Weak Poincaré Inequalities on Path Spaces *

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Abstract

The weak Poincaré inequality is established on the finite time-interval Brownian path space over a class of Riemannian manifolds with unbounded Ricci curvatures. Compatible sufficient and necessary curvature conditions are also presented for the inequality to hold on infinite time-interval path spaces. Consequently, the convergence rates of the corresponding Ornstein-Uhlenbech semigroups are described.

Keywords: Path space, weak Poincaré inequality, Riemannian manifold, Brownian motion.

1 Introduction and Main Results

It was in 1994 when Fang [8] realized the Poincaré inequality on the finite time-interval Brownian path space over a compact Riemannian manifold. Fang’s observation, in particular his version of Clark-Ocone-Haussmann formula, stimulated a series sequel papers concerning Poincaré and log-Sobolev inequalities on path and loop spaces. It is now very clear that the log-Sobolev (and hence the Poincaré) inequality holds on the finite time-interval Brownian path space provided the based Riemannian manifold has bounded Ricci curvature, see e.g. [4, 10, 11, 6, 19] and references therein. The main purpose of this paper is to prove the weak Poincaré inequality on path spaces either with infinite time-interval or with finite time-interval but over unbounded Ricci curvature manifolds.

Let \((M, g)\) be a (metric and stochastic) complete Riemannian manifold of dimension \(d\). Let \(x_0 \in M\) and \(T > 0\) be fixed. Consider the path space

\[ M^T := \{ \gamma \in C([0, T]; M) : \gamma_0 = x_0 \} \]

equipped with the product \(\sigma\)-field induced by the class of cylindrical functions

\[ \mathcal{F}C^1_b(T) := \{ F : F(\gamma) = f(\gamma_{s_1}, \ldots, \gamma_{s_n}), n \geq 1, f \in C^1_b(\mathbb{R}^n), 0 < s_1 < \cdots < s_n \leq T \}. \]

Let \(\mu^T\) denote the distribution of the Brownian motion starting at \(x_0\) up to time \(T\). Then \(\mu^T\) is a probability measure on \(M^T\).

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Let \( \{U_t\}_{t \geq 0} \) be the horizontal Brownian motion on \( O(M) \), the bundle of orthonormal frames, with \( U_0 = u_0 \in O_{x_0}(M) \). More precisely, let \( \{U_t\}_{t \geq 0} \) solve the stochastic differential equation
\[
dU_t = \sum_{i=1}^d H_i \circ dB^i_t, \quad U_0 = u_0,
\]
where \( \{H_i\}_{i=1}^d \) are orthonormal horizontal vector fields on \( O(M) \), and \( B_t := (B^i_t)_{i=1}^d \) is the Brownian motion on \( \mathbb{R}^d \). It is well-known that \( \gamma_t := \pi(U_t) \) provides the Brownian motion starting at \( x_0 \), where \( \pi : O(M) \rightarrow M \) is the canonical projection.

For \( h \in \mathbf{H} := \{ h \in C([0, T]; \mathbb{R}) : \| h \|_{\mathbf{H}}^2 := \int_0^T h'(s)^2 ds < \infty \} \), let \( D_h \) denote the derivative along direction \( h \). For \( F \in \mathcal{F}C^1_b(T) \) with \( F(\gamma) = f(\gamma_1, \ldots, \gamma_n) \), one has
\[
D_h F(\gamma) = \sum_{i=1}^n \langle \nabla^i h, U_{s_i}(s) \rangle\gamma(i),
\]
where \( \nabla^i \) is the gradient w.r.t. the \( i \)-th component. Then the gradient \( DF(\gamma) \in \mathbf{H} \) is defined by \( \langle DF(\gamma), h \rangle_{\mathbf{H}} := D_h F(\gamma) \) for \( h \in \mathbf{H} \). This, together with the measure \( \mu^T \), gives rise to a natural quadric form defined on \( \mathcal{F}C^1_b(T) \)
\[
\mathcal{E}(F, G) := \mu^T(\langle DF, DG \rangle_{\mathbf{H}}) = E(\langle DF, DG \rangle_{\mathbf{H}}).
\]

We say that the weak poincaré inequality holds if there exists \( \alpha : (0, \infty) \rightarrow (0, \infty) \) such that
\[
(1.1) \quad E F^2 \leq \alpha(\gamma) \mathcal{E}(F, F) + r \| F \|_{\infty}^2, \quad r > 0, \quad EF = 0, \quad F \in \mathcal{F}C^1_b(T).
\]

This inequality is introduced by Röckner and the author in [16] to describe the convergence rate of the corresponding Markov semigroup, and has been applied and studied in a series of sequel papers, see e.g. [2, 3, 21, 22, 23]. The reference function \( \alpha \) appeared in (1.1) is exactly corresponding to the convergence rate of the associated semigroup, see [16] for details.

In general, if
\[
\int_0^T E |\text{Ric}_{\gamma_s}|^2 ds < \infty,
\]
where \( |\text{Ric}_{\gamma_s}| \) denotes the operator norm of the Ricci curvature \( \text{Ric}_{\gamma_s} : T_{\gamma_s}M \rightarrow T_{\gamma_s}M \), then Driver’s integration by parts formula introduced in [7] holds, and hence \( D \) is closable in \( L^2(M^T, \mu^T; \mathbf{H}) \) (see e.g. [6, 12] for a standard argument). Moreover, the form \( (\mathcal{E}, \mathcal{D}C^1_b(T)) \) is closable in \( L^2(\mu^T) \) and its closure \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) is a Dirichlet form on \( L^2(\mu^T) \). Let \( (P^T_t)_{t \geq 0} \) be the associated (Ornstein-Uhlenbech) Markov Semigroup.

As mentioned above, when \( M \) is compact, Fang proved the following Poincaré inequality
\[
(1.2) \quad E F^2 \leq C \mathcal{E}(F, F), \quad F \in \mathcal{F}C^1_b(T), \quad EF = 0,
\]
where \( C > 0 \) is a constant. His proof is based on the following Clark-Oscone-Haussmann type formula observed by himself in [8]:
\[
(1.3) \quad F = EF + \int_0^T \langle H_s^F, dB_s \rangle, \quad F \in \mathcal{F}C^1_b(T),
\]
where
\[ H^F_s := E \left( D_s F + \frac{1}{2} \Phi^{-1}_s \int_s^T \Phi_t \text{Ric}_U \left( D_t F \right) dt \bigg| \mathcal{F}_s \right). \]

Here, \( D_s F := \frac{d}{ds}(DF)_s \), \( \text{Ric}_U(v) \in \mathbb{R}^d \) for \( v \in \mathbb{R}^d \) is defined by \( \langle \text{Ric}_U(v), v' \rangle := \text{Ric}(U_v, U_{v'}) \) for all \( v' \in \mathbb{R}^d \), \( \mathcal{F}_s \) is the natural \((\mu_T\text{-complete})\) filtration of \( B_s \) and \( \Phi_t \) solves the equation
\[ \frac{d\Phi_t}{dt} + \frac{1}{2} \Phi_t \text{Ric}_U = 0, \quad \Phi_0 = I. \]

From (1.3) one sees that the boundedness of \( |\text{Ric}| \) is a natural condition for (1.2) to hold. As far as we know, there is no any example where \( |\text{Ric}| \) is unbounded but (1.2) holds. In this paper, we allow \( |\text{Ric}| \) to be unbounded and search for reasonable conditions for the weak Poincaré inequality (1.1) to hold.

Let \( K \) be an increasing function such that
\[ |\text{Ric}| \leq K(\rho(x)), \quad x \in M, \]
where \( \rho(x) := \text{dist}(x, x_0) \) is the Riemannian distance. Moreover, let \( K_1 \) be nonnegative and increasing such that
\[ \text{Ric}(X, X) \geq -K_1(\rho(x)), \quad x \in M, X \in T_x M, |X| = 1. \]
Finally, for \( R > 0 \), let
\[ \tau_R := \inf \{ t \geq 0 : \rho(\gamma_t) \geq R \}. \]
Then we are ready to state our first result.

**Theorem 1.1.** If
\[ \lim_{R \to \infty} \frac{1}{\sqrt{P(\tau_R \leq T)}} \int_0^\infty ds \frac{1}{e^{TK_1(s)/2}K(s)} = \infty. \]
Then (1.1) holds for
\[ \alpha(r) := \inf_{R \in A_r} \left\{ 8 + e^{TK_1(R)}K(R)^2 \right\} < \infty, \quad r > 0, \]
where
\[ A_r := \left\{ R > 0 : \inf_{R_1 \in (0, R)} P(\tau_{R_1} \leq T) \left( 3 + \frac{8T + T^3}{\int_{R_1}^R \frac{1}{K(s)}e^{-TK_1(s)/2}ds} \right) \leq r \right\}. \]

The following is an alternative to Theorem 1.1.

**Theorem 1.2.** Let \( \xi := \int_0^T e^{\int_0^t K_1(\rho(\gamma_u))du} K(\rho(\gamma_t))^2 dt. \) If \( E\xi < \infty \) then (1.1) holds for
\[ \alpha(r) := \inf \left\{ 2 + \lambda T : P\left( \max_{s \in [0, T]} E(\xi | \mathcal{F}_s) > 2\lambda \right) \leq r \right\}, \quad r > 0. \]
In particular, (1.1) holds for
\[ \alpha(r) := 2 + \frac{T}{2r} E\xi. \]
As consequences of Theorems 1.1 and 1.2, we have the following results with specific upper bounds of $K$.

**Corollary 1.3.** In the following three cases $(\mathcal{E}, \mathcal{F}C^1_0(T))$ is closable in $L^2(\mu^T)$ and let $(P^T_t)_{t \geq 0}$ denote the associated Ornstein-Uhlenbeck semigroup.

1. If there exist $c > 0$ and $\delta \in (0, 2)$ such that $K(r) \leq cr^\delta$ for big $r$, then (1.1) holds for

$$\alpha(r) = \exp[\theta_1 + \theta_2(\log^+ r^{-1})^{\delta/2}]$$

for some $\theta_1, \theta_2 > 0$ and all $r > 0$. Consequently, there exist $c_1, c_2 > 0$ such that

$$\|P^T_t - \mu^T\|_{\infty \to 2} \leq c_1 \exp[-c_2(\log^+ t)^{2/\delta}], \quad t > 0,$$

where $\| \cdot \|_{\infty \to 2}$ is the operator norm from $L^\infty(\mu^T)$ to $L^2(\mu^T)$.

2. If there exists $c > 0$ such that $K(r) \leq c r^2$ for big $r$ and $2e^2T^2e^{1+cT\sqrt{d-1}} < 1$, then (1.1) holds for

$$\alpha(r) = c_1 r^{-1}$$

for some $c_1 > 0$ and all $r > 0$. Consequently,

$$\|P^T_t - \mu^T\|_{\infty \to 2}^2 \leq c_2 t^{-1}$$

for some $c_2 > 0$ and all $t > 0$.

3. If $K_1$ is bounded, i.e. the Ricci curvature is bounded below, and if $K(r) \leq e^{cr^2}$ for some $c \in (0, \frac{1}{2T^2})$ and big $r$. Then (1.1) holds for

$$\alpha(r) = c_1 \left( 1 + K \left( c_2 \sqrt{\log^+ r^{-1}} \right) \right)$$

for some $c_1, c_2 > 0$.

**Remark** (1) The proof of Theorem 1.1 is based on a local version of formula (1.3), which makes sense as $|\text{Ric}|$ is locally bounded (see (2.1) below).

(2) Since the Poincaré inequality holds for bounded $|\text{Ric}|$ and the Ricci curvature is locally bounded, one may hope that the weak Poincaré inequality holds on $M^T$ as soon as $M$ is stochastic complete. To realize such an intuition, one would write

$$\mathbf{E}F^2 \leq \mathbf{E}F^2 \mathbf{1}_{\{\tau R > T\}} + \|F\|^2_{\infty} P(\tau R \leq T).$$

Obviously, for any $r > 0$ there exists $R$ such that $P(\tau R \leq T) \leq r$. Then it remains to show that

$$\mathbf{E}F^2 \mathbf{1}_{\{\tau R > T\}} \leq C(R) \mathbf{E}(F, F)$$

for some $C(R) > 0$. To prove this inequality from (1.3), a key trouble is that the indicator function $\mathbf{1}_{\{\tau R > T\}}$ can not be moved into the conditional expectation, i.e.

$$\mathbf{1}_{\{\tau R > T\}} \mathbf{E}(\Phi^{-1}_s \Phi_1 \text{Ric}_{U_1}(D_t F) || \mathcal{F}_s) \neq \mathbf{E}(\mathbf{1}_{\{\tau R > T\}} \Phi^{-1}_s \Phi_1 \text{Ric}_{U_1}(D_t F) || \mathcal{F}_s).$$
Hence it is not clear how $1_{\{\tau_{R} > T\}}$ can control the curvature involved in (1.3). So, it seems that the non-explosion itself is not enough to imply the weak Poincaré inequality. But we do not have any counterexample to support this opinion.

Next, let us consider the infinite time-interval path space $M^\infty$. In this case Theorems 1.1 and 1.2 are no longer valid. On the other hand, it is proved by Aida [1] that the log-Sobolev inequality holds on $M^\infty$ provided the Ricci curvature is uniformly positive (and hence $M$ has to be compact). This condition is slightly relaxed by the author [20] for the Poincaré inequality. As for the weak Poincaré inequality, we hope that some negatively curved Riemannian manifolds will be allowed. But according to the following result the weak Poincaré inequality implies very likely that the curvature is less negative than $-c\rho^{-2}$ for big $\rho$. Let $\mathcal{F}_b^1 := \cup_{T>0}\mathcal{F}_b^1(T)$ and $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ the Friedrichs extension of $(\mathcal{E}, \mathcal{F}_b^1)$ in the $L^2$-space w.r.t the distribution of the Brownian motion starting at $x_0$.

**Theorem 1.4.** Let $M$ be connected. If $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ is irreducible then $M$ is a Liouville manifold, i.e. all bounded harmonic functions are constant. Consequently, if $M$ is a Cartan-Hadamard manifold with sectional curvature $S$ satisfying
\[
-c_1\rho^2 \leq S \leq -c_2\rho^{-2}
\]
for some $c_1, c_2 > 0$ and big $\rho$, then $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ is reducible.

Theorem 1.4 is a slight improvement of a result in [1] which says that Poincaré inequality on $M^\infty$ implies the Liouville property. Moreover, for a Cartan-Hadamard manifold, it is conjectured by Greene and Wu [9] that $M$ is not a Liouville manifold if
\[
S \leq -c\rho^{-2}
\]
for some $c > 0$ and big $\rho$. This conjecture is proved by Hus and Kendall [13] for $d = 2$ and by Le [15] in general but with the slightly stronger condition (1.7). See also [12] for more comments and results in this direction.

It might be useful to mention that the converse of the first assertion in Theorem 1.4 is not true, i.e. the Liouville property of $M$ is not enough to imply the irreducibility of $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}))$. Indeed, let $M$ be a compact connected Riemannian manifold with sectional curvature pinched by two negative numbers, then its universal cover is simply connected with the same curvature property and hence has non-trivial bounded harmonic functions according to [5]. Therefore, by Proposition 3.1 below the Dirichlet form $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ is reducible. But $M$ is a Liouville manifold because of the compactness. Such a Riemannian manifold possesses some special analysis properties, see e.g. [17] for a lower bound depending on the volume (rather than the diameter as usual) of the first eigenvalue.

According to Theorem 1.4, to prove the weak Poincaré inequality it would be natural to consider the following condition
\[
|\text{Ric}| \leq K(\rho) := \frac{c}{(1 + \rho)^\delta}
\]
for some $c > 0$ and some $\delta > 2$. Under this condition, if $\rho(\gamma_t) \to \infty$ fast enough as $t \to \infty$, then the influence of $|\text{Ric}_t|$ will be under control in order to get a weak Poincaré inequality from (1.3). To study the escape rate of the Brownian motion, we need a condition on
the lower bound of $\Delta \rho$, which leads to the drift term of $\rho(r_t)$.

**Theorem 1.5.** Assume that $x_0$ is a pole with $\Delta \rho \geq \frac{\theta}{\rho}$ for some $\theta > 5$ and (1.8) holds for some $\delta > 2$ and some $c \in (0, \theta - 5)$. Then for any $p > \frac{2}{\theta - 1}$, there exists $c(p) > 0$ such that

$$\mathbb{E} F^2 \leq \alpha(r) \mathcal{E}(F, F) + r \|F\|_\infty^2, \quad r > 0, \quad F \in \mathcal{F} C^1_b, \quad \mathbb{E} F = 0,$$

holds for $\alpha(r) = c(p)r^{-p}$. Consequently,

$$\liminf_{t \to \infty} \frac{\log \|P_t^\infty - \mu^\infty\|_{\infty-2}}{\log t} \geq \frac{\theta - 1}{4}.$$

**Remark.** (a) Results stated in this section can be extended to the Brownian motion with drift generated by $\frac{1}{2}(\Delta + Z)$ for a $C^1$-vector field $Z$ by using a variant of (1.3) appeared in [6] for a more general case. Indeed, for the case with drift our results hold with $K_1$ and $K$ the corresponding bounds of $\text{Ric} - \nabla Z$ instead of $\text{Ric}$, and in Theorem 1.4 the harmonic functions are with respect to the operator $\Delta + Z$.

(b) If $M$ is a Cartan-Hadamard manifold then $\Delta \rho \geq \frac{d-1}{\rho}$ and hence one may take $\theta = d - 1$ in Theorem 1.5 so that the result works for $d > 6$. One may hope to find out a manifold such that $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ is irreducible but (1.9) does not hold for any $\alpha$. Such examples already existed in [16] for Dirichlet forms on configuration spaces, but are unknown for the present setting.

## 2 Proofs of Theorems 1.1, 1.2 and Corollary 1.3

**Proof of Theorem 1.1.** (a) Let us give a local version of (1.3) with unbounded $|\text{Ric}|$. For $R_1 > 0$, let

$$\phi(r) := \frac{1}{\sqrt{P(\tau_{R_1} \leq T)}} \int_{R_1}^{R_1 \vee r} \frac{ds}{e^{TK_1(s)/2}K(s)}, \quad r \geq 0.$$

For $R > R_1$,

$$h_R(r) := \left(1 - \frac{\phi(r)}{\phi(R)}\right)^+, \quad r \geq 0.$$

Then $h_R(r) = 1$ if $r \leq R_1$ and $h_R(r) = 0$ for $r \geq R$.

Next, let $f_R \in C^\infty(\mathbb{M})$ be nonnegative such that $f_R = 1$ on $B(R) := \{\rho \leq R\}$. Let $M_R := \{f_R > 0\}, g_R := f_R^{-2}g$ on $M_R$. Then $(M_R, g_R)$ is a stochastic complete Riemannian manifold with Ricci curvature $\text{Ric}_R$ bounded from below, see [18, Propositions 2.1 and 2.3]. Let $M^T_R := \{\gamma \in C([0, T] ; M_R) : \gamma_0 = x_0\}$. Then we have the following Fang’s formula on $M^T_R$:

$$F = \mathbb{E}_R F + \int_0^T \langle H_{R,s} F, dB_s \rangle, \quad F \in \mathcal{F} C^1_b(T),$$

(2.1)
where $\mathbf{E}_R$ is the expectation taken w.r.t. the Brownian motion on $(M_R, g_R)$ driven by the same $B_s$, and
\[
H_{R,s}^F := \mathbf{E}_R \left( D_{R,s}F + \frac{1}{2} \Phi_{R,s}^{-1} \int_s^T \Phi_{R,t} \text{Ric}_{U_{R,t}}^R (D_{R,t}F) dt \right)_{|F_s} \]
with $D_{R,s}, \Phi_{R,s}, \text{Ric}^R$ and $U_{R,t}$ the corresponding quantities defined on $(M_R, g_R)$.

(b) Let $\rho_R$ denote the Riemannian distance function to $x_0$ on $M_R$. For any $m \geq 1$, let
\[
\rho_{R,m}(\gamma) := \max_{0 \leq i \leq 2^m} \rho_R(\gamma_{i/T/2^m}), \quad \gamma \in M^T_R.
\]
We have (see e.g. [12, p.252])
\[
D_{R,s} \rho_{R,m} = \frac{d}{ds} \sum_{i=0}^{2^m} (s \wedge s_i) U_{R,s_i}^{-1} \nabla_i \rho_{R,m}, \quad s_i := \frac{iT}{2^m},
\]
where $\nabla_i$ is the gradient operator on $(M_R, g_R)$ w.r.t. the $i$-th component. Since (almost surely) there is only one of gradients $\{\nabla_i \rho_{R,m}\}_{0 \leq i \leq 2^m}$ is not zero, one has
\[
|D_{R,s} \rho_{R,m}| \leq 1, \quad s \in [0, T], \quad m \geq 1,
\]
Since $h_R(r) = 0$ for $r \geq R$ and $h_R(r) = 1$ for $r \leq R_1$, we obtain from (2.2) that
\[
|D_{R,s} (F \cdot h_R \circ \rho_{R,m})| \leq |D_{R,s} F| 1_{\{\rho_{R,m} < R\}} + \|F\|_\infty h'_R(\rho_{R,m}) 1_{\{R > \rho_{R,m} > R_1\}} =: I_{R,m}(s).
\]
Therefore, it follows from (2.1) that
\[
\mathbf{E}_R[F^2 \cdot (h_R \circ \rho_{R,m})^2] \\
\leq 2(\mathbf{E}_R[F \cdot (h_R \circ \rho_{R,m})]^2)^2 \\
+ 2 \mathbf{E}_R \int_0^T \left\{ 2I_{R,m}(s)^2 + \frac{1}{2} \left( \int_s^T |\Phi_{R,s}^{-1} \Phi_{R,t}| \cdot |\text{Ric}_{U_{R,t}}^R| I_{R,m}(t) \right) dt \right\} ds.
\]
Noting that $\rho_{R,m}(\gamma) \uparrow \tilde{\rho}_R(\gamma) := \sup_{t \in [0,T]} \rho_R(\gamma_t)$, and that on $B(R)$ one has $g_R = g$ so that the new Brownian path coincides with the original one as soon as $\tilde{\rho}_R(\gamma) \leq R$ (i.e. $\tau_R \geq T$), we obtain by letting $m \uparrow \infty$ in (2.3) that $(\bar{\rho}(\gamma) := \sup_{t \in [0,T]} \rho(\gamma_t))$
\[
\mathbf{E}[F^2 \cdot (h_R \circ \bar{\rho})^2] \\
\leq 2(\mathbf{E}[F \cdot h_R \circ \bar{\rho}])^2 \\
+ 2 \mathbf{E} \left\{ \left[ \int_0^T \left( |D_{\bar{\rho}} F| 1_{\{\bar{\rho} \leq R\}} + \|F\|_\infty h'_R(\bar{\rho}) 1_{\{R \geq \bar{\rho} \geq R_1\}} \right)^2 ds \right] \\
\cdot \left( 2 + \frac{1}{2} \int_0^T ds \int_s^T |\Phi_{s}^{-1} \Phi_t|^2 \cdot |\text{Ric}_{U_t}|^2 1_{\{\bar{\rho} \leq R\}} dt \right) \right\} \\
\leq 2(\mathbf{E}[F \cdot h_R \circ \bar{\rho}])^2 + 4\mathbf{E}[F, F] \left( 2 + \frac{1}{2} \int_0^T ds \int_s^T e^{TK_1(R)} K(R)^2 \right) \\
+ 4T \|F\|_\infty^2 \mathbf{E} \left( 2h'_R(\bar{\rho})^2 1_{\{\tau_R \leq T\}} \right) \\
+ \frac{1}{2} \int_0^T ds \int_s^T h'_R(\bar{\rho})^2 |\Phi_{s}^{-1} \Phi_t|^2 \cdot |\text{Ric}_{U_t}|^2 1_{\{R_1 \leq \bar{\rho} \}} ds.
\]
Since
\[ h_R'(\bar{\rho})^2 \leq \frac{\phi(R)^{-2} e^{-TK_1(\bar{\rho})}}{P(\tau_{R_1} \leq T) K(\bar{\rho})^2}, \quad |\Phi_s^{-1}\Phi|^{2} \cdot |\text{Ric}_{U_t}|^2 \leq e^{TK_1(\bar{\rho})} K(\bar{\rho})^2, \quad T \geq t \geq s, \]
(2.4) implies
\[
\mathbb{E}[F^2 \cdot (h_R \circ \bar{\rho})^2] \\
\leq \mathcal{E}(F, F)(8 + T^2 e^{TK_1(R)} K(R)^2) + \frac{\|F\|^2_\infty (8T + T^3)}{\phi(R)^2} + 2(\mathbb{E}[F \cdot h_R \circ \bar{\rho}])^2.
\]
Therefore, if \( \mathbb{E}F = 0 \) then
\[
\mathbb{E}F^2 \leq \mathbb{E}[F^2 (h_R \circ \bar{\rho})^2] + \|F\|^2_\infty P(\tau_{R_1} \leq T) \\
\leq \mathcal{E}(F, F)(8 + T^2 e^{TK_1(R)} K(R)^2) + \|F\|^2_\infty (3P(\tau_{R_1} \leq T) + (8T + T^3)\phi(R)^{-2}).
\]
Hence, (1.1) holds for the desired \( \alpha \). Finally, since \( \tau_k \to \infty \) as \( k \to \infty \) and (1.6) holds, for any \( r > 0 \) one may find \( R > R_1 > 0 \) such that
\[
\phi(R)^{-2} T(8 + T^2) + 3P(\tau_{R_1} \leq T) \leq r.
\]
This means that \( A_r \neq \emptyset \) and hence \( \alpha(r) \) determined in the theorem is finite for any \( r > 0 \).

\( \square \)

**Proof of Theorem 1.2.** If \( \mathbb{E} \xi < \infty \) then (1.3) holds. For \( F \in \mathcal{F}_k^1(T) \) with \( \mathbb{E}F = 0 \), we have
\[
|H_s^F|^2 \leq 2\mathbb{E}(|D_sF|^2|\mathcal{F}_s) + \frac{1}{2} \left( \int_s^T \mathbb{E}(|D_tF|^2|\mathcal{F}_s) \right) \\
\times \left( \int_s^T \mathbb{E}(e^{\int_t^s \mathbb{E}(K(\rho(t)))) dt} K(\rho(t)))^2 |\mathcal{F}_s) \right) dt \\
\leq 2\mathbb{E}(|D_sF|^2|\mathcal{F}_s) + \frac{1}{2} \left( \int_s^T \mathbb{E}(|D_tF|^2|\mathcal{F}_s) \right) \mathbb{E}(\xi|\mathcal{F}_s).
\]
Combining (1.3) with (2.5) and letting \( \xi_s := \mathbb{E}(\xi|\mathcal{F}_s) \), for any \( \lambda > 0 \) such that \( P(\sup_{s \in [0,T]} \xi_s > 2\lambda) \leq r \) we have
\[
\mathbb{E}F^2 \leq \mathbb{E}F^2 1_{\{\sup_{s \in [0,T]} \xi_s \leq 2\lambda\}} + r\|F\|^2_\infty \\
\leq \int_0^T \mathbb{E}|H_s^F|^2 1_{\{\sup_{s \in [0,T]} \xi_s \leq 2\lambda\}} ds + r\|F\|^2_\infty \leq (2 + \lambda T)\mathcal{E}(F, F) + r\|F\|^2_\infty.
\]
This implies the first assertion. Finally, since \( \xi_s \) is a nonnegative martingale, one has
\[
P\left( \sup_{s \in [0,T]} \xi_s \geq \lambda \right) \leq \frac{1}{\lambda} \mathbb{E} \xi_0 = \frac{1}{\lambda} \mathbb{E} \xi.
\]
Hence the second assertion follows from the first. \hfill \Box

To prove Corollary 1.3 from Theorems 1.1 and 1.2, we need to estimate either \( P(\tau_R \leq T) \) or \( P(\sup_{s \in [0,T]} \xi_s > \lambda) \). To this end, we will compare the radial part with an one-dimensional diffusion process. So, let us first present the following simple lemma for one-dimensional diffusion processes.

**Lemma 2.1.** Consider the reflecting diffusion process \( \{\rho_t\} \) generated by \( L := \frac{d}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \) on \( [0, \infty) \) with \( \rho_0 = 0 \), where \( b \in C(0, \infty) \) is nonnegative. If there exists \( c_1 > 0 \) such that

\[
c_2 := 2 \sup_{x > 0} x(b(x) - c_1 x) < \infty,
\]

then for any \( r_0 > 0 \) and \( h(t) := r_0 e^{-2(c_1 + r_0) t} \), we have

\[
E e^{h(t)\rho_t^2} \leq \exp \left[ \frac{1+c_2}{2(c_1 + r_0)} (1 - e^{-2(c_1 + r_0) t}) \right], \quad t > 0.
\]

Moreover, for any \( \lambda, T > 0 \),

\[
P \left( \max_{t \in [0,T]} \rho_t \geq \lambda \right) \leq \exp \left[ \frac{1+c_2}{2(c_1 + r_0)} (1 - e^{-2(c_1 + r_0) T}) - h(T) \lambda^2 \right].
\]

**Proof.** By Itô’s formula one has

\[
de^{h(t)\rho_t^2} = 2\rho_t h(t) e^{h(t)\rho_t^2} dB_t + h(t) e^{h(t)\rho_t^2} \left\{ 1 + 2b(\rho_t)\rho_t + 2h(t)\rho_t^2 + \frac{h'(t)}{h(t)}\rho_t^2 \right\} dt
\]

\[
\leq 2\rho_t h(t) e^{h(t)\rho_t^2} dB_t + h(t)(1 + c_2) e^{h(t)\rho_t^2} dt,
\]

where and in the sequel \( b_t \) is the one-dimensional Brownian motion. This implies (2.6) by a standard argument by using Gronwall lemma.

Next, since \( \{e^{h(T)\rho_t^2}\}_{t > 0} \) is a submartingale (note that \( b \geq 0 \) ), we have

\[
P \left( \max_{t \in [0,T]} \rho_t \geq \lambda \right) = P \left( \max_{t \in [0,T]} e^{h(T)\rho_t^2} \geq e^{h(T)\lambda^2} \right) \leq e^{-h(T)\lambda^2} E e^{h(T)\rho_T^2}.
\]

Then (2.7) follows from (2.6). \hfill \Box

Now, under the assumption (1.5), the Laplacian comparison theorem implies

\[
\Delta \rho \leq \sqrt{(d - 1)K_1(\rho)} \coth \left( \sqrt{K_1(\rho)/(d - 1)} \right)
\]

\[
\leq \frac{d - 1}{\rho} + \sqrt{(d - 1)K_1(\rho)}
\]

outside the point \( x_0 \) and its cut-locus \( \text{cut}(x_0) \). On the other hand, by Kendall [14] we have

\[
d\rho(\gamma_t) = dB_t + \frac{1}{2} \Delta \rho(\gamma_t) dt - dL_t + dL_t' + dL_t''.
\]
where $L'_t$ is an increasing process with support contained in $\text{cut}(x_0)$, and $L''_t$ is the local time of $\rho(\gamma_t)$ at 0. Therefore, letting $\rho_t$ be the reflecting diffusion process in Lemma 2.1 with $b(x) = \frac{1}{2}(\frac{d-1}{x} + \sqrt{(d-1)K_1(x)})$. We have

$$\rho(\gamma_t) \leq \rho_t, \quad a.s.$$ 

Thus, Lemma 2.1 gives the following estimate of $P(\tau_R \leq T)$.

**Lemma 2.2.** Assume there exists $c_1 > 0$ such that

$$c_2 := \sup_{x > 0} \left\{ x\sqrt{(d-1)K_1(x)} - 2c_1x^2 \right\} < \infty.$$ 

Let $\delta(T) = \frac{1}{2T}e^{-1-2c_1T}$. We have

$$P(\tau_R \leq T) = P\left( \max_{t \in [0,T]} \rho(r_t) \geq R \right) \leq e^{d+c_2-\delta(T)R^2}, \quad R > 0.$$ 

**Proof.** Let $r_0 = \frac{1}{2T}$, we have $\delta(T) = r_0 e^{-2(c_1+r_0)T} = h(T)$. Put

$$b(x) = \frac{1}{2}\left( \frac{d-1}{x} + \sqrt{(d-1)K_1(x)} \right),$$

then $2\sup_{x > 0} x(b(x) - c_1x) = d - 1 + c_2$. By Lemma 2.1 and the fact that $\rho(\gamma_t) \leq \rho_t$, we arrive at

$$P\left( \max_{t \in [0,T]} \rho(\gamma_t) \geq R \right) \leq P\left( \max_{t \in [0,T]} \rho_t \geq R \right) \leq \exp \left[ \frac{(d+c_2)r_0}{c_1+r_0} - \delta(T)R^2 \right] \leq e^{d+c_2-\delta(T)R^2}. \quad \square$$

**Proof of Corollary 1.3.** By Lemma 2.2 we see that $\mathbb{E}K(\rho(\gamma_t))^2 < \infty$ in the specific three cases, and hence $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mu^T)$. We now prove for the three cases respectively.

(a) Let $K(r) \leq cr^\delta$ for some $c > 0$ and $\delta \in (0,2)$. Since (1.5) holds also for $K$ in place of $K_1$, and since

$$\sup_{x > 0} \left\{ x\sqrt{(d-1)K(x)} - 2c_1x^2 \right\} < \infty$$

for any $c_1 > 0$, it follows from Lemma 2.2 that for any $\theta < \frac{1}{2Te}$, there exists $c(\theta) > 0$ such that

$$(2.8) \quad P(\tau_R \leq T) \leq c(\theta)e^{-\theta R^2}, \quad R > 0.$$ 

On the other hand,

$$\int_R^{R+1} e^{-TK(s)/2}K(s)ds \geq c'e^{-TcR^\delta}$$
for some $c' > 0$ and big $R$. Combining this and (2.8), there exist $\theta_0, R_0 > 0$ such that

$$\frac{1}{\sqrt{P(\tau_R \leq T)}} \int_R^{R+1} \frac{e^{-TK(s)/2}}{K(s)} ds \geq e^{\theta_0 R^2},$$

$$P(\tau_R \leq T) \leq e^{-2\theta_0 R^2}, \quad R \geq R_0.$$ 

Then $R + 1 \in A_r$ provided $R > R_0$ and

$$e^{-2\theta_0 R^2}(T(8 + T^2) + 3) \leq r.$$ 

Therefore, for $\alpha$ given in Theorem 1.1 we have

$$\alpha(r) \leq \exp[\theta_1 + \theta_2(\log^+ r^{-1})^{\delta/2}] =: \tilde{\alpha}(r)$$

for some $\theta_1, \theta_2$ and all $r > 0$.

To finish the proof of (1), it remains to estimate $\|P_T^T - \mu^T\|_{\infty \to 2}$ by using [16, Theorem 2.1]. To do this, let

$$\eta(t) := \inf \{r > 0 : \tilde{\alpha}(r) \log r^{-1} \leq 2t \}, \quad t > 0.$$ 

Assume that $\eta(t)$ is reached by $r_0 \in (0, 1)$. If $r_0 > e^{-\sqrt{t}}$ then

$$\tilde{\alpha}(r_0) \sqrt{t} \geq \tilde{\alpha}(r_0) \log r_0^{-1} = 2t.$$ 

Thus

$$r_0^{-1} \geq \exp \left[ \left\{ \frac{1}{\theta_2} (\log(2\sqrt{t}) - \theta_1)^+ \right\}^{2/\delta} \right] =: \eta_2(t).$$

Therefore,

$$\eta(t) = r_0 \leq \max \{e^{-\sqrt{t}}, \eta_2(t)^{-1} \} \leq c_1 \exp[-c_2(\log^+ t)^{2/\delta}]$$

for some $c_1, c_2 > 0$ and all $t > 0$. Then the desired result follows from $\|P_T^T - \mu^T\|_{\infty \to 2} \leq \eta(t)$ according to [16].

(b) Let $K(r) \leq c^2 r^2$ for some $c > 0$ and big $r$. Then Lemma 2.2 applies with $2c_1 := c\sqrt{d - 1}$, i.e., there exists $c' > 0$ such that

$$P \left( \max_{s \in [0,T]} \rho(\gamma_s) \geq R \right) = P(\tau_R \leq T) \leq c'e^{-\delta(T)R^2}, \quad R > 0.$$ 

If $2c^2 T^2 e^{1+cT\sqrt{T-1}} < 1$, then $\delta(T) := \frac{1}{2T} e^{-1-2c_1 T} > c^2 T$. Hence, letting $\bar{\rho} := \max_{s \in [0,T]} \rho(\gamma_s)$, we obtain

$$E\xi \leq T E e^{2T\bar{\rho}^2} \bar{\rho}^4 < \infty.$$ 

Therefore, the desired result follows from Theorem 1.2.
(c) If \( K_1 \) is bounded, then (2.8) holds for any \( \theta < \frac{1}{27e} \). If \( K(r) \leq e^{cr^2} \) for some \( c < \frac{1}{27e} \) and big \( r \), then there exists \( \theta_1 > 0 \) such that

\[
\frac{1}{\sqrt{\mathcal{P}(\tau_R \leq T)}} \left( 1 + \int_{R}^{R+1} \frac{e^{-TK_1(s)/2}}{K(s)} \, ds \right) \geq e^{\theta_1 R^2}
\]

for big \( R \). Thus, \( \frac{1}{\theta_1} (\log r^{-1})^{\frac{1}{2}} \in A_r \) for small \( r \geq 0 \). Therefore, by Theorem 1.1, (1.1) holds some \( \alpha \) with

\[
\alpha(r) := 8 + e^{TK_1(\infty)} K \left( \frac{1}{\theta_1} \sqrt{\log r^{-1}} \right)
\]

for small \( r > 0 \). But one may take \( \alpha \) to be decreasing in (1.1), then (1.1) holds for

\[
\alpha(r) := c_1 \left( 1 + K \left( c_2 \sqrt{\log^+ r^{-1}} \right) \right)
\]

for some \( c_1, c_2 > 0 \) and all \( r > 0 \).

3 Proofs of Theorem 1.4 and 1.5

Proof of Theorem 1.4. The proof is modified from Aida [1], where the Poincaré inequality is considered. Let \( \mu \) be a nonconstant bounded harmonic function. Let \( F_t \in \mathcal{F}C_{b}^{1} \) with \( F_t(\gamma) := u(\gamma_t), \gamma \in \Gamma \). We have

\[
F_t^2 = \mathbf{E} u(\gamma_t)^2, \quad \mathcal{E}(F_t, F_t) = t \mathbf{E} |\nabla u|^2(\gamma_t), \quad t > 0.
\]

Since \( u \) is harmonic,

\[
\mathbf{E} F_t = u(x_0), \quad \int_0^T \mathbf{E} |\nabla u|^2(\gamma_t) \, dt = u(\gamma_T)^2 - u(x_0)^2, \quad T > 0.
\]

Thus, \( \mathbf{E} u(\gamma_t)^2 \) is increasing in \( t \). Since \( u \) is harmonic and bounded, \( F_t \) is a bounded continuous martingale. Then by the martingale convergence theorem there exists \( F_\infty \) such that \( F_t \) converges to \( F_\infty \) a.s. Moreover, by (3.1) and (3.2) we have

\[
\int_1^\infty \frac{\mathcal{E}(F_t, F_t)}{t} \, dt = \int_1^\infty \mathbf{E} |\nabla u|^2(\gamma_t) \, dt \leq \|u\|^2_{2} < \infty.
\]

Then there exists a sequence \( t_n \uparrow \infty \) such that \( \mathcal{E}(\infty) = \mathcal{E}(F_{t_n}, F_{t_n}) \rightarrow 0 \) as \( n \rightarrow \infty \). Thus, \( F_\infty \in \mathcal{D}(\mathcal{E}(\infty)) \) and \( \mathcal{E}(\infty) = 0 \). But by (3.2) one has

\[
\mathbf{E} F_\infty^2 - (\mathbf{E} F_\infty)^2 = \int_0^\infty \mathbf{E} |\nabla u|^2(\gamma_t) \, dt > 0
\]

since \( u \) is a nonconstant harmonic function. Therefore, \( (\mathcal{E}(\infty), \mathcal{D}(\mathcal{E}(\infty))) \) is reducible.

We remark that Aida [1] also considered compact manifolds with non-Liouville Riemannian covers. Let \( M \) be compact and let \( \tilde{M} \) be a Riemannian cover of \( M \). Let \( \gamma_t \) be
the lift of the Brownian motion $\gamma_t$ to $\tilde{M}$ with $\tilde{\gamma}_0 = x_0$. For $f \in C^1_b(\tilde{M})$, by \cite[Lemma 4.1]{1} one has $\tilde{F} \in \mathcal{D}(\mathcal{E})$, where $\tilde{F}_t(\gamma) := f(\tilde{\gamma}_t)$, and

$$\mathcal{E}(\tilde{F}_t, \tilde{F}_t) = tE|\nabla f|^2(\tilde{\gamma}_t), \quad t > 0.$$ 

Thus the proof of Theorem 1.4 leads to the following result.

**Proposition 3.1.** Let $M$ be a compact connected Riemannian manifold. If $M$ has a non-Liouville Riemannian cover, then $(\mathcal{E}^\infty, \mathcal{D}(\mathcal{E}^\infty))$ is reducible.

**Proof of Theorem 1.5.** (a) Note that

$$\lim_{\delta \to 0} \frac{\delta(\theta - 2\delta - 1)}{2} = \theta - 5 > c > 0$$

for any $p > \frac{2}{\theta - 1}$, there exists $\delta' \in (2, \delta \wedge \frac{\theta - 2}{2})$ and $v \in (\delta', \theta - 1)$ such that $\frac{1}{2}\delta'(\theta - 2\delta' - 1) > c$ and $\frac{\theta}{\theta - 1} < p$. Since $c/(1 + \rho)^\delta$ is decreasing in $\delta$, replacing $\delta$ by $\delta'$ in (1.8) we may assume that $\delta$ itself satisfies

$$\frac{\delta(\theta - 2\delta - 1)}{2} > c, \quad \delta \in (2, \frac{\theta - 2}{2}), \quad \delta < p \quad \text{for some} \quad v \in (\delta, \theta - 1).$$

Let us fix $F \in \mathcal{F}C^1_b$ with $EF = 0$. For any $t_0 > 0$, let $T > t_0$ be such that $F \in \mathcal{F}C^1_b(T)$. Since in our case $|\text{Ric}|$ is bounded so that (1.3) holds, we have

$$\mathbf{E}F^2 \leq \int_0^T \mathbf{E}|H_s^F|^2 ds \leq 2\mathcal{E}(F, F) + \frac{1}{2} \int_0^T I_s ds,$$

where

$$I_s := \left( \int_s^T \mathbf{E}\left( e^{\frac{1}{2} \int_t^s K(\rho(\gamma_u)) du} K(\rho(\gamma_t)) |D_t F| \big| \mathcal{F}_s \right) dt \right)^2 \leq \left\{ \int_s^T \mathbf{E}\left( |D_t F|^2 \big| \mathcal{F}_s \right) dt \right\} \left\{ \int_s^T \mathbf{E}\left( e^{\frac{1}{2} \int_t^s K(\rho(\gamma_u)) du} K(\rho(\gamma_t))^2 \big| \mathcal{F}_s \right) dt \right\} \frac{1}{2} \cdot \left( \mathbf{E}\left( K(\rho(\gamma_t))^2 \big| \mathcal{F}_s \right) \right)^{\frac{1}{2}} dt.$$

(b) Given $s > 0$, one has

$$d\{e^{2c \int_s^t (1 + \rho(\gamma_u))^{-2} du} (1 + \rho(\gamma_t))^{-2\delta} \} = dM_t + e^{2c \int_s^t (1 + \rho(\gamma_u))^{-2} du} \left\{ \frac{2c}{(1 + \rho(\gamma_t))^{3\delta}} + \frac{1}{2} \Delta (1 + \rho)^{-2\delta}(\gamma_t) \right\} dt.$$
for some martingale $M_t$. By $\Delta \rho \geq \frac{\theta}{\rho}$ and (3.3), we have
\[
\frac{1}{2} \Delta (1 + \rho)^{-2 \delta} \leq \frac{-\delta (\theta - 2 \delta - 1)}{(1 + \rho)^{2 \delta + 2}} \leq \frac{-2c}{(1 + \rho)^{3 \delta}}.
\]
Then $\{e^{2c f_t(1 + \rho(\gamma_t))^{-1}}d\mu(1 + \rho(\gamma_t))^{-2 \delta}\}_{t \geq s}$ is a supermartingale. Hence by (1.8) we obtain
\[
\mathbb{E}(e^{2c f_t K(\rho(\gamma_t))d\mu K(\rho(\gamma_t))^2 | \mathcal{F}_s}) \leq \frac{c^2}{(1 + \rho(\gamma_s))^{2 \delta}}, \quad t \geq s.
\]
Moreover, we have
\[
d(1 + \rho(\gamma_t))^{-2 \delta} \leq dM_t - \frac{\delta (\theta - 2 \delta - 1)}{(1 + \rho(\gamma_t))^{2 \delta + 2}} dt
\]
for a martingale $M_t$. Letting $h(t) := \mathbb{E}((1 + \rho(\gamma_t))^{-2 \delta} | \mathcal{F}_s)$ we obtain (note that $\theta \geq 2 \delta + 1$)
\[
h'(t) \leq -\delta (\theta - 2 \delta - 1) h(t)^{(1 + \delta)/\delta}, \quad t \geq s.
\]
This implies
\[
h(t) \leq c_1 (t - s + h(s)^{-1/\delta})^{-\delta} \leq c_1 (1 + t - s)^{-\delta}
\]
for some $c_1 > 0$ and all $t \geq s$. Thus
\[
\mathbb{E}(K(\rho(\gamma_t))^2 | \mathcal{F}_s) \leq c_2 (1 + t - s)^{-\delta}, \quad t \geq s
\]
for some $c_2 > 0$.

Combining (3.4),(3.5),(3.6) and (3.7), we arrive at
\[
\mathbb{E}F^2 \leq 2 \delta (F, F) + c_3 \mathbb{E} \int_0^T \left\{ \int_s^T \mathbb{E}(|D_t F|^2 | \mathcal{F}_s) \ dt \right\} \frac{ds}{(1 + \rho(\gamma_s))^{\delta}}
\]
for some constant $c_3 > 0$.

(c) Let $\varepsilon \in \left(\frac{\nu - \delta}{\nu + 2}, \frac{\nu - 2}{\nu} \right)$, $\varepsilon' = \frac{\nu}{2} - 1 - \varepsilon > 0$, $a > 0$. Consider
\[
\xi_s := \frac{(1 + s)^{(1 + \varepsilon)}}{(1 + \rho(\gamma_s))^\nu} + \frac{a}{(1 + s)^{\varepsilon'}}, \quad s \geq 0.
\]
We have
\[
d\xi_s = dM_s + \left\{ \frac{(1 + \varepsilon)(1 + s)^{\varepsilon}}{(1 + \rho(\gamma_s))^\nu} - \frac{a\varepsilon'}{(1 + s)^{1 + \varepsilon'}} + \frac{(1 + s)^{1 + \varepsilon}}{2} \Delta \frac{1}{2} \frac{1}{(1 + \rho)^{\nu}(\gamma_s)} \right\} ds
\]
\[
\leq dM_s + \frac{1}{(1 + s)^{1 + \varepsilon'}} \left\{ -a\varepsilon' - \frac{(1 + s)^{2 + \varepsilon + \varepsilon'} \nu(\theta - \nu - 1)}{2(1 + \rho)^{\nu + 2}} + \frac{(1 + \varepsilon)(1 + s)^{\varepsilon + \varepsilon' + 1}}{(1 + \rho(\gamma_s))^\nu} \right\} ds
\]
\[
= dM_s + \frac{1}{(1 + s)^{1 + \varepsilon'}} \left\{ -a\varepsilon' - \frac{\nu}{2} (\theta - \nu - 1) \eta_s^{\varepsilon' + 2} + (1 + \varepsilon) \eta_s \right\} ds,
\]

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where
\[
\eta_s := \frac{(1 + s)^{r+\epsilon+1}}{(1 + \rho(\gamma s))^v} = \frac{(1 + s)^{(r-2)/2}}{(1 + \rho(\gamma s))^v}.
\]

Letting
\[
a := \frac{1}{\epsilon'} \sup_{s \geq 0} \left\{ (1 + \epsilon) \eta - \frac{v}{2} (\theta - v - 1) \eta^{(v+2)/v} \right\} \in (0, \infty),
\]
we see that \(\xi_s(\geq 0)\) is a supermartingale. Then
\[
P\left( \sup_{s \geq 0} \frac{(1 + s)^{1+\epsilon}}{(1 + \rho(\gamma s))^v} \geq \lambda \right) \leq \frac{E \xi_0}{\lambda} = \frac{1 + a}{\lambda}, \quad \lambda > 0.
\]

Thus
\[
P\left( \sup_{s \geq 0} \frac{(1 + s)^{(1+\epsilon)\delta/v}}{(1 + \rho(\gamma s))^\delta} \geq \lambda \right) \leq \lambda^{-\delta/v}(1 + a), \quad \lambda > 0.
\]

Combining this with (3.8), we obtain
\[
E F^2 \leq E F^2 1_{\{(1 + s)^{(1+\epsilon)\delta/v} \leq \lambda (1 + \rho(\gamma s))^\delta, s \geq 0\}} + \frac{1 + a}{\lambda^{\delta/v}} \| F \|_\infty^2
\]
\[
\leq 1 + a \| F \|_\infty^2 + \delta'(F, F) \left( 2 + c_3 \int_0^T \frac{\lambda ds}{(1 + s)^{(1+\epsilon)\delta/v}} \right).
\]

Noting that \((1+\epsilon)\delta/v > 1\), there exists \(c_4 > 0\) such that
\[
E F^2 \leq (1 + a) \lambda^{-\delta/v} \| F \|_\infty^2 + (2 + c_4 \lambda) \delta'(F, F), \quad \lambda > 0.
\]

This imply (1.9) for
\[
\alpha(r) = c r^{-\delta/v}
\]
for some \(c > 0\) and all \(r \in (0, 1]\). Since \(\delta/v < p\) and since one may take \(\alpha(r) = 1\) in (1.9) for \(r \geq 1\), there exists \(c(p) > 0\) such that (1.9) holds for \(\alpha(r) := c(p)r^{-p}\). Thus, by [16, Corollary 2.3], we obtain
\[
\| P_t^\infty - \mu_\infty \|_{2^{-\delta}}^2 \leq c' t^{-1/p}
\]
for some \(c' > 0\). Since \(p > \frac{2}{\theta-1}\) is arbitrary, we complete the proof.

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