HEAT KERNEL ESTIMATES WITH APPLICATION TO COMPACTNESS OF MANIFOLDS *

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Abstract

Li-Yau’s type two-side heat kernel bounds are obtained for symmetric diffusions under a curvature-dimension condition, where the heat kernel upper bound is established for a more general case. As an application, the compactness of manifolds is studied by using heat kernels. Especially, a conjecture by Bueler is proved.

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1 Introduction

Let $M$ be a connected complete Riemannian manifold of dimension $d$. Let $P_t$ be the diffusion semigroup generated by $L = \Delta + \nabla V$, and $p_t(x, y)$ its kernel with respect to $\mu(dx) = \exp[V(x)]dx$, where $V \in C^2(M)$ and $dx$ denotes the Riemannian volume element. The first aim of this note is to estimate $p_t$ by using the measure of balls.

It is well known by Li-Yau [16], for the case that $V = 0$ and the Ricci curvature is nonnegative,

$$\frac{\exp[-\frac{\rho(x,y)^2}{(4-\varepsilon)t}]}{C(\varepsilon) \mu(B(y, \sqrt{t}))} \leq p_t(x, y) \leq \frac{C(\varepsilon) \exp[-\frac{\rho(x,y)^2}{(4+\varepsilon)t}]}{\mu(B(y, \sqrt{t}))}$$

(1.1)

for any $\varepsilon \in (0, 4)$ and some $C(\varepsilon) > 1$, where $\rho(x, y)$ denotes the Riemannian distance between $x$ and $y$ and $B(y, r)$ is the closed geodesic ball with center $y$ and radius $r$. This estimate has been extended by Sturm [23] to the case that the Ricci curvature is bounded from below ($V = 0$):

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We note that (1.2) improves an upper bound given by Davies [9] in which
\[ \mu(B(y, \sqrt{t})) \] is replaced by \[ \sqrt{\mu(B(y, \sqrt{t}))\mu(B(x, \sqrt{t}))}. \]

The case that \( \nabla V \neq 0 \) is also studied by Qian [19], who obtained the same type of
estimates in (1.1) under the assumption
\[ \text{Ric}^\alpha_V(X, X) := (\text{Ric} - \text{Hess}_V)(X, X) - \frac{1}{\alpha} \langle X, \nabla V \rangle^2 \geq 0 \] (1.3)
for some \( \alpha \geq 1 \) and all \( X \in TM \).

To state our result, we introduce Bakry-Emery’s curvature-dimension condition. Let
\[ \Gamma_2(f, g) = \frac{1}{2} \left\{ L \langle \nabla f, \nabla g \rangle - \langle \nabla f, \nabla Lg \rangle - \langle \nabla g, \nabla Lf \rangle \right\}. \]
We say that the curvature-dimension condition holds if there exist \( K \geq 0 \) and \( n > 0 \) such that
\[ \Gamma_2(f, f) \geq -K |\nabla f|^2 + \frac{1}{n} (Lf)^2. \] (1.4)
It is easy to check that (1.4) is equivalent to \( \text{Ric}^\alpha_V \geq -C \) for some \( C \geq 0 \) and \( \alpha \geq 1 \), see e.g. [2].

**Theorem 1.1.** (1) If (1.4) holds, then for any \( \varepsilon > 0 \), there exist \( C_1(\varepsilon), C_2(\varepsilon) \geq 0 \) such that
\[ p_t(x, y) \geq \frac{C_1(\varepsilon)}{\mu(B(y, \sqrt{t}))} \exp \left[ -\frac{(1 + \varepsilon)\rho(x, y)^2}{4t} - C_2(\varepsilon)t \right], \quad t > 0, x, y \in M. \] (1.5)

(2) If \( \nabla V = Z_1 + Z_2 \) for some vector fields \( Z_1, Z_2 \) such that \( L = \Delta + Z_1 \) satisfies (1.4) and \( Z_2 \) is bounded, then for any \( \varepsilon > 0 \), there exist \( C_1(\varepsilon), C_2(\varepsilon) \geq 0 \) such that
\[ p_t(x, y) \leq \frac{C_1(\varepsilon)}{\mu(B(y, \sqrt{t}))} \exp \left[ -\frac{\rho(x, y)^2}{(4 + \varepsilon)t} + C_2(\varepsilon)t \right], \quad t > 0, x, y \in M. \] (1.6)

**Remark.** By Theorem 1.1, we have extended Sturm’s two-side estimates (1.2) to operators satisfying the curvature-dimension condition (1.4). Qian’s result mentioned above is also extended since (1.4) is equivalent to \( \text{Ric}^\alpha_V \geq -C \) for some \( C \geq 0 \).

Our next goal is to apply the above heat kernel bounds to the study of compactness of \( M \). The study goes back to Meyers’ theorem [18] which implies the compactness of \( M \) if the Ricci curvature is bounded from below by a positive constant. This result has been extended to more relaxed curvature conditions. By Saloff-Coste [22] (see also [15]), if \( \mu(M) < \infty \), then \( M \) is compact provided the log-Sobolev inequality holds for \( V = 0 \) (recall that the uniform positivity of Ricci curvature is stronger than the log-Sobolev
inequality). Along a different line, Li [17] proved the compactness of $M$ under a condition called “stochastic positivity” of the Ricci curvature. Very recently, Bakry-Ledoux-Wang [3] shown that $M$ is compact if there exists $V \in C^2(M)$ such that $\mu(M) < \infty$, $V = V_1 + V_2$ with $\nabla V_2$ bounded and $\Delta + \nabla V_1$ satisfying (1.4), and a $F$-Sobolev inequality holds for some increasing $F$ with $\lim_{r \to \infty} F(r) = \infty$:

$$\mu(f^2 F(f^2)) \leq c_1 \mu(|\nabla f|^2) + c_2, \quad \mu(f^2) = 1. \quad (1.7)$$

Our next result provides two criteria for the compactness of $M$ by using heat kernel.

**Theorem 1.2.** $M$ is compact provided there exists $V \in C^2(M)$ such that one of the following holds:

1. $L = \Delta + \nabla V$ satisfies (1.4) and

$$\int_M p_t(x, x) \mu(dx) < \infty \text{ for some } t > 0. \quad (1.8)$$

2. There exists $V_1, V_2 \in C^2(M)$ such that $V = V_1 + V_2$, $\Delta + \nabla V_1$ satisfies (1.4), $\|\nabla V_2\|_\infty < \infty$, and

$$\int_M \frac{1}{p_t(x, y)^s} \mu(dy) < \infty \quad (1.9)$$

for some $s, t > 0$ and $x \in M$.

We now go to compare Theorem 1.2 with previous related results mentioned above. Since when $M$ is compact, the log-Sobolev inequality holds and the heat kernel is bounded away from 0 and $\infty$ for each $t > 0$, we see that (1.7), (1.8) and (1.9) are also necessary for the compactness of $M$. It turns out that these three conditions are equivalent provided $L$ satisfies (1.4), but each of them should have its own advantage in application. Moreover, curvature conditions in [18, 17] mentioned above are not necessary (but very useful) for compactness, and our situation is slightly more general than in [22] where $V = 0$ is considered.

The following conjecture by Bueler [5] (Conjecture 6.3) is a direct consequence of Theorem 1.2.

**Corollary 1.3.** Assume that the Ricci curvature is bounded from below and $V = 0$. $M$ is compact if and only if there exist $x \in M$ and $t > 0$ such that

$$\int p_t(x, y)^{-1} dy < \infty. \quad (1.10)$$

**Proof.** If $M$ is compact, then for any $t > 0$ and $x \in M$, $p_t(x, \cdot)^{-1}$ is bounded and hence (1.10) holds. The inverse result follows from Theorem 1.2 (2) with $s = 1$, just note that in the present case (1.4) holds for $n = d$ by Bochner-Lichnerowicz-Weitzenbock formula (see e.g. Lemma 5.3.1 in [10]).

Following the line of [5], we may apply Corollary 1.3 to the $L^2$-harmonic $d$-forms of the “heat kernel weighted Hodge Laplacian”. We fix $x_0 \in M$ and $t > 0$, let $d\nu = p_t(x_0, x) dx$. 

Let $L^2_\nu \Omega^n$ be the Hilbert space of square integrable (w.r.t. $\nu$) differential $n$-forms, where $0 \leq n \leq d$. The heat kernel weighted Hodge Laplacian is defined by

$$\Delta_\nu = d^* \nu d + d d^* \nu,$$

where $d^* \nu$ is the adjoint of the exterior derivative $d$ on $L^2_\nu \Omega^n$. For $0 \leq n \leq d$, let $H^n_\nu = \ker \Delta^n_\nu$ denote the space of harmonic $n$-forms in $L^2_\nu \Omega^n$.

**Corollary 1.4.** Assume that the Ricci curvature is bounded from below. $M$ is compact if and only if there exists $0 \neq \omega \in H^d_\nu$.

**Proof.** Let $V = 0$ and $\alpha = \int p_t(x_0, x)^{-1} \, dx$. By Theorem 6.2 in [5], $\dim H^d_\nu = 1_{(0, \infty)}(\alpha)$. The proof is completed by combining this with Corollary 1.3. □

## 2 Proof of Theorem 1.1

We shall go with the line of [23] to derive the lower bound estimate, and use the Harnack inequality due to [24] to get the upper bound estimate. The proof is based on the following three lemmas, where the first one extends Theorem 4.5 in [23], the second one is a generalization of Cheeger-Gromov-Taylor’s volume comparison theorem [7], and the final one is a slight extension of a result in [2] which was proved for symmetric diffusions with curvature bounded from below.

**Lemma 2.1.** Let $y \in M$ be fixed and let $\rho_y(x) = \rho(x, y)$. If there exists $k > 0$ and $m \in \mathbb{N}$ such that

$$L \rho_y(x) \leq mk \coth[k \rho_y(x)], \quad x \in M \setminus \text{cut}(y),$$

where cut$(y)$ is the cut-locus of $y$. Then there exists $C(m) > 0$ such that for all $x \in M$ and $t > 0$,

$$P_t 1_{B(y, \sqrt{t})}(x) \geq \frac{C(m)}{k(m+1)} \exp \left[ -\frac{(\rho(x, y) + \sqrt{t})^2}{4t} - \frac{m^2 k^2 t}{4} - \frac{m(\rho(x, y) + \sqrt{t})}{2} \right] \cdot \left(1 + (\rho(x, y) + \sqrt{t}) k + k^2 t \right)^{\frac{m}{2} - 1} \left(1 + (\rho(x, y) + \sqrt{t}) k \right).$$

**Proof.** The proof is based on an argument due to [12, 8]. Let $\mathbb{H}^{m+1}_k$ be the $(m+1)$-dimensional hyperbolic space with sectional curvature $-k^2$. Let

$$\tilde{\rho}_R(t, r) = \tilde{P}_1 1_{B(o_1, R)}(o_2)$$

for $R, r > 0$ and $o_1, o_2 \in \mathbb{H}^{m+1}_k$ with $\text{dist}(o_1, o_2) = r$, where $\tilde{B}(o, r)$ denotes the geodesic ball in $\mathbb{H}^{m+1}_k$ with center $o$ and radius $r$, and $\tilde{P}_t$ is the semigroup generated by the Laplacian $\Delta$ on $\mathbb{H}^{m+1}_k$. By Lemma 3.2 in [23], we have $\frac{\partial}{\partial r} \tilde{\rho}_R(t, r) \leq 0$. Then (2.1) yields
Lemma 2.2. The proof is completed by noting that \( \text{vol}(\tilde{\omega}) \) for some

\[ L\tilde{p}_R(t, \rho_y)(x) = \left( \frac{\partial}{\partial r} \tilde{p}_R(t, r) \big|_{r=\rho_y(x)} \right) L\rho_y(x) + \frac{\partial^2}{\partial r^2} \tilde{p}_R(t, r) \big|_{r=\rho_y(x)} \]

\[ \geq \Delta \tilde{p}_1 B(o_1, R)(o_1)_{\text{dist}(o_1, o_2) = \rho_y(x)} = \frac{\partial}{\partial t} \tilde{p}_R(t, \rho_y)(x), \quad x \notin \text{cut}(y). \]

Since \( \tilde{p}_R(t, r) \) is decreasing in \( r \), by an argument in [6] (c.f. Appendix in [26]), \( L\tilde{p}_R(t, \rho_y) \geq \frac{\partial}{\partial t} \tilde{p}_R(t, \rho_y) \) on \( M \) in the distribution sense. Then, as claimed in [23], one obtains by using the parabolic maximal principle

\[ P_1 B(y, R)(x) \geq \tilde{p}_R(t, \rho_y(x)), \quad x \in M, t > 0. \]  

(2.2)

We present below a probabilistic proof of (2.2) for our own interest. Let \( x \) be the \( L \)-diffusion process. Since \( \tilde{p}_R(t, r) \) is decreasing in \( r \), by the Itô’s inequality for the radial part due to Kendall [14], we have

\[ d\tilde{p}_R(t-s, \rho_y)(x_s) = \sqrt{2} \left[ \frac{\partial \tilde{p}_R(t-s, r)}{\partial r} \right]_{r=\rho_y(x_s)} \, db_s \]

\[ + \left[ L\tilde{p}_R(t-s, \rho_y)(x_s) - \frac{\partial \tilde{p}_R(u, \rho_y(x_s))}{\partial u} \big|_{u=t-s} \right] \, ds + dL_s \]

for \( s \in [0, t] \), where \( b_s \) is the one-dimensional Brownian motion, \( L_s \) is an increasing process with support contained in \( \{ s : x_s \in \text{cut}(y) \} \), and \( L\tilde{p}_R(t-s, \rho_y) = 0 \) on \( \text{cut}(y) \). Since \( L\tilde{p}_R(t-s, \rho_y) \geq \frac{\partial}{\partial u} \tilde{p}_R(u, \rho_y) \) outside of the zero-measured set \( \text{cut}(y) \), we obtain

\[ P_1 B(y, R)(x) \geq P \lim_{s \to t} \tilde{p}_R(t-s, \rho_y)(x) \]

\[ = \tilde{p}_R(t, \rho_y(x)) + E^x \int_0^t d\tilde{p}_R(t-s, \rho_y)(x_s) \geq \tilde{p}_R(t, \rho_y(x)). \]

Next, by Theorem 3.1 in [11] (or Theorem 5.7.2 in [10]), the heat kernel on \( \mathbb{H}^{m+1}_1 \) (denoted by \( k_m(t, \rho) \)) has the lower bound

\[ \frac{C_1(m)}{(4\pi t)^{(m+1)/2}} \exp \left[ -\frac{\rho^2}{4t} - \frac{m^2 t}{4} - \frac{m\rho}{2} \right] (1 + \rho + t)^{m-1}(1 + \rho) \]

for some \( C_1(m) > 0 \). Moreover, by a conformal change of the metric (c.f. [25]), and noting that \( k_m(t, r) \) is decreasing in \( r \), we obtain

\[ \tilde{p}_\sqrt{t}(t, r) \geq \text{vol}(B(o_1, \sqrt{t})) k_m(k^2 t, k(r + \sqrt{t})). \]

The proof is completed by noting that \( \text{vol}(B(o_1, r)) \geq C_2(m) r^{m+1} \) for some \( C_2(m) > 0 \) and all \( r > 0 \).

Lemma 2.2. If (2.1) holds, then

\[ \mu(B(y, \alpha r)) \leq \mu(B(y, r)) \alpha^{m+1} \exp[mk(\alpha - 1)r], \quad r > 0, \alpha > 1. \]  

(2.3)
Proof. For \( r > 0 \) and \( \alpha > 1 \), we have
\[
\frac{\mu(B(y, \alpha r))}{\mu(B(y, r))} = \frac{\int_{S^{d-1}} d\theta \int_{0}^{(\alpha r)^{\wedge}(\theta)} g(s, \theta)ds}{\int_{S^{d-1}} d\theta \int_{0}^{r^{\wedge}(\theta)} g(s, \theta)ds},
\]
where \((s, \theta) \in \mathbb{R}_+ \times S^{d-1}\) is the polar coordinate around \( y \), \( r(\theta) = \inf\{r > 0 : \exp[r \theta] \in \text{cut}(y)\} \), and \( g(s, \theta) = \frac{du}{dsd\theta} \). By (2.1) we have
\[
\frac{\partial}{\partial s} \log g(s, \theta) = L\rho_y(x) \leq mk \coth[k s], \quad x = \exp[s \theta], s \leq r(\theta).
\]
This implies
\[
g(\alpha s, \theta) \leq g(s, \theta) \exp \left[ \int_{s}^{\alpha s} mk \coth[k t]dt \right] = \frac{\sinh[m k s]}{\sinh[m k]} g(s, \theta) \leq g(s, \theta)\alpha^m \exp[m k (\alpha - 1)s], \quad \alpha s \leq r(\theta).
\]
Therefore, (2.4) implies
\[
\frac{\mu(B(y, \alpha r))}{\mu(B(y, r))} \leq \frac{\alpha \int_{S^{d-1}} d\theta \int_{0}^{r^{\wedge}(\theta)} g(\alpha s, \theta)ds}{\int_{S^{d-1}} d\theta \int_{0}^{r^{\wedge}(\theta)} g(s, \theta)ds}
\]
\[
\leq \frac{\alpha^{m+1} \int_{S^{d-1}} d\theta \int_{0}^{r^{\wedge}(\theta)} g(s, \theta)ds}{\int_{S^{d-1}} d\theta \int_{0}^{r^{\wedge}(\theta)} g(s, \theta)ds} \exp[m k (\alpha - 1)r]
\]
\[
\leq \alpha^{m+1} \exp[m k (\alpha - 1)r].
\]

**Lemma 2.3.** Assume that there exist two \( C^1 \)-vector fields \( Z_1, Z_2 \) such that \( Z_2 \) is bounded, and \( \text{Ric}(X, X) - \langle \nabla_X Z_1, X \rangle \geq -K|X|^2 \) for some \( K \geq 0 \) and all \( X \in TM \). Then for any \( \varepsilon > 0 \), there exist \( C_1(\varepsilon), C_2(\varepsilon) > 0 \) such that
\[
p_t(x, y) \leq C_1(\varepsilon) \exp\left[ -\frac{\rho(x, y)^2}{4+\varepsilon} + C_2(\varepsilon)t \right] \mu(\langle B(x, \sqrt{t}) \rangle \mu(B(y, \sqrt{t}))^{-1}, \quad x, y \in M, t > 0.
\]

*Proof.* Let \( P_t^1 \) be the semigroup generated by \( \Delta + Z_1 \). By the curvature condition we have (see Lemma 2.1 in [24])
\[
[P_t^1 f(x)]^\alpha \leq P_t^{\alpha}(f^\alpha(y)) \exp \left[ -\frac{\alpha K\rho(x, y)^2}{2(\alpha - 1) \left( 1 - \exp[-2Kt] \right)} \right]
\]
for any nonnegative \( f \in C_0(M), x, y \in M, t > 0 \) and \( \alpha > 1 \). Following the proof of this Harnack inequality in [24], we may obtain
\[
[P_t f]^\alpha \vee [P_t^1 f]^\alpha \leq \left( (P_t f^\alpha) \wedge (P_t^1 f^\alpha) \right) \exp[\|Z_2\|_{\infty}^2 t/(4(\alpha - 1))], \quad t > 0.
\]
Actually, let
\[ \phi(s) = \log P_s(P^1_{t-s}f)^\alpha, \quad \psi(s) = \log P_s(P^1_{t-s}f)^\alpha, \quad s \in [0, t]. \]

It is easy to check that \( \phi'(s) \wedge \psi'(s) \geq -\frac{\alpha \| Z_2 \|^2}{4(\alpha - 1)} \) for \( s \in (0, t) \), which implies (2.7). By (2.6) and (2.7) we obtain

\[
[P_t f(x)]^{\alpha^3} \leq [P^1_t f(x)]^{\alpha^2} \exp[\alpha^2 \| Z_2 \|^2 t/4(\alpha - 1)]
\leq [P^1_t f(x)]^{\alpha^2} (y)^{\alpha^2 \exp[\alpha^2 \| Z_2 \|^2 t/4(\alpha - 1)]} + \frac{\alpha^2 K \rho(x, y)^2}{2(\alpha - 1)(1 - \exp[-2Kt])} \tag{2.8}
\]

Then the proof is completed by an argument in [2]. We survey their proof here for readers’ convenience.

For \( T > 0, x \in M \) and \( p > 1 \), let \( q = p/(p - 1) \) and \( \xi(s, y) = -\frac{\rho(x, y)^2}{2(T - qs)} \), \( s, y < T \). For any nonnegative \( f \in C_b(M) \), we go to prove

\[
\int (P_t f(y))^p \exp[\xi(t, y)] \mu(dy) \leq \int f(y)^p \exp[\xi(0, y)] \mu(dy), \quad qs < T. \tag{2.9}
\]

By an approximation argument, it suffices to prove for the case that \( \mu \) is finite. Actually, we may take \( \{V_n\} \subset C^2(M) \) such that \( V_n \uparrow V \) and for each \( n \), \( \mu_n(dx) := \exp[V_n(x)] dx \) is finite and \( V_n = V \) on \( B(x, n) \). If (2.9) holds for \( \mu_n \) and \( P^1_t \), generated by \( \Delta + \nabla V_n \), then it holds for \( P_t \) and \( \mu \) too. Since \( \mu \) is finite, we may assume that \( f \geq c > 0 \) for some constant \( c \). Put

\[
I(s) = \int_M (P_s f(y))^p \exp[\xi(s, y)] \mu(dy).
\]

It is easy to see that

\[
I'(s) \leq -\int p(p - 1)(P_t f)^p \exp[\xi(s, \cdot)] \left[ \frac{\nabla P_s f}{P_s f} - \frac{\rho(x, \cdot)}{2(p - 1)(T - qs)} \right] \mu(dy) \leq 0, \quad qs < T.
\]

This implies (2.9).

Let \( p, \alpha > 1 \) be such that \( \alpha^3 p = 2 \), by (2.8) and (2.9), there exist \( c_1, c_2 > 0 \) such that

\[
\mu(B(x, \sqrt{2t})) \exp[-c_1 (1 + t) - t/(T - qt)] (P_t f(x))^2 \leq (P_t f(x))^2 \int_{B(x, \sqrt{2t})} \exp \left[ -c_2 t - \frac{\alpha^2 K \rho(x, y)^2}{2(\alpha - 1)(1 - \exp[-2Kt])} - \frac{\rho(x, y)^2}{2(T - qt)} \right] \mu(dy)
\leq \int (P_t f^{\alpha^3}(y))^p \exp \left[ -\frac{\rho(x, y)^2}{2(T - qt)} \right] \mu(dy)
\leq \int f(y)^2 \exp \left[ -\frac{\rho(x, y)^2}{2T} \right] \mu(dy).
\]
Taking \( f(y) = [n \wedge p_t(x, y)] \exp[n \wedge \frac{\rho(x, y)^2}{2t}] \) we obtain
\[
\int [n \wedge p_t(x, y)]^2 \exp[n \wedge \frac{\rho(x, y)^2}{2t}] \mu(dy) \leq \frac{\exp[c_1(1 + t) + \frac{t}{T - q}] \mu(B(x, \sqrt{2t})]}{\mu(B(x, \sqrt{2t}))}.
\] (2.10)

Since \( q \to 1 \) as \( p \to 2 \), we see from (2.10) that for any \( \delta > 2 \) there exist \( C_1(\delta), C_2(\delta) > 0 \) such that
\[
E_\delta(x, t) := \int p_t(x, y)^2 \exp[\frac{\rho(x, y)^2}{\delta t}] \mu(dy) \leq \frac{C_1(\delta) \exp[C_2(\delta)t]}{\mu(B(x, \sqrt{2t}))}.
\]
The proof is completed by observing (see (3.4) in [13])
\[
\mu(B(y, \sqrt{t})) \geq \mu(B(y, \sqrt{t})) \exp[-c(1 + \rho(x, y)/\sqrt{t})].
\] (2.13)

**Proof of Theorem 1.1.** If (1.4) holds, then for any \( \delta \in (0, 1) \) there exists \( c(\delta) > 0 \) such that (see e.g. [4])
\[
p_t(x, y) \geq p_{\delta t}(x, z) \exp[-c(\delta)(1 + t)], \quad t > 0, z \in B(y, \sqrt{t}).
\]

Then
\[
p_t(x, y) = \frac{1}{\mu(B(y, \sqrt{t}))} \int_{B(y, \sqrt{t})} p_t(x, y) \mu(dz) \geq \frac{\exp[-c(\delta)(1 + t)]}{\mu(B(y, \sqrt{t}))} P_{\delta t}1_{B(y, \sqrt{t})}(x).
\] (2.11)

Next, by a generalization Laplacian comparison theorem due to [21], there exist \( m, k \) such that (2.1) holds for any \( y \in M \). By Lemma 2.1, for any \( \varepsilon \in (0, 1) \) there exist \( C_1(\varepsilon), C_2(\varepsilon) > 0 \) such that
\[
P_{\delta t}1_{B(y, \sqrt{t})}(x) \geq C_1(\varepsilon) \exp[-\frac{(1 + \varepsilon)\rho(x, y)^2}{4\delta t} - \delta C_2(\varepsilon)t]
\] (2.12)
for any \( t > 0 \) and any \( x, y \in M \). Putting (2.11) and (2.12) together we prove the first assertion.

Finally, the second assertion follows immediately from Lemma 2.2 since it implies (see the proof of Lemma 2.2 b) in [23])
\[
\mu(B(x, \sqrt{t})) \geq \mu(B(y, \sqrt{t})) \exp[-c(1 + \rho(x, y)/\sqrt{t})]
\] (2.13)
for some \( c > 0 \). \(\square\)
3 Proof of Theorem 1.2

If (1.4) holds and \( \int p_{t_0}(x, x) \mu(dx) < \infty \) for some \( t_0 > 0 \), by Theorem 1.1 we have

\[
\int \frac{1}{\mu(B(x, \sqrt{t_0}))} \mu(dx) < \infty. \tag{3.1}
\]

By Lemma 2.2, (3.1) implies that

\[
\int \frac{1}{\mu(B(x, r))} \mu(dx) < \infty, \quad r > 0. \tag{3.2}
\]

It then follows from Lemma 2.3 that

\[
\int p_t(x, x) \mu(dx) < \infty, \quad t > 0. \tag{3.3}
\]

By an argument in [3] we prove that \( M \) is compact. We put the argument below for readers' convenience.

By [1], we have the following semigroup Poincaré inequality

\[
P_tf^2 \leq \frac{\exp[Kt] - 1}{K} P_t|\nabla f|^2 + (P_tf)^2. \tag{3.4}
\]

Let \( o \in M \) be fixed. For any \( r > 0 \) and \( f \in C_0^\infty(B(o, r)^c) \), (3.4) yields that

\[
\mu(f^2) \leq \frac{\exp[Kt] - 1}{K} \mu(|\nabla f|^2) + \mu(f^2) \int_{M \times B(o, r)^c} p_t(x, y) \mu(dx) \mu(dy)
\]

\[
\leq \frac{\exp[Kt] - 1}{K} \mu(|\nabla f|^2) + \mu(f^2) \int_{B(o, r)^c} p_{2t}(x, x) \mu(dx). \tag{3.5}
\]

By (3.3), \( \varepsilon(r, t) := \int_{B(o, r)^c} p_{2t}(x, x) \mu(dx) \to 0 \) as \( r \to \infty \). Let

\[
\lambda(r) = \inf\{\mu(|\nabla f|^2) : f \in C_0^\infty(B(o, r)^c), \mu(f^2) = 1\}.
\]

By (3.5) we obtain

\[
\lim_{r \to \infty} \lambda(r) \geq \sup_{t > 0} \frac{K}{\exp[Kt] - 1} = \infty. \tag{3.6}
\]

On the other hand, however, (1.4) implies (2.1) for some \( m, k > 0 \) and all \( x, y \in M \). Therefore, there exists \( c > 0 \) such that \( \lambda(B(x, 1)) \leq c \) for any \( x \in M \), where \( \lambda(B(x, 1)) \) denotes the first Dirichlet eigenvalue of \( L \) on \( B(x, 1) \). If \( M \) is not compact, for any \( r > 0 \) there exists \( x \) such that \( B(x, 1) \subset B(o, r)^c \). Then \( \lambda(r) \leq \lambda(B(x, 1)) \leq c \). This is a contradiction to (3.6).

Next, assume that (2) holds. By (1.6) and noting that \( p_t(x, y) = p_t(y, x) \),

\[
\int \exp[c\rho(x, y)^2] \mu(dy) < \infty \tag{3.7}
\]
for some $c > 0$. If $M$ is not compact, we may take $\{x_n\} \subset M$ such that $\rho(x, x_n) = n$. By (2.3) or (2.13), there exists $c_1 > 0$ such that

$$\mu(B(x_n, 1/2)) \geq \exp[-c_1(1 + n)], \quad n \geq 1.$$  

Therefore

$$\int \exp[c\rho(x, y)^2]\mu(dy) \geq \sum_{n=1}^{\infty} \exp[c(n - 1/2)^2 - c_1(1 + n)] = \infty.$$  

This is a contradiction to (3.7).

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References


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