Functional Inequalities for Uniformly Integrable Semigroups and Application to Essential Spectrums *

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Abstract

Let \((E, \mathcal{F}, \mu)\) be a probability space, \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) a (not necessarily symmetric) Dirichlet form on \(L^2(\mu)\), and \(P_t\) the associated sub-Markov semigroup. The equivalence of the following eight properties is studied: (i) the \(L^2\)-uniform integrability of the unit ball in the Sobolev space; (ii) the super-Poincaré inequality (1.2); (iii) the \(F\)-Sobolev inequality (1.3); (iv) the \(L^2\)-uniform integrability of \(P_t\); (v) the \(L^2\)-uniform integrability of the associated resolvents; (vi) the compactness of \(P_t\); (vii) the compactness of the associated resolvents; (viii) empty essential spectrum of the associated generator.

The main results can be summarized as follows. In general, (i), (ii) and (iii) are equivalent to each other, and they imply (iv) which is equivalent to (v). If \(P_t\) has transition density and \(\mathcal{F}\) is \(\mu\)-separable, then the first seven properties from (i) to (vii) are equivalent. If in addition \((E, \mathcal{D}(E))\) is symmetric, then all the above eight properties are equivalent.

Moreover, the essential spectrum of the generator is estimated by using weaker functional inequalities.

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1 Introduction

Let \((E, \mathcal{F}, \mu)\) be a probability space, \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) a Dirichlet form on (real) \(L^2(\mu)\). Denote by \(R_\lambda, P_t\) and \(L\) respectively the associated resolvent, sub-Markov semigroup and generator. It is well known that, when \(\mathcal{E}\) is symmetric, \(L\) has a spectral gap if and only if the following Poincaré inequality holds:

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\[
\mu(f^2) \leq c\mathcal{E}(f, f) + \mu(f)^2, \quad (1.1)
\]

where and throughout the paper, the test functions in an inequality are in \(\mathcal{D}(\mathcal{E})\).

To describe the essential spectrum of \(L\), the following super-Poincaré inequality has been introduced in the recent work [18]:

\[
\mu(f^2) \leq r\mathcal{E}(f, f) + \beta(r)\mu(|f|)^2, \quad r > 0,
\]

where \(\beta : (0, \infty) \to (0, \infty)\) is a decreasing function. It was shown in [18] that, for Dirichlet forms of symmetric diffusions on a Riemannian manifold, (1.2) is equivalent to that the essential spectrum of \(L\) is empty, they are also equivalent to the following \(F\)-Sobolev inequality:

\[
\mu(f^2F(f^2)) \leq \mathcal{E}(f, f), \quad \mu(f^2) = 1,
\]

where \(F : (0, \infty) \to \mathbb{R}\) is increasing such that \(\sup_{r \in (0,1)}|rF(r)| < \infty\) and \(\lim_{r \to \infty} F(r) = \infty\). In general (Theorem 3.1 in [18]), (1.3) implies (1.2). Some estimates of \(\beta\) in (1.2) and \(F\) in (1.3) are also available in [18].

The first aim of this paper is to extend the above results to general framework. Our another motivation comes from the famous work [8] (see also [9], [2] and [5]), which proved the equivalence of the log-Sobolev inequalities and the hypercontractivity of corresponding semigroups. It is then very natural for us to study semigroup properties for the above general inequalities (1.2) and (1.3). To see what are possibly the desired semigroup properties, we first recall the following result claimed in [19] (theorems 5 and 6). We say that \(A \subset L^p(\mu)\) is \(L^p\)-uniformly integrable, if \(\sup_{f \in A} \mu(|f|^{p1_{\{|f| \geq r\}}}) \to 0\) as \(r \to \infty\).

**Theorem 1.1.** Let \(E\) be a Polish space with \(\mathcal{F}\) the Borel \(\sigma\)-field and \(\mu\) a probability measure. Let \(P_t\) be the semigroup of a Markov process on \(E\) and \(L\) its generator in \(L^2(\mu)\). If \(P_t\) is symmetric in \(L^2(\mu)\), then the \(L^2\)-uniform integrability of \(\{f \in \mathcal{D}(L) : \mu(f^2) = 1, -\mu(fLf) \leq r\}\) for all \(r > 0\) is equivalent to that of \(P_t\) for all \(t > 0\), i.e., \(\{P_t f : \mu(f^2) \leq 1\}\) is uniformly integrable in \(L^2(\mu)\) for any \(t > 0\). If in addition \(P_t\) has transition density for each \(t > 0\), then they are also equivalent to the compactness of \(P_t\) for all \(t > 0\).

It is easy to see that the \(L^2\)-uniform integrability of the Sobolev unit ball \(B := \{f : \mu(f^2) + \mathcal{E}(f, f) \leq 1\}\) is equivalent to that of \(A_r := \{f : \mu(f^2) = 1, \mathcal{E}(f, f) \leq r\}\) for all \(r > 0\). Actually, it suffices to show the \(L^2\)-uniform integrability of \(B\) from that of \(A_r\) for all \(r > 0\). For any \(\varepsilon \in (0, 1)\), we have \(\{f/\sqrt{\mu(f^2)} : f \in B, \mu(f^2) \geq \varepsilon\} \subset A_{\varepsilon^{-1}}\), which is \(L^2\)-uniformly integrable. Since \(f^2/\mu(f^2) \geq f^2\) for \(f \in B\), we see that \(\{f : f \in B, \mu(f^2) \geq \varepsilon\}\) is also \(L^2\)-uniformly integrable. Then, there exists \(n_\varepsilon > 0\) such that \(\mu(f^21_{\{|f^2| \geq n_\varepsilon\}}) \leq \varepsilon\) for any \(f \in B\). This means that \(B\) is \(L^2\)-uniformly integrable.

Since the \(F\)-Sobolev inequality implies the \(L^2\)-uniform integrability of \(A_r\) for any \(r > 0\), it also implies, in the situation of Theorem 1.1, the \(L^2\)-uniform integrability of \(P_t\) for any \(t > 0\). Actually, in this paper, we prove that all of them are equivalent in a general setting. From this, the equivalence of the above functional inequalities, the compactness of semigroups (and resolvents), and the empty essential spectrum of generators is also proved. The main results are summarized as follows.
Theorem 1.2. If $\mathcal{F}$ is $\mu$-separable, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric, and $P_t$ has transition density w.r.t. $\mu$ for any $t > 0$, then the following eight statements are equivalent to each other:

(i) The Sobolev unit ball is uniformly integrable in $L^2(\mu)$;
(ii) The super-Poincaré inequality (1.2) holds for some $\beta$;
(iii) The $F$-Sobolev inequality (1.3) holds for some $F$;
(iv) $P_t$ is uniformly integrable in $L^2(\mu)$ for each $t > 0$;
(v) $R_\lambda$ is uniformly integrable in $L^2(\mu)$ for each $\lambda$ in the resolvent set;
(vi) $P_t$ is compact in $L^2(\mu)$ for each $t > 0$;
(vii) $R_\lambda$ is compact in $L^2(\mu)$ for each $\lambda$ in the resolvent set;
(viii) The essential spectrum of $L$ is empty.

More precisely, for any probability space $(E, \mathcal{F}, \mu)$ and any Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mu)$, we have (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v), where “(a) $\iff$ (b)” means “(a) is equivalent to (b)” while “(a) $\implies$ (b)” means “(a) implies (b)”. If in addition $P_t$ has transition density for each $t > 0$ and $\mathcal{F}$ is $\mu$-separable, then the first seven statements are equivalent. If moreover $L$ is normal, i.e., $L^*L = LL^*$, where $L^*$ is the adjoint of $L$, then all the above eight statements are equivalent.

Obviously, the framework in Theorem 1.2 is more general than that in Theorem 1.1. Especially, Theorem 1.2 holds without the existence of Markov processes. The proof of Theorem 1.2 is contained in the next two sections, which consist of more precise results and their proofs.

Next, in section 4 we estimate the essential spectrum of $L$ by using the following weaker inequality:

$$\mu(f^2) \leq r \mathcal{E}(f, f) + \beta(r)\mu(|f|^2), \quad r > r_0,$$

where $r_0 \geq 0$ and $\beta : (r_0, \infty) \to (0, \infty)$ is a decreasing function. Obviously, for $\beta \equiv 1$ in $(r_0, \infty)$, (1.4) is equivalent to the Poincaré inequality (1.1) with $c = r_0$. Actually, (1.4) with $\beta \equiv 1$ implies (1.1) for nonnegative $f \in \mathcal{D}(\mathcal{E})$, and hence for any $f \in \mathcal{D}(\mathcal{E})$ by first replacing $f$ with $(f + n)^+$ and then letting $n \to \infty$.

It has been shown in [18] that, for Dirichlet forms of symmetric diffusions on a Riemannian manifold and a class of jump processes, (1.4) is equivalent to $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$, where $\sigma_{\text{ess}}(-L)$ denotes the essential spectrum of $-L$. The proof is based on Donnelly-Li’s decomposition principle (see [6]). Recently, such sort of principle has been established for more general diffusions. Let $E$ be a locally compact Hausdorff topological space, $\mathcal{F}$ the Borel $\sigma$-field, and $\mu$ a probability measure on $(E, \mathcal{F})$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local, regular, irreducible symmetric Dirichlet form on $L^2(\mu)$ (see [7] for detailed definitions). Assume that the metric induced by $\mathcal{E}$ is equivalent to the original topology (see [7] for the definition of the metric). Then

$$\inf \sigma_{\text{ess}}(-L) = \sup_{A \text{ is compact}} \inf \{\mathcal{E}(f, f) : f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1, \text{supp} f \subset A^c\}.$$
Theorem 2.1. For $f, g \in L^2(\mu)$, then a measure space. For any $\nu$, then $\mu - a.e. x, y \in E$. 

Finally, some further discussions, including methods to prove the functional inequalities and an example in infinite dimension, are presented in the last section.

2 Poincaré-Sobolev type inequalities and uniform integrability

In this section, we prove the equivalence of the first five statements in Theorem 1.2. The proof follows from the following four theorems which include more precise results.

From now on, we use $\| \cdot \|_p$ and $\| \cdot \|_{p \rightarrow q}$ to denote, respectively, the norm in $L^p(\mu)$ and the norm of an operator from $L^p(\mu)$ to $L^q(\mu)$. Moreover, let $\langle f, g \rangle_{\mu} = \int_E f g \, d\mu$ for $f, g \in L^2(\mu)$. If there is no specific explanation, $(E, \mathcal{F}, \mu)$ is a probability space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mu)$.

Theorem 2.1. Let $(E, \mathcal{F}, \mu)$ be a measure space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a bilinear form on $L^2(\mu)$. Then (iii) implies (i). If (i) holds, then for any $\varepsilon \in (0, 1)$, (1.2) holds with

$$ \beta(r) = \frac{1}{(1 - \varepsilon)^2} \inf \{ s > 0 : \mu(f^21_{\{f^2 \leq s\}}) \leq \varepsilon \text{ for any } f \in A_{r^{-1}} \}. \quad (2.1) $$

Proof. The first assertion is obvious. Denote by $n(r)$ the left-hand side of (2.1). We see that $n(r)$ is finite for any $r > 0$ since $A_{r^{-1}}$ is uniformly integrable in $L^2(\mu)$. Then, for any $f \in A_{r^{-1}}$,

$$ 1 = \mu(f^2) = \int_{\{f^2 \leq (1-\varepsilon)^2 n(r)\}} f^2 \, d\mu + \int_{\{f^2 > (1-\varepsilon)^2 n(r)\}} f^2 \, d\mu $$

$$ \leq \varepsilon + (1 - \varepsilon)\sqrt{n(r)}\mu(|f|). $$

Therefore,

$$ \mu(f^2) = 1 \leq n(r)\mu(|f|)^2, \quad f \in A_{r^{-1}}. \quad (2.2) $$

On the other hand, if $f \notin A_{r^{-1}}$ and $\mu(f^2) = 1$, then $\mu(f^2) \leq r\mathcal{E}(f, f)$. The proof is completed by combining this with (2.2).

To prove that (ii) implies (iii), we need the following lemma.

Lemma 2.2. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form on $L^2(\mu)$, where $(E, \mathcal{F}, \mu)$ is a measure space. For any $N \geq 1$ and any $f \in \mathcal{D}(\mathcal{E})$, if $f_0, \cdots, f_N \in L^2(\mu)$ satisfy

$$ \sum_{n=0}^N |f_n(x) - f_n(y)| \leq |f(x) - f(y)|, \quad \sum_{n=0}^N |f_n(x)| \leq |f(x)|, \quad \mu - a.e. \ x, y \in E, \quad (2.3) $$

then $f_n \in \mathcal{D}(\mathcal{E})$ for each $n$ and $\sum_{n=0}^N \mathcal{E}(f_n, f_n) \leq \mathcal{E}(f, f)$.
Proof. By Proposition I-4.11 in [12], (2.3) implies that \( f_n \in D(E), \ n = 0, \ldots , N \). We then modify the proof of Theorem I-4.12 in [12]. Let \( \{ R_\alpha : \alpha > 0 \} \) be the resolvents of \( L \). Then \( R_\alpha \) is symmetric and bounded in \( L^2(\mu) \) for each \( \alpha > 0 \). Put

\[
E^\alpha(h, g) = \alpha \langle h - \alpha R_\alpha h, g \rangle_\mu, \ h, g \in L^2(\mu).
\]

By Theorem I-2.13 in [12],

\[
E(g, g) = \lim_{\alpha \to \infty} E^\alpha(g, g), \ g \in D(E).
\]

Therefore, it suffices to show that

\[
\sum_{n=0}^N E^\alpha(f_n, f_n) \leq E^\alpha(f, f), \ \alpha > 0. \tag{2.4}
\]

Noting that the space of simple functions is dense in \( L^2(\mu) \) and \( R_\alpha \) is bounded, we need only to prove (2.4) for simple functions. Let

\[
f_n = \sum_{i=1}^m \alpha_{n_i} 1_{A_i}, \ f = \sum_{i=1}^m \alpha_i 1_{A_i}
\]

for some \( m \geq 1, \alpha_{n_i}, \alpha_i \in \mathbb{R} \) and measurable sets \( \{ A_i \} \) with \( A_i \cap A_j = \emptyset \) for any \( i \neq j \) and \( 0 < \lambda_i := \mu(A_i) < \infty \) for any \( i \). Then (2.3) implies that

\[
\sum_{n=0}^N |\alpha_{n_i} - \alpha_{n_j}| \leq |\alpha_i - \alpha_j|, \ \sum_{n=0}^N |\alpha_{n_j}| \leq |\alpha_j|, \ 1 \leq i, j \leq m. \tag{2.5}
\]

Let \( b_{ij} = E^\alpha(1_{A_i}, 1_{A_j}) \), we have

\[
E^\alpha(f_n, f_n) = \alpha \sum_{i,j=1}^m \alpha_{n_i} \alpha_{n_j} b_{ij}, \ E^\alpha(f, f) = \alpha \sum_{i,j=1}^m \alpha_i \alpha_j b_{ij}. \tag{2.6}
\]

Next, let \( a_{ij} = \alpha \langle 1_{A_i}, R_\alpha 1_{A_j} \rangle_\mu \), then \( b_{ij} = \lambda_j \delta_{ij} - a_{ij} \) and \( a_{ij} = a_{ji} \geq 0 \). Since \( \alpha R_\alpha \) is sub-Markovian,

\[
m_j := \lambda_j - \sum_{i+1}^m a_{ij} = \lambda_j - \alpha \sum_{i=1}^m \langle R_\alpha 1_{A_i}, 1_{A_j} \rangle_\mu \geq 0.
\]

Then, for any \( z_1, \ldots , z_m \in \mathbb{R} \),

\[
\sum_{i,j=1}^m b_{ij} z_i z_j = \sum_{i<j} a_{ij} (z_i - z_j)^2 + \sum_{j=1}^m m_j z_j^2.
\]

By combining this with (2.5) and (2.6), we obtain
\[ \sum_{n=0}^{N} \mathcal{E}^\alpha(f_n, f_n) = \alpha \sum_{i<j} a_{ij} \sum_{n=0}^{N} (\alpha_{ni} - \alpha_{nj})^2 + \alpha \sum_{j=1}^{m} m_j \left( \sum_{n=0}^{N} \alpha_{nj}^2 \right) \]

\[ \leq \alpha \sum_{i<j} a_{ij} \left( \sum_{n=0}^{N} |\alpha_{ni} - \alpha_{nj}|^2 \right) + \alpha \sum_{j=1}^{m} m_j \left( \sum_{n=0}^{N} |\alpha_{nj}|^2 \right) \]

\[ \leq \alpha \sum_{i<j} a_{ij} (\alpha_i - \alpha_j)^2 + \alpha \sum_{j=1}^{m} m_j \alpha_j^2 = \mathcal{E}^\alpha(f, f). \]

This proves (2.4).

Lemma 2.3 enables us to extend Theorem 3.2 in [18] to general case.

**Theorem 2.3.** Let \((E, \mathcal{F}, \mu)\) be a measure space and \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) a Dirichlet from on \(L^2(\mu)\). If (1.2) holds, then (1.3) holds with

\[ F(r) = \frac{c_1(\varepsilon)}{r} \int_{0}^{r} \xi(\varepsilon t) dt - c_2(\varepsilon) \quad (2.7) \]

for any \(\varepsilon \in (0, 1)\) and some \(c_1(\varepsilon), c_2(\varepsilon) > 0\), where

\[ \xi(t) = \sup_{r > 0} \left( \frac{1}{r} - \frac{\beta(r)}{rt} \right) \geq 0, \quad t > 0. \quad (2.8) \]

Moreover, \(\xi(t)\) is increasing in \(t\) and is finite for \(t < N := \sup\{\|f\|_\infty^2 : \mu(f^2) = 1\}\), so (1.3) makes sense for \(F\) determined by (2.7).

**Proof.** Since the inequalities (1.2) and (1.3) depend only on the symmetric part of \(\mathcal{E}\), we may assume that \(\mathcal{E}\) itself is symmetric. Let \(f \in \mathcal{D}(\mathcal{E})\) with \(\mu(f^2) = 1\). For any \(\delta > 1\) and any \(n \geq 0\), define

\[ f_n = (|f| - \delta^n/2)^+ \wedge (\delta^{(n+1)/2} - \delta^{n/2}). \]

Obviously, for any \(x, y \in E\), we have \(f_n(x) \geq f_n(y)\) (resp. \(f_n(x) \leq f_n(y)\)) for each \(n\) if \(|f(x)| \geq |f(y)|\) (resp. \(|f(x)| \leq |f(y)|\)). Then, for any \(N \geq 1\),

\[ \sum_{n=0}^{N} |f_n(x) - f_n(y)| = \left| \sum_{n=0}^{N} [f_n(x) - f_n(y)] \right| \]

\[ = |(|f(x)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1) - (|f(y)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1)| \]

\[ \leq |f(x) - f(y)| \]

and

\[ \sum_{n=0}^{N} |f_n(x)| = \sum_{n=0}^{N} f_n(x) = (|f(x)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1) \leq |f(x)|. \]
By Lemma 2.2, we have
\[ \mathcal{E}(f, f) \geq \sum_{n=0}^{N} \mathcal{E}(f_n, f_n). \] (2.9)

Now, if (1.2) holds, then
\[ \mu(f_n^2) \leq r \mathcal{E}(f_n, f_n) + \beta(r) \mu(|f_n|)^2, \quad r > 0. \] (2.10)
Noting that
\[ \mu(|f_n|)^2 = (\mu(f_n 1_{\{|f_n| \leq \delta_n\}}))^2 \leq \mu(f_n^2) \mu(f^2 > \delta_n) \leq \delta^{-n} \mu(f_n^2), \]
it follows from (2.10) that
\[ \mathcal{E}(f_n, f_n) \geq \sup_{r>0} \left( \frac{1}{r} - \frac{\beta(r)}{r \delta^n} \right) \mu(f_n^2) = \xi(\delta^n) \mu(f_n^2). \]
This implies that \( \|f\|_{\infty}^2 \leq \delta^n \) if \( \xi(\delta^n) = \infty \). Therefore \( \xi(t) < \infty \) for \( t < N \) since \( \delta^n > 1 \) is arbitrary. Combining this with (2.9), we obtain
\[ \mathcal{E}(f, f) \geq \sum_{n=0}^{\infty} \xi(\delta^n) \mu(f_n^2) \geq \sum_{n=0}^{\infty} \xi(\delta^n) \mu(f^2 \geq \delta^{n+1}[\delta^{(n+1)/2} - \delta^{n/2}]^2
\]
\[ \geq \frac{(\sqrt{\delta} - 1)^2}{1 - \delta^{-1}} \sum_{n=0}^{\infty} \int_{\delta^{n-1}}^{\delta^n} \xi(t) \mu(f^2 \geq \delta^2 t) dt
\]
\[ \geq c_1(\delta) \int_{0}^{\infty} \xi(t) \mu(f^2 \geq \delta^2 t) dt - c_2(\delta)
\]
\[ = \delta^2 c_1(\delta) \int_{0}^{\infty} \xi(\delta^{-2} t) \mu(f^2 \geq t) dt - c_2(\delta)
\]
\[ = \delta^2 c_1(\delta) \int_{E} d\mu \int_{0}^{f^2} \xi(\delta^{-2} t) dt - c_2(\delta) \]
for some \( c_1(\delta), c_2(\delta) > 0 \). This proves the Theorem since \( \delta > 1 \) is arbitrary. \( \square \)

**Theorem 2.4.** Let \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be a coercive closed form on \( L^2(\mu) \).

1. If (1.3) holds, then \( P_t \) is uniformly integrable in \( L^2(\mu) \) for any \( t > 0 \). More precisely, let \( \overline{F}(r) = \inf_{s \geq r} F(s) \), then
\[ \sup_{\mu(f^2) \leq 1} \mu((P_t f)^2 \overline{F}((P_t f)^2)) < \infty, \quad t > 0. \] (2.11)
2. The statement (iv) is equivalent to (v).

**Proof.** a) For a linear operator \( (T, \mathcal{D}(T)) \) on \( L^2(\mu) \), we denote by \( (T^c, \mathcal{D}(T^c)) \) its complexification on the complex \( L^2 \) space \( L^2_c(\mu) \), see e.g. [11] and [12] for details. Since \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is a coercive closed form, by the proof of Corollary I.2.21 in [12], \( L' := (1 - L)^c \)
satisfies the strong sector condition (see pages 15 and 16 in [12]). By Theorem I-2.20 and Corollary I-2.21 in [12], we have \( e^{-tP_f^C L_\mu^2} \subset \mathcal{D}(L') \) and

\[
\| L'e^{-tP_f^C} u \|_2 \leq \frac{c}{t} \| u \|_2 \tag{2.12}
\]

for some \( c > 0 \), any \( t > 0 \) and any \( u \in L_\mu^2 \). Since \( \mathcal{D}(L') = \mathcal{D}(L^C) = \{ u : \text{Im} u, \text{Re} u \in \mathcal{D}(L) \} \) and \( L_\mu^2 \subset L_\mu^2 \), for any \( f \in L_\mu^2 \), we have \( P_tf \in \mathcal{D}(L) \). It then follows from (2.12) that

\[
\|(L - 1)P_t f \| \leq \frac{c e^t}{t} \| f \|_2, \quad t > 0.
\]

Noting that \( \mathcal{D}(L) \subset \mathcal{D}(E) \) and \( E(f, f) = -\mu(fL f) \) for \( f \in \mathcal{D}(L) \), we obtain

\[
E(P_t f, P_t f) = \langle (1 - L)P_t f, P_t f \rangle_\mu - \mu((P_t f)^2) \leq \frac{c e^t}{t} \| f \|_2 \cdot \| P_t f \|_2 \leq \frac{c e^t}{t} \mu(f^2).
\]

Combining this with the \( F \)-Sobolev inequality, we obtain for any \( t > 0 \) and any \( f \) with \( \mu(f^2) \leq 1 \) and \( \mu((P_t f)^2) > 0 \),

\[
\mu((P_t f)^2 F((P_t f)^2)) \leq \mu\left((P_t f)^2 F\left((P_t f)^2 / \mu((P_t f)^2)\right)\right) \leq \mu\left((P_t f)^2 F\left((P_t f)^2 / \mu((P_t f)^2)\right)\right) \leq E(P_t f, P_t f) \leq \frac{c e^t}{t}.
\]

This means that \( P_t \) is uniformly integrable in \( L_\mu^2 \) for any \( t > 0 \).

b) It remains to prove the equivalence of (iv) and (v). Observing that the under the \( L_\mu^2 \)-operator norm of a sequence of \( L_\mu^2 \)-uniformly integrable linear operators is still \( L_\mu^2 \)-uniformly integrable. If (iv) holds, then \( \int_0^N e^{-\alpha t} P_t \, dt \) is uniformly integrable in \( L_\mu^2 \) for any \( N > 0 \). Since for any \( \alpha > 0 \),

\[
R_\alpha = \lim_{N \to \infty} \int_0^N e^{-\alpha t} P_t \, dt
\]

in the \( L_\mu^2 \)-operator norm (see e.g. the proof of Proposition I-1.10 in [12]), \( R_\alpha \) is uniformly integrable in \( L_\mu^2 \) (and hence in \( L_\mu^2 \)) for any \( \alpha > 0 \). For any \( \lambda \in \rho(L) \), the resolvent set of \( L \), by the resolvent equation

\[
R_\lambda = R_\alpha (1 + (\lambda - \alpha)R_\lambda)
\]

and noting that \( R_\lambda \) is bounded, we prove that \( R_\lambda \) is uniformly integrable in \( L_\mu^2 \).

Conversely, assume that \( R_\lambda \) is \( L_\mu^2 \)-uniformly integrable for each \( \lambda \in \rho(L) \). Let \( R'_\lambda \) be the resolvent of \(-L'\), then \( R'_\lambda = R_{\lambda+1} \). Since \( L' \) satisfies the strong sector condition, the spectrum of \( L' \) is contained in a sector set

\[
S(K) = \{ z \in \mathbb{C} : |\text{Im} z| \leq K \text{Re} z \}
\]
for some $K > 0$. For any $t > 0$ we have (see the first paragraph on page 490 in [11])

$$e^{-t}P_t^c = \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda t} R_d \lambda \, d\lambda$$

in the $L^2$-operator norm topology, where $\Gamma$ is a positively oriented closed curve in $\mathbb{C} \setminus S(K)$ which encloses the spectrum of $L'$. Therefore, for any $t > 0$, $e^{-t}P_t^c$ is uniformly integrable in $L^2_c(\mu)$. Consequently, $P_t$ is $L^2$-uniformly integrable for each $t > 0$. \hfill $\Box$

**Theorem 2.5.** If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric closed form on $L^2(\mu)$, then (v) implies (i).

**Proof.** Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric, $-L$ is nonnegative and self-adjoint. Moreover, $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-L})$ and $\mathcal{E}(f, f) = \langle \sqrt{-L}f, \sqrt{-L}f \rangle_\mu$, see e.g. page 28 in [12]. By the proof of Theorem V-3.49 in [11] (just replace “compact” by “uniformly integrable”), we prove that (v) implies the $L^2$-uniform integrability of $R_{\alpha}^\mu$, the resolvent of $\sqrt{-L}$. Observing that for any $f \in \mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-L})$ and any $\alpha < 0$,

$$\left\| \left( \alpha - \sqrt{-L} \right) f \right\|_2^2 \leq 2 \left[ \alpha^2 \left\| f \right\|_2^2 + \left\| \sqrt{-L} f \right\|_2^2 \right],$$

we obtain for any $r > 0$ and $\alpha < 0$,

$$\{ f : \mu(f^2) \leq 1, \mathcal{E}(f, f) \leq r \} \subset \{ R_{\alpha}^\mu g : \mu(g^2) \leq 2(\alpha^2 + r) \}$$

which is uniformly integrable in $L^2(\mu)$. \hfill $\Box$

3 Applications to the compactness of $P_t$ and the essential spectrum of $L$

The aim of this section is to complete the proof of Theorem 1.2. We first present the following lemma which is an extension of a result due to Wu [19] and where uniformly integrable operators on $L^p(\mu)$ were introduced and deeply studied. This lemma enables one to prove the compactness of semigroups on a Hilbert space-valued $L^p$-space.

**Lemma 3.1.** Let $p \in [1, \infty)$ be fixed. Let $H$ be a separable Hilbert space, and $L^p_H(\mu)$ the $L^p$-space (w.r.t. $\mu$) of $H$-valued functions. Let $P$ be a bounded linear operator on $L^p_H(\mu)$ with kernel $p(x, y)$, i.e., for $\mu$-a.e. $x, y \in E$, $p(x, y)$ is a bounded linear operator on $H$ such that

$$Pf(x) = \int_E p(x, y)f(y)\mu(dy), \quad f \in L^p_H(\mu), \quad \mu\text{-a.e. } x \in E.$$

Suppose that for $\mu$-a.e. $x \in E$ and an orthonormal basis $\{e_j\}$ of $H$,

$$\sum_k \left( \int_E \left\| \sum_j \langle p(x, y)e_j, e_k \rangle_H \mu(dy) \right\|_H^2 \right)^2 < \infty \quad (3.1)$$

and $\{ |Pf|_H^p : \|f\|_\infty \leq 1 \}$ is uniformly integrable w.r.t. $\mu$. Then for any $L^p$-uniformly integrable $A \subset L^p_H(\mu)$, $P(A) := \{ Pf : f \in A \}$ is relatively compact in $L^p_H(\mu)$. 

Proof. If \( P(A) \) is not relatively compact in \( L^p_H(\mu) \), then there exist \( \varepsilon > 0 \) and a sequence \( \{f_n\} \subset A \) such that \( \|P f_n - P f_m\|_p \geq \varepsilon, \ n \neq m \). Since \( P \) is bounded in \( L^p_H(\mu) \) and \( \{f^p_H : f \in A\} \) is uniformly integrable, we may choose \( K > 0 \) such that

\[
\|P f_n, K - P f_m, K\|_p \geq \frac{\varepsilon}{2}, \ n \neq m, \tag{3.2}
\]

where \( f_n, K = f_n 1_{\{f_n, H \leq K\}} \). We fix a measurable version of \( f_n \) for each \( n \). Let \( \sigma := \sigma\{f_n, K : n \geq 1\} \). Then \( L^1(\sigma; \mu) \) is separable. By Bourbaki theorem (see Chapter IV, page 112 in [3]), the set \( \{f_n, K\} \subset L^\infty(\sigma; \mu) \) is weakly compact and metrisable with respect to the weak topology \( \sigma(L^\infty(\sigma; \mu), L^1(\sigma; \mu)) \). Therefore, there exist \( f \in L^\infty(\sigma; \mu) \) and a subsequence \( \{f_{n, K}\} \) such that for any \( g \in L^1(\sigma; \mu) \) one has \( \mu(\langle f_{n, K}, g \rangle_H) \rightarrow \mu(\langle f, g \rangle_H) \) as \( i \rightarrow \infty \). Noting that for any \( g \in L^1_H(\mu) \) and \( f' \in L^\infty(\sigma; \mu) \), there holds \( \mu(\langle f, g \rangle_H) = \mu(\langle f', \mu(g|\sigma) \rangle_H) \), we obtain \( \mu(\langle f_{n, K}, g \rangle_H) \rightarrow \mu(\langle f, g \rangle_H) \) for all \( g \in L^1_H(\mu) \). On the other hand, for \( \mu\text{-a.e. } x \in E, \)

\[
|P f(x) - P f_{n, K}(x)|^2_H = \sum_k \langle P f_{n, K}(x) - P f(x), e_k \rangle^2_H
\]

\[
= \sum_k \left( \int_E \langle f_{n, K}(y) - f(y), \sum_j \langle p(x, y)e_j, e_k \rangle_H e_j \rangle_H \mu(dy) \right)^2.
\]

Noting that

\[
\left( \int_E \langle f_{n, K}(y) - f(y), \sum_j \langle p(x, y)e_j, e_k \rangle_H e_j \rangle_H \mu(dy) \right)^2 \leq 4K^2 \left( \int_E \sqrt{\sum_j \langle p(x, y)e_j, e_k \rangle^2_H} \mu(dy) \right)^2
\]

which is sumable w.r.t. \( k \) according to (3.1), by the weak convergence of \( f_{n, K} \) and using the dominated convergence theorem, we prove that \( P f_{n, K}(x) \rightarrow P f(x) \) in \( H \) for \( \mu\text{-a.e. } x \in E \). Since \( \{||P f_{n, K}||_H^p : i \geq 1\} \) is uniformly integrable, \( P f_{n, K} \rightarrow P f \) in \( L^p_H(\mu) \). This is a contradiction to (3.2).

\[\square\]

**Theorem 3.2.** Assume that \( \mathcal{F} \) is \( \mu\)-separable and \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is a coercive closed form on \( L^2(\mu) \). If \( P_t \) has transition density w.r.t. \( \mu \), then (iv) \( \Leftrightarrow \) (vi) \( \Leftrightarrow \) (vii) \( \Rightarrow \) (viii) and (i). Consequently, if in addition \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is a Dirichlet form, then the first seven statements in Theorem 1.2 are equivalent.

**Proof.** Noting that the essential spectrum of \( L \) is empty if \( R_1 \) is compact (see e.g. Theorem III-6.29 in [11]), it suffices to prove that (iv) \( \Rightarrow \) (vi) \( \Leftrightarrow \) (vii) \( \Rightarrow \) (i). The proof of (vi) \( \Leftrightarrow \) (vii) is similar to that of (iv) \( \Leftrightarrow \) (v), and (iv) \( \Rightarrow \) (vi) follows immediately from Lemma 3.1 by taking \( H = \mathbb{R}, P = P_{t/2} \) and \( A = \{P_{t/2}f : \mu(f^2) \leq 1\} \). Hence we need only to show that (vii) \( \Rightarrow \) (i).

Let \( \hat{L} \) be the self-adjoint operator associated to the symmetric part of \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \), and let \( L' = (1 - L)^C \). Then \( L' \) is sectorial and \( \text{Re} L' = (1 - \hat{L})^C \) (see pages 309 and 310 in
[11]). By (vii), the resolvents of $L'$ are compact. Then (see Theorem VI-3.3 in [11]), the resolvents of $\text{Re} L'$ are compact and hence so are those of $L$. Therefore, the resolvents of $L$ are $L^2$-uniformly integrable. By Theorem 2.5, we prove (i). Finally, the second assertion follows from the first one and Theorems 2.1, 2.3 and 2.4.

**Theorem 3.3.** Assume that $L$ is normal, i.e., $L^* L = LL^*$, then (viii) implies (vi).

**Proof.** Let $P_t^c$ be the semigroup generated by $L^c$. Since $L$ is normal, so is $L^c$. By the Note after Theorem 13.38 in [15], we have

$$P_t^c = \int_{\sigma(L)} e^{\lambda t} dE_{\lambda}, \quad t > 0 \tag{3.3}$$

under the operator norm topology, where $\sigma(L)$ denotes the spectrum of $L$, and $(E_{\lambda})$ the corresponding resolution of the identity. Let $\{\lambda_i : i \geq 1\}$ be all the eigenvalues (including multiplicity) of $L$, and $\{f_i : i \geq 1\}$ the corresponding normalized eigenfunctions. It follows from (3.3) that

$$P_t^c = \sum_{i=1}^{\infty} e^{\lambda_i t} \langle \cdot, f_i \rangle \mu f_i, \quad f \in L^2_c(\mu), \quad t > 0,$$

where the series converges in the operator norm. Since $\sum_{i=1}^{N} e^{\lambda_i t} \langle \cdot, f_i \rangle \mu f_i$ is compact for any $N \geq 1$, $P_t^c$ is compact too. Therefore, $P_t$ is compact for any $t > 0$. \qed

## 4 Estimates of the essential spectrum

The aim of this section is to prove the equivalence of (1.4) and $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$. More general, let $(E, D(E))$ be a closed form on $L^2_H(\mu)$, where $H$ is a separable Hilbert space. Consider the inequality

$$\mu(|f|_H^2) \leq r \mathcal{E}(f, f) + \beta(r) \mu(|f|_H)^2, \quad f \in D(E), \quad r > r_0 \tag{4.1}$$

for some $r_0 \geq 0$ and positive decreasing $\beta \in C(r_0, \infty)$.

**Theorem 4.1.** Assume that $\mathcal{F}$ is $\mu$-separable, $(E, D(E))$ is a coercive closed form on $L^2_H(\mu)$ and the generator $L$ is normal. Let $\theta \in [0, \frac{\pi}{2})$ be such that

$$\mathcal{E}_1(f, g) \leq (1 + \tan \theta) \mathcal{E}_1(f, f)^{1/2} \mathcal{E}_1(g, g)^{1/2}, \quad f, g \in D(E) \tag{4.2}$$

where $\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \mu((f, g)_H)$. If $P_t$ has kernel satisfying (3.1) for some $t > 0$, then (4.1) implies

$$\sigma_{\text{ess}}(-L) \subset \left\{ z \in \mathbb{C} : |\text{Im } z| \leq \theta(1 + \text{Re } z), \text{Re } z \geq \frac{r_0 + 1}{r_0 \sqrt{1 + \theta^2} - 1} \right\} \tag{4.3}$$

**Proof.** From pages 310 and 311 in [11] we know that (4.2) implies

$$\sigma(1 - L) \subset \{ z \in \mathbb{C} : |\text{Im } z| \leq \theta \text{Re } z \}, \tag{4.4}$$
where $\sigma(1-L)$ denotes the spectrum of $1-L$. For any $\delta > 0$, let $D_\delta = \{ z \in \sigma(1-L) : \Re z \leq \delta \}$. By (4.4), $D_\delta$ is bounded and closed, and

$$D_\delta \subset \{ z \in \mathbb{C} : |\Im z| \leq \theta \Re z, \Re z \leq \delta \}.$$ (4.5)

Put

$$Q_\delta = \int_{D_\delta} dE_{\lambda}' = \int_{\sigma(1-L)} 1_{D_\delta}(\lambda) dE_{\lambda}'$$

where $E_{\lambda}'$ is the resolution of the identity of $L'$, the complexification of $1-L$ on $L^2_{H^C}(\mu)$, where $H^C$ is the complexification Hilbert space of $H$. Similarly to the last paragraph on page 374 in [15], we have

$$L'Q_\delta = \int_{\sigma(L')} \lambda 1_{D_\delta}(\lambda) dE_{\lambda}' = \int_{D_\delta} \lambda dE_{\lambda}'$$ (4.6)

which is bounded and normal, and $Q_\delta L' \subset L'Q_\delta$. Next, by the remark after Theorem 13.25 in [15], we have $P_t'Q_\delta = Q_\delta P_t'$, where $P_t' = e^{-tP_t^C}$ is the semigroup generated by $L'$. Then, for $H_\delta := Q_\delta(L^2_{H^C}(\mu))$, we have $P_t'H_\delta \subset H_\delta$ for any $t \geq 0$ and $(P_t'|_{H_\delta})_{t \geq 0}$ is the semigroup on $H_\delta$ generated by $L'|_{H_\delta}$, the closure of $L'|_{H_\delta}$. From (4.6) we see (c.f. Theorem 12.21 and the last paragraph on page 374 in [15])

$$\|L'|_{H_\delta}\|_{2 \rightarrow 2} = \|L'Q_\delta\|_{2 \rightarrow 2} \leq \sup_{\lambda \in D_\delta} |\lambda| \leq \delta \sqrt{1+\theta^2}.$$ (4.7)

Moreover, since $Q_\delta L' \subset L'Q_\delta$, we have $Q_\delta(D(L')) \subset D(L')$. Then $L'|_{H_\delta}$ is densely defined and hence is well defined on the whole space $H_\delta$ since it is bounded. For any $f \in H_\delta$, we have

$$\mathcal{E}_1(\Re f, \Re f) = \mu((\Re f, (1-L)\Re f)_{H'}) \leq s^2 \mu(|\Re f|_{H'}^2) + \frac{1}{2s} \mu(|L'\Re f|_{H'}^2), \quad s > 0.$$

The same holds for $\Im f$. Then, it follows from (4.7) that

$$\mathcal{E}_1(\Re f, \Re f) + \mathcal{E}_1(\Im f, \Im f) \leq \frac{1}{2} [s + s^{-1}\delta^2(1+\theta^2)] \mu(|f|_{H^C}^2), \quad s > 0.$$

Taking $s = \sqrt{\delta^2(1+\theta^2)}$ and applying (4.1) to $\Re f$ and $\Im f$, we obtain

$$\mu(|f|_{H^C}^2) \leq \frac{\delta r \sqrt{1+\theta^2}}{1+r} \mu(|f|_{H^C}^2) + \frac{2\beta(r)}{1+r} \mu(|f|_{H^C}^2)^2, \quad r > r_0, f \in H_\delta.$$ (4.8)

If $\delta < \frac{r_{0}+1}{r_0\sqrt{1+\theta^2}}$, then there exists $r_\delta > r_0$ such that $\frac{\delta r_\delta \sqrt{1+\theta^2}}{1+r_\delta} < 1$. Therefore, (4.8) implies that the $L^1$-unit ball in $H_\delta$ is $L^1$-uniformly integrable and that the $L^1$-norm and the $L^2$-norm are equivalent on $H_\delta$. By Lemma 3.1, $P_t'|_{H_\delta}$ is $L^1$-compact and hence also $L^2$-compact. By Theorems 1 and 2 in [10], the essential spectrum of $L'|_{H_\delta}$ is empty. By (4.6) and noting that $(L'Q_\delta)|_{H_\delta} = L'|_{H_\delta}$ since $Q_\delta$ is a projection, we have

$$\emptyset = \sigma_{ess}(L'|_{H_\delta}) = \sigma_{ess}(L') \cap D_\delta.$$
Therefore,

\[ \sigma_{\text{ess}}(1 - L) = \sigma_{\text{ess}}(L') \subset D_\delta', \quad \delta < \frac{r_0 + 1}{r_0 \sqrt{1 + \theta^2}}. \]

By Combining this with (4.4), we complete the proof.

**Corollary 4.2.** Assume that \( \mathcal{F} \) is \( \mu \)-separable, \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is a symmetric closed form on \( L^2_H(\mu) \), and \( P_t \) has kernel satisfying (3.1) for some \( t > 0 \). If (4.1) holds, then \( \sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty) \).

**Proof.** In this case we have \( \theta = 0 \), then the assertion follows from Theorem 4.1 immediately. Below we present a different proof using Weyl’s criterion.

Assume that there exists \( \lambda \in [0, r_0^{-1}) \cap \sigma_{\text{ess}}(-L) \). Let \( r_1 > r_0 \) be such that \( r_1 \lambda < 1 \) and take

\[ \varepsilon = \min \left\{ \frac{1 - r_1 \lambda}{2 \sqrt{2} r_1}, \frac{e^{-2\lambda t} (1 - \lambda r_1)}{8 t \beta(r_1)} \right\} \]

which is positive. By Weyl’s criterion (see [13], Theorem VII.12 and comments on page 264), there exists a sequence \( \{f_n\} \subset \mathcal{D}(L) \) such that \( \|f_n\|_2 = 1 \), \( \mu(\langle f_n, f_m \rangle_H) = 0 \) for \( n \neq m \), and

\[ \|\langle \lambda + L \rangle f_n \|_2 \leq \varepsilon, \quad n \geq 1. \tag{4.9} \]

For any \( m, n \geq 1 \), let \( h(s) = \|P_s(f_n - f_m) - e^{-\lambda s}(f_n - f_m)\|_2^2 \geq 0 \). By (4.9),

\[ h'(s) = 2 \mu(\langle (P_s - e^{-\lambda s})(f_n - f_m), (P_s L + \lambda e^{-\lambda s})(f_n - f_m) \rangle_H) \]
\[ = -2 \lambda h(s) + 2 \mu(\langle (P_s - e^{-\lambda s})(f_n - f_m), P_s(\lambda + L)(f_n - f_m) \rangle_H) \tag{4.10} \]
\[ \leq 2 \varepsilon \sqrt{2h(s)} < 8 \varepsilon. \]

Then

\[ \|P_t(f_n - f_m) - e^{-\lambda t}(f_n - f_m)\|_2 \leq \sqrt{8 \varepsilon} t, \quad n, m \geq 1. \]

Therefore,

\[ \|P_t(f_n - f_m)\|_1 \geq e^{-\lambda t} \|f_n - f_m\|_1 - \|P_t(f_n - f_m) - e^{-\lambda t}(f_n - f_m)\|_2 \tag{4.11} \]

\[ \geq e^{-\lambda t} \|f_n - f_m\|_1 - \sqrt{8 \varepsilon} t. \]

Next, by (4.1) and (4.9), for \( n \neq m \) we have

\[ 2 = \|f_n - f_m\|_2^2 \leq r_1 \mu(\langle f_n - f_m, L(f_m - f_n) \rangle_H) + \beta(r_1) \|f_n - f_m\|_1^2 \]
\[ = 2r_1 \lambda + r_1 \mu(\langle f_n - f_m, (L + \lambda)(f_m - f_n) \rangle_H) + \beta(r_1) \|f_n - f_m\|_1^2 \]
\[ \leq 2r_1 \lambda + 2 \sqrt{2} r_1 \varepsilon + \beta(r_1) \|f_n - f_m\|_1^2. \]

This implies
\[ \|f_n - f_m\|_1 \geq \sqrt{\frac{2(1 - r_1 \lambda) - 2\sqrt{2r_1 \varepsilon}}{\beta(r_1)}} \geq \sqrt{\frac{1 - r_1 \lambda}{\beta(r_1)}}, \quad n \neq m. \tag{4.12} \]

Noting that \( A = \{f_n : n \geq 1\} \) is \( L^1 \)-uniformly integrable, by Lemma 3.1 there exists \( n_i \uparrow \infty \) such that

\[ \lim_{i \to \infty} \|P_t(f_n - f_{n_i+1})\|_1 = 0. \]

But combining (4.11) with (4.12) one obtains

\[ \|P_t(f_n - f_m)\|_1 \geq e^{-\lambda t} \sqrt{\frac{1 - r_1 \lambda}{\beta(r_1)}} - \sqrt{8\varepsilon t} > 0 \]

for any \( n \neq m \) and small \( \varepsilon \), this is a contradiction. \( \square \)

The following is a converse result of Theorem 4.1 and Corollary 4.2.

**Theorem 4.3.** Let \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be a closed form on \( L_H^2(\mu) \) such that \( L \) is normal. If \( \sigma_{ess}(-L) \subset \{z \in \mathbb{C} : \Re z \geq r_0^{-1}\} \), then (4.1) holds.

**Proof.** For any \( r > r_0 \), let \( r_1 = \frac{1}{2}(r + r_0) \). Then \( \sigma_{ess}(-L) \cap \{z \in \mathbb{C} : \Re z \leq r_1^{-1}\} = \emptyset \). Let \( \{\lambda_1, \cdots, \lambda_n\} \) be all the eigenvalues of \(-L\) (including multiplicity) satisfying \( \lambda_1 \leq r_1^{-1} \), and \( \{f_1, \cdots, f_n\} \) the corresponding normalized eigenfunctions. Let \( \mathcal{H}_r = \text{spn}\{f_1, \cdots, f_n\} \), we have \( \mathcal{H}_r = Q_r(L_H^2(\mu)) \), where \( Q_r = \int_{\{\Re z \leq r_1^{-1}\}} dE_\lambda \) and \( E_\lambda \) is the resolution of the identity of \(-L\). For any \( f \in \mathcal{D}(L) \), let \( g = Q_r(f) \) and \( h = f - g \). We have \( g = \sum_{i=1}^{n_r} \mu(f_i) H \), \( f_i \in \mathcal{D}(L^C) \) and hence \( h \in \mathcal{D}(L^C) \). It is easy to see that

\[ \mu(|h|_{H^C}^2) \leq -r_1 \mu(|h, L^2 h|_{H^C}) \leq -r_1 \mu(f L f) = r_1 \mathcal{E}(f, f). \tag{4.13} \]

Next, since \( \mathcal{H}_r \) is a finite-dimensional space,

\[ \beta_1(r) := \sup_{g \in \mathcal{H}_r} \frac{\mu(|g|_{H^C}^2)}{\mu(|g|_{H^C})^2} \in [1, \infty). \]

We have

\[ \mu(|g|_{H^C}^2) \leq \beta_1(r) \mu(|g|_{H^C})^2. \tag{4.14} \]

Let \( \varepsilon_r = \frac{r - r_1}{2r \beta_1(r) n_r^2} \), and let \( c_r > 0 \) be the smallest constant such that

\[ \mu(|f_i|_{H^C}^2 1_{|f_i|_{H^C} \geq c_r}) \leq \varepsilon_r, \quad i = 1, \cdots, n_r. \]

Then

\[ \mu(|g|_{H^C}) \leq \sum_{i=1}^{n_r} \mu(|f_i|_{H^C} \cdot |f|_H) \leq \sum_{i=1}^{n_r} (c_r \mu(|f|_H) + \mu(|f|_H \cdot |f_i|_{H^C} 1_{|f_i|_{H^C} \geq c_r}))) \]

\[ \leq n_r c_r \mu(|f|_H) + n_r \varepsilon_r \mu(|f|_H^2). \]
Therefore, (4.14) implies that
\[
\mu(|g|_{HF}^2) \leq 2\beta_1(r)n_r^2\varepsilon_r \mu(|f|_H^2) + 2\beta_1(r)n_r^2c_r^2\mu(|f|_H)^2.
\]
Combining this with (4.13) we obtain
\[
\mu(|f|^2_H) = \mu(|g|_{HF}^2) + \mu(|h|_{HF}^2)
\leq r_1\mathcal{E}(f, f) + 2\beta_1(r)n_r^2c_r^2\mu(|f|_H)^2 + 2\beta_1(r)n_r^2\varepsilon_r \mu(|f|_H^2).
\]
This proves (4.1) with decreasing \( \beta \in C(r_0, \infty) \) such that
\[
\beta(r) \geq \frac{2\beta_1(r)n_r^2c_r^2r}{r_1} = \frac{4\beta_1(r)n_r^2c_r^2r}{r_0 + r}.
\]
\[\square\]

One may also study the essential spectrum and semigroup properties on \( L^p_H(\mu) \) using the \( F \)-Sobolev inequality. In this case, \(|f|\) and \( f^2 \) in the literature are understood respectively as \(|f|_H\) and \(|f|^2_H\).

**Theorem 4.4.** Let \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be a coercive closed form on \( L^2_H(\mu) \). If (1.3) holds, which obviously implies (i) and (ii), then \( P_t \) (equivalently, \( R_\lambda \)) is uniformly integrable in \( L^2_H(\mu) \) for each \( t > 0(\text{each } \lambda \in \rho(L)) \). If in addition \( \mathcal{F} \) is \( \mu \)-separable and \( P = P_t \) satisfies the condition in Lemma 3.1 for each \( t > 0 \), then \( P_t \) (equivalently, \( R_\lambda \)) is compact for each \( t > 0(\text{each } \lambda \in \rho(L)) \), and hence the essential spectrum of \( L \) is empty.

**Proof.** The first assertion follows from the proof of Theorem 2.4 with \( f^2 \) replaced by \(|f|^2_H\). The second assertion follows from Lemma 3.1 and Theorem VI-3.3 in [11]. \[\square\]

As an application of Theorems 4.1, 4.3 and 4.4, we consider the second-order differential operators on differential forms. Let \( M \) be a \( n \)-dimensional complete Riemannian manifold. We fix a point \( o \in M \) and let \( \{X_1, \ldots, X_n\} \) a family of smooth vector fields outside of \( \text{cut}(o) \), the cut locus of \( o \), such that for each \( x \notin \text{cut}(o) \), \( \{X_1(x), \ldots, X_n(x)\} \) is an orthonormal basis of the tangent vector space at \( x \). Let \( \omega_i \) be the dual one-form of \( X_i, 1 \leq i \leq n \), then for each \( 1 \leq p \leq n \), \( \{\omega_i \wedge \cdots \wedge \omega_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\} \) is an orthonormal basis of the space of differential \( p \)-forms \( \Omega^p \) outside of \( \text{cut}(o) \). Since the volume of \( \text{cut}(o) \) is zero, we may consider the space of \( \mu \)-square integrable differential \( p \)-forms as a dense sub-space of \( L^2_{g_M}(\mu) \) via the above basis, where \( \mu \) is a probability measure on \( M \) with positive density w.r.t. the volume element, and \( d = \frac{n!}{p!(n-p)!} \). Then we are able to apply Theorems 4.1, 4.3 and 4.4 to study second order differential operators on the space of \( L^2 \)-integrable differential \( p \)-forms which are generators of coercive closed forms.

Finally, we remark that if \( P_t \) is compact for some \( t > 0 \), then the essential spectrum of \( L \) is empty. Actually, this result is true for any strongly continuous semigroup with \( \|P_t\| \leq c_1e^{-ct} \) for some \( c_1, c_2 > 0 \) and all \( t > 0 \), see Theorems 1 and 2 in [10]. Therefore, to ensure the empty essential spectrum, it suffices to prove the uniform integrability of \( P_t \) for some \( t > 0 \) according to Lemma 3.1. This can be done by using Harnack inequalities of \( P_t \), see the next section. It would be also useful to note that the \( L^p \)-uniform integrability of the sub-Markov operator \( P_t \) on \( L^2(\mu) \) for some \( p \in (1, \infty) \) is equivalent to that for any \( p \in (1, \infty) \). Then one may also prove the empty essential spectrum on \( L^p(\mu) \) for \( p \in (1, \infty) \).
5 Further discussions

We first recall some known criteria for the super-Poincaré inequality. It has been proved in [18] for $L = \Delta + \nabla V$ on a Riemannian manifold with $\mu(dx) = e^Vdx$ a probability measure, the super-Poincaré inequality (1.2) holds provided

$$\lim_{\rho \to \infty} L\rho = -\infty,$$

where $\rho$ is the Riemannian distance function from a fixed point, and the limit is taken outside of the cut-locus of this point. More generally, if $\lim_{\rho \to \infty} L\rho = -c < 0$, then by Cheeger’s inequality and Donnely-Li’s decomposition principle, we have $\sigma_{ess}(-L) \subset [c^2/4, \infty)$. By Theorem 4.3, (1.4) holds for $r_0 = 4c^2$. The assertions remain true if $\rho$ is a smooth compact function with $|\nabla \rho| \leq 1$. By “compact” we mean that $\{\rho \leq r\}$ is compact for any $r > 0$. An analogue sufficient condition for (1.2) was also presented there for a class of jump processes, including irreducible countable Markov chains.

An alternative way to prove (1.2) is to use the isoperimetric inequalities. Consider once again the above $L$ for symmetric diffusion, then (1.2) holds provided (Theorem 3.4 in [18])

$$k(r) := \inf_{\mu(A) \leq r} \frac{\mu(\partial A)}{\mu(A) \mu(\partial A)} \to \infty \text{ as } r \downarrow 0,$$

where $\mu_\partial$ is the $(d-1)$-dimensional measure induced by $\mu$, and $A$ runs over all connected smooth domains. The converse result holds if the curvature of $L$ is bounded from below. Corresponding results were also proved in [17] for general jump processes.

By Theorem 1.2, we may prove the super-Poincaré inequality by checking the uniform integrability of $P_t$. For this it is convenient to apply a Harnack inequality. We shall only consider the above $L$ on a manifold, but the Harnack inequalities are also available for other cases, see e.g. [1] for diffusion semigroups on abstract Wiener spaces and [14] for generalized Mehler semigroups.

Let $L = \Delta + Z$ for some $C^1$-vector field $Z$. Assume that $\text{Ric} - \langle \nabla Z, \cdot \rangle$ is bounded from below. By the proof of Lemma 2.1 in [16], for any $t > 0$ there exists $c_t > 0$(see the original statement for the specific choice of $c_t$) such that

$$(P_t f(x))^2 \leq P_t f^2(y) \exp[c_t \rho(x, y)^2],$$

(5.1)

where $\rho(x, y)$ denotes the Riemannian distance between $x$ and $y$. This implies the following result.

**Proposition 5.1.** Let $P_t$ be the diffusion semigroup generated by $L$, and let $B(x, r)$ denote the geodesic ball with center $x$ and radius $r$. If there exists a positive measurable function $\gamma$ such that

$$\int_M \frac{\exp[c_t \gamma(x)^2]}{\mu(B(x, \gamma(x)))} \mu(dx) < \infty,$$

(5.2)

then $P_t$ is uniformly integrable in $L^2(\mu)$. Especially, it is the case if
\[
\int_M \frac{\mu(dx)}{\mu(B(x,r))} < \infty \text{ for some } r > 0.
\]

**Proof.** Let \( f \) be such that \( \mu(f^2) = 1 \). By (5.1),

\[
1 = \int_M P_t f^2(y)\mu(dy) \geq [P_t f(x)]^2 \int_M \exp[-c_t \rho(x,y)^2] \mu(dy)
\geq [P_t f(x)]^2 \mu(B(x, \gamma(x))) \exp[-c_t \gamma(x)^2].
\]

Then

\[
[P_t f(x)]^2 \leq \frac{\exp[c_t \gamma(x)^2]}{\mu(B(x, \gamma(x)))}.
\]

Therefore, if (5.2) holds, then \( P_t \) is uniformly integrable in \( L^2(\mu) \).

Finally, we present below an example to show that our results apply also to some infinite-dimensional cases.

Let \( T_t \) be a strongly continuous semigroup on a separable Hilbert space \( H \) with \( \|T_t\| \to 0 \) as \( t \to \infty \). Let \( Q \) be a continuous linear self-adjoint nonnegative operator on \( H \). Define

\[
Q_t = \int_0^t T_s Q T_s^* ds, \quad t > 0.
\]

We assume that \( \sup_{t>0} \text{Tr}Q_t < \infty \) and \( \ker(Q_T) = \{0\} \) for any \( t > 0 \). Put

\[
Q_\infty = \int_0^\infty T_t Q T_t^* ds,
\]

then \( Q_\infty \) is well-defined with finite trace. Consider the generalized Mehler semigroup

\[
P_t f(x) = \int_H f(y) N(T_t x, Q_t)(dy), \quad x \in H, \ t \geq 0,
\]

where \( N(T_t x, Q_t) \) denotes the Gaussian measure on \( H \) with mean \( T_t x \) and covariance operator \( Q_t \). Then the unique invariant measure of \( P_t \) is \( \mu = N(0, Q_\infty) \). According to [4], \( P_t \) has transition density if and only if

\[
S_t S_t^* \text{ is a Hilbert-Schmidt operator with } \|S_t S_t^*\| < 1, \quad (5.3)
\]

where \( S_t := Q_\infty^{-1/2} T_t Q_\infty^{1/2} \).

**Proposition 5.2.** Assume that (5.3) holds for some \( t > 0 \). Then \( P_s \) is compact in \( L^2(\mu) \) for any \( s \geq t \) and the essential spectrum of the generator is empty.

**Proof.** By Theorem 4 in [4], we have \( \|P_t\|_{p \to q} < \infty \) for \( g > p > 1 \) with \( \sqrt{p-1} > \|T_t T_t^*\| \sqrt{q-1} \). Then (5.3) implies that \( P_t \) is uniformly integrable in \( L^2(\mu) \). The proof is completed by Lemma 3.1 and [10].
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