is the Chemin–Lerner's mixed space–time Besov space $\tilde{L}^\rho([0,T], B^s_{p,r})$, abbreviated by $\tilde{L}^\rho_T B^s_{p,r}$, which is the set of tempered distribution f satisfying

$$\|f\|_{\tilde{L}^\rho_T B^s_{p,r}} \triangleq \|\{2^{qs} \|\Delta_q f\|_{L^\rho_T L^p}\}_{q \geqslant -1}\|_{\ell^r} < \infty.$$

Owing to Minkowiski's inequality, we immediately obtain

$$L^\rho_T B^s_{p,r} \hookrightarrow \tilde{L}^\rho_T B^s_{p,r} \quad \text{if } r \geqslant \rho \quad \text{and} \quad \tilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r} \quad \text{if } \rho \geqslant r.$$

These can similarly extend to the homogeneous ones $L^\rho_T \dot{B}^s_{p,r}$ and $\tilde{L}^\rho_T \dot{B}^s_{p,r}$.

Bernstein's inequality is very fundamental in the analysis involving Besov spaces.

LEMMA 2.1. *Let a, b, p and q be positive numbers satisfying $0 < a \leqslant b < \infty$ and $1 \leqslant p \leqslant q \leqslant \infty$, $k \geqslant 0$, $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$ with $n \in \mathbb{Z}^+$. Then there exist positive constants C and c independent of λ such that*

$$\text{if } \operatorname{supp} \hat{f} \subset \{\xi : |\xi| \leqslant \lambda b\} \Longrightarrow \||D|^k f\|_{L^q(\mathbb{R}^n)} \leqslant C \lambda^{k + n(1/p - 1/q)} \|f\|_{L^p(\mathbb{R}^n)}$$

and

$$\text{if } \operatorname{supp} \hat{f} \subset \{\xi : a\lambda \leqslant |\xi| \leqslant b\lambda\} \Longrightarrow c\lambda^k \|f\|_{L^p(\mathbb{R}^n)} \leqslant \||D|^k f\|_{L^p(\mathbb{R}^n)} \leqslant C\lambda^k \|f\|_{L^p(\mathbb{R}^n)}.$$

When $k \in \mathbb{N}$, similar estimates hold if $|D|^k$ is replaced by $\sup_{|\gamma|=k} \partial^\gamma$.

3. *Strichartz-type estimates for the corresponding linear equation*

In this section, we are devoted to show the Strichartz-type estimates of the following linear dispersive dissipative equation:

$$\begin{cases} \partial_t \tilde{\theta} + \nu |D|^\alpha \tilde{\theta} + A \mathcal{R}_1 \tilde{\theta} = f, \\ \tilde{\theta}|_{t=0} = \tilde{\theta}_0. \end{cases} \tag{3.1}$$

Applying the spatial Fourier transformation to the upper equation, we obtain

$$\begin{cases} \partial_t \hat{\tilde{\theta}} + \nu|\xi|^\alpha \hat{\tilde{\theta}} - Ai\dfrac{\xi_1}{|\xi|}\hat{\tilde{\theta}} = \hat{f}, \\ \hat{\tilde{\theta}}|_{t=0} = \widehat{\tilde{\theta}_0}. \end{cases}$$

Furthermore,

$$\hat{\tilde{\theta}}(t,\xi) = e^{iAt(\xi_1/|\xi|)-\nu t|\xi|^\alpha}\widehat{\tilde{\theta}_0}(\xi) + \int_0^t e^{iA(t-\tau)(\xi_1/|\xi|)-\nu(t-\tau)|\xi|^\alpha}\hat{f}(\tau,\xi)\,d\tau.$$

Thus by setting

$$\mathcal{G}^A(t) : g \longmapsto \int_{\mathbb{R}^2_\xi} e^{iAta(\xi)-\nu t|\xi|^\alpha+ix\cdot\xi}\hat{g}(\xi)\,d\xi,$$

with

$$a(\xi) \triangleq \xi_1/|\xi|,$$

we have

$$\tilde{\theta}(t) = \mathcal{G}^A(t)\tilde{\theta}_0 + \int_0^t \mathcal{G}^A(t-\tau)f(\tau)\,d\tau. \tag{3.2}$$

Hence, it reduces to consider the Strichartz-type estimate of $\mathcal{G}^A(t)g$, and because the phase function $a(\xi)$ is somewhat 'singular', we shall study the case when $\hat{g}$ is supported in the set $\mathcal{B}_{r,R}$ for some $0 < r < R$, with

$$\mathcal{B}_{r,R} \triangleq \{\xi \in \mathbb{R}^2 : |\xi_1| \geqslant r,\ |\xi| \leqslant R\}.$$

The main result of this section is as follows.

Proposition 3.1. *Let r and R be two positive numbers satisfying $r < R$, and $g \in L^2(\mathbb{R}^2)$ satisfying* $\operatorname{supp}\hat{g} \subset \mathcal{B}_{r,R}$. *Then for every $p \in [1,\infty]$ and $q \in [2,\infty]$, there exists a positive constant $C = C_{r,R,p,q,\nu}$ such that*

$$\|\mathcal{G}^A(t)g\|_{L^p(\mathbb{R}^+;L^q(\mathbb{R}^2))} \leqslant CA^{-1/8p(1-2/q)}\|g\|_{L^2(\mathbb{R}^2)}. \tag{3.3}$$

Proposition 3.1 combined with (3.2) and Minkowiski's inequality (cf. (4.8) below) implies the following Strichartz-type estimates for the linear system (3.1).

Corollary 3.2. *Let r and R be two positive numbers satisfying $r < R$. Assume that $\tilde{\theta}_0 \in L^2(\mathbb{R}^2)$ and $f \in L^1(\mathbb{R}^+;L^2(\mathbb{R}^2))$ satisfying that*

$$\operatorname{supp}\widehat{\tilde{\theta}_0} \cup \left(\bigcup_{t\geqslant 0}\operatorname{supp}\hat{f}(t,\cdot)\right) \subset \mathcal{B}_{r,R},$$

and $\tilde{\theta}$ solves the corresponding linear dispersive equation (3.1). Then for every $p \in [1,\infty]$ and $q \in [2,\infty]$, there exists a positive constant $C = C_{r,R,p,q,\nu}$ such that

$$\|\tilde{\theta}\|_{L^p(\mathbb{R}^+;L^q(\mathbb{R}^2))} \leqslant CA^{-1/8p(1-2/q)}(\|\tilde{\theta}_0\|_{L^2(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^+;L^2(\mathbb{R}^2))}).$$

In order to prove Proposition 3.1, we introduce the following kernel function:

$$K(t,\mu,z) \triangleq \int_{\mathbb{R}^2_\xi} \Psi(\xi)\,e^{i\mu a(\xi)+iz\cdot\xi-\nu t|\xi|^\alpha}\,d\xi, \tag{3.4}$$

where $t > 0$, $\mu > 0$, $z \in \mathbb{R}^2$, $\Psi \in \mathcal{D}(\mathbb{R}^2)$ is a smooth cut-off function such that $\Psi \equiv 1$ on $\mathcal{B}_{r,R}$ and is supported in $\mathcal{B}_{r/2,2R}$, and, for instance, we can explicitly define

$$\Psi(\xi) = \chi\left(\frac{|\xi|}{R}\right)\left(1 - \chi\left(\frac{2|\xi_1|}{r}\right)\right) \tag{3.5}$$

with $\chi \in \mathcal{D}(]-2,2[)$ satisfying $\chi(x) \equiv 1$ for $|x| \leqslant 1$.

As a first step, we show the following basic dispersive estimate of K.

Lemma 3.3. *Let r and R be two positive numbers satisfying $r < R$, and K be defined by* (3.4)*. Then there exists an absolute constant $C = C_{r,R}$ such that for every $z \in \mathbb{R}^2$,*

$$|K(t,\mu,z)| \leqslant C \min\{1, \mu^{-1/4}\}\, e^{-r^\alpha \nu t/4}.$$

Proof of Lemma 3.3. We shall use the method of stationary phase to show this formula. Denoting by

$$\Phi(\xi, z) \triangleq \nabla_\xi \left(a(\xi) + \frac{z}{\mu}\cdot\xi\right) = -\frac{\xi_2}{|\xi|^3}\xi^\perp + \frac{z}{\mu}$$

with $\xi^\perp = (-\xi_2, \xi_1)$, we introduce the following differential operator:

$$\mathcal{L} \triangleq \frac{\mathrm{Id} - i\Phi(\xi,z)\cdot\nabla_\xi}{1 + \mu|\Phi(\xi,z)|^2},$$

and we see that $\mathcal{L}\, e^{i\mu a(\xi) + iz\cdot\xi} = e^{i\mu a(\xi)+iz\cdot\xi}$. From integration by parts, we have

$$K(t,\mu,z) = \int_{\mathbb{R}^2} e^{i\mu a(\xi)+iz\cdot\xi}\mathcal{L}^t(\Psi(\xi)\, e^{-\nu t|\xi|^\alpha})\, d\xi,$$

where $\mathcal{L}^t$ is given by

$$\mathcal{L}^t \triangleq \frac{1}{1+\mu|\Phi(\xi,z)|^2}\left(1 + i\nabla\cdot\Phi - i\frac{2\mu\sum_{j,k}\Phi^j\Phi^k\partial_{\xi_j}\Phi^k}{1+\mu|\Phi|^2}\right)\mathrm{Id} + \frac{\Phi(\xi,z)\cdot\nabla_\xi}{1+\mu|\Phi(\xi,z)|^2}.$$

Since ξ is supported in $\mathcal{B}_{r/2,2R}$, we find

$$|\nabla\Phi(\xi,z)| = \left|\nabla\left(\frac{\xi_2\xi^\perp}{|\xi|^3}\right)\right| \lesssim \frac{1}{r^2},$$

thus

$$\left|1 + i\nabla\cdot\Phi - i\frac{2\mu\sum_{j,k}\Phi^j\Phi^k\partial_{\xi_j}\Phi^k}{1+\mu|\Phi|^2}\right| \lesssim 1 + \frac{1}{r^2}.$$

Since $\Psi \in \mathcal{D}(\mathbb{R}^2)$ satisfies $|\nabla\Psi| \lesssim 1/r$, we infer that

$$\begin{aligned}|\nabla(\Psi(\xi)\, e^{-\nu t|\xi|^\alpha})| &\leqslant |\nabla\Psi|\, e^{-\nu t|\xi|^\alpha} + |\Psi|(\nu t|\xi|^\alpha\, e^{-\nu t|\xi|^\alpha/2})|\xi|^{-1}\, e^{-\nu t|\xi|^\alpha/2}\\ &\lesssim \frac{1}{r}e^{-\nu t r^\alpha/4}.\end{aligned}$$

If $|\Phi(\xi,z)| \geqslant 1$, this is the case of non-stationary phase, and collecting the upper estimates and noting that $\mu^{1/2}|\Phi| \leqslant 1 + \mu|\Phi|^2$, we have

$$|\mathcal{L}^t(\Psi(\xi)\, e^{-\nu t|\xi|^\alpha})| \lesssim \frac{1}{r^2}\min\{1, \mu^{-1/2}\}\, e^{-\nu t r^\alpha/4}.$$

This yields

$$|K(t,\mu,z)| \leqslant \int_{\mathcal{B}_{r/2,2R}} |\mathcal{L}^t(\Psi(\xi)\, e^{-\nu t|\xi|^\alpha})|\, d\xi \lesssim \frac{R^2}{r^2} \min\{1,\mu^{-1/2}\}\, e^{-\nu t r^\alpha/4}.$$

If $|\Phi(\xi,z)| \leqslant 1$, this corresponds to the case of stationary case and is more delicate. Gathering the necessary estimates as above, we have

$$|\mathcal{L}^t(\Psi(\xi)\, e^{-\nu t|\xi|^\alpha})| \lesssim \frac{1}{r^2} \frac{1}{1+\mu|\Phi(\xi,z)|^2}\, e^{-\nu t r^\alpha/4},$$

and it leads to

$$|K(t,\mu,z)| \lesssim \frac{1}{r^2}\, e^{-\nu t r^\alpha/4} \int_{\mathcal{B}_{r/2,2R}} \frac{1}{1+\mu|\Phi(\xi,z)|^2}\, d\xi. \tag{3.6}$$

For the case $z=0$, we see that $\Phi(\xi,0) = -(\xi_2/|\xi|^3)\xi^\perp$ and $|\Phi(\xi,0)| = |\xi_2|/|\xi|^2$, thus

$$|K(t,\mu,0)| \lesssim \frac{R}{r^2}\, e^{-\nu t r^\alpha/4} \int_0^{2R} \frac{1}{1+\mu\xi_2^2/(4R^2)}\, d\xi_2 \lesssim \frac{R^2}{r^2}\, e^{-\nu t r^\alpha/4} \mu^{-1/2}. \tag{3.7}$$

For every $\xi \in \mathcal{B}_{r/2,2R}$ and $z \in \mathbb{R}^2 \setminus \{0\}$, we have the following orthogonal decomposition:

$$\xi = (\xi_1,\xi_2) = \xi_{z,\parallel} e_z + \xi_{z,\perp} e_z^\perp,$$

where $e_z = z/|z|$, $e_z^\perp = z^\perp/|z|$,

$$\xi_{z,\parallel} \triangleq \xi \cdot e_z \quad \text{and} \quad \xi_{z,\perp} \triangleq \xi \cdot e_z^\perp.$$

Noting that $\xi^\perp = (-\xi_{z,\perp})e_z + \xi_{z,\parallel} e_z^\perp$, we have

$$|\Phi(\xi,z)| = \left| -\frac{\xi_2}{|\xi|^3}\xi^\perp + \frac{z}{\mu} \right| = \left| \left(\frac{\xi_2 \xi_{z,\perp}}{|\xi|^3} + \frac{|z|}{\mu} \right) e_z - \frac{\xi_2 \xi_{z,\parallel}}{|\xi|^3} e_z^\perp \right| \geqslant \frac{|\xi_2||\xi_{z,\parallel}|}{|\xi|^3},$$

and thus for every $z \neq 0$, we have $|K(t,\mu,z)| \lesssim r^{-2}\, e^{-\nu t r^\alpha/4} H(\mu,z)$ with

$$H(\mu,z) \triangleq \int_{\mathcal{B}_{r/2,2R}} \frac{1}{1+\mu\xi_2^2\xi_{z,\parallel}^2/(8R^3)}\, d\xi.$$

With no loss of generality, we assume that $z = |z|e_z = |z|(\cos\phi, \sin\phi)$ with $\phi \in [0,\pi/2]$. Then for every $\xi = (\xi_1,\xi_2)$, we have $\xi_{z,\parallel} = \xi \cdot e_z = \xi_1 \cos\phi + \xi_2 \sin\phi$, thus if $\xi_1\xi_2 \geqslant 0$, we observe

$$\xi_{z,\parallel}^2 \geqslant \xi_1^2 \cos^2\phi + \xi_2^2 \sin^2\phi \geqslant \min\{\xi_1^2, \xi_2^2\}.$$

Hence

$$\begin{aligned}
&\int_{\mathcal{B}_{\frac{r}{2},2R} \cap \{\xi_1\xi_2 \geqslant 0\}} \frac{1}{1+\mu\xi_2^2\xi_{z,\parallel}^2/(8R^3)}\, d\xi_1\, d\xi_2 \\
&\quad \leqslant \int_{\mathcal{B}_{r/2,2R} \cap \{\xi_1\xi_2 \geqslant 0\}} \frac{1}{1+\mu\xi_2^2 \min\{\xi_1^2,\xi_2^2\}/(8R^3)}\, d\xi_1\, d\xi_2 \\
&\quad \leqslant R \int_{-2R}^{2R} \frac{1}{1+\mu\xi_2^2 \min\{r^2/4,\xi_2^2\}/(8R^3)}\, d\xi_2 \\
&\quad \lesssim \max\left\{ \frac{R^{5/2}}{r}\mu^{-1/2}, R^{7/4}\mu^{-1/4} \right\}.
\end{aligned}$$

Noting that

$$\int_{\mathcal{B}_{r/2,2R} \cap \{\xi_1\xi_2 \leqslant 0\}} \frac{1}{1+\mu\xi_2^2\xi_{z,\parallel}^2/(8R^3)}\, d\xi_1\, d\xi_2 = 2\int_{\mathcal{B}_{r/2,2R} \cap \{\xi_1 \geqslant 0, \xi_2 \leqslant 0\}} \frac{1}{1+\mu\xi_2^2\xi_{z,\parallel}^2/(8R^3)}\, d\xi_1\, d\xi_2,$$

we find that

$$
\begin{aligned}
H(\mu,z) &\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2}, R^{7/4}\mu^{-1/4}\right\} + \int_{\mathcal{B}_{r/2,2R}\cap\{\xi_1\geqslant 0,\xi_2\leqslant 0\}} \frac{1}{1+\mu\xi_2^2\xi_{z,\|}^2/(8R^3)}\,d\xi_1\,d\xi_2 \\
&\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2}, R^{7/4}\mu^{-1/4}\right\} + \int_{-2R}^{0}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cos\phi+\xi_2\sin\phi)^2/(8R^3)}\,d\xi_1\,d\xi_2 \\
&\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2}, R^{7/4}\mu^{-1/4}\right\} + \tilde{H}(\mu,\phi),
\end{aligned}
$$

where

$$
\tilde{H}(\mu,\phi) \triangleq \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cos\phi-\xi_2\sin\phi)^2/(8R^3)}\,d\xi_1\,d\xi_2 \quad \forall\phi\in[0,\pi/2].
$$

Now it suffices to treat $\tilde{H}(\mu,\phi)$, and we shall divide into several cases according to ϕ. First for the endpoint case $\phi=0$, we directly have

$$
\begin{aligned}
\tilde{H}(\mu,0) &= \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2\xi_1^2/(8R^3)}\,d\xi_1\,d\xi_2 \\
&\leqslant 2R\int_0^{2R} \frac{1}{1+\mu\xi_2^2(r^2/4)/(8R^3)}\,d\xi_2 \\
&\lesssim R\left(\frac{\mu r^2}{R^3}\right)^{-1/2}\int_0^\infty \frac{1}{1+\tilde{\xi}_2^2}\,d\tilde{\xi}_2 \lesssim \frac{R^{5/2}}{r}\mu^{-1/2}.
\end{aligned} \tag{3.8}
$$

If ϕ is close to 0 so that $(r/2)\cos\phi - 2R\sin\phi \geqslant (r/4)\cos\phi$, that is, $\phi\in]0,\arctan(r/8R)]$, we similarly obtain

$$
\begin{aligned}
\tilde{H}(\mu,\phi) &\leqslant \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(r\cos\phi/4)^2/(8R^3)}\,d\xi_1\,d\xi_2 \\
&\lesssim \frac{R^{5/2}}{r}\mu^{-1/2}.
\end{aligned} \tag{3.9}
$$

For every $\phi\in[\arctan(r/8R),\pi/4]$, if $\xi_2\in[0,r/4]$, we find that $\xi_1-\xi_2\tan\phi\geqslant r/2-(r/4)\tan(\pi/4)=r/4$, thus we obtain

$$
\begin{aligned}
\tilde{H}(\mu,\phi) &\leqslant \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\cos\phi)^2(\xi_1-\xi_2\tan\phi)^2/(8R^3)}\,d\xi_1\,d\xi_2 \\
&\lesssim R\int_0^{r/4} \frac{1}{1+\mu\xi_2^2(r/4)^2/(16R^3)}\,d\xi_2 \\
&\quad + \int_{r/4}^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu(r/4)^2(\xi_1-\xi_2\tan\phi)^2/(16R)^3}\,d\xi_1\,d\xi_2 \\
&\lesssim \frac{R^{5/2}}{r}\mu^{-1/2} + R\int_{\mathbb{R}} \frac{1}{1+\mu(r/4)^2\tilde{\xi}_1^2/(16R)^3}\,d\tilde{\xi}_1 \lesssim \frac{R^{5/2}}{r}\mu^{-1/2}.
\end{aligned} \tag{3.10}
$$

For the other endpoint case $\phi=\pi/2$, we directly obtain

$$
\tilde{H}\left(\mu,\frac{\pi}{2}\right) \leqslant \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^4/(8R^3)}\,d\xi_1\,d\xi_2 \lesssim R\left(\frac{\mu}{R^3}\right)^{-1/4}\int_{\mathbb{R}} \frac{1}{1+\tilde{\xi}_2^4}\,d\tilde{\xi}_2 \lesssim R^{7/4}\mu^{-1/4}. \tag{3.11}
$$

Now we consider the case $\phi \in [\phi_0, \pi/2[$, where $\phi_0 \in [\pi/4, \pi/2[$ is a number chosen later. Noticing that

$$\tilde{H}(\mu,\phi) \leqslant \int_0^{2R}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2, \tag{3.12}$$

if $\xi_2 \leqslant (\cot\phi)r/4$, we observe that $\xi_1\cot\phi - \xi_2 \geqslant (\cot\phi)(r/4) > 0$, thus

$$\begin{aligned}
\mathrm{I} &\triangleq \int_0^{(r/4)\cot\phi}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2\\
&\leqslant 2R\int_0^{(r/4)\cot\phi} \frac{1}{1+\mu\xi_2^2(\cot\phi)^2(r/4)^2/(16R^3)}\,d\xi_2\\
&\lesssim \frac{R^{5/2}}{r\mu^{1/2}\cot\phi}\int_0^{\mu^{1/2}r^2(\cot\phi)^2/64R^{3/2}} \frac{1}{1+\tilde{\xi}_2^2}\,d\tilde{\xi}_2\\
&\lesssim R^{7/4}\mu^{-1/4}\frac{8R^{3/4}}{\mu^{1/4}r\cot\phi}\arctan\left(\frac{\mu^{1/2}r^2(\cot\phi)^2}{64R^{3/2}}\right).
\end{aligned}$$

Since $\lim_{x\to 0+}\arctan(x^2)/x = \lim_{x\to 0+} 2x/(1+x^2) = 0$, there exists an absolute positive constant c_0 such that for every $x \in]0, c_0]$, we obtain $\arctan(x^2)/x \leqslant 1$. Thus in order to find some $\phi_0 \in [\pi/4, \pi/2[$ satisfying $r\mu^{1/4}\cot(\phi_0)/8R^{3/4} \leqslant c_0$, we only need to choose

$$\phi_0 \triangleq \max\left\{\frac{\pi}{4}, \arctan\left(\frac{\mu^{1/4}r}{8c_0R^{3/4}}\right)\right\},$$

then, for every $\phi \in [\phi_0, \pi/2[$, we have

$$\mathrm{I} \lesssim R^{7/4}\mu^{-1/4}.$$

If $\xi_2 \geqslant 4R\cot\phi$, then we find that $|\xi_1\cot\phi - \xi_2| \geqslant \xi_2 - 2R\cot\phi \geqslant \xi_2/2$, thus

$$\begin{aligned}
\mathrm{II} &\triangleq \int_{4R\cot\phi}^{\infty}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2\\
&\leqslant 2R\int_0^{\infty}\frac{1}{1+\mu\tilde{\xi}_2^4/(64R^3)}\,d\tilde{\xi}_2 \lesssim R^{7/4}\mu^{-1/4}.
\end{aligned}$$

If $\xi_2 \in [(\cot\phi)r/4, 4R\cot\phi]$, noting that $\cot\phi \leqslant \cot\phi_0 \leqslant (8c_0R^{3/4}/r)\mu^{-1/4}$, we have

$$\begin{aligned}
\mathrm{III} &\triangleq \int_{r(\cot\phi)/4}^{4R\cot\phi}\int_{r/2}^{2R} \frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2\\
&\leqslant (2R)(4R\cot\phi) \lesssim \frac{R^{11/4}}{r}\mu^{-1/4}.
\end{aligned}$$

Hence in the case of $\phi \in [\phi_0, \pi/2[$, we have

$$\tilde{H}(\mu,\phi) \leqslant \mathrm{I}+\mathrm{II}+\mathrm{III} \lesssim \frac{R^{11/4}}{r}\mu^{-1/4}. \tag{3.13}$$

Finally, it remains to consider the case $\phi \in [\pi/4, \phi_0]$. Also by virtue of (3.12), if $\xi_2 \leqslant r(\cot\phi_0)/4$, we know that $\xi_1\cot\phi - \xi_2 \geqslant (r/2)\cot\phi_0 - r(\cot\phi_0)/4 = r(\cot\phi_0)/4$, and combining with the fact that $r\cot\phi_0=\min\{r,8c_0R^{3/4}\mu^{-1/4}\}$, we have

$$\begin{aligned}
\mathrm{I} &\triangleq \int_0^{r(\cot\phi_0)/4}\int_{r/2}^{2R}\frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2\\
&\leqslant 2R\int_0^{\infty}\frac{1}{1+\mu\xi_2^2(\min\{r/4,2c_0R^{3/4}\mu^{-1/4}\})^2/(16R^3)}\,d\xi_2\\
&\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2},R^{7/4}\mu^{-1/4}\right\}.
\end{aligned}$$

Otherwise, if $\xi_2\geqslant r(\cot\phi_0)/4=\min\{r/4,2c_0R^{3/4}\mu^{-1/4}\}$, we infer that

$$\begin{aligned}
\mathrm{II} &\triangleq \int_{r(\cot\phi_0)/4}^{2R}\int_{r/2}^{2R}\frac{1}{1+\mu\xi_2^2(\xi_1\cot\phi-\xi_2)^2/(16R^3)}\,d\xi_1\,d\xi_2\\
&\leqslant R\int_{\mathbb{R}}\frac{1}{1+\mu(r(\cot\phi_0)/4)^2\tilde{\xi}_2^2/(16R^3)}\,d\tilde{\xi}_2\\
&\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2},R^{7/4}\mu^{-1/4}\right\}.
\end{aligned}$$

Therefore in the case of $\phi\in[\pi/4,\phi_0]$, we have

$$\tilde{H}(\mu,\phi)\leqslant \mathrm{I}+\mathrm{II}\lesssim \max\left\{\frac{R^{5/2}}{r}\mu^{-1/2},R^{7/4}\mu^{-1/4}\right\}. \tag{3.14}$$

Collecting the above estimates, this finishes the proof of this lemma. □

Next we are devoted to proving Proposition 3.1 based on Lemma 3.3.

Proof of Proposition 3.1. Noting that

$$\begin{aligned}
\mathcal{G}^A(t)g(x) &= \int_{\mathbb{R}^2}\Psi(\xi)\hat{g}(\xi)\,e^{iAta(\xi)-\nu t|\xi|^\alpha+ix\cdot\xi}\,d\xi\\
&= K(t,At,\cdot)*g(x),
\end{aligned}$$

where K is defined by (3.4), we apply Lemma 3.3 to obtain

$$\|\mathcal{G}^A(t)g\|_{L^\infty_x}\lesssim_{r,R}(At)^{-1/4}\,e^{-\nu r^\alpha t/4}\|g\|_{L^1}.$$

On the other hand, by the Planchrel theorem, we find

$$\|\mathcal{G}^A(t)g\|_{L^2_x}\leqslant e^{-\nu r^\alpha t/2}\|g\|_{L^2}.$$

Thus from interpolation, we have the following dispersive estimates that for every $q\in[2,\infty]$ and $t\in\mathbb{R}^+$,

$$\|\mathcal{G}^A(t)g\|_{L^q_x}=\|K(t,At,\cdot)*g(x)\|_{L^q_x}\lesssim_{r,R}(At)^{-(q-2)/4q}\,e^{-r^\alpha\nu t/4}\|g\|_{L^{q'}}, \tag{3.15}$$

where $q'\triangleq q/(q-1)$ is the dual number of q.

Now we shall use the classical duality method, also called as TT^*-method, to show the expected estimates. For every $q\in[2,\infty]$, denoting by

$$\mathcal{U}_q\triangleq\{\varphi\in\mathcal{D}(\mathbb{R}^+\times\mathbb{R}^2):\|\varphi\|_{L^\infty(\mathbb{R}^+;L^{q'}(\mathbb{R}^2))}\leqslant 1\},$$

we have

$$\begin{aligned}\|\mathcal{G}^A(t)g\|_{L^1(\mathbb{R}^+;L^q)} &= \sup_{\varphi\in\mathcal{U}_q}\left|\int_{\mathbb{R}^+}\langle \mathcal{G}^A(t)g(x),\varphi(t,x)\rangle_{L^2_x}\,dt\right|\\ &= \sup_{\varphi\in\mathcal{U}_q}\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\hat{g}(\xi)\Psi(\xi)\,e^{iAta(\xi)-\nu t|\xi|^\alpha}\,\bar{\hat{\varphi}}(t,\xi)\,dt\,d\xi\right|\\ &\leqslant \|g\|_{L^2}\sup_{\varphi\in\mathcal{U}_q}\left\|\int_{\mathbb{R}^+}\Psi(\xi)\bar{\hat{\varphi}}(t,\xi)\,e^{iAta(\xi)-\nu t|\xi|^\alpha}\,dt\right\|_{L^2_\xi}.\end{aligned}$$

Taking advantage of the Plancherel theorem, the Hölder inequality and (3.15), we obtain

$$\begin{aligned}&\left\|\int_{\mathbb{R}^+}\Psi(\xi)\bar{\hat{\varphi}}(t,\xi)\,e^{iAta(\xi)-\nu t|\xi|^\alpha}\,dt\right\|^2_{L^2_\xi}\\ &\quad=\int_{\mathbb{R}^2_\xi}\int_{(\mathbb{R}^+)^2}\Psi(\xi)\bar{\hat{\varphi}}(t,\xi)\,e^{itAa(\xi)-\nu t|\xi|^\alpha}\,\bar{\Psi}(\xi)\hat{\varphi}(\tau,\xi)\,e^{-i\tau Aa(\xi)-\nu\tau|\xi|^\alpha}\,dt\,d\tau\,d\xi\\ &\quad=\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\langle\Psi(\xi)\hat{\varphi}(\tau,\xi)\,e^{i(t-\tau)Aa(\xi)-\nu(t+\tau)|\xi|^\alpha},\Psi(\xi)\hat{\varphi}(t,\xi)\rangle_{L^2_\xi}\,d\tau\,dt\\ &\quad=\int_{\mathbb{R}^+}\int_0^t\langle K(t+\tau,(t-\tau)A,\cdot)*\varphi(\tau,x),(\mathcal{F}^{-1}\Psi)*\varphi(t,x)\rangle_{L^2_x}\,d\tau\,dt\\ &\qquad+\int_{\mathbb{R}^+}\int_t^\infty\langle(\mathcal{F}^{-1}\Psi)*\varphi(\tau,x),K(t+\tau,(\tau-t)A,\cdot)*\varphi(t,x)\rangle_{L^2_x}\,d\tau\,dt\\ &\quad\leqslant C_{r,R}\int_{(\mathbb{R}^+)^2}\left(\frac{1}{A|t-\tau|}\right)^{(q-2)/4q}e^{-r^\alpha\nu(t+\tau)/4}\|\varphi\|^2_{L^\infty(\mathbb{R}^+;L^{q'}(\mathbb{R}^2))}\,d\tau\,dt.\end{aligned}$$

Since, for every $q\in[2,\infty]$, $\varphi\in\mathcal{U}_q$ and

$$\int_{(\mathbb{R}^+)^2}\left(\frac{1}{|t-\tau|}\right)^{(q-2)/4q}e^{-r^\alpha\nu(t+\tau)/4}\,d\tau\,dt\leqslant C_{\nu,q},$$

we obtain

$$\|\mathcal{G}^A(t)g\|_{L^1(\mathbb{R}^+;L^q)}\lesssim_{r,R,q,\nu}A^{-(q-2)/8q}\|g\|_{L^2}\quad\forall q\in[2,\infty].$$

By the Bernstein inequality and the Plancherel theorem, we also have

$$\|\mathcal{G}^A(t)g\|_{L^\infty(\mathbb{R}^+;L^q)}\lesssim_R\|\mathcal{G}^A(t)g\|_{L^\infty(\mathbb{R}^+;L^2)}\lesssim_R\|g\|_{L^2}.$$

From interpolation, we infer that for every $p\in[1,\infty]$ and $q\in[2,\infty]$

$$\|\mathcal{G}^A(t)g\|_{L^p(\mathbb{R}^+;L^q)}\lesssim_{r,R,q,p,\nu}A^{-(1/8p)((q-2)/q)}\|g\|_{L^2}.\qquad\square$$

4. *Proof of Theorem* 1.3

This section is dedicated to the proof of the global existence and convergence of weak solutions to the dispersive dissipative QG equation (1.1).

Since $\bar{\theta}(t,x_2)$ solving (1.5) is globally and uniquely defined, we only need to consider the difference $\Theta^A(t,x)\triangleq\theta^A(t,x)-\bar{\theta}(t,x_2)$, with the associated difference equation formally given by

$$\begin{cases}\partial_t\Theta^A+(\mathcal{R}^\perp\Theta^A)\cdot\nabla\Theta^A-(\mathcal{H}\bar{\theta})\partial_1\Theta^A+\nu|D|^\alpha\Theta^A+A(\mathcal{R}_1\Theta^A)=-(\mathcal{R}_1\Theta^A)\partial_2\bar{\theta},\\ \Theta^A(0,x)=\tilde{\theta}_0(x),\end{cases}\tag{4.1}$$

where $\mathcal{H}$ is the usual Hilbert transform in $\mathbb{R}_{x_2}$. Note that we have used the following facts that $\partial_1\bar\theta = 0$, $\mathcal{R}_1\bar\theta = (-|D|^{-1}\partial_1)\bar\theta = 0$ and

$$\mathcal{R}_2\bar\theta(x) = \int_{\mathbb{R}^2_\xi} e^{ix\cdot\xi}\left(-i\frac{\xi_2}{|\xi|}\right)\hat{\bar\theta}(\xi_2)\delta(\xi_1)\,d\xi = \int_{\mathbb{R}} e^{ix_2\xi_2}\left(-i\frac{\xi_2}{|\xi_2|}\right)\hat{\bar\theta}(\xi_2)\,d\xi_2 = \mathcal{H}\bar\theta(x_2),$$

and

$$|D|^\alpha\bar\theta(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}|\xi|^\alpha\hat{\bar\theta}(\xi_2)\delta(\xi_1)\,d\xi = \int_{\mathbb{R}} e^{ix_2\xi_2}|\xi_2|^\alpha\hat{\bar\theta}(\xi_2)\,d\xi_2 = |D_2|^\alpha\bar\theta(x_2),$$

with $\delta(\cdot)$ the Dirac-δ function.

4.1. *Existence of solutions to the perturbed equation* (4.1)

We first consider the *a priori* estimates. By taking the L^2 inner product of (4.1) with Θ^A, integration by parts, and from (1.4) and the fact that $\nabla\cdot(\mathcal{R}^\perp\Theta^A) = 0$ and $\partial_1(\mathcal{H}\bar\theta(x_2)) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Theta^A(t)\|^2_{L^2} + \nu\||D|^{\alpha/2}\Theta^A(t)\|^2_{L^2} = -\int_{\mathbb{R}^2}\mathcal{R}_1\Theta^A(t,x)\partial_2\bar\theta(t,x_2)\cdot\Theta^A(t,x)\,dx.$$

From the Hölder inequality, Sobolev embedding ($\dot H^{\alpha/2}(\mathbb{R}) \hookrightarrow L^{2/(1-\alpha)}(\mathbb{R})$) and the Calderón–Zygmund theorem, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta^A(t)\|^2_{L^2} + \nu\||D|^{\alpha/2}\Theta^A(t)\|^2_{L^2} &\leqslant \|\mathcal{R}_1\Theta^A(t)\|_{L^{2,2/(1-\alpha)}_{x_1,x_2}}\|\partial_2\bar\theta(t)\|_{L^{2/\alpha}_{x_2}}\|\Theta^A(t)\|_{L^2}\\ &\leqslant C\||D_2|^{\alpha/2}\Theta^A(t)\|_{L^2}\|\partial_2\bar\theta(t)\|_{L^{2/\alpha}_{x_2}}\|\Theta^A(t)\|_{L^2}\\ &\leqslant C\||D|^{\alpha/2}\Theta^A(t)\|_{L^2}\|\partial_2\bar\theta(t)\|_{L^{2/\alpha}_{x_2}}\|\Theta^A(t)\|_{L^2}.\end{aligned}$$

Using the Young inequality, we further have

$$\frac{1}{2}\frac{d}{dt}\|\Theta^A(t)\|^2_{L^2} + \frac{\nu}{2}\||D|^{\alpha/2}\Theta^A(t)\|^2_{L^2} \leqslant \frac{C}{\nu}\|\partial_2\bar\theta(t)\|^2_{L^{2/\alpha}_{x_2}}\|\Theta^A(t)\|^2_{L^2}.$$

Gronwall's inequality ensures that

$$\|\Theta^A(t)\|^2_{L^2} + \nu\int_0^t\|\Theta^A(\tau)\|^2_{\dot H^{\alpha/2}}\,d\tau \leqslant \|\tilde\theta_0\|^2_{L^2}\exp\left\{\frac{C}{\nu}\|\partial_2\bar\theta\|^2_{L^2_tL^{2/\alpha}}\right\}.$$

From the Sobolev embedding ($\dot H^{(1-\alpha)/2}(\mathbb{R}) \hookrightarrow L^{2/\alpha}(\mathbb{R})$) and the energy-type estimate of the linear dissipative equation (1.5), we find

$$\|\partial_2\bar\theta\|^2_{L^2_tL^{2/\alpha}} \lesssim \|\bar\theta\|^2_{L^2_t\dot H^{3/2-\alpha/2}} \lesssim \nu^{-1}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}. \tag{4.2}$$

Hence, we finally obtain that for every $t\in\mathbb{R}^+$

$$\|\Theta^A(t)\|^2_{L^2} + \nu\int_0^t\|\Theta^A(\tau)\|^2_{\dot H^{\alpha/2}}\,d\tau \leqslant \|\tilde\theta_0\|^2_{L^2}\exp\left\{\frac{C}{\nu^2}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}\right\}. \tag{4.3}$$

Next we sketch the proof of the global existence of solution to (4.1). We have the following approximate system:

$$\begin{cases}\partial_t\Theta^A_\epsilon + (\mathcal{R}^\perp\Theta^A_\epsilon)\cdot\nabla\Theta^A_\epsilon - (\mathcal{H}\bar\theta_\epsilon)\,\partial_1\Theta^A_\epsilon + \nu|D|^\alpha\Theta^A_\epsilon + A(\mathcal{R}_1\Theta^A_\epsilon)\\ \quad -\epsilon\Delta\Theta^A_\epsilon = -(\mathcal{R}_1\Theta^A_\epsilon)\,\partial_2\bar\theta_\epsilon,\\ \partial_t\bar\theta_\epsilon + \nu|D_2|^\alpha\bar\theta_\epsilon = 0,\\ \Theta^A(0,x) = \varphi_\epsilon * \tilde\theta_0(x), \quad \bar\theta_\epsilon(0,x_2) = \tilde\varphi_\epsilon * \bar\theta_0(x_2),\end{cases} \tag{4.4}$$

where $\varphi_\epsilon(x) = \epsilon^{-2}\varphi(x/\epsilon)$ and $\varphi\in\mathcal{D}(\mathbb{R}^2)$ satisfies $\int_{\mathbb{R}^2}\varphi = 1$, while $\tilde\varphi_\epsilon(x_2) = \epsilon^{-1}\tilde\varphi(x_2/\epsilon)$ and $\tilde\varphi\in\mathcal{D}(\mathbb{R})$ satisfies $\int_{\mathbb{R}}\tilde\varphi = 1$. Let $m>2$ and $m\in\mathbb{Z}^+$, and fix $\epsilon>0$. Since $\|\varphi_\epsilon * \tilde\theta_0\|_{H^m} \lesssim_\epsilon \|\tilde\theta_0\|_{L^2}$ and $\|\tilde\varphi_\epsilon * \bar\theta_0\|_{H^m_{x_2}} \lesssim_\epsilon \|\bar\theta_0\|_{L^2_{x_2}}$, and since $-\epsilon\Delta\Theta^A_\epsilon$ is the subcritical dissipation, from the standard energy method we find that for all $T > 0$

$$\sup_{t\in[0,T]} \|\Theta^A_\epsilon(t)\|_{H^m} \leqslant C(\epsilon, T, \varphi, \tilde{\varphi}, \|\tilde{\theta}_0\|_{L^2}, \|\bar{\theta}_0\|_{L^2}).$$

This estimate combined with a Galerkin approximation process yields the global existence of a strong solution $(\Theta^A_\epsilon, \bar{\theta}_\epsilon)$ to (4.4). Furthermore, from (4.3) and the estimation $\|\varphi_\epsilon * f\|_{H^s} \leqslant \|f\|_{H^s}$, $\forall s \in \mathbb{R}$, we have the uniform energy inequality with respect to ϵ that for all $T > 0$

$$\begin{aligned} \|\Theta^A_\epsilon(T)\|^2_{L^2} + \nu \int_0^T \|\Theta^A_\epsilon(s)\|^2_{\dot{H}^{\alpha/2}}\, ds &\leqslant \|\varphi_\epsilon * \tilde{\theta}_0\|^2_{L^2} \exp\left\{\frac{C}{\nu^2}\|\tilde{\varphi}_\epsilon * \bar{\theta}_0\|^2_{H^{3/2-\alpha}}\right\} \\ &\leqslant \|\tilde{\theta}_0\|^2_{L^2} \exp\left\{\frac{C}{\nu^2}\|\bar{\theta}_0\|^2_{H^{3/2-\alpha}}\right\}. \end{aligned} \tag{4.5}$$

Hence this ensures that, up to a subsequence, Θ^A_ϵ converges weakly (or weakly-$*$) to a function Θ^A in $L^\infty_T L^2 \cap L^2_T \dot{H}^{\alpha/2}$. Similarly as the case of the dissipative QG equation, from the compactness argument, we further obtain that as ϵ tends to 0,

$$\begin{cases} \Theta^A_\epsilon \longrightarrow \Theta^A, \\ \mathcal{R}_j \Theta^A_\epsilon \longrightarrow \mathcal{R}_j \Theta^A, \quad j = 1, 2, \end{cases} \quad \text{strongly in } L^2([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^2)).$$

Since $\bar{\theta}_0 \in H^{3/2-\alpha}(\mathbb{R})$, it is clear to see that $\bar{\theta}_\epsilon$ strongly converges to $\bar{\theta} = e^{-\nu t|D_2|}\bar{\theta}_0$ in $L^\infty([0,T]; H^{3/2-\alpha}(\mathbb{R}))$. Therefore, we can pass to the limit in (4.4) to show that Θ^A is a weak solution of (4.1).

4.2. *Proof of estimation* (1.6)

Now we show the strong convergence of Θ^A by using the Strichartz-type estimate (3.3). To this end, we introduce the following cut-off operator:

$$\mathcal{I}_{r,R} = \mathcal{I}_{r,R}(D) \triangleq \chi\left(\frac{|D|}{R}\right)\left(\mathrm{Id} - \frac{\chi(|D_1|)}{r}\right), \tag{4.6}$$

where $0 < r < R$ and $\chi \in \mathcal{D}(\mathbb{R})$ satisfies that $\chi(x) \equiv 1$ for all $|x| \leqslant 1$ and χ is compactly supported in $\{x : |x| < 2\}$. Then for the term $\mathcal{I}_{r,R}\Theta^A$, we have the following estimation (with its proof placed at the end of this subsection).

LEMMA 4.1. *Let r and R be two positive numbers satisfying $r < R$. Then for every $T > 0$ and $\sigma \in]2, \infty[$, there exists an absolute constant $\tilde{C}$ depending on r, R, T, σ, ν, $\|\bar{\theta}_0\|_{H^{3/2-\alpha}}$ and $\|\tilde{\theta}_0\|_{L^2}$ but independent of A such that*

$$\|\mathcal{I}_{r,R}\Theta^A\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} \leqslant \tilde{C} A^{-1/16(1-2/\sigma)}. \tag{4.7}$$

Now we consider the contribution from the part of high frequency and the part of low frequency in ξ_1. From the Sobolev embedding, Berenstein inequality and the energy estimate (4.3), we obtain, for every $\sigma \in [2, 4/(2-\alpha)[$,

$$\begin{aligned} \|(\mathrm{Id} - \chi(|D|/R))\Theta^A\|_{L^2(\mathbb{R}^+;L^\sigma(\mathbb{R}^2))} &\lesssim \|(\mathrm{Id} - \chi(|D|/R))\Theta^A\|_{L^2(\mathbb{R}^+;\dot{H}^{1-2/\sigma}(\mathbb{R}^2))} \\ &\lesssim R^{-(\alpha/2+2/\sigma-1)}\|(\mathrm{Id} - \chi(|D|/R))\Theta^A\|_{L^2(\mathbb{R}^+;\dot{H}^{\alpha/2}(\mathbb{R}^2))} \\ &\lesssim R^{-(2/\sigma-(2-\alpha)/2)}\|\tilde{\theta}_0\|_{L^2} \exp\left\{\frac{C}{\nu^2}\|\bar{\theta}_0\|^2_{H^{3/2-\alpha}}\right\}. \end{aligned}$$

Also thanks to the Bernstein inequality (in x_1 and x_2 separately), we find that for every $T > 0$ and $\sigma \in]2, \infty]$,

$$\begin{aligned}\|\chi(|D_1|/r)\chi(|D|/R)\Theta^A\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} &\lesssim T^{1/2} r^{1/2-1/\sigma} R^{1/2-1/\sigma} \|\Theta^A\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \\ &\lesssim_{T,R} r^{1/2-1/\sigma} \|\tilde\theta_0\|_{L^2} \exp\left\{\frac{C}{\nu^2}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}\right\}.\end{aligned}$$

Collecting the upper estimates, we have that for every $A, r, R, T > 0$ and $\sigma \in]2, 4/(2-\alpha)[$,

$$\|\Theta^A\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} \leqslant C_0 R^{-(2/\sigma-(2-\alpha)/2)} + C_{R,T} r^{1/2-1/\sigma} + \tilde C A^{-1/16(1-2/\sigma)},$$

where $\tilde C$ depends on r, R, T, $\|\tilde\theta_0\|_{L^2}$ and $\|\bar\theta_0\|_{H^{3/2-\alpha}}$ but not on A. Hence, passing A to ∞, then r to 0 and then R to ∞ yields the desired estimate (1.6).

At last it suffices to prove Lemma 4.1.

Proof of Lemma 4.1. By virtue of Duhamel's formula, we have

$$\begin{aligned}\mathcal{I}_{r,R}\Theta^A &= \mathcal{G}^A(t)\mathcal{I}_{r,R}\tilde\theta_0 - \int_0^t \mathcal{G}^A(t-\tau)\mathcal{I}_{r,R}(\mathcal{R}^\perp\Theta^A\cdot\nabla\Theta^A)(\tau)\,d\tau \\ &\quad + \int_0^t \mathcal{G}^A(t-\tau)\mathcal{I}_{r,R}(\mathcal{H}\bar\theta\partial_1\Theta^A - \mathcal{R}_1\Theta^A\partial_2\bar\theta)(\tau)\,d\tau \\ &\triangleq \Gamma_1 - \Gamma_2 + \Gamma_3.\end{aligned}$$

From the Strichartz-type estimate (3.3), we know that for every $\sigma \in]2, \infty[$

$$\|\Gamma_1\|_{L^2(\mathbb{R}^+;L^\sigma(\mathbb{R}^2))} \lesssim_{\sigma,r,R} A^{-1/16(1-2/\sigma)}\|\tilde\theta_0\|_{L^2(\mathbb{R}^2)}.$$

Applying the Minkowski inequality and again (3.3) to Γ_2, we infer that for every $\sigma \in]2, \infty[$ and $T > 0$,

$$\begin{aligned}\|\Gamma_2\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} &\leqslant \left(\int_0^T \left|\int_0^T 1_{[0,t]}(\tau)\|\mathcal{G}^A(t-\tau)\mathcal{I}_{r,R}(\mathcal{R}^\perp\Theta^A\cdot\nabla\Theta^A)(\tau)\|_{L^\sigma}\,d\tau\right|^2 dt\right)^{1/2} \\ &\leqslant \int_0^T \left(\int_\tau^T \|\mathcal{G}^A(t-\tau)\mathcal{I}_{r,R}(\mathcal{R}^\perp\Theta^A\cdot\nabla\Theta^A)(\tau)\|^2_{L^\sigma}\,dt\right)^{1/2} d\tau \\ &\lesssim_{r,R,\sigma} A^{-1/16(1-2/\sigma)}\int_0^T \|\mathcal{I}_{r,R}(\mathcal{R}^\perp\Theta^A\cdot\nabla\Theta^A)(\tau)\|_{L^2}\,d\tau.\end{aligned} \tag{4.8}$$

From Bernstein's inequality and the energy estimate (4.3), we further obtain

$$\begin{aligned}\|\mathcal{I}_{r,R}(\mathcal{R}^\perp\Theta^A\cdot\nabla\Theta^A)\|_{L^1([0,T];L^2(\mathbb{R}^2))} &\lesssim R^2\|(\mathcal{R}^\perp\Theta^A)\Theta^A\|_{L^1([0,T];L^1(\mathbb{R}^2))} \\ &\lesssim R^2 T\|\Theta^A\|^2_{L^\infty([0,T];L^2(\mathbb{R}^2))} \\ &\lesssim R^2 T\|\tilde\theta_0\|^2_{L^2}\exp\left\{\frac{C}{\nu^2}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}\right\}.\end{aligned}$$

Thus we have

$$\|\Gamma_2\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} \lesssim_{r,R,\sigma} A^{-1/16(1-2/\sigma)} T\|\tilde\theta_0\|^2_{L^2}\exp\left\{\frac{C}{\nu^2}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}\right\}.$$

For Γ_3, similarly as above, especially from Bernstein's inequality in x_2-variable and the Calderón–Zygmund theorem, we infer that for every $\sigma \in]2,\infty[$ and $T>0$,

$$\begin{aligned}\|\Gamma_3\|_{L^2([0,T];L^\sigma(\mathbb{R}^2))} &\lesssim A^{-1/16(1-2/\sigma)}\|\mathcal{I}_{r,R}((\mathcal{H}\bar\theta)\partial_1\Theta^A-(\mathcal{R}_1\Theta^A)\partial_2\bar\theta)\|_{L^1([0,T];L^2(\mathbb{R}^2))}\\ &\lesssim A^{-1/16(1-2/\sigma)}(R^{3/2}\|(\mathcal{H}\bar\theta)\Theta^A\|_{L^1([0,T];L^{2,1}_{x_1,x_2})}\\ &\quad+R^{3/2}\|(\mathcal{R}_1\Theta^A)\bar\theta\|_{L^1([0,T];L^{2,1}_{x_1,x_2})})\\ &\lesssim A^{-1/16(1-2/\sigma)}R^{3/2}T\|\bar\theta\|_{L^\infty([0,T];L^2(\mathbb{R}))}\|\Theta^A\|_{L^\infty([0,T];L^2(\mathbb{R}^2))}\\ &\lesssim A^{-1/16(1-2/\sigma)}R^{3/2}T\|\bar\theta_0\|_{L^2}\|\tilde\theta_0\|_{L^2}\exp\left\{\frac{C}{\nu^2}\|\bar\theta_0\|^2_{H^{3/2-\alpha}}\right\}.\end{aligned}$$

Hence, gathering the upper estimates leads to the expected estimate (4.7). □

5. *Proof of Theorem* 1.4

Now we show the global existence of θ^A as stated in Theorem 1.4. If we consider only equation (1.1) to obtain the $H^{2-\alpha}$ estimates of θ^A, due to (1.4), it seems impossible to derive an estimate global in time unless the data θ_0 are small enough (just as Proposition 1.1). Thus we shall adopt an idea from the work of Chemin, Desjardins, Gallagher and Grenier [**6**], that is, to subtract from equation (1.1) the solution $\tilde\theta^A$ of the linear equation (1.7) (or its main part $\mathcal{I}_{r,R}\tilde\theta^A$ with $\mathcal{I}_{r,R}$ defined in (4.6)). Roughly speaking, since from the Strichartz-type estimate (3.3), $\tilde\theta^A$ can be sufficiently small for A large enough, thus the equation of $\theta^A-\tilde\theta^A$ will have small initial data and small forcing terms, and we can hope to obtain the global existence result.

More precisely, we first introduce $\tilde\theta^A_m \triangleq \mathcal{I}_{r,R}\tilde\theta^A$ as the main part of $\tilde\theta^A$ which solves the following equation:

$$\partial_t\tilde\theta^A_m+\nu|D|^\alpha\tilde\theta^A_m+A\mathcal{R}_1\tilde\theta^A_m=0,\quad \tilde\theta^A_m|_{t=0}=\mathcal{I}_{r,R}\theta_0, \tag{5.1}$$

and since $\mathcal{I}_{r,R}\theta_0$ strongly converges to θ_0 in $H^{2-\alpha}(\mathbb{R}^2)$ as r tends to 0 and R tends to ∞, the difference $\tilde\theta^A-\tilde\theta^A_m$ is globally defined and can be made arbitrarily small in the functional spaces stated in Theorem 1.4. Hence, in what follows, we shall focus on the difference $\eta^A\triangleq\theta^A-\tilde\theta^A_m$ with r small enough and R large enough chosen later, and we shall be devoted to show the global existence of η^A. The corresponding equation can be written as

$$\begin{aligned}\partial_t\eta^A+\nu|D|^\alpha\eta^A+A\mathcal{R}_1\eta^A+(\mathcal{R}^\perp\eta^A)\cdot\nabla\eta^A+(\mathcal{R}^\perp\tilde\theta^A_m)\cdot\nabla\eta^A=F(\eta^A,\tilde\theta^A_m),\\ \eta^A|_{t=0}=(\mathrm{Id}-\mathcal{I}_{r,R})\theta_0.\end{aligned} \tag{5.2}$$

with the forcing term

$$F(\eta^A,\tilde\theta^A_m)\triangleq-(\mathcal{R}^\perp\tilde\theta^A_m)\cdot\nabla\tilde\theta^A_m-(\mathcal{R}^\perp\eta^A)\cdot\nabla\tilde\theta^A_m. \tag{5.3}$$

Note that for brevity, we have omitted the dependence of r,R in the notation of η^A and $\tilde\theta^A_m$.

5.1. *A priori estimates*

In this subsection, we mainly focus on the *a priori* estimates. The main result is the following claim: for any smooth solution η^A to (5.2) and for every $\epsilon>0$ small enough, there exist three positive absolute constants r_0, R_0 and A_0 such that for every $A\geqslant A_0$, we have

$$\sup_{t\geqslant0}\|\eta^A(t)\|^2_{H^{2-\alpha}(\mathbb{R}^2)}+\frac{\nu}{2}\int_{\mathbb{R}^+}\|\eta^A(t)\|^2_{\dot H^{2-\alpha/2}(\mathbb{R}^2)}\,dt\leqslant\epsilon. \tag{5.4}$$

For every $q\in\mathbb{N}$, applying Δ_q to equation (5.2) and denoting $\eta^A_q\triangleq\Delta_q\eta^A$, $F_q\triangleq\Delta_qF$, we obtain

$$\partial_t\eta^A_q+\nu|D|^\alpha\eta^A_q+A(\mathcal{R}_1\eta^A_q)+(\mathcal{R}^\perp\eta^A)\cdot\nabla\eta^A_q+(\mathcal{R}^\perp\tilde\theta^A_m)\cdot\nabla\eta^A_q=\widetilde F_q(\eta^A,\tilde\theta^A_m),$$

with

$$\tilde{F}_q(\eta^A, \tilde{\theta}^A_m) \triangleq -[\Delta_q, \mathcal{R}^\perp \eta^A] \cdot \nabla \eta^A - [\Delta_q, \mathcal{R}^\perp \tilde{\theta}^A_m] \cdot \nabla \eta^A + F_q(\eta^A, \tilde{\theta}^A_m).$$

Since η^A is real-valued, we know that η^A_q is also real-valued, thus taking L^2 inner product of the upper equation with η^A_q, and from the Bernstein inequality and the integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta^A_q(t)\|^2_{L^2} + \nu \||D|^{\alpha/2}\eta^A_q(t)\|^2_{L^2} &= \int_{\mathbb{R}^2} \tilde{F}_q(\eta^A, \tilde{\theta}^A_m)\eta^A_q(t,x)\,dx \\ &\leqslant 2^{-q(\alpha/2)}\|\tilde{F}_q(\eta^A, \tilde{\theta}^A_m)(t)\|_{L^2} 2^{q(\alpha/2)}\|\eta^A_q(t)\|_{L^2} \\ &\leqslant C_0 2^{-q(\alpha/2)}\|\tilde{F}_q(\eta^A, \tilde{\theta}^A_m)(t)\|_{L^2}\||D|^{\alpha/2}\eta^A_q(t)\|_{L^2}.\end{aligned}$$

From Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|\eta^A_q(t)\|^2_{L^2} + \frac{\nu}{2}\||D|^{\alpha/2}\eta^A_q(t)\|^2_{L^2} \leqslant \frac{C_0}{\nu}(2^{-q(\alpha/2)}\|\tilde{F}_q(\eta^A, \tilde{\theta}^A_m)(t)\|_{L^2})^2.$$

Integrating in time leads to

$$\|\eta^A_q(t)\|^2_{L^2} + \nu\||D|^{\alpha/2}\eta^A_q\|^2_{L^2_t L^2} \leqslant \|\eta^A_q(0)\|^2_{L^2} + \frac{C_0}{\nu}\int_0^t 2^{-q\alpha}\|\tilde{F}_q(\eta^A, \tilde{\theta}^A_m)(\tau)\|^2_{L^2}\,d\tau.$$

By multiplying both sides of the upper inequality by $2^{2q(2-\alpha)}$ and summing over all $q \in \mathbb{N}$, we obtain

$$\begin{aligned}&\sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\|\eta^A_q(t)\|^2_{L^2} + \nu\sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\||D|^{\alpha/2}\eta^A_q\|^2_{L^2_t L^2} \\ &\quad\leqslant \sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\|\eta^A_q(0)\|^2_{L^2} + \frac{C_0}{\nu}\int_0^t\left(\sum_{q\in\mathbb{N}} 2^{2q(2-3\alpha/2)}\|\tilde{F}_q(\eta^A, \tilde{\theta}^A_m)(\tau)\|^2_{L^2}\right)d\tau. \qquad (5.5)\end{aligned}$$

Using Lemma A.1 with $s = 2-\alpha$ and $\beta = \alpha/2$ yields that

$$\begin{aligned}\sum_{q\in\mathbb{N}} 2^{2q(2-(3/2)\alpha)}\|[\Delta_q, \mathcal{R}^\perp(\eta^A + \tilde{\theta}^A_m)] \cdot \nabla\eta^A\|^2_{L^2} &\lesssim_\alpha (\|\eta^A\|^2_{\dot{B}^{2-\alpha/2}_{2,2}} + \|\tilde{\theta}^A_m\|^2_{\dot{B}^{2-\alpha/2}_{2,2}})\|\eta^A\|^2_{B^{2-\alpha}_{2,2}} \\ &\lesssim_\alpha (\||D|^{\alpha/2}\eta^A\|^2_{B^{2-\alpha}_{2,2}} \\ &\qquad + \||D|^{\alpha/2}\tilde{\theta}^A_m\|^2_{B^{2-\alpha}_{2,2}})\|\eta^A\|^2_{B^{2-\alpha}_{2,2}},\end{aligned}$$

where in the second line we used the embedding $B^{2-\alpha}_{2,2} \hookrightarrow \dot{B}^{2-\alpha}_{2,2}$. From Lemma A.2-(1), we infer that

$$\sum_{q\in\mathbb{N}} 2^{2q(2-(3/2)\alpha)}\|\Delta_q(\mathcal{R}^\perp\eta^A \cdot \nabla\tilde{\theta}^A_m)\|^2_{L^2} \lesssim R^{6-3\alpha}\|\eta^A\|^2_{L^2}\|\tilde{\theta}^A_m\|^2_{L^\infty} + \|\eta^A\|^2_{B^{2-\alpha}_{2,2}}\||D|^{\alpha/2}\tilde{\theta}^A_m\|^2_{B^{2-\alpha}_{2,2}}$$

and

$$\sum_{q\in\mathbb{N}} 2^{2q(2-(3/2)\alpha)}\|\Delta_q(\mathcal{R}^\perp\tilde{\theta}^A_m \cdot \nabla\tilde{\theta}^A_m)\|^2_{L^2} \lesssim R^{6-3\alpha}\|\tilde{\theta}^A_m\|^2_{L^2}\|\tilde{\theta}^A_m\|^2_{L^\infty}.$$

Inserting the upper estimates to (5.5), we obtain

$$\begin{aligned}
&\sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\|\eta_q^A(t)\|_{L^2}^2 + \nu\sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\||D|^{\alpha/2}\eta_q^A\|_{L_t^2L^2}^2 \\
&\quad\leqslant \sum_{q\in\mathbb{N}} 2^{2q(2-\alpha)}\|\eta_q^A(0)\|_{L^2}^2 + \frac{C_\alpha}{\nu}\int_0^t (\||D|^{\alpha/2}\eta^A(\tau)\|_{B_{2,2}^{2-\alpha}}^2 \\
&\qquad + \||D|^{\alpha/2}\tilde\theta_m^A(\tau)\|_{B_{2,2}^{2-\alpha}}^2)\|\eta^A(\tau)\|_{B_{2,2}^{2-\alpha}}^2\, d\tau \\
&\qquad + \frac{C_\alpha}{\nu}R^{6-3\alpha}\int_0^t (\|\tilde\theta_m^A(\tau)\|_{L^2}^2 + \|\eta^A(\tau)\|_{L^2}^2)\|\tilde\theta_m^A(\tau)\|_{L^\infty}^2\, d\tau.
\end{aligned} \tag{5.6}$$

Now we consider the low-frequency part. Applying Δ_{-1} to equation (5.2), we have

$$\partial_t(\Delta_{-1}\eta^A) + \nu|D|^\alpha(\Delta_{-1}\eta^A) + A\mathcal{R}_1(\Delta_{-1}\eta^A) = \Delta_{-1}G(\eta^A, \tilde\theta_m^A),$$

where

$$G(\eta^A, \tilde\theta_m^A) \triangleq -(\mathcal{R}^\perp\eta^A\cdot\nabla\eta^A) - (\mathcal{R}^\perp\tilde\theta_m^A\cdot\nabla\eta^A) - (\mathcal{R}^\perp\eta^A\cdot\nabla\tilde\theta_m^A) - (\mathcal{R}^\perp\tilde\theta_m^A\cdot\nabla\tilde\theta_m^A).$$

By using the L^2 energy method, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_{-1}\eta^A(t)\|_{L^2}^2 + \nu\||D|^{\alpha/2}\Delta_{-1}\eta^A(t)\|_{L^2}^2 \leqslant \|\Delta_{-1}G(\eta^A, \tilde\theta_m^A)(t)\|_{\dot H^{-\alpha/2}}\||D|^{\alpha/2}\Delta_{-1}\eta^A(t)\|_{L^2}.$$

By virtue of the Young inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_{-1}\eta^A(t)\|_{L^2}^2 + \frac{\nu}{2}\||D|^{\alpha/2}\Delta_{-1}\eta^A(t)\|_{L^2}^2 \leqslant \frac{C_0}{\nu}\|\Delta_{-1}G(\eta^A, \tilde\theta_m^A)(t)\|_{\dot H^{-\alpha/2}}^2.$$

Integrating in time yields that

$$\begin{aligned}
&\|\Delta_{-1}\eta^A(t)\|_{L^2}^2 + \nu\||D|^{\alpha/2}\Delta_{-1}\eta^A\|_{L_t^2L^2}^2 \\
&\quad\leqslant \|\Delta_{-1}\eta^A(0)\|_{L^2}^2 + \frac{C_0}{\nu}\int_0^t \|\Delta_{-1}G(\eta^A, \tilde\theta_m^A)(\tau)\|_{\dot H^{-\alpha/2}}^2\, d\tau.
\end{aligned} \tag{5.7}$$

From Lemma A.2-(2), we deduce that

$$\begin{aligned}
\|\Delta_{-1}(\mathcal{R}^\perp\eta^A\cdot\nabla\eta^A)\|_{\dot H^{-\alpha/2}}^2 &\lesssim \sum_{-\infty<q\leqslant 0} 2^{-q\alpha}\|\dot\Delta_q(\mathcal{R}^\perp\eta^A\cdot\nabla\eta^A)\|_{L^2}^2 \\
&\lesssim \sum_{-\infty<q\leqslant 0} 2^{2q(2-\alpha)}\||D|^{\alpha/2}\eta^A\|_{L^2}^2\|\eta^A\|_{L^2}^2 \\
&\lesssim \||D|^{\alpha/2}\eta^A\|_{L^2}^2\|\eta^A\|_{L^2}^2,
\end{aligned}$$

and

$$\|\Delta_{-1}(\mathcal{R}^\perp\tilde\theta_m^A\cdot\nabla\eta^A)\|_{\dot H^{-\frac{\alpha}{2}}}^2 + \|\Delta_{-1}(\mathcal{R}^\perp\eta^A\cdot\nabla\tilde\theta_m^A)\|_{\dot H^{-\frac{\alpha}{2}}}^2 \lesssim \||D|^{\alpha/2}\tilde\theta_m^A\|_{L^2}^2\|\eta^A\|_{L^2}^2.$$

It is also obvious to see that

$$\|\Delta_{-1}(\mathcal{R}^\perp\tilde\theta_m^A\cdot\nabla\tilde\theta_m^A)\|_{\dot H^{-\frac{\alpha}{2}}}^2 \lesssim \|\Delta_{-1}((\mathcal{R}^\perp\tilde\theta_m^A)\,\tilde\theta_m^A)\|_{\dot H^{1-\alpha/2}}^2 \lesssim \|\tilde\theta_m^A\|_{L^2}^2\|\tilde\theta_m^A\|_{L^\infty}^2.$$

Inserting the upper estimates into (5.7), we obtain

$$\begin{aligned}&\|\Delta_{-1}\eta^A(t)\|_{L^2}^2+\nu\||D|^{\alpha/2}\Delta_{-1}\eta^A\|_{L_t^2L^2}^2\\&\quad\leqslant\|\Delta_{-1}\eta^A(0)\|_{L^2}^2+\frac{C_\alpha}{\nu}\int_0^t(\||D|^{\alpha/2}\eta^A(\tau)\|_{L^2}^2+\||D|^{\alpha/2}\tilde\theta_m^A(\tau)\|_{L^2}^2)\|\eta^A(\tau)\|_{L^2}^2\,d\tau\\&\qquad+\frac{C_\alpha}{\nu}\int_0^t\|\tilde\theta_m^A(\tau)\|_{L^2}^2\|\tilde\theta_m^A(\tau)\|_{L^\infty}^2\,d\tau.\end{aligned}\tag{5.8}$$

Combining this estimate with (5.6) leads to

$$\begin{aligned}&\|\eta^A(t)\|_{B^{2-\alpha}_{2,2}}^2+\nu\||D|^{\alpha/2}\eta^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2\\&\quad\leqslant\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2+\frac{C_\alpha}{\nu}\int_0^t(\||D|^{\alpha/2}\eta^A(\tau)\|_{B^{2-\alpha}_{2,2}}^2+\||D|^{\alpha/2}\tilde\theta_m^A(\tau)\|_{B^{2-\alpha}_{2,2}}^2)\|\eta^A(\tau)\|_{B^{2-\alpha}_{2,2}}^2\,d\tau\\&\qquad+\frac{C_\alpha}{\nu}R^{6-3\alpha}\int_0^t(\|\tilde\theta_m^A(\tau)\|_{L^2}^2+\|\eta^A(\tau)\|_{L^2}^2)\|\tilde\theta_m^A(\tau)\|_{L^\infty}^2\,d\tau.\end{aligned}$$

From the fact that $\|\eta^A\|_{L_t^\infty L^2}\leqslant\|\tilde\theta_m^A\|_{L_t^\infty L^2}+\|\theta^A\|_{L_t^\infty L^2}\leqslant 2\|\theta_0\|_{L^2}$, we moreover find that

$$\begin{aligned}&|\eta^A\|_{L_t^\infty B^{2-\alpha}_{2,2}}^2+\nu\||D|^{\alpha/2}\eta^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2\\&\quad\leqslant\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2+\frac{C_\alpha}{\nu}\int_0^t\||D|^{\alpha/2}\tilde\theta_m^A(\tau)\|_{B^{2-\alpha}_{2,2}}^2\|\eta^A\|_{L_\tau^\infty B^{2-\alpha}_{2,2}}^2\,d\tau\\&\qquad+\frac{C_\alpha}{\nu}\|\eta^A\|_{L_t^\infty B^{2-\alpha}_{2,2}}^2\||D|^{\alpha/2}\eta^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2+\frac{C_\alpha}{\nu}R^{6-3\alpha}\|\theta_0\|_{L^2}^2\int_0^t\|\tilde\theta_m^A(\tau)\|_{L^\infty}^2\,d\tau.\end{aligned}\tag{5.9}$$

Set

$$T_A^*\triangleq\sup\left\{t\geqslant 0;\ \|\eta^A\|_{L_t^\infty B^{2-\alpha}_{2,2}}^2<\frac{\nu^2}{2C_\alpha}\right\},$$

due to $\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2=\|(\mathrm{Id}-\mathcal{I}_{r,R})\theta_0\|_{B^{2-\alpha}_{2,2}}^2$, and by the Lebesgue theorem, we can choose some small number r and large number R such that $\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2\leqslant\nu^2/4C_\alpha$, thus $T_A^*>0$ follows from that η^A is a (continuous in time) smooth solution. Then, through the Strichartz-type estimate (3.3), we obtain that for every $t\in[0,T_A^*[$,

$$\begin{aligned}\|\eta^A\|_{L_t^\infty B^{2-\alpha}_{2,2}}^2+\frac{\nu}{2}\||D|^{\alpha/2}\eta^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2&\leqslant\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2+\frac{C_\alpha}{\nu}R^{6-3\alpha}C_{r,R}A^{-1/8}\|\theta_0\|_{L^2}^4\\&\quad+\frac{C_\alpha}{\nu}\int_0^t\||D|^{\alpha/2}\tilde\theta_m^A(\tau)\|_{B^{2-\alpha}_{2,2}}^2\|\eta^A\|_{L_\tau^\infty B^{2-\alpha}_{2,2}}^2\,d\tau.\end{aligned}$$

Gronwall's inequality yields that for every $t\in[0,T_A^*[$

$$\begin{aligned}\|\eta^A\|_{L_t^\infty B^{2-\alpha}_{2,2}}^2+\frac{\nu}{2}\||D|^{\alpha/2}\eta^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2&\leqslant\exp\left\{\frac{C_\alpha}{\nu^2}\|\theta_0\|_{H^{2-\alpha}}^2\right\}\\&\quad\times\left(\|\eta^A(0)\|_{B^{2-\alpha}_{2,2}}^2+\frac{C_{r,R,\alpha}}{\nu}A^{-1/8}\|\theta_0\|_{L^2}^4\right),\end{aligned}$$

where we have used the following fact that

$$\|\tilde\theta_m^A(t)\|_{B^{2-\alpha}_{2,2}}^2+\nu\||D|^{\alpha/2}\tilde\theta_m^A\|_{L_t^2B^{2-\alpha}_{2,2}}^2\leqslant\|\theta_0\|_{B^{2-\alpha}_{2,2}}^2\leqslant C_0\|\theta_0\|_{H^{2-\alpha}}^2.$$

For every $\epsilon>0$, we can further choose some small number r and large number R such that

$$\|(\mathrm{Id}-\mathcal{I}_{r,R})\theta_0\|_{B^{2-\alpha}_{2,2}}^2\exp\left\{\frac{C_\alpha}{\nu^2}\|\theta_0\|_{H^{2-\alpha}}^2\right\}\leqslant\frac{\epsilon}{2C_0},$$

where C_0 is the absolute constant from the relation $(1/C_0)\|f\|^2_{B^s_{2,2}} \leqslant \|f\|^2_{H^s} \leqslant C_0\|f\|^2_{B^s_{2,2}}$, $\forall s \in \mathbb{R}$. For fixed r and R, we can choose A large enough so that

$$A^{-1/8}\frac{C_{r,R,\alpha}}{\nu}\|\theta_0\|^4_{L^2}\exp\left\{\frac{C_\alpha}{\nu^2}\|\theta_0\|^2_{H^{2-\alpha}}\right\} \leqslant \frac{\epsilon}{2C_0}.$$

Hence for every $\epsilon > 0$ and for the appropriate r, R, A (that is, $r_0, R_0, A \geqslant A_0$), we have

$$\sup_{t\in[0,T^*_A[}\|\eta^A(t)\|^2_{B^{2-\alpha}_{2,2}} + \frac{\nu}{2}\int_0^{T^*_A}\||D|^{\alpha/2}\eta^A(t)\|^2_{B^{2-\alpha}_{2,2}}\,dt \leqslant \frac{\epsilon}{C_0}.$$

Furthermore, for every $\epsilon \leqslant C_0\nu^2/4C_\alpha$, we have $T^*_A = \infty$ and

$$\sup_{t\in\mathbb{R}^+}\|\eta^A(t)\|^2_{B^{2-\alpha}_{2,2}} + \frac{\nu}{2}\int_0^{\infty}\||D|^{\alpha/2}\eta^A(t)\|^2_{B^{2-\alpha}_{2,2}}\,dt \leqslant \frac{\epsilon}{C_0}.$$

Therefore (5.4) follows.

5.2. *Uniqueness*

For every $T > 0$, let θ^A_1 and θ^A_2 belonging to

$$L^\infty([0,T]; H^{2-\alpha}(\mathbb{R}^2)) \cap L^2([0,T]; \dot{H}^{2-\alpha/2}(\mathbb{R}^2))$$

be two solutions to (1.1) with the same initial data $\theta_0 \in H^{2-\alpha/2}(\mathbb{R}^2)$. Thus set $\delta\theta^A \triangleq \theta^A_1 - \theta^A_2$, and then the difference equation writes

$$\partial_t\delta\theta^A + (\mathcal{R}^\perp\theta^A_1)\cdot\nabla\delta\theta^A + \nu|D|^\alpha\delta\theta^A + A(\mathcal{R}_1\delta\theta^A) = -(\mathcal{R}^\perp\delta\theta^A)\cdot\nabla\theta^A_2,$$
$$\delta\theta^A|_{t=0} = \delta\theta^A_0(=0).$$

We use the L^2 energy argument to obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta\theta^A(t)\|^2_{L^2} + \frac{\nu}{2}\||D|^{\alpha/2}\delta\theta^A(t)\|^2_{L^2} \leqslant \|(\mathcal{R}^\perp\delta\theta^A)\cdot\nabla\theta^A_2(t)\|_{\dot{H}^{-\alpha/2}}\||D|^{\alpha/2}\delta\theta^A(t)\|_{L^2}.$$

From the following classical product estimate that for every divergence-free vector field $f \in \dot{H}^{s_1}(\mathbb{R}^2)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^2)$ with $s_1, s_2 < 1$ and $s_1 + s_2 > -1$,

$$\|f\cdot\nabla g\|_{\dot{H}^{s_1+s_2-1}(\mathbb{R}^2)} \lesssim_{s_1,s_2} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^2)}\|\nabla g\|_{H^{s_2}(\mathbb{R}^2)},$$

we know that

$$\|(\mathcal{R}^\perp\delta\theta^A)\cdot\nabla\theta^A_2(t)\|_{\dot{H}^{-\alpha/2}} \lesssim_\alpha \|\delta\theta^A\|_{L^2}\|\nabla\theta^A_2\|_{\dot{H}^{1-\frac{\alpha}{2}}}.$$

Thanks to the Young inequality, we further find

$$\frac{d}{dt}\|\delta\theta^A\|^2_{L^2} + \nu\|\delta\theta^A\|^2_{\dot{H}^{\alpha/2}} \lesssim_{\alpha,\nu} \|\nabla\theta^A_2(t)\|^2_{\dot{H}^{1-\alpha/2}}\|\delta\theta^A(t)\|^2_{L^2}. \tag{5.10}$$

Gronwall's inequality yields

$$\|\delta\theta^A(t)\|^2_{L^2} \leqslant \|\delta\theta^A_0\|^2_{L^2}\exp\left\{C\|\theta^A_2\|^2_{L^2([0,T];\dot{H}^{2-\alpha/2}(\mathbb{R}^2))}\right\} \lesssim \|\delta\theta_0\|^2_{L^2}\exp\left\{\frac{C}{\nu^2}\|\theta_0\|^2_{H^{2-\alpha}}\right\}.$$

Hence the uniqueness is guaranteed.

5.3. *Global existence*

From the Friedrich method, we consider the following approximate system:

$$\begin{cases} \partial_t\eta^A_k + \nu|D|^\alpha\eta^A_k + A\mathcal{R}_1\eta^A_k + J_k(\mathcal{R}^\perp\eta^A_k\cdot\nabla\eta^A_k) + J_k(\mathcal{R}^\perp\tilde{\theta}^A_m\cdot\nabla\eta^A_k) \\ \quad = J_kF(\eta^A_k,\tilde{\theta}^A_m), \ \ \eta^A_k|_{t=0} = J_k(\mathrm{Id}-\mathcal{I}_{r,R})\theta_0, \end{cases} \tag{5.11}$$

where $J_k : L^2 \mapsto J_k L^2$, $k \in \mathbb{N}$ is the projection operator such that $J_k f \triangleq \mathcal{F}^{-1}(1_{B(0,k)}(\xi)\hat{f}(\xi))$ and $\tilde{\theta}^A_m$ solving (5.1) is the main part of $\tilde{\theta}^A$. Indeed system (5.11) becomes an ordinary differential equation on the space $J_k L^2 \triangleq \{f \in L^2 : \operatorname{supp} \hat{f} \subset B(0,k)\}$ with the L^2 norm. Since

$$\|J_k(\mathcal{R}^\perp \eta^A_k \cdot \nabla \eta^A_k)\|_{L^2} \lesssim k\|\mathcal{R}^\perp \eta^A_k\|_{L^2}\|\nabla \eta^A_k\|_{L^2} \lesssim k^2\|\eta^A_k\|^2_{L^2}$$

and

$$\|J_k(\mathcal{R}^\perp \tilde{\theta}^A_m \cdot \nabla \eta^A_k + \mathcal{R}^\perp \eta^A_k \cdot \nabla \tilde{\theta}^A_m)\|_{L^2} \lesssim k^2\|\theta_0\|_{L^2}\|\eta^A_k\|_{L^2},$$

and $\|J_k(\mathcal{R}^\perp \tilde{\theta}^A_m \cdot \nabla \tilde{\theta}^A_m)\|_{L^2} \lesssim k^2\|\theta_0\|^2_{L^2}$, we have that for every $r, R > 0$ and $k \in \mathbb{N}$, there exists a unique solution $\eta^A_k \in \mathcal{C}^\infty([0, T_k[; J_k L^2)$ to system (5.11), with $T_k > 0$ the maximal existence time. Moreover, from the L^2 energy method and in a similar way as obtaining (5.10), we obtain

$$\frac{d}{dt}\|\eta^A_k\|^2_{L^2} + \nu\|\eta^A_k\|^2_{\dot{H}^{\alpha/2}} \lesssim_\nu \|\eta^A_k\|^2_{L^2}\|\tilde{\theta}^A_m\|^2_{\dot{H}^{2-\alpha/2}} + \|\tilde{\theta}^A_m\|^2_{L^2}\|\tilde{\theta}^A_m\|^2_{\dot{H}^{2-\alpha/2}}.$$

Gronwall's inequality and the energy-type estimate of the linear equation (1.7) yield that

$$\begin{aligned}\|\eta^A_k(t)\|^2_{L^2} &\leqslant \exp\{C_\nu\|\tilde{\theta}^A_m\|_{L^2_t \dot{H}^{2-\alpha/2}}\}(\|\eta^A_k(0)\|^2_{L^2} + \|\tilde{\theta}^A_m\|^2_{L^\infty_t L^2}\|\tilde{\theta}^A_m\|^2_{L^2_t \dot{H}^{2-\alpha/2}}) \\ &\leqslant \exp\{C_\nu\|\theta_0\|_{H^{2-\alpha}}\}(\|\theta_0\|^2_{L^2} + \|\theta_0\|^2_{L^2}\|\theta_0\|^2_{H^{2-\alpha}}).\end{aligned}$$

Hence the classical continuation criterion ensures that $T_k = \infty$ and $\eta^A_k \in \mathcal{C}^\infty(\mathbb{R}^+; J_k L^2)$ is a global solution to the system (5.11). This further guarantees the *a priori* estimate in Section 5.1, that is, we obtain that there exist positive absolute constants ϵ_0, r_0, R_0 and A_0 independent of k such that for every $0 < \epsilon < \epsilon_0$ and $A > A_0$,

$$\sup_{t\in\mathbb{R}^+} \|\eta^A_k(t)\|^2_{H^{2-\alpha}} + \nu \int_{\mathbb{R}^+} \|\eta^A_k(t)\|^2_{\dot{H}^{2-\alpha/2}}\, dt \leqslant \epsilon.$$

On the basis of this uniform estimate, it is not hard to show that $(\eta^A_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}^+; L^2(\mathbb{R}^2))$, and thus it converges strongly to a function $\eta^A \in \mathcal{C}(\mathbb{R}^+; L^2(\mathbb{R}^2))$. By a standard process, one can prove that η^A solves system (5.2) and $\eta^A \in L^\infty(\mathbb{R}^+; H^{2-\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^{2-\alpha/2}(\mathbb{R}^2))$. Moreover, from the proof in Section 5.1 and by replacing $\|\eta^A\|_{L^\infty_t B^{2-\alpha}_{2,2}}$ with $\|\eta^A\|_{\tilde{L}^\infty_t B^{2-\alpha}_{2,2}}$ in (5.9), one indeed can prove that $\eta^A \in \tilde{L}^\infty(\mathbb{R}^+; B^{2-\alpha}_{2,2}(\mathbb{R}^2))$, and this implies that $\eta^A \in \mathcal{C}(\mathbb{R}^+; H^{2-\alpha}(\mathbb{R}^2))$. Finally, let $\theta^A = \eta^A + \tilde{\theta}^A_m$, then for A large enough θ^A is the unique solution to the dispersive dissipative QG equation (1.1), and as $A \to \infty$, $r \to 0$, $R \to \infty$ and $\epsilon \to 0$ one-by-one, we obtain the expected convergence (1.8).

Appendix

We first consider some commutator estimates.

LEMMA A.1. *Let $v = (v_1, \ldots, v_n)$ be a smooth divergence-free vector field over $\mathbb{R}^n$ and f be a smooth scalar function of $\mathbb{R}^n$. Then, for every $q \in \mathbb{N}$, $\beta \in]0, 1 + n/2[$ and $s \in]\beta - 1 - n/2, 1 + n/2[$, there exists a positive absolute constant C depending only on β, s and n such that*

$$2^{q(s-\beta)}\|[\Delta_q, v] \cdot \nabla f\|_{L^2(\mathbb{R}^n)} \leqslant Cc_q\|v\|_{\dot{B}^{1+\frac{n}{2}-\beta}_{2,2}(\mathbb{R}^n)}\|f\|_{B^s_{2,2}(\mathbb{R}^n)},$$

where $(c_q)_{q\in\mathbb{N}}$ satisfies $\sum_{q\in\mathbb{N}}(c_q)^2 \leqslant 1$. Especially, if $n = 2$ and $v = \mathcal{R}^\perp f$, we also have that for every $\beta > 0$ and $s > \beta - 1 - n/2$,

$$2^{q(s-\beta)}\|[\Delta_q, v] \cdot \nabla f\|_{L^2(\mathbb{R}^n)} \leqslant Cc_q\|f\|_{\dot{B}^{1+\frac{n}{2}-\beta}_{2,2}(\mathbb{R}^n)}\|f\|_{B^s_{2,2}(\mathbb{R}^n)},$$

with $(c_q)_{q\in\mathbb{N}}$ satisfying $\sum_{q\in\mathbb{N}}(c_q)^2 \leqslant 1$.

*Proof of Lemma A.*1. From Bony's decomposition, we have

$$\begin{aligned}[\Delta_q, v]\cdot\nabla f &= \sum_{|k-q|\leqslant 4}[\Delta_q, S_{k-1}v]\cdot\nabla\Delta_k f + \sum_{|k-q|\leqslant 4}[\Delta_q, \Delta_k v]\cdot\nabla S_{k-1}f\\ &\quad + \sum_{k\geqslant q-3}[\Delta_q, \Delta_k v]\cdot\nabla\widetilde{\Delta}_k f\\ &\triangleq \mathrm{I}_q + \mathrm{II}_q + \mathrm{III}_q.\end{aligned}$$

For I_q, thanks to the expression $\Delta_q = h_q(\cdot)* = 2^{qn}h(2^q\cdot)*$ with $h \triangleq \mathcal{F}^{-1}(\psi)\in\mathcal{S}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}2^{q(s-\beta)}\|\mathrm{I}_q\|_{L^2} &\lesssim \sum_{|k-q|\leqslant 4}\|xh_q\|_{L^1}2^{q(s-\beta)}\|\nabla S_{k-1}v\|_{L^\infty}\|\nabla\Delta_k f\|_{L^2}\\ &\lesssim \sum_{|k-q|\leqslant 4}2^{q(s-\beta-1)}2^{k(1-s)}\left(\sum_{k_1\leqslant k-2}2^{k_1\beta}2^{k_1(1+n/2-\beta)}\|\dot{\Delta}_{k_1}v\|_{L^2}(2^{ks}\|\Delta_k f\|_{L^2})\right)\\ &\lesssim \sum_{-\infty<k_1\leqslant q+2}2^{(k_1-q)\beta}2^{k_1(1+n/2-\beta)}\|\dot{\Delta}_{k_1}v\|_{L^2}\|f\|_{B^s_{2,2}}\\ &\lesssim c_q\|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\|f\|_{B^s_{2,2}},\end{aligned}$$

with $(c_q)_{q\in\mathbb{N}}$ satisfying $\sum_{q\in\mathbb{N}}(c_q)^2\leqslant 1$. For II_q, we directly obtain that for every $s<1+n/2$

$$\begin{aligned}2^{q(s-\beta)}\|\mathrm{II}_q\|_{L^2} &\lesssim \sum_{|k-q|\leqslant 4;k\in\mathbb{N}}2^{q(s-\beta)}\|\Delta_k v\|_{L^2}\|\nabla S_{k-1}f\|_{L^\infty}\\ &\lesssim \sum_{|k-q|\leqslant 4;k\in\mathbb{N}}2^{k(s-\beta)}\|\Delta_k v\|_{L^2}\left(\sum_{k_1\leqslant k-2}2^{k_1(1+n/2-s)}(2^{k_1 s}\|\Delta_{k_1}f\|_{L^2})\right)\\ &\lesssim \|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\left(\sum_{k_1\leqslant q+2}2^{(k_1-q)(1+n/2-s)}(2^{k_1 s}\|\Delta_{k_1}f\|_{L^2})\right)\\ &\lesssim c_q\|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\|f\|_{B^s_{2,2}}.\end{aligned}$$

In particular, when $n=2$ and $v=\mathcal{R}^\perp f$, using the Calderón–Zygmund theorem we obtain

$$\begin{aligned}2^{q(s-\beta)}\|\mathrm{II}_q\|_{L^2} &\lesssim \sum_{|k-q|\leqslant 4;k\in\mathbb{N}}2^{q(s-\beta)}\|\Delta_k v\|_{L^2}\|\nabla S_{k-1}f\|_{L^\infty}\\ &\lesssim \sum_{|k-q|\leqslant 4;k\in\mathbb{N}}2^{ks}\|\Delta_k f\|_{L^2}2^{-k\beta}\left(\sum_{-\infty<k_1\leqslant k-2}2^{k_1\beta}2^{k_1(1+n/2-\beta)}\|\dot{\Delta}_{k_1}f\|_{L^2}\right)\\ &\lesssim c_q\|f\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\|f\|_{B^s_{2,2}}.\end{aligned}$$

From the divergence-free property of v, we further decompose III_q as follows:

$$\mathrm{III}_q = \sum_{k\geqslant q-3;k\in\mathbb{N};i}[\partial_i\Delta_q, \Delta_k v_i]\widetilde{\Delta}_k f + [\Delta_q, \Delta_{-1}v]\cdot\nabla\tilde{\Delta}_{-1}f \triangleq \mathrm{III}^1_q + \mathrm{III}^2_q.$$

For III_q^1, from direct computation we find that

$$\begin{aligned}
&2^{q(s-\beta)}\|\mathrm{III}_q^1\|_{L^2}\\
&\lesssim 2^{q(s-\beta)}\left(\sum_{k\geqslant q-3;k\in\mathbb{N};i}\|\partial_i\Delta_q(\Delta_k v_i\tilde{\Delta}_k f)\|_{L^2}+\sum_{|k-q|\leqslant 2;k\in\mathbb{N}}\|\Delta_k v\cdot\nabla\Delta_q\tilde{\Delta}_k f\|_{L^2}\right)\\
&\lesssim 2^{q(s-\beta)}\left(\sum_{k\geqslant q-3;k\in\mathbb{N}}2^{q(1+n/2)}\|\Delta_k v\|_{L^2}\|\tilde{\Delta}_k f\|_{L^2}+\sum_{|k-q|\leqslant 2;k\in\mathbb{N}}2^{k(n/2)}\|\Delta_k v\|_{L^2}2^q\|\Delta_q f\|_{L^2}\right)\\
&\lesssim \|f\|_{B^s_{2,2}}\sum_{k\geqslant q-3;k\in\mathbb{N}}2^{(q-k)(s-\beta+1+n/2)}2^{k(1+n/2-\beta)}\|\Delta_k v\|_{L^2}+\|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}2^{qs}\|\Delta_q f\|_{L^2}\\
&\lesssim c_q\|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\|f\|_{B^s_{2,2}}.
\end{aligned}$$

For III_q^2, due to that $\mathrm{III}_q^2=0$ for all $q\geqslant 3$, and similarly as estimating I_q we obtain

$$\begin{aligned}
2^{q(s-\beta)}\|\mathrm{III}_q^2\|_{L^2}&\lesssim 1_{q\in\{0,1,2\}}\|xh_q\|_{L^1}\|\nabla\Delta_{-1}v\|_{L^\infty}\|\tilde{\Delta}_{-1}f\|_{L^2}\\
&\lesssim 1_{q\in\{0,1,2\}}\left(\sum_{-\infty<k_1\leqslant 0}2^{k_1\beta}2^{k_1(1+n/2-\beta)}\|\dot{\Delta}_{k'}v\|_{L^2}\|f\|_{L^2}\right)\\
&\lesssim 1_{q\in\{0,1,2\}}\|v\|_{\dot{B}^{1+n/2-\beta}_{2,2}}\|f\|_{B^s_{2,2}}.
\end{aligned}$$

Gathering the upper estimates leads to the expected results. □

We also treat some product estimates.

Lemma A.2. *Let v be a smooth divergence-free vector field over $\mathbb{R}^n$ and f be a smooth scalar function of $\mathbb{R}^n$. Then we have the following.*

(1) *If f satisfies $\operatorname{supp}\hat{f}\subset\{\xi:|\xi|\leqslant R\}$, a positive absolute constant C can be found such that for every $q\in\mathbb{N}$, $\beta\in]0,n/2[$ and $s>\beta-1-n/2$,*

$$2^{q(s-\beta)}\|\Delta_q(v\cdot\nabla f)\|_{L^2}\leqslant CR^{1+s-\beta}c_q\|v\|_{L^2}\|f\|_{L^\infty}+Cc_q\|v\|_{B^s_{2,2}}\|f\|_{\dot{B}^{1+n/2-\beta}_{2,2}},\tag{A.1}$$

with $(c_q)_{q\in\mathbb{N}}$ satisfying $\sum_{q\in\mathbb{N}}(c_q)^2\leqslant 1$. Especially, if $n=2$ and $v=\mathcal{R}^\perp f$, for all β,s satisfying $s+1-\beta>0$ we also have

$$2^{q(s-\beta)}\|\Delta_q(v\cdot\nabla f)\|_{L^2}\leqslant CR^{1+s-\beta}c_q\|f\|_{L^2}\|f\|_{L^\infty}.\tag{A.2}$$

(2) *For every $q\in\mathbb{Z}^-\cup\{0\}$, $\beta\in]0,n/2[$, there exists a positive absolute constant C such that*

$$\|\dot{\Delta}_q(v\cdot\nabla f)\|_{L^2}\leqslant C2^{q(1+n/2-\beta)}\||D|^\beta v\|_{L^2}\|f\|_{L^2}\tag{A.3}$$

and

$$\|\dot{\Delta}_q(v\cdot\nabla f)\|_{L^2}\leqslant C2^{q(1+n/2-\beta)}\|v\|_{L^2}\||D|^\beta f\|_{L^2}.\tag{A.4}$$

Proof of Lemma A.2. (1) We first prove (A.1). Thanks to Bony's decomposition, we have

$$\begin{aligned}
\Delta_q(v\cdot\nabla f)&=\sum_{|k-q|\leqslant 4}\Delta_q(S_{k-1}v\cdot\nabla\Delta_k f)+\sum_{|k-q|\leqslant 4}\Delta_q(\Delta_k v\cdot\nabla S_{k-1}f)+\sum_{k\geqslant q-3}\nabla\cdot\Delta_q(\tilde{\Delta}_k v\Delta_k f)\\
&\triangleq \mathrm{I}_q+\mathrm{II}_q+\mathrm{III}_q.
\end{aligned}$$

For I_q, from the support property of $\hat{f}$, we have

$$2^{q(s-\beta)}\|\mathrm{I}_q\|_{L^2} \lesssim \sum_{|k-q|\leqslant 4;2^k\lesssim R} 2^{q(s-\beta)}\|S_{k-1}v\|_{L^2}2^k\|\Delta_k f\|_{L^\infty} \lesssim R^{1+s-\beta}c_q\|v\|_{L^2}\|f\|_{L^\infty},$$

with $(c_q)_{q\in\mathbb{N}}$ satisfying $\sum_{q\in\mathbb{N}}(c_q)^2 \leqslant 1$. For the other two terms, in a similar and simpler way as the treatment of II_q and III_q, we obtain that

$$2^{q(s-\beta)}\|\mathrm{II}_q + \mathrm{III}_q\|_{L^2} \lesssim c_q\|v\|_{B^s_{2,2}}\|f\|_{\dot{B}^{1+n/2-\beta}_{2,2}}.$$

Next we treat (A.2). Since $\operatorname{supp}\widehat{v\cdot\nabla f} \subset \{\xi : |\xi| \leqslant 2R\}$, we find

$$\begin{aligned}2^{q(s-\beta)}\|\Delta_q(v\cdot\nabla f)\|_{L^2} &\lesssim 1_{\{q;\,2^q\lesssim R\}}2^{q(s-\beta+1)}\|\Delta_q(vf)\|_{L^2}\\ &\lesssim 1_{\{q;2^q\lesssim R\}}R^{s-\beta+1}\|f\|_{L^2}\|f\|_{L^\infty},\end{aligned}$$

and it clearly implies (A.2).

(2) We then prove (A.3). We also have the decomposition

$$\begin{aligned}\dot{\Delta}_q(v\cdot\nabla f) &= \sum_{|k-q|\leqslant 4}\dot{\Delta}_q(\dot{S}_{k-1}v\cdot\nabla\dot{\Delta}_k f) + \sum_{|k-q|\leqslant 4}\dot{\Delta}_q(\dot{\Delta}_k v\cdot\nabla\dot{S}_{k-1}f) + \sum_{k\geqslant q-3}\nabla\cdot\dot{\Delta}_q(\dot{\Delta}_k v\tilde{\dot{\Delta}}_k f)\\ &\triangleq \dot{\mathrm{I}}_q + \dot{\mathrm{II}}_q + \dot{\mathrm{III}}_q.\end{aligned}$$

For $\dot{\mathrm{I}}_q$, we directly have

$$\begin{aligned}\|\dot{\mathrm{I}}_q\|_{L^2} &\lesssim \sum_{|k-q|\leqslant 4}\|\dot{S}_{k-1}v\|_{L^\infty}2^k\|\dot{\Delta}_k f\|_{L^2}\\ &\lesssim \sum_{|k-q|\leqslant 4}\sum_{-\infty<k_1\leqslant k-2}2^{k_1(n/2-\beta)}2^{k_1\beta}\|\dot{\Delta}_{k_1}v\|_{L^2}2^k\|\dot{\Delta}_k f\|_{L^2}\\ &\lesssim 2^{q(1+n/2-\beta)}\||D|^\beta v\|_{L^2}\|f\|_{L^2}.\end{aligned}$$

For $\dot{\mathrm{II}}_q$, from Bernstein's inequality we similarly obtain

$$\|\dot{\mathrm{II}}_q\|_{L^2} \lesssim \sum_{|k-q|\leqslant 4}2^{k(n/2)}\|\dot{\Delta}_k v\|_{L^2}2^k\|\dot{S}_{k-1}f\|_{L^2} \lesssim 2^{q(1+n/2-\beta)}\||D|^\beta v\|_{L^2}\|f\|_{L^2}.$$

We treat $\dot{\mathrm{III}}_q$ as follows:

$$\begin{aligned}\|\dot{\mathrm{III}}_q\|_{L^2} &\lesssim \sum_{k\geqslant q-3}2^{q(1+n/2)}\|\dot{\Delta}_k v\|_{L^2}\|\tilde{\dot{\Delta}}_k f\|_{L^2}\\ &\lesssim 2^{q(1+n/2)}\left(\sum_{k\geqslant q-3}2^{-k\beta}2^{k\beta}\|\dot{\Delta}_k v\|_{L^2}\|f\|_{L^2}\right)\\ &\lesssim 2^{q(1+n/2-\beta)}\||D|^\beta v\|_{L^2}\|f\|_{L^2}.\end{aligned}$$

Collecting the upper estimates yields (A.3). The proof of (A.4) is almost identical to the above process, and we omit it. □

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