Least energy solutions for indefinite biharmonic problems via modified Nehari-Pankov manifold

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Abstract

In this paper, by using a modified Nehari-Pankov manifold, we prove the existence and the asymptotic behavior of least energy solutions for the following indefinite biharmonic equation:

\[ \Delta^2 u + (\lambda V(x) - \delta(x))u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \]

\((P_\lambda)\)

where \(N \geq 5, 2 < p \leq \frac{2N}{N-4}, \lambda > 0\) is a parameter, \(V(x)\) is a nonnegative potential function with nonempty zero set \(\text{int}V^{-1}(0)\), \(\delta(x)\) is a positive function such that the operator \(\Delta^2 + \lambda V(x) - \delta(x)\) is indefinite and non-degenerate for \(\lambda\) large. We show that both in subcritical and critical cases, equation \((P_\lambda)\) admits a least energy solution which for \(\lambda > 0\) localized near the zero set \(\text{int}V^{-1}(0)\).

Keywords: Least energy solutions; modified Nehari-Pankov manifold; biharmonic equations; indefinite potential.

AMS Subject Classification: 35Q55, 35J65

1 Introduction and main results

In the present paper, we are considering the following biharmonic equation:

\[
\begin{cases}
\Delta^2 u + (\lambda V(x) - \delta(x))u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\
u \in H^2(\mathbb{R}^N),
\end{cases}
\]

\((1.1)\)

where \(N \geq 5, \lambda > 0, 2 < p \leq 2**\), \(2** := \frac{2N}{N-4}\). We are interested in the case that the operator \(\Delta^2 + \lambda V(x) - \delta(x)\) is indefinite and non-degenerate for \(\lambda\) large, we study the existence and the

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asymptotic behavior of least energy solutions to problem (1.1) both in subcritical and critical cases.

As mathematical model, biharmonic equations can be used to describe some phenomena appeared in physics and engineering, such as, the problems of nonlinear oscillation in a suspension bridge (see Lazer and McKenna [23], McKenna and Walter [26]) and the problems of the static deflection of an elastic plate in a fluid (see Abrahams and Davis [1]). More precisely, when we consider the compatibility equations of elastic mechanics under small deviation of the thin plates, or the Von Karma system describing the mechanic behaviors under large deviation of thin plates, we are forced to study a class of higher order equations or systems with biharmonic operator $\Delta^2$.

Mathematically, the biharmonic operator is closely related to Paneitz operator, which has been found considerable interests because of its geometry roots.

Recently, Ghergu and Taliaferro [13] proved the nonexistence of positive super-solutions to some nonlinear biharmonic equations. Although the results they obtained can be seen as an extension of Armstrong and Sirakov [3] from Laplacian equations to Biharmonic equations, the methods in [13] and [3] are different. The results in [3] are mainly based on a method which depends only on properties related to the maximum principle, while the results in [13] are due to a new representation formula and an a priori point-wise bound of nonnegative super-solutions of bi-harmonic equations.

We also refer the readers to the paper by Alves and Nóbrega (see [2]), where the authors considered the following problem

$$\begin{cases}
\Delta^2 u = f(u), & \text{in } \Omega, \\
u = Bu = 0, & \text{on } \partial \Omega
\end{cases}$$

and $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$, $f$ is a $C^1$ function with subcritical growth. They obtained the existence of nodal solutions for problem (1.2) in the cases $Bu = \Delta u$ (Navier boundary condition) and $Bu = \frac{\partial u}{\partial \nu}$ (Dirichlet boundary condition) with the unit outer norm $\nu$.

There are also some other investigations for the biharmonic problems, for example in the work of Liu and Chen [20], they obtained the existence of ground state solutions for a class of biharmonic equation involving critical exponent. In [19], Karachik, Sadybekov and Torebok proved the uniqueness of solutions to boundary value problems for the biharmonic equation in a ball. In [22], Luo proved the uniqueness of the weak extremal solution to biharmonic equation with logarithmically convex nonlinearities.

We also want to introduce the works by Guo and Wei in [16] and [17], where the authors firstly discussed the Liouville type results and regularity of the extremal solutions of biharmonic equation with negative exponents. They also obtained some qualitative properties of entire radial solutions for a biharmonic equation with supercritical nonlinearity.

For more results related to biharmonic problems, please see [8, 9, 12, 15, 18, 27, 29, 31, 34] and the references therein.

The study for the Schrödinger equations involving Laplacian with indefinite potentials, we firstly refer to the paper by Y. Ding and J. Wei [10]. In that paper, the authors considered the
following problem

\[
\begin{cases}
-\Delta u(x) + \lambda V(x)u(x) = \lambda |u(x)|^{p-2}u(x) + \lambda g(x, u), & x \in \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]  

(1.3)

where \( V(x) \) can be negative in some domains in \( \mathbb{R}^N \) and \( g(x, u) \) is a perturbation term. By using variational methods, the authors proved that there exists \( \Lambda > 0 \) such that for \( \lambda > \Lambda \), (1.3) admits at least one nontrivial solution both for subcritical case and critical case.

For the indefinite potentials involving Laplacian, we also refer to the work by A. Szulkin and T. Weth [30], where the authors gave a new minimax characterization of the corresponding critical value and hence reduced the indefinite problem to a definite one. They also presented a precise description to the Nehari-Pankov manifold which is useful even for other problems. For the study of indefinite potential Schrödinger equations, we also refer the readers to T. Bartsch and the second author [5], where the multi-bump solutions was considered.

More recently, the second author and the third author together with the other coauthor (see Y. Guo, Z. Tang and L. Wang [14]) considered the existence and the asymptotic behavior of least energy solutions to problem (1.1) in the case when the operator \( \Delta^2 + \lambda V(x) - \delta(x) \) is positively definite.

The aim of the present paper is to study the existence and the asymptotic behavior of least energy solutions to (1.1) in the indefinite case. More precisely, we assume that \( V(x) \) and \( \delta(x) \) satisfy the following conditions:

\((V_1)\) \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( V(x) \geq 0 \) and \( 0 < V_\infty := \liminf_{|x| \to \infty} V(x) < +\infty \);

\((V_2)\) \( \Omega := \text{int } V^{-1}(0) \) is a non-empty bounded domain in \( \mathbb{R}^N \) with smooth boundary and \( \overline{\Omega} = V^{-1}(0) \);

\((V_3)\) The operator \( \Delta^2 - \delta(x) \) defined in \( H^2(\Omega) \cap H^1_0(\Omega) \) is indefinite and non-degenerate, that is \( \mu_k < \delta(x) < \mu_{k+1} \) for some \( k \geq 1 \). \( \{\mu_i\} \) is the class of all eigenvalues of the operator \( \Delta^2 \) in \( H^2(\Omega) \cap H^1_0(\Omega) \).

**Remark 1.1** According to conditions \((V_1)\), \((V_2)\) and \((V_3)\), we see that

\( (i) \) Condition \((V_1)\) can be replaced by the following:

\( \hat{(V_1)} \) \( V(x) \in C(\mathbb{R}^N, \mathbb{R}), V(x) \geq 0 \) and the set \( \{x \in \mathbb{R}^N : 0 \leq V(x) \leq M_0\} \) is bounded in \( \mathbb{R}^N \) for some \( M_0 > 0 \).

Indeed, take \( M_0 = \frac{1}{2} V_\infty \), according to conditions \((V_1)\) and \((V_2)\), there exists \( R > 0 \) such that

\[ \Omega \subset \{x \in \mathbb{R}^N : V(x) \leq M_0\} \subset B_R(0), \]

(1.4)

where \( B_R(0) \) (or \( B_R \)) denotes the ball centered at 0 with radius \( R \).
(ii) The regularity of the boundary $\partial \Omega$ in $(V_2)$ can be replaced by a weaker one: $\partial \Omega$ is Lipschitz continuous and satisfies uniformly outer ball condition. Moreover, we can define a new norm in $H(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$ by

$$
\|u\|_0 = \left( \int_{\Omega} |\Delta u|^2 \, dx \right)^{1/2}
$$

which is equivalent to the standard one. For more details, please see F. Gazzola, H.-Ch. Grunau and G. Sweers [11, Theorem 2.31].

(iii) Under condition $(V_3)$, as proved in Lemma 2.5 in the next section, we will see that the operator $\Delta^2 + \lambda V(x) - \delta(x)$ is non-degenerate and indefinite in $H^2(\mathbb{R}^N)$ for $\lambda$ large. Namely, for $\lambda$ large enough, $0$ is not an eigenvalue of $\Delta^2 + \lambda V(x) - \delta(x)$ and the principle eigenvalue of $\Delta^2 + \lambda V(x) - \delta(x)$ is negative.

Before stating our main result, we present some notations first. Let $H(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$ be the Hilbert space endowed with the norm

$$
\|u\|_0 := \left( \int_{\Omega} |\Delta u|^2 \, dx \right)^{1/2}.
$$

We denote $\lambda V(x) - \delta(x)$ by $V_\lambda(x)$ and define

$$
X := \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 \, dx < +\infty \right\}.
$$

Let us denote

$$
\|u\|_\lambda := \left[ \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+(x) u^2) \, dx \right]^{1/2},
$$

where $V_\lambda^+ = \max\{V_\lambda, 0\}$. It is easy to see that $(X, \| \cdot \|_\lambda)$ is a Banach space for each $\lambda > 0$ and we denote it by $X_\lambda$ for simplicity.

We define the functional $J_\lambda(u)$ on $X_\lambda$ by:

$$
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda(x) u^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx.
$$

(1.5)

It is not difficult to verify that the functional $J_\lambda(u)$ is $C^1$ in $X_\lambda$ and for every $w \in X_\lambda$,

$$
J'_\lambda(u)w = \int_{\mathbb{R}^N} (\Delta u \Delta w + V_\lambda uw) \, dx - \int_{\mathbb{R}^N} |u|^{p-2} uw \, dx.
$$

(1.6)

Let us also denote

$$
L_\lambda := \Delta^2 + V_\lambda(x), \quad L_0 := \Delta^2 - \delta(x)
$$

and $\{e_k\}_{k \geq 1}$ be the eigenfunctions of the operator $L_0$ defined in $H(\Omega)$, which is an orthogonal base of $H(\Omega)$ and $L^2(\Omega)$. By the assumption $(V_3)$, $H(\Omega)$ can be split to orthogonal sum $H^-(\Omega) \oplus H^+(\Omega)$ according to the positive and negative eigenfunction spaces of $L_0$, i.e.

$$
H(\Omega) = H^-(\Omega) \oplus H^+(\Omega),
$$
where
\[ H^-(\Omega) = \text{span}\{e_1, e_2, \ldots, e_k\}, \quad H^+(\Omega) = \text{span}\{e_{k+1}, e_{k+2}, \ldots\}. \]

As proved in Lemma 2.3, we will see that the essential spectrum \( \sigma_{\text{ess}}(L_\lambda) \) of \( L_\lambda \) satisfies
\[ \inf \sigma_{\text{ess}}(L_\lambda) \geq \lambda M_0 - \|\delta(x)\|_{L^\infty}. \]
Hence \( L_\lambda \) has finite Morse index in \( X_\lambda \) for \( \lambda \) large and \( L_\lambda \) has finite eigenvalues below \( \inf \sigma_{\text{ess}}(L_\lambda) \). Since 0 is not an eigenvalue of \( L_\lambda \) for \( \lambda \) large enough (see Lemma 2.5). Thus \( X_\lambda \) can be split to an orthogonal sum \( X_\lambda = X^-_\lambda \oplus X^+_\lambda \) according to the negative and positive eigenfunction spaces of \( L_\lambda \) for \( \lambda \) large enough. Instead of using the classical Nehari-Pankov manifold which is defined as
\[ \hat{N}_\lambda = \{ u \in X_\lambda \backslash \{0\} : P^-_\lambda \nabla J_\lambda(u) = 0, J_\lambda'(u) \cdot u = 0 \}, \quad (1.7) \]
where \( P^-_\lambda \) is the orthogonal projection from \( X_\lambda \) to \( X^-_\lambda \). We will define a modified Nehari-Pankov manifold, to do that, let us denote first
\[ G_0(u) = J_\lambda'(u)u, \quad G_i(u) = J_\lambda'(u)e_i, \quad i = 1, 2, \ldots, k. \]
We define a modified Nehari-Pankov manifold \( \mathcal{N}_\lambda \) by:
\[ \mathcal{N}_\lambda = \{ u \in X_\lambda \backslash \{0\} : G_i(u) = 0, i = 0, 1, 2, \ldots, k \}. \]
and the corresponding level value
\[ c_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u). \]

Let us denote \( A_\lambda \) be the set of all weak solutions to (1.1), then we say \( u \in A_\lambda \backslash \{0\} \) is a least energy solution of (1.1) if
\[ J_\lambda(u) \leq J_\lambda(v) \text{ for any } v \in A_\lambda \backslash \{0\}. \]

**Remark 1.2** It is easy to see that all weak solutions to (1.1) belong to \( \mathcal{N}_\lambda \), i.e. \( A_\lambda \subset \mathcal{N}_\lambda \). We will prove later that the minimizer for \( c_\lambda \) in \( \mathcal{N}_\lambda \) is indeed a weak solution to (1.1), thus \( u \) is a least energy solution if and only if \( J_\lambda(u) = c_\lambda \) with \( u \in \mathcal{N}_\lambda \).

Now we consider the following problem defined on \( \Omega = \text{int} V^{-1}(0) \),
\[ \begin{cases} \Delta^2 u - \delta(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u \neq 0, & \text{in } \Omega, \\ u = 0, \Delta u = 0, & \text{on } \partial \Omega, \end{cases} \quad (1.8) \]
which is a kind of limit problem of the original problem (1.1). The corresponding energy functional to (1.8) is defined on \( H(\Omega) \) by
\[ J_\Omega(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 - \delta(x)u^2)dx - \frac{1}{p} \int_\Omega |u|^p dx. \]
Moreover for any \( v \in H(\Omega) \),
\[ J'_\Omega(u)v = \int_\Omega (\Delta u \Delta v - \delta(x)uv)dx - \int_\Omega |u|^{p-2}uv dx. \]
We want to point out that in the case of \( p = 2^{**} \), problem (1.8) is close to the famous Brezis-Nirenberg problem and our method to prove the existence of least energy solutions to problem (1.8) also follows the methods developed by Brezis and Nirenberg (see [7]).

Let \( P_0 \) denote the orthogonal projection from \( H(\Omega) \) to \( H^-(\Omega) \), we define the following Nehari-Pankov manifold \( \mathcal{N}_\Omega \) by

\[
\mathcal{N}_\Omega := \left\{ u \in H_\Omega \setminus H^-(\Omega) : P_0 \nabla J_\Omega(u) = 0, J'_\Omega(u) u = 0 \right\}
\]

where as mentioned, \( e_i(i = 1, 2, \ldots, k) \) denote the negative eigenfunctions of the operator \( L_0 \). The corresponding level \( c(\Omega) \) is defined by

\[
c(\Omega) := \inf_{\mathcal{N}_\Omega} J_\Omega(u).
\]

According to A. Szulkin and T. Weth [30], we knew that \( u \) is a least energy solution to (1.8) if \( J_\Omega(u) = c(\Omega) \) with \( u \in \mathcal{N}_\Omega \).

**Remark 1.3** The reason why we introduce a modified Nehari-Pankov manifold \( \mathcal{N}_\lambda \) for the functional \( J_\lambda \) instead of using the Nehari-Pankov manifold \( \hat{\mathcal{N}}_\lambda \) of \( J_\lambda \) directly is that for any \( u \in \mathcal{N}_\Omega \) which is the Nehari-Pankov manifold related to the limit functional \( J_\Omega \), one can not say that \( u \in \hat{\mathcal{N}}_\lambda \) which is the Nehari-Pankov manifold related to the functional \( J_\lambda \). Thus to consider the asymptotic behavior of the least energy solution of (1.1), we introduce a modified Nehari-Pankov manifold \( \mathcal{N}_\lambda \) and it is easy to see that \( \mathcal{N}_\Omega \subset \mathcal{N}_\lambda \).

Our main result is:

**Theorem 1.4** Suppose \((V_1), (V_2)\) and \((V_3)\) hold, \(2 < p < 2^{**} := \frac{2N}{N - 4}\) for \( N \geq 5\) or \( p = 2^{**}\) for \( N \geq 8\). Then for \( \lambda \) large, (1.1) has a least energy solution \( u_\lambda(x) \) which achieves \( c_\lambda \). Moreover, for any sequence \( \lambda_n \to \infty \), there exists a subsequence of \( \{u_{\lambda_n}(x)\} \) (still denoted by \( \{u_{\lambda_n}(x)\} \)) such that \( u_{\lambda_n}(x) \) converges in \( H^2(\mathbb{R}^N) \) to a least energy solution \( u(x) \) of (1.8).

The paper is organized as follows: In Section 2, we give some preliminary results. In Section 3, we study the limit equation (1.8) and prove the existence of least energy solutions. In Section 4, we prove the existence of least energy solutions to (1.1) for \( \lambda \) large enough. In Section 5, we study the limit of \( c_\lambda \) as \( \lambda \to +\infty \) and finalize the paper by proving Theorem 1.4.

## 2 Preliminary results

In this section, we present some preliminary results which we need in proving our main result and we divided them into two subsections. More precisely, in Subsection 2.1, we introduce the spectrum of the operators \( \Delta^2 + \lambda V - \delta \) and \( \Delta^2 - \delta \). In Subsection 2.2, we give some properties of the Nehari-Pankov manifold \( \mathcal{N}_\Omega \) and also the modified Nehari-Pankov manifold \( \mathcal{N}_\lambda \).
2.1 Eigenvalues and eigenfunction spaces

In this subsection, we mainly discuss the eigenvalues and eigenfunction spaces of the operator $L_\lambda$ defined in $X_\lambda$. To do that we firstly give the following embedding result which is

**Lemma 2.1** Assume $(V_1), (V_2)$ and $(V_3)$ hold, then there exists $\Lambda_0 > 0$ such that for each $\lambda > \Lambda_0$ and $u \in X_\lambda$, we have

$$\|u\|_{H^2(\mathbb{R}^N)} \leq C\|u\|_\lambda$$

for some $C > 0$ which does not depend on $\lambda$.

**Proof:** Let $M_0 = \frac{1}{2} V_\infty$, by (1.4), we know that $V_\lambda(x) \geq M_0, \forall x \in \mathbb{R}^N \setminus B_R(0)$ and supp$V_\lambda^- \subset B_R(0), \forall \lambda > \frac{\mu_{k+1}}{M_0}$, (2.2)

where supp$V_\lambda^-$ denotes the support set of $V_\lambda^-$. Thus for each $u \in X_\lambda$ and $\lambda > \frac{M_0 + \mu_{k+1}}{M_0}$, by (2.2), we have

$$\int_{\mathbb{R}^N \setminus B_R(0)} u^2 dx \leq \frac{1}{M_0} \int_{\mathbb{R}^N \setminus B_R(0)} (\lambda V_\lambda(x) - \delta) u^2 dx$$

$$\leq \frac{1}{M_0} \int_{\mathbb{R}^N \setminus B_R(0)} V_\lambda^+ u^2 dx$$

$$\leq \frac{1}{M_0} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.$$ (2.3)

By Hölder’s inequality and Sobolev inequality, we obtain that

$$\int_{B_R(0)} u^2 dx \leq \left( \int_{B_R} |u|^{\frac{2N}{N-4}} dx \right)^{\frac{N-4}{N}} |B_R|^\frac{4}{N}$$

$$\leq C_1 |B_R|^\frac{4}{N} \int_{\mathbb{R}^N} |\Delta u|^2 dx$$

$$\leq C_1 |B_R|^\frac{4}{N} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.$$ (2.4)

Combining (2.3) and (2.4), we have

$$\int_{\mathbb{R}^N} (|\Delta u|^2 + u^2) dx \leq \left( \frac{M_0 + 1}{M_0} + C_1 |B_R|^\frac{4}{N} \right) \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.$$

Thus (2.1) holds for $\Lambda_0 = \frac{M_0 + \mu_{k+1}}{M_0}$ and $C = \frac{M_0 + 1}{M_0} + C_1 |B_R|^\frac{4}{N}$. This completes the proof of this lemma. \qed

**Remark 2.2** As a result of Lemma 2.1, one can see that $X_\lambda$ can be continuously imbedded into $L^p(\mathbb{R}^N)$ for $2 < p \leq 2^{* *} := \frac{2N}{N-4}$ and the embedding $X_\lambda \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compact for
2 \leq p < \frac{2N}{N-4} \text{ when } \lambda > \Lambda_0. \text{ Moreover, for any } u \in X_\lambda, \text{ when } \lambda > \Lambda_0 \text{ and } 2 \leq p \leq 2^*, \text{ there exists a } C > 0 \text{ independent of } \lambda \text{ such that }
\left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{2}{p}} \leq C \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.

Now we come to study the eigenvalue problems for the operator \( L_\lambda \) as \( \lambda \) large, we have the following lemma.

**Lemma 2.3** Under the conditions \((V_1), (V_2)\) and \((V_3)\), for each \( \lambda > \Lambda_0 \), we have
\[ \sigma_{\text{ess}}(L_\lambda) \subset [\lambda M_0 - \|\delta(x)\|_{L^\infty}, +\infty). \]
Furthermore, \( \inf \sigma_{\text{ess}}(L_\lambda) \to +\infty \) as \( \lambda \to +\infty \).

**Proof:** The proof of this lemma is similar to the proof of Proposition 2.3 in [4]. For readers’ convenience, we give the details.

We set \( W_\lambda = V_\lambda - \lambda M_0 + \delta = \lambda(V(x) - M_0) \) and write \( W_\lambda^1 = \max\{W_\lambda, 0\}, \) \( W_\lambda^2 = \min\{W_\lambda, 0\}. \) Obviously, for \( \lambda > \Lambda_0, \)
\[ \sigma(\Delta^2 + W_\lambda^1 + \lambda M_0 - \delta) \subset [\lambda M_0 - \|\delta(x)\|_{L^\infty}, +\infty) \] (2.5)
for \( W_\lambda^1 \geq 0. \) Let \( H_\lambda = \Delta^2 + W_\lambda^1 + \lambda M_0 - \delta \), then \( L_\lambda = H_\lambda + W_\lambda^2. \)

We claim that
\( W_\lambda^2 \) is a relative form compact perturbation of \( L_\lambda \) for \( \lambda > \Lambda_0. \)

Indeed, since \( W_\lambda^2 \) is bounded, then the form domain of \( H_\lambda \) is the same as the form domain \( X_\lambda \) of \( L_\lambda. \) Thus we have to show that
\[ X_\lambda \hookrightarrow X_\lambda^*, \text{ } u \mapsto W_\lambda^2 \cdot u \text{ is compact.} \]

Here \( X_\lambda^* \) is the dual space of \( X_\lambda. \) Take a bounded sequence \( \{u_n\}_{n\geq 1} \) in \( X_\lambda, \) then according to Lemma 2.1, \( \{u_n\}_{n\geq 1} \) is also a bounded sequence in \( H^2(\mathbb{R}^N). \) Thus for some \( u \in H^2(\mathbb{R}^N), \) up to a subsequence,
\[
\begin{align*}
& u_n \rightharpoonup u \text{ weakly in } H^2(\mathbb{R}^N), \\
& u_n \to u \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^N), \\
& u_n \to u \text{ a.e. in } \mathbb{R}^N
\end{align*}
\] (2.6)
as \( n \to +\infty \). According to (2.2), we know that \( \text{supp} W^2_\lambda \subset B_R \) for any \( \lambda > \Lambda_0 \). Thus by Hölder’s inequality, Sobolev inequality and Lemma 2.1, for any \( \lambda > \Lambda_0 \), \( v \in X_\lambda \), we have

\[
\left| \int_{\mathbb{R}^N} W^2_\lambda (u_n - u)v \, dx \right| = \left| \int_{B_R} W^2_\lambda (u_n - u)v \, dx \right| \\
\leq \| \delta \|_{L^\infty} \int_{B_R} |(u_n - u)v| \, dx \\
\leq \mu_{k+1} \left( \int_{B_R} |u_n - u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_R} v^2 \, dx \right)^{\frac{1}{2}} \\
\leq \mu_{k+1} \left( \int_{B_R} |u_n - u|^2 \, dx \right)^{\frac{1}{2}} \| v \|_{H^2(\mathbb{R}^N)} \\
\leq C \left( \int_{B_R} |u_n - u|^2 \, dx \right)^{\frac{1}{2}} \| v \|_\lambda.
\]

(2.7)

Hence by (2.6) and (2.7), we have as \( n \to +\infty \),

\[
\| W^2_\lambda u_n - W^2_\lambda u \|_{X^*_\lambda} \leq C \left( \int_{B_R} |u_n - u|^2 \, dx \right)^{\frac{1}{2}} \to 0.
\]

Thus \( W^2_\lambda \) is a relative form compact perturbation of \( L_\lambda \).

According to the classical Weyl theorem (see Example 3 in [28], page 117), \( \sigma_{\text{ess}}(L_\lambda) = \sigma_{\text{ess}}(H_\lambda) \). Thus by (2.5), for \( \lambda > \Lambda_0 \), we have

\[
\sigma_{\text{ess}}(L_\lambda) \subset [\lambda M_0 - \| \delta \|_{L^\infty}, +\infty).
\]

Moreover,

\[
\inf \sigma_{\text{ess}}(L_\lambda) \to +\infty \text{ as } \lambda \to +\infty.
\]

Thus the proof of this lemma is completed. \( \square \)

**Remark 2.4** Let

\[
\mu_n(L_0) := \max_{S \in \Sigma_{0,n-1}} \min_{u \in S} \frac{\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) \, dx}{\int_{\mathbb{R}^N} u^2 \, dx}
\]

and

\[
\mu_n(L_\lambda) := \max_{S \in \Sigma_{\lambda,n-1}} \min_{u \in S} \frac{\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) \, dx}{\int_{\mathbb{R}^N} u^2 \, dx},
\]

where \( \Sigma_{0,n-1} \) and \( \Sigma_{\lambda,n-1} \) denote the collection of \((n - 1)\)–dimensional subspaces in \( H(\Omega) \) and \( X_\lambda \) respectively. It is easy to see that \( \mu_n(L_\lambda) \leq \mu_n(L_0) \), according to the above lemma and the min-max principle in spectral analysis (see Theorem XIII.1 and Theorem XIII.2 in [28]), we obtain that \( \mu_n(L_\lambda) \) is indeed an eigenvalue of \( L_\lambda \) for \( \lambda \) large enough.
Finally, let \( \{\mu_i(L_\lambda)\} \) be the class of all distinct eigenvalues of \( L_\lambda := \Delta^2 + V_\lambda \) in \( X_\lambda \) and \( \{\mu_i(L_0)\} \) be the class of all distinct eigenvalues of \( L_0 := \Delta^2 - \delta(x) \) in \( H(\Omega) \). Without loss of generality, we may assume that

\[
\mu_1(L_\lambda) < \mu_2(L_\lambda) < \mu_3(L_\lambda) < \cdots < \mu_{k_\lambda}(L_\lambda) < \inf \sigma_{ess}(L_\lambda),
\]

and

\[
\mu_1(L_0) < \mu_2(L_0) < \mu_3(L_0) < \cdots < \mu_k(L_0) < 0 < \mu_{k+1}(L_0) < \cdots.
\]

Moreover, \( \mu_{k_\lambda}(L_\lambda) \to +\infty \) as \( \lambda \to +\infty \) and \( \mu_i(L_0) \to +\infty \) as \( i \to +\infty \).

Let \( V_i(L_\lambda) \) be the eigenfunction space of \( \mu_i(L_\lambda) \) and \( V_i(L_0) \) be the eigenfunction space of \( \mu_i(L_0) \). We say that \( V_i(L_\lambda) \) converges to \( V_i(L_0) \), i.e. \( V_i(L_\lambda) \to V_i(L_0) \) as \( \lambda \to +\infty \), if for any sequence \( \lambda_n \to \infty \) and normalized eigenfunctions \( \psi_n \in V_i(L_{\lambda_n}) \), there exists a normalized eigenfunction \( \psi \in V_i(L_0) \) such that \( \psi_n \to \psi \) strongly in \( H^2(\mathbb{R}^N) \) along a subsequence.

The following Lemma concerns the asymptotic behavior of \( \mu_i(L_\lambda) \) and \( V_i(L_\lambda) \) as \( \lambda \to +\infty \).

**Lemma 2.5** For \( i = 1, 2, \ldots \), we have

\[
\mu_i(L_\lambda) \to \mu_i(L_0) \quad \text{and} \quad V_i(L_\lambda) \to V_i(L_0), \quad \text{as} \quad \lambda \to +\infty.
\]

Moreover by assumption \((V_\delta)\), there exists \( \Lambda_1 > \Lambda_0 \) such that for any \( \lambda > \Lambda_1 \), we have

\[
\mu_1(L_\lambda) < \mu_2(L_\lambda) < \cdots < \mu_k(L_\lambda) < 0 < \mu_{k+1}(L_\lambda) < \cdots < \mu_{k_\lambda}(L_\lambda) < \inf \sigma_{ess}(L_\lambda).
\]

**Proof:** We prove this lemma by induction.

**Step 1:** We prove the case for \( i = 1 \), i.e.

\[
\mu_1(L_\lambda) \to \mu_1(L_0) \quad \text{and} \quad V_1(L_\lambda) \to V_1(L_0) \quad \text{as} \quad \lambda \to +\infty.
\]

Let \( \psi_n \in X_{\lambda_n} \) be an eigenfunction corresponding to \( \mu_1(L_{\lambda_n}) \) which satisfies

\[
\int_{\mathbb{R}^N} \psi_n^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (|\Delta \psi_n|^2 + V_{\lambda_n} \psi_n^2) dx = \mu_1(L_{\lambda_n}). \tag{2.8}
\]

As \( \mu_1(L_{\lambda_n}) \) is increasing in \( \lambda_n \) and \( \mu_1(L_{\lambda_n}) \leq \mu_1(L_0) \), by (2.8) we have

\[
||\psi_n||^2_{L_\lambda} = \int_{\mathbb{R}^N} (|\Delta \psi_n|^2 + V_{\lambda_n} \psi_n^2) dx + \int_{\mathbb{R}^N} V_{\lambda_n} \psi_n^2 dx = \mu_1(L_{\lambda_n}) + \int_{\mathbb{R}^N} V_{\lambda_n} \psi_n^2 dx \leq \mu_1(L_0) + ||\delta(x)||_{L^\infty}
\]

\[
\leq \mu_1(L_0) + \mu_{k+1}.
\]

According to Lemma 2.1, \( \{\psi_n\} \) is bounded in \( H^2(\mathbb{R}^N) \). Up to a subsequence, there is \( \psi \in H^2(\mathbb{R}^N) \) such that as \( \lambda_n \to +\infty \), we have

\[
\begin{align*}
\psi_n &\to \psi \quad \text{weakly in} \ H^2(\mathbb{R}^N), \\
\psi_n &\to \psi \quad \text{strongly in} \ L^2_{loc}(\mathbb{R}^N), \\
\psi_n &\to \psi \quad \text{a.e. in} \ \mathbb{R}^N. \tag{2.9}
\end{align*}
\]
Firstly, we prove that
\[ \psi \in H^2(\Omega) \cap H^1_0(\Omega). \]

In fact, we just need to verify that
\[ \psi(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega. \]

For each integer \( m \geq 1 \), we denote
\[ C_m := \left\{ x \in \mathbb{R}^N : V(x) > \frac{1}{m} \right\}. \]

Fix \( m \), by (2.8), as \( \lambda_n \to +\infty \), we have
\[ \int_{C_m} \psi_n^2 \, dx \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x) \psi_n^2 \, dx \]
\[ \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (|\Delta \psi_n|^2 + \lambda_n V(x) \psi_n^2) \, dx \]
\[ \leq \frac{m}{\lambda_n} (\mu_1(L_{\lambda_n}) + \|\delta(x)\|_{L^\infty}) \leq \frac{m}{\lambda_n} (\mu_1(L_0) + \mu_{k+1}) \to 0. \]

Thus \( \psi(x) = 0 \) a.e. in \( C_m \). Note that \( \bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \Omega \), we have \( \psi(x) = 0 \) a.e. in \( \mathbb{R}^N \setminus \Omega \).

Secondly, we prove that
\[ \int_{\Omega} \psi^2 \, dx = 1. \]

In fact, according to (2.2) and (2.8), we have
\[ \int_{\mathbb{R}^N \setminus B_R(0)} \psi_n^2 \, dx \leq \frac{1}{M_0 \lambda_n} \int_{\mathbb{R}^N \setminus B_R(0)} \lambda_n V(x) \psi_n^2 \, dx \]
\[ \leq \frac{1}{M_0 \lambda_n} \int_{\mathbb{R}^N} (|\Delta \psi_n|^2 + \lambda_n V(x) \psi_n^2) \, dx \]
\[ \leq \frac{1}{M_0 \lambda_n} (\mu_1(L_0) + \|\delta(x)\|_{L^\infty}) \to 0 \]
as \( \lambda_n \to +\infty \). Thus
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \psi_n^2 \, dx = 0. \]  \hspace{1cm} \text{(2.10)}

Combining (2.8), (2.9) and (2.10), we have
\[ \int_{\Omega} \psi^2 \, dx = \lim_{n \to +\infty} \int_{B_R(0)} \psi_n^2 \, dx \]
\[ = \lim_{n \to +\infty} \int_{\mathbb{R}^N} \psi_n^2 \, dx - \lim_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \psi_n^2 \, dx = 1. \]

Finally, we prove that
\[ \mu_1(L_{\lambda_n}) \to \mu_1(L_0) \text{ as } n \to +\infty. \]
In fact, $\psi_n \to \psi$ strongly in $L^2(\mathbb{R}^N)$ as $n \to +\infty$. Thus by (2.8), we have
\[
\mu_1(L_0) := \inf \left\{ \int_{\Omega} (|\Delta u|^2 - \delta(x)u^2)dx : u \in H^2(\Omega) \cap H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}
\]
\[
\leq \int_{\Omega} (|\Delta \psi|^2 - \delta(x)\psi^2)dx
\]
\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} [|\Delta \psi_n|^2 + (\lambda_n V(x) - \delta(x))\psi_n^2] dx
\]
\[
= \lim_{n \to \infty} \mu_1(L_{\lambda_n}) \leq \mu_1(L_0),
\]
which implies that $\mu_1(L_{\lambda_n}) \to \mu_1(L_0)$ as $n \to \infty$. Since $\mu_1(L_{\lambda})$ is increasing in $\lambda$, then $\mu_1(L_{\lambda}) \to \mu_1(L_0)$ as $\lambda \to +\infty$.

**Step 2:** Suppose that the results hold up to $k - 1$ for $k \geq 2$, we want to prove that the same result is true for the $k$-th eigenvalue.

Since $L_{\lambda}\psi = L_0\psi$ for any $\psi \in H(\Omega)$, then by the $k$-th Rayleigh quotient descriptions of $\mu_k(L_{\lambda})$ and $\mu_k(L_0)$, we have
\[
\limsup_{\lambda \to +\infty} \mu_k(L_{\lambda}) \leq \mu_k(L_0).
\]

Just like the case when $k = 1$, we can take $\lambda_n \to +\infty$ and the normalized eigenfunctions $\psi_n \in V_k(L_{\lambda_n})$ which is the eigenfunction space corresponding to $\mu_k(L_{\lambda_n})$, such that
\[
\int_{\mathbb{R}^N} (|\Delta \psi_n|^2 + V_{\lambda_n}\psi_n^2)dx = \mu_k(L_{\lambda_n}),
\]
\[
\int_{\mathbb{R}^N} \psi_n^2dx = 1, \quad \psi_n \perp V_j(L_{\lambda_n}), \quad j = 1, 2, 3 \cdots, k - 1.
\]

Similar to the proof in **Step 1**, we have for some $\psi \in H(\Omega)$ with $\int_{\Omega} |\psi|^2dx = 1$,
\[
\begin{cases}
\psi_n \rightharpoonup \psi & \text{weakly in } H^2(\mathbb{R}^N), \\
\psi_n \to \psi & \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N), \\
\psi_n \to \psi & \text{a.e. in } \mathbb{R}^N.
\end{cases}
\]

Since $\psi_n \perp V_j(L_{\lambda_n}), \quad j = 1, 2, \cdots, k - 1$, and $V_j(L_{\lambda_n}) \to V_j(L_0)$ as $n \to +\infty$, then $\psi \perp V_j(L_0), \quad j = 1, 2, \cdots, k - 1$ and
\[
\mu_k(L_0) \leq \int_{\Omega} (|\Delta \psi|^2 - \delta(x)\psi^2)dx
\]
\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} [|\Delta \psi_n|^2 + (\lambda_n V(x) - \delta(x))\psi_n^2] dx
\]
\[
\leq \lim_{n \to \infty} \mu_k(L_{\lambda_n}) \leq \mu_k(L_0).
\]
This induces that $\mu_k(L_{\lambda_n}) \to \mu_k(L_0)$ and $V_k(L_{\lambda_n}) \to V_k(L_0)$ as $n \to +\infty$. \hfill \square

**Remark 2.6** By assumption $(V_3)$, for $\lambda$ large enough, the operator $\Delta^2 + V_{\lambda}$ defined in $X_{\lambda}$ is non-degenerate and indefinite whose Morse index is $d_j = \dim X_{\lambda}$ uniformly in $\lambda$. 

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2.2 The modified Nehari-Pankov manifold

In this subsection, we consider the modified Nehari-Pankov manifold $\mathcal{N}_\lambda$ and the corresponding level value $c_\lambda$.

Firstly, we use the following lemma to collect some properties of the Nehari-Pankov manifold $\mathcal{N}_\Omega$ and the corresponding level $c_0$, which are

**Lemma 2.7** Let $\Omega := \operatorname{int} V^{-1}(0)$, for any $w \in H(\Omega) \setminus H^-(\Omega)$, set

$$H_w := \{ v + tw : v \in H^-(\Omega), t > 0 \}.$$ 

Then the following properties hold:

(i) $\mathcal{N}_\Omega = \{ w \in H(\Omega) \setminus H^-(\Omega) : \nabla (J_\Omega (w)|H_w) = 0 \}$.

(ii) For every $w \in H^+(\Omega) \setminus \{0\}$ there exists $t_w > 0$ and $\varphi (w) \in H^-(\Omega)$ such that

$$H_w \cap \mathcal{N}_\Omega = \{ \varphi (w) + t_w \cdot w \}.$$

(iii) For every $w \in \mathcal{N}_\Omega$ and every $u \in H_w \setminus \{w\}$ there holds $J_\Omega (u) < J_\Omega (w)$.

(iv) $c(\Omega) = \inf_{u \in \mathcal{N}_\Omega} J_\Omega (u) > 0$.

**Proof:** The similar proof can be found in the paper by A. Szulkin and T. Weth [30] which is concerned about the Laplacian operator. For the completeness of the paper, we give the detail of the proof.

(i) Take $\omega \in \mathcal{N}_\Omega$, according to the definition of $\mathcal{N}_\Omega$, we have

$$\omega \in H(\Omega) \setminus H^-(\Omega), P_0^- \nabla J_\Omega (\omega) = 0, J_\Omega'(\omega) \omega = 0.$$ 

For any $\phi = \psi + t \omega \in H_\omega$, we obtain

$$\langle \nabla J_\Omega (\omega), \phi \rangle_{H(\Omega)} = \langle \nabla J_\Omega (\omega), P_0^- \nabla J_\Omega (\omega) \rangle_{H(\Omega)} + t \langle \nabla J_\Omega (\omega), \omega \rangle_{H(\Omega)} = \langle P_0^- \nabla J_\Omega (\omega), \psi \rangle_{H(\Omega)} + 0 = 0.$$ 

Thus $\nabla (J_\Omega (\omega)|H_\omega) = 0$.

Next, we take $\omega \in H(\Omega) \setminus H^-(\Omega)$ and $\nabla (J_\Omega (\omega)|H_\omega) = 0$. For any $\phi = \psi + t \omega \in H_\omega$, we have

$$\langle \nabla J_\Omega (\omega), \phi \rangle_{H(\Omega)} = 0.$$ 

Let $\phi = \omega$, we obtain that $J_\Omega (\omega) = 0$. Note that $P_0^- \psi = \psi \in H^-(\Omega)$, then we can easily get that $P_0^- \nabla J_\Omega (\omega) = 0$. Thus $\omega \in \mathcal{N}_\Omega$.

(iii) Take $u = v + t \omega \in H_\omega \setminus \{\omega\}$, a direct computation shows that

$$J_\Omega (u) - J_\Omega (\omega) = \frac{1}{2} \left[ \int_{\Omega} (|\Delta u|^2 - \delta u^2) dx - \int_{\Omega} (|\Delta \omega|^2 - \delta \omega^2) dx \right] + \frac{1}{p} \int_{\Omega} (|\omega|^p - |u|^p) dx$$ 

$$= \frac{1}{2} \int_{\Omega} (|\Delta v|^2 - \delta v^2) dx + \frac{t^2 - 1}{2} \int_{\Omega} (|\Delta \omega|^2 - \delta \omega^2) dx + t \int_{\Omega} (\Delta v \Delta \omega - \delta v \omega) dx$$

$$+ \frac{1}{p} \int_{\Omega} (|\omega|^p - |v + t \omega|^p) dx.$$
Thus
\[ \phi(J) \] and we can also obtain that which leads to a contradiction. Hence
\[ \phi(J) \]

Since \( \omega \in \mathcal{N}_\Omega \), then
\[ \int_\Omega (|\Delta \omega|^2 - \delta \omega^2) dx = \int_\Omega |\omega|^p dx, \int_\Omega (\Delta \omega \Delta v - \delta \omega v) dx = \int_\Omega |\omega|^{p-2} \omega v dx. \]

Thus
\[ J_\Omega(u) - J_\Omega(\omega) = \frac{1}{2} \int_\Omega (|\Delta v|^2 - \delta v^2) dx + \int_\Omega \left( \frac{t^2}{2} - 1 \right) |\omega|^p + t|\omega|^{p-2} \omega v + \frac{1}{p} |\omega|^p - \frac{1}{p} v + t \omega |^p \right) dx. \]

Let
\[ \phi(t) = \frac{t^2}{2} - 1 \right) |\omega|^p + t|\omega|^{p-2} \omega v + \frac{1}{p} |\omega|^p - \frac{1}{p} v + t \omega |^p. \]

Then \( \phi(t) \leq 0 \) if \( \omega(v + t \omega) \leq 0 \). For \( \omega(v + t \omega) > 0 \), i.e. \( t > -\frac{v}{\omega} \), it is easy to see that
\[ \phi(0) = -\left( \frac{1}{2} - \frac{1}{p} \right) |\omega|^p - \frac{1}{p} |v|^p \leq 0, \lim_{t \to +\infty} \phi(t) = -\infty. \]

Assume \( \phi(t_0) = \sup_{t \geq max\{0, -\frac{v}{\omega}\}} \phi(t) > 0 \) for some \( t_0 \geq max\{0, -\frac{v}{\omega}\} \), then \( t_0 > max\{0, -\frac{v}{\omega}\} \).

Thus \( \phi'(t_0) = 0 \) implies that
\[ |\omega|^{p-2} \omega(v + t_0 \omega) - |v + t_0 \omega|^{p-2}(v + t_0 \omega) \omega = 0, \text{ i.e. } |\omega| = |v + t_0 \omega|. \]

Therefore,
\[ \phi(t_0) = \frac{t_0^2}{2} - 1 \right) |\omega|^p + t_0 |\omega|^{p-2} \omega v = -\frac{|\omega|^{p-2} v^2}{2} \leq 0, \]

which leads to a contradiction. Hence \( \phi(t) \leq 0 \) for \( t > 0 \).

If \( v \neq 0 \), since \( v \in X_\lambda \), then we can easily obtain that \( J_\Omega(u) < J_\Omega(\omega) \). If \( v = 0 \), then \( t \neq 1 \) and we can also obtain that \( J_\Omega(u) < J_\Omega(\omega) \). Thus \( J_\Omega(u) < J_\Omega(\omega) \) for any \( u \in H_\omega \setminus \{\omega\} \) where \( \omega \in \mathcal{N}_\Omega \).

(iv) Denote
\[ S_\alpha = \left\{ u \in H^+ (\Omega) : \int_\Omega (|\Delta u|^2 - \delta(x) u^2) dx = \alpha^2 \right\}. \]

Note that for any \( u \in S_\alpha \), we have
\[ \int_\Omega |\Delta u|^2 dx = \int_\Omega (|\Delta u|^2 - \delta(x) u^2) dx + \int_\Omega \delta(x) u^2 dx \leq \left( 1 + \left( \frac{1}{\mu_{k+1}(L_0)} \right) \right) \int_\Omega (|\Delta u|^2 - \delta(x) u^2) dx = \left( 1 + \left( \frac{1}{\mu_{k+1}(L_0)} \right) \right) \alpha^2, \]

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where $\mu_{k+1}(L_\lambda) > 0$. Then by Sobolev imbedding theorem, we have

$$J_\Omega(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 - \delta(x)u^2)dx - \frac{1}{p} \int_\Omega |u|^pdx$$

$$\geq \frac{1}{2} \int_\Omega (|\Delta u|^2 - \delta(x)u^2)dx - \frac{1}{p} \cdot C \left( \int_\Omega |\Delta u|^2dx \right) \frac{\alpha^2}{2}$$

$$\geq \frac{1}{2} \alpha^2 - C \left( 1 + \frac{\|\delta(x)\|_{L_\infty}}{\mu_{k+1}(L_\lambda)} \right) \frac{\alpha^p}{2} > 0$$

for $\alpha > 0$ small enough. Let $u \in \mathcal{N}_\Omega$, then $u = u^- + u^+$, where $u^- \in H^-(\Omega)$ and $0 \not= u^+ \in H^+(\Omega)$. Since $tu^+ \in H_u \cap S_\alpha$ for some $t > 0$, then according to $(iii)$ in this Lemma, we have

$$J_\lambda(u) > J_\lambda(tu^+) \geq \inf_{S_\alpha} J_\lambda(u) > 0.$$  

$(ii)$ Take $\omega \in H^+(\Omega) \setminus \{0\}$. If $\omega \in \mathcal{N}_\Omega$, then by $(iii)$, we have $\phi(\omega) = 0$ and $t_\omega = 1$. If $\omega \not\in \mathcal{N}_\Omega$, we may assume that

$$\int_\Omega (|\Delta \omega|^2 - \delta(x)\omega^2)dx = 1.$$ 

For any $u \in H_\omega$ which is contained in a finite space, we have

$$J_\Omega(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 - \delta(x)u^2)dx - \frac{1}{p} \int_\Omega |u|^pdx \to -\infty.$$ 

as $\|u\|_0 \to +\infty$ due to the fact that all norms in a finite space are equivalent. Thus for any $u \in H_\omega \setminus B_R(0)$, we have $J_\Omega(u) < 0$ for some large $R > 0$. According to the proof of $(iv)$, we obtain that $J_\Omega(t_\omega) > 0$ for some small $t > 0$. Note that $H_\omega$ is contained in a finite dimensional space, then there exists some $u_0 \in H_\omega \cap B_R(0)$ such that $J_\Omega(u_0) = \sup_{H_\omega} J_\Omega(u)$. Thus $\nabla J_\Omega(u_0) |_{H_\omega} = 0$ which implies that $u_0 \in \mathcal{N}_\Omega \cap H_\omega$ due to $(i)$. According to $(iii)$, we know that $u_0$ is unique. Put $\phi(\omega) = u_0^-$ and $t_\omega \omega = u_0^+$, we have $H_\omega \cap \mathcal{N}_\Omega = \{ \phi(\omega) + t_\omega \cdot \omega \}$. □

**Remark 2.8** By Lemma 2.7, we conclude that for each $w \in H(\Omega) \setminus H^-(\Omega)$, the set $\mathcal{N}_\Omega$ intersects $H_\omega$ in exactly one point $\tau(w) := \phi(w) + t_w \cdot w$ which is the unique global maximum point of $J_\Omega \setminus H_\omega$. Moreover, similar to the proof in A. Szulkin and T. Weth [30], the map $\tau : w \mapsto \phi(w) + t_w \cdot w$ is continuous and the restriction of $\tau$ to the unit sphere $S^2_{\Omega} \in H^+(\Omega)$ is a homeomorphism between $S^2_{\Omega}$ to $\mathcal{N}_\Omega$. Thus the least energy level $c(\Omega)$ has a minimax characterization given by

$$c(\Omega) = \inf_{w \in H^+(\Omega) \setminus \{0\}} \max_{w \in H_\omega} J_\Omega(u).$$

Now we are going to give some properties of the modified Nehari-Pankov manifold $\mathcal{N}_\lambda$ and the corresponding level $c_\lambda$. 

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Lemma 2.9 For $w \in X_{\lambda} \setminus H^{-}(\Omega)$, set
$$
\hat{H}_w := \{v + tw : v \in H^{-}(\Omega), t > 0\}.
$$

Then there exists a constant $\Lambda_2 (\Lambda_2 > \Lambda_1 > 0)$ such that for any $\lambda > \Lambda_2$ we have the following properties hold:

(i) $N_\lambda = \{w \in X_{\lambda} \setminus H^{-}(\Omega) : \nabla(J_\lambda(w)|\hat{H}_w) = 0\}$.

(ii) Let
$$
E_\lambda^+ := \left\{ w \in E_\lambda^+ : \int_{\mathbb{R}^N} w_i dx = 0, i = 1, 2, \cdots, k \right\}.
$$

Then for every $w \in E_\lambda^+ \setminus \{0\}$ there exists $t_w > 0$ and $\varphi(w) \in H^{-}(\Omega)$ such that
$$
\hat{H}_w \cap N_\lambda = \{\varphi(w) + t_w \cdot w\}.
$$

(iii) For every $w \in N_\lambda$ and every $u \in \hat{H}_w \setminus \{w\}$ there holds $J_\lambda(u) < J_\lambda(w)$.

(iv) $c_\lambda = \inf_{u \in N_\lambda} J_\lambda(u) \geq \tau > 0$ for some small $\tau > 0$ which is independent on $\lambda$.

**Proof:** The proofs of (i) – (iii) are similar to the corresponding proofs of Lemma 2.7, we omit them and we only need to prove (iv).

Firstly, we claim that there exists a $\Lambda_2 (\Lambda_2 > \Lambda_1 > 0)$ such that for any $\lambda > \Lambda_2$ and $u \in E_\lambda^+$, we have the following inequality
$$
\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx \geq C \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx
$$
holds for some $C > 0$ which is independent on $\lambda$.

In fact, for any $u \in E_\lambda^+$, we have
$$
u = u_\lambda^+ + u_\lambda^- \in X_\lambda^+, u_\lambda^- \in X_\lambda^-.$$

Note that
$$
\int_{\mathbb{R}^N} (|\Delta u_\lambda^+|^2 + V_\lambda^+(u_\lambda^+)^2) dx \leq \int_{\mathbb{R}^N} (|\Delta u_\lambda|^2 + V_\lambda(u_\lambda^-)^2) dx + \int_{\mathbb{R}^N} V_\lambda^-(u_\lambda^+)^2 dx.
$$
Then a direct computation gives us that

\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) \, dx
\]

\[
= \int_{\mathbb{R}^N} (|\Delta u^+_\lambda|^2 + V_\lambda (u^+_\lambda)^2) \, dx + \int_{\mathbb{R}^N} (|\Delta u^-_\lambda|^2 + V_\lambda (u^-_\lambda)^2) \, dx
\]

\[
\geq \frac{\mu_{k+1}(L_\lambda)}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} (|\Delta u^+_\lambda|^2 + V_\lambda (u^+_\lambda)^2) \, dx + \int_{\mathbb{R}^N} (|\Delta u^-_\lambda|^2 + V_\lambda (u^-_\lambda)^2) \, dx
\]

\[
\geq \frac{\mu_{k+1}(L_\lambda)}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} (|\Delta u^+_\lambda|^2 + V_\lambda (u^+_\lambda)^2) \, dx \\
+ \left\{ \frac{\|\delta(x)\|_{L^\infty}}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} (|\Delta u^-_\lambda|^2 + V_\lambda (u^-_\lambda)^2) \, dx \\
- \frac{\mu_{k+1}(L_\lambda)}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} V^- (u^-_\lambda)^2 + 2u^+_\lambda u^-_\lambda \, dx \right\}.
\]

Since

\[
u^-_\lambda = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} u e_{\lambda,i} \, dx \right) e_{\lambda,i} = \sum_{i=1}^{k} \left[ \int_{\mathbb{R}^N} u (e_{\lambda,i} - e_{i}) \, dx \right] e_{\lambda,i},
\]

where \( e_{i} \) and \( e_{\lambda,i} \) are the eigenfunction of \( L_0 \) and \( L_\lambda \) corresponding to \( \mu_{i}(L_0) \) and \( \mu_{i}(L_\lambda) \) respectively and \( e_{\lambda,i} \to e_{i} \) in \( H^2(\mathbb{R}^N) \). Then as \( \lambda \to +\infty \), we have

\[
\left| \int_{\mathbb{R}^N} (|\Delta u^\pm_\lambda|^2 + V_\lambda (u^\pm_\lambda)^2) \, dx \right|
\]

\[
= \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} u e_{\lambda,i} \, dx \right)^2 \int_{\mathbb{R}^N} (|\Delta e_{\lambda,i}|^2 + V_\lambda e_{\lambda,i}^2) \, dx
\]

\[
= \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} u (e_{\lambda,i} - e_{i}) \, dx \right)^2 \mu_{i}(L_\lambda)
\]

\[
\leq o(1)\|u\|_{L^2}^2.
\]

Thus as \( \lambda \to +\infty \), we have

\[
\left( \frac{\|\delta(x)\|_{L^\infty}}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} (|\Delta u^-_\lambda|^2 + V_\lambda (u^-_\lambda)^2) \, dx \\
- \frac{\mu_{k+1}(L_\lambda)}{\|\delta(x)\|_{L^\infty} + \mu_{k+1}(L_\lambda)} \int_{\mathbb{R}^N} V^- (u^-_\lambda)^2 + 2u^+_\lambda u^-_\lambda \, dx \right) = o(1)\|u\|_{L^2}^2.
\]
Therefore, there exists a constant \( \Lambda_2 > \Lambda_1 \) such that
\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx
\geq \frac{\mu_{k+1}(L_0)}{2(||\delta(x)||_{L^\infty} + \mu_{k+1}(L_0))} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx
- \frac{\mu_{k+1}(L_0)}{4(||\delta(x)||_{L^\infty} + \mu_{k+1}(L_0))} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx
= \frac{\mu_{k+1}(L_0)}{4(||\delta(x)||_{L^\infty} + \mu_{k+1}(L_0))} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.
\]

Thus taking \( C = \frac{\mu_{k+1}(L_0)}{4(||\delta(x)||_{L^\infty} + \mu_{k+1}(L_0))} > 0 \), we have
\[
\int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx \geq C \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx.
\]

Secondly, let
\[
S_\alpha := \left\{ u \in E_\lambda^+ : \int_{\mathbb{R}^N} (|\Delta u|^2 dx + V_\lambda u^2) dx = \alpha^2 \right\}.
\]

Then for any \( u \in S_\alpha \), by Sobolev inequality, we have
\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx
\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx - \frac{C}{p} \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda^+ u^2) dx \right)^{\frac{p}{2}}
\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx - C \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + V_\lambda u^2) dx \right)^{\frac{p}{2}}
= \frac{1}{2} \alpha^2 - C \alpha^p \geq \frac{1}{4} \alpha^2 > 0
\]
for \( \alpha > 0 \) small enough. Thus \( \inf_{S_\alpha} J_\lambda(u) > 0 \).

Finally, for any \( w \in \mathcal{N}_\lambda \), \( w^+ = -w^- + w \in H_w \), we take \( t > 0 \) small enough such that \( tw^+ \in H_w \cap S_\alpha \), thus by taking \( \tau = \frac{1}{4} \alpha^2 \) and \( (iii) \) in this Lemma, we have
\[
J_\lambda(w) > J_\lambda(tw^+) \geq \inf_{S_\alpha} J_\lambda(u) \geq \tau > 0,
\]
which implies \( c_\lambda \geq \tau > 0 \).

At the end of this section, we use the following Lemma to prove that the minimizer for \( c_\lambda \) in \( \mathcal{N}_\lambda \) is indeed a weak solution (also a least energy solution) of (1.1).

**Lemma 2.10** For \( \lambda > \Lambda_2 \), assume \( w \) is an achieved function for \( c_\lambda \) in \( \mathcal{N}_\lambda \), i.e. \( c_\lambda = J_\lambda(u) \) and \( u \in \mathcal{N}_\lambda \). Then \( u \) is a least energy solution to (1.1).
We just need to verify that $u$ is a weak solution to (1.1), i.e. $J'_A(u) = 0$ in $X$. In fact, according to Lagrange multiplier theorem, there exist $(\lambda_0, \lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k$ such that

$$J'_A(u) + \lambda_0 G'_0(u) + \lambda_1 G'_1(u) + \cdots + \lambda_k G'_k(u) = 0.$$ 

Multiplying $u$ and $e_i (i = 1, 2, \cdots, k)$ on both sides of the above equation respectively, we have the following system holds:

$$
\begin{align*}
\begin{cases}
a_{00} \lambda_0 + a_{01} \lambda_1 + a_{02} \lambda_2 + \cdots + a_{0k} \lambda_k &= 0, \\
a_{10} \lambda_0 + a_{11} \lambda_1 + a_{12} \lambda_2 + \cdots + a_{1k} \lambda_k &= 0, \\
a_{20} \lambda_0 + a_{21} \lambda_1 + a_{22} \lambda_2 + \cdots + a_{2k} \lambda_k &= 0, \\
\cdots \\
a_{k0} \lambda_0 + a_{k1} \lambda_1 + a_{k2} \lambda_2 + \cdots + a_{kk} \lambda_k &= 0,
\end{cases}
\end{align*}
$$

where

$$
\begin{align*}
a_{00} &= (p - 2) \int_{\mathbb{R}^N} |u|^p dx, \\
a_{i0} &= a_{i0} = (p - 2) \int_{\mathbb{R}^N} |u|^{p-2} u e_i dx, i = 1, 2, \cdots, k, \\
a_{ij} &= a_{ji} = (p - 1) \int_{\mathbb{R}^N} |u|^{p-2} e_i e_j dx, i, j = 1, 2, \cdots, k, \\
a_{ii} &= (p - 1) \int_{\mathbb{R}^N} |u|^{p-2} e_i^2 dx - \mu_i(L_0) \int_{\mathbb{R}^N} e_i^2 dx, i = 1, 2, \cdots, k.
\end{align*}
$$

Denote the coefficient matrix of the above system by $A = (a_{ij})_{0 \leq i, j \leq k}$, we define $f(y) = y^T A y$ for any $y \in \mathbb{R}^{k+1}$, where $y^T$ denotes the transposition of the vector $y$ in $\mathbb{R}^{k+1}$. For any $y \in \mathbb{R}^{k+1}$, by a simple computation, we have

$$
f(y) = (p - 2) \left( \int_{\mathbb{R}^N} |u|^p dx \right) y_0^2 + 2 \sum_{i=1}^k (p - 2) \left( \int_{\mathbb{R}^N} |u|^{p-2} u e_i dx \right) y_0 y_i \\
+ \sum_{i=1}^k (p - 1) \int_{\mathbb{R}^N} |u|^{p-2} e_i^2 dx - \mu_i(L_0) \int_{\mathbb{R}^N} e_i^2 dx \right) y_i^2 \\
+ 2 \sum_{i=1}^k \sum_{j>i} (p - 1) \left( \int_{\mathbb{R}^N} |u|^{p-2} e_i e_j dx \right) y_i y_j \\
= (p - 2) \int_{\mathbb{R}^N} |u|^{p-2} \left( y_0 u + \sum_{i=1}^k y_i e_i \right)^2 dx \\
+ \int_{\mathbb{R}^N} |u|^{p-2} \left( \sum_{i=1}^k y_i e_i \right)^2 dx - \sum_{i=1}^k \mu_i(L_0) \left( \int_{\mathbb{R}^N} e_i^2 dx \right) y_i^2.
$$

Note that $p > 2$, $\mu_i(L_0) < 0$, $i = 1, 2, \cdots, k$ and $u, e_1, e_2, \cdots, e_k$ are linear independent, then for any $y \in \mathbb{R}^{k+1}$, we have $f(y) > 0$. Thus the matrix $A$ is positively definite. Therefore the
solution of the above system is \((\lambda_0, \lambda_1, \cdots, \lambda_k) = 0\) which implies that \(J'_\lambda(u) = 0\) in \(X_\lambda\), i.e. \(u\) is a least energy solution to (1.1).

### 3 Limit problem

In this section, we consider the limit problem defined in \(\Omega\), where \(\Omega\) is the interior part of the zero set \(V^{-1}(0)\):

\[
\begin{aligned}
\Delta^2 u - \delta u &= |u|^{p-2}u, \\
u &= 0, \Delta u = 0,
\end{aligned}
\]

(3.1)

Recall that the corresponding functional to (3.1) is

\[
J_\Omega(u) = \frac{1}{2} \int_{\Omega} (|\Delta u|^2 - \delta u^2) dx - \frac{1}{p} \int_{\Omega} |u|^p dx,
\]

(3.2)

the Nehari-Pankov manifold is

\[
N_\Omega := \{ u \in H(\Omega) \setminus \{ 0 \} : P_0^{-} \nabla J_\Omega(u) = 0, J'_\Omega(u)u = 0 \}
\]

and the corresponding level value is defined by

\[
c(\Omega) = \inf_{N_\Omega} J_\Omega(u).
\]

We say that \(\{ u_n \}\) is a \((PS)_c\) sequence of \(J_\Omega\) if \(J_\Omega(u_n) \to c\) and \(J'_\Omega(u_n) \to 0\) in \(H'(\Omega)\), the dual space of \(H(\Omega)\), as \(n \to +\infty\). \(J_\Omega\) satisfies the \((PS)_c\) condition if any \((PS)_c\) sequence \(\{ u_n \}\) contains a convergent subsequence.

**Lemma 3.1** For \(2 < p \leq 2^{**}\), \(N \geq 5\), \(\{ u_n \}\) is a \((PS)_{c(\Omega)}\) sequence, i.e. as \(n \to +\infty\),

\[
J_\Omega(u_n) \to c(\Omega), \quad J'_\Omega(u_n) \to 0 \quad \text{in} \ H'(\Omega),
\]

where \(H'(\Omega)\) is the dual space of \(H(\Omega)\). Then \(\{ u_n \}\) is bounded in \(H(\Omega)\).

**Proof:** For \(n\) large enough, we have

\[
c(\Omega) + 1 + \| u_n \|_0 \geq J_\Omega(u_n) - \frac{1}{p} J'_\Omega(u_n) u_n = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} (|\Delta u_n|^2 - \delta u_n^2) dx
\]

and

\[
c(\Omega) + 1 + \| u_n \|_0 \geq J_\Omega(u_n) - \frac{1}{2} J'_\Omega(u_n) u_n = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u_n|^p dx.
\]

By Hölder’s inequality, we have

\[
\int_{\Omega} |u_n|^2 dx \leq |\Omega|^{1-\frac{2}{p}} \left( \int_{\Omega} |u_n|^p dx \right)^{\frac{2}{p}}.
\]

Thus \(\{ u_n \}\) is bounded in \(H(\Omega)\).
Lemma 3.2 For $2 < p < 2^{**}$ and $N \geq 5$, $c(\Omega)$ is achieved by a nontrivial solution $u$ of (3.1) in $\mathcal{N}_\Omega$.

Proof: Since the proof is quite standard, for readers’ convenience, we give the sketch of the proof.

Indeed, from the definition of $c(\Omega)$ and thanks to Ekeland’s Variational Principle, we know that there exists a sequence $\{u_n\} \subset \mathcal{N}_\Omega$ such that

$$J_{\Omega}(u_n) \to c(\Omega) \quad \text{and} \quad J'_{\Omega}(u_n) \to 0 \quad \text{in} \quad H'(\Omega). \quad (3.3)$$

Thus by Lemma 3.1 and the fact that $H(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we immediately obtain that $J_{\Omega}(u_n)$ satisfies Palais-Smale condition. Namely (3.3) indicate that there is a subsequence of $\{u_n\}$ (still denote it as itself) and $u \in \mathcal{N}_\Omega$ such that $u_n \to u$ in $H(\Omega)$ and

$$J_{\Omega}(u) = c(\Omega) > 0, \quad J'_{\Omega}(u) = 0.$$

Thus we complete the proof of this lemma. $\square$

Now we focus on the existence of least energy solution of (3.1) in the critical case. We want to point out that in this case problem (3.1) is close to the famous Brezis-Nirenberg problem

$$\begin{aligned}
-\Delta u - \delta u &= |u|^{2^*-2}u, & x \in \Omega, \\
0 &= u, & x \in \partial \Omega.
\end{aligned} \quad (3.4)$$

where $2^*$ is the critical Sobolev exponent which is $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$. And our method to prove the existence of least energy solutions to problem (3.1) in critical case also follows the methods developed by Brezis and Nirenberg (see [7]).

Firstly we have the following estimate for the least energy $c(\Omega)$ when $p = 2^{**}$.

Lemma 3.3 For $N \geq 8$, $p = 2^{**}$, we have

$$0 < c(\Omega) < \frac{2}{N} S^{\frac{N}{2}},$$

where

$$S = \inf \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx : u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{2^{**}} \, dx = 1 \right\}.$$

Proof: It was shown by P.L.Lions (see Corollary I.2 in [25]) that there is an nonnegative minimizer for $S$ which is radial symmetric and decreasing in $|x|$. In 1998, C.S. Lin [21] showed that any positive extremal function of $S$ has the form

$$U_\varepsilon = \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-4}{2}},$$

for each $\varepsilon > 0$. One also can refer to the paper by J.Wei and X.Xu [32], where the authors extended C.S.Lin’s results to more general case.
We may assume that \(0 \in \Omega\). Let \(\eta\) be a smooth cutoff function satisfying
\[
\eta(x) = 1 \quad \text{for} \quad x \in B_r(0) \quad \text{and} \quad \text{supp} \, \eta \subset \Omega.
\]
Defining \(u_\varepsilon(x) = \eta(x)U_\varepsilon(x) \in H(\Omega)\). By a direct calculation, we have
\[
\int_\Omega |\Delta u_\varepsilon|^2 \, dx = \int_{B_r(0)} |\Delta U_\varepsilon|^2 \, dx + \int_{\mathbb{R}^N \setminus B_r(0)} |\Delta u_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^{N-4}),
\]
\[
\int_\Omega u_\varepsilon \, dx = \int_{B_r(0)} \eta U_\varepsilon \, dx = \int_\Omega \eta \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-4}{2}} \, dx = O(\varepsilon^{\frac{N-4}{2}}),
\]
\[
\int_\Omega |u_\varepsilon|^2 dx = \int_{B_r(0)} |U_\varepsilon|^2 dx + \int_{\mathbb{R}^N \setminus B_r(0)} |u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^N),
\]
and
\[
\int_\Omega |u_\varepsilon|^2 \, dx \geq c^2 \int_{B_r(0)} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{N-4} \, dx + c^2 \int_{B_r(0) \setminus B_{r\varepsilon}(0)} \frac{1}{|x|^2 + |x|^2} \, dx + c^2 \varepsilon^{N-4} \int_{\Omega \setminus B_r(0)} \eta^2 \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{N-4} \, dx
\]
\[
\geq \begin{cases} 
\varepsilon^4 |\ln \varepsilon| + O(\varepsilon^4), & \text{if } N = 8, \\
\varepsilon^4 + O(\varepsilon^{N-4}), & \text{if } N \geq 9.
\end{cases}
\]
Let us define
\[
M_\varepsilon := \max \left\{ \int_\Omega (|\Delta u|^2 - \delta(x)u^2) \, dx : u \in H_{a_\varepsilon} \setminus \{0\}, \int_\Omega |u|^{2^*} \, dx = 1 \right\}.
\]
We claim that
\[
\text{for } \varepsilon > 0 \text{ small enough and } N \geq 8, \text{ we have } M_\varepsilon < S.
\]
In fact, take
\[
z_\varepsilon = u_\varepsilon - \sum_{i=1}^k \left( \int_\Omega u_\varepsilon e_i dx \right) e_i,
\]
and we may assume that \(u = y + tu_\varepsilon = \overline{y} + tz_\varepsilon \) with \(t > 0\) and \(\|u\|_{L^{2^*}(\Omega)} = 1\) such that
\[
\int_\Omega (|\Delta u|^2 - \delta(x)u^2) \, dx = M_\varepsilon.
\]
By Hölder’s inequality, we have
\[
\|\overline{y}\|_{L^2(\Omega)} \leq c_1 \|u\|_{L^{2^*}(\Omega)} = c_1.
\]
Since \( \dim H^-(\Omega) < +\infty \), then
\[
\| \overline{y} \|_{L^{2*}(\Omega)} \leq C \| \overline{y} \|_{L^2(\Omega)} \leq c_2.
\]
Note that
\[
\| t z_\varepsilon \|_{L^{2*}(\Omega)} = \| u - \overline{y} \|_{L^{2*}(\Omega)} \leq \| u \|_{L^{2*}(\Omega)} + \| \overline{y} \|_{L^{2*}(\Omega)} \leq 1 + c_2
\]
and
\[
\| z_\varepsilon \|_{L^{2*}(\Omega)} \geq \| u_\varepsilon \|_{L^{2*}(\Omega)} - \left\| \sum_{i=1}^k \left( \int_{\mathbb{R}^N} u_\varepsilon e_i dx \right) e_i \right\|_{L^{2*}(\Omega)} \geq \frac{1}{2} S^{\frac{N-4}{4}} > 0
\]
for \( \varepsilon > 0 \) small enough. Then we can easily obtain that \( 0 < t \leq c_3 \). Thus
\[
\int_{\Omega} |y|^2 dx = \int_{\Omega} \left| \overline{y} - t \sum_{i=1}^k \left( \int_{\Omega} u_\varepsilon e_i dx \right) e_i \right|^2 dx
\]
\[
\leq 2 \int_{\Omega} |\overline{y}|^2 dx + 2t^2 \sum_{i=1}^k \left( \int_{\Omega} u_\varepsilon e_i dx \right)^2
\]
\[
\leq c_4.
\]
Again by \( \dim H^-(\Omega) < +\infty \), we get that
\[
|y| \leq \| y \|_{L^\infty(\Omega)} \leq C \| y \|_{L^2(\Omega)} \leq c_5.
\]
Thus we have
\[
1 = \| u \|_{L^{2*}(\Omega)}^{2*} \geq t^{2*} \| u_\varepsilon \|_{L^{2*}(\Omega)}^{2*} + 2^{*}t^{2*}-1 \int_{\Omega} u_\varepsilon^{2*}-1yd\varepsilon
\]
\[
\geq t^{2*} \| u_\varepsilon \|_{L^{2*}(\Omega)}^{2*} - c_6 \int_{\Omega} |u_\varepsilon|^{2*}-1d\varepsilon \| y \|_{L^2(\Omega)}.
\]
Note that
\[
\frac{\| u_\varepsilon \|_{L^{2*}(\Omega)}^2 - \mu_k \| u_\varepsilon \|_{L^2(\Omega)}^2}{\| u_\varepsilon \|_{L^{2*}(\Omega)}^2} = \begin{cases} 
\frac{S^4 - \mu_k d\varepsilon^4 | \ln \varepsilon | + O(\varepsilon^4)}{(S^2 + O(\varepsilon^4))^2} , & \text{if } N = 8, \\
\frac{S_N^4 - \mu_k d\varepsilon^4 + O(\varepsilon^{N-4})}{(S_N^2 + O(\varepsilon^N))^2} , & \text{if } N \geq 9
\end{cases}
\]
\[
= \begin{cases} 
S - \mu_k dS^{-1}\varepsilon^4 | \ln \varepsilon | + O(\varepsilon^4) , & \text{if } N = 8, \\
S - \mu_k dS^{\frac{4-N}{4}}\varepsilon^4 + O(\varepsilon^{N-4}) , & \text{if } N \geq 9
\end{cases}
\]
Proof: By Ekeland’s Variational Principle and the definition of implies that 

\[ M_e \leq \mu_k(L_0)\|y\|_{L^2(\Omega)}^2 + \frac{\|u_\varepsilon\|^2_0 - \mu_k\|u_\varepsilon\|_{L^2(\Omega)}^2}{\|u_\varepsilon\|_{L^{2*}(\Omega)}}\|tu_\varepsilon\|_{L^{2*}(\Omega)}^2 + c_\gamma \|u_\varepsilon\|_{L^1(\Omega)}\|y\|_{L^2(\Omega)} \]

\[ \leq \mu_k(L_0)\|y\|_{L^2(\Omega)}^2 + \frac{\|u_\varepsilon\|^2_0 - \mu_k\|u_\varepsilon\|_{L^2(\Omega)}^2}{\|u_\varepsilon\|_{L^{2*}(\Omega)}} \left(1 + c_6 \int_\Omega |v_\varepsilon|^{2* - 1} dx\|y\|_{L^2(\Omega)}\right)^{2*} \]

\[ + c_\gamma \|u_\varepsilon\|_{L^1(\Omega)}\|y\|_{L^2(\Omega)} \]

\[ \leq \begin{cases} (S - \mu_k\varepsilon^4S^{-1})(1 + O(\varepsilon^2)\|y\|_{L^2(\Omega)}) + O(\varepsilon^4), & \text{if } N = 8, \\ (S - \mu_k\varepsilon^4S^{\frac{4-N}{4}})(1 + O(\varepsilon^{N-4}))\|y\|_{L^2(\Omega)}) + O(\varepsilon^N), & \text{if } N \geq 9, \end{cases} \]

< S.

Since for each \( u \in H_{u_\varepsilon} \setminus \{0\}, \)

\[ J_\Omega(u) \leq \max_{t \geq 0} J_\Omega(tu) = \frac{2}{N} \left( \|u\|^2_0 - \frac{\int_\Omega \delta(x)u^2 dx}{\|u\|_{L^{2*}(\Omega)}^2} \right). \]

Then \( \max_{u \in H_{u_\varepsilon}} J_\Omega(u) \leq \frac{2}{N} M_e^{N} < \frac{2}{N} S^{\frac{N}{2}} \) for \( \varepsilon > 0 \) small and \( N \geq 8 \). Remark 2.8 immediately implies that \( c(\Omega) < \frac{2}{N} S^{\frac{N}{2}} \) for \( N \geq 8 \).

Lemma 3.4 For \( p = 2^{**}, N \geq 8 \), \( c(\Omega) \) is achieved by a nontrivial solution \( u \) of (3.1) in \( N_{\Omega} \).

Proof: By Ekeland’s Variational Principle and the definition of \( c(\Omega) \), we can easily get a \((PS)_{c(\Omega)}\) sequence \( \{u_n\} \). Moreover, \( \{u_n\} \) is bounded in \( H(\Omega) \). Then up to a subsequence, we may assume that

\[ \begin{cases} u_n \rightharpoonup u & \text{in } H(\Omega), \\ u_n \to u & \text{in } L^{2*}(\Omega), \\ u_n \to u & \text{in } L^2(\Omega). \end{cases} \]

Let \( v_n = u_n - u \), by Brézis-Lieb’s Lemma, we have

\[ \int_\Omega |\Delta u_n|^2 dx = \int_\Omega |\Delta u|^2 dx + \int_\Omega |\Delta v_n|^2 dx + o(1), \]

\[ \int_\Omega |u_n|^{2*} dx = \int_\Omega |u|^{2*} dx + \int_\Omega |v_n|^{2*} dx + o(1). \]

A direct computation shows that

\[ J_\Omega(u_n) = J_\Omega(u) + \frac{1}{2} \int_\Omega |\Delta v_n|^2 dx - \frac{1}{2^{**}} \int_\Omega |v_n|^{2*} dx + o(1) \]

and

\[ J_\Omega'(u_n)u_n = J_\Omega'(u)u + \int_\Omega |\Delta v_n|^2 dx - \frac{1}{2^{**}} \int_\Omega |v_n|^{2*} dx + o(1). \]
It is easy to see that \( J'_\Omega(u) = 0 \) and \( J_\Omega'(u) \geq 0 \). We may assume that

\[
b = \lim_{n \to +\infty} \int_\Omega |\Delta v_n|^2 \, dx = \lim_{n \to +\infty} \int_\Omega |v_n|^{2^*_s} \, dx > 0.
\]

On one hand,

\[
b = \lim_{n \to +\infty} \int_\Omega |v_n|^{2^*_s} \, dx = \lim_{n \to +\infty} \int_\Omega |\Delta v_n|^2 \, dx \geq S \lim_{n \to +\infty} \left( \int_\Omega |v_n|^{2^*_s} \, dx \right)^{2^*_s} = Sb^{2^*_s}.
\]

Thus \( b \geq S^{\frac{2}{N}} \). But on the other hand,

\[
\frac{2}{N} S^{\frac{N}{2}} > c(\Omega) \geq \frac{1}{2} \lambda \lim_{n \to +\infty} \int_\Omega |\Delta v_n|^2 \, dx - \frac{1}{2^*_s} \lim_{n \to +\infty} \int_\Omega |v_n|^{2^*_s} \, dx = \frac{2}{N} b.
\]

Thus \( b < S^{\frac{N}{2}} \) which leads to a contradiction. Therefore, \( u_n \to u \) in \( H(\Omega) \) and \( u \) is a minimizer for \( c(\Omega) \).

\[\tag{4.4}\]

4 Existence of least energy solutions

In this section, we consider the existence of least energy solutions for (1.1). We use the same notations as in Section 1. Recall that \( \{u_n\} \subset X_\lambda \) is called a Palais-Smale \( c \) sequence (\( (PS)_c \) sequence in short) for functional \( J_\lambda(u) \) if

\[
J_\lambda(u_n) \to c \text{ and } J'_\lambda(u_n) \to 0 \text{ in } X'_\lambda, \text{ as } n \to +\infty
\]

where \( X'_\lambda \) is the dual space of \( X_\lambda \). We say that the functional \( J_\lambda(u) \) satisfies \( (PS)_c \) condition if any of the \( (PS)_c \) sequence (up to a subsequence, if necessary)\( \{u_n\} \) converges strongly in \( X_\lambda \).

In the following subsections, we firstly present some properties of the \( (PS)_c \) sequence of \( J_\lambda(u) \) and then we prove the existence of least energy solutions of (1.1) both in subcritical and critical cases.

4.1 Properties of \( (PS)_c \) sequence

Lemma 4.1 For \( 2 < p \leq 2^*_s, \lambda > \Lambda_2 \), if \( \{u_n\} \) is a \( (PS)_c \) sequence for \( J_\lambda(u) \), then \( \{u_n\} \) is bounded in \( X_\lambda \). Furthermore, if \( u_n \to 0 \) in \( X_\lambda \), then up to a subsequence,

\[
\limsup_{n \to +\infty} \|u_n\|_\lambda^2 \leq \frac{2p}{p - 2} c. \tag{4.1}
\]

Proof: Since \( \{u_n\} \) is a \( (PS)_c \) sequence of \( J_\lambda(u) \), then for \( \lambda > \Lambda_2 \), we have

\[
c + o(1) + o(\|u_n\|_\lambda) = J_\lambda(u_n) - \frac{1}{p} J'_\lambda(u_n) u_n
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_\lambda u_n^2) \, dx \tag{4.2}
\]
and\[c + o(1) + o(\|u_n\|_\lambda) = J_\lambda(u_n) - \frac{1}{2} J'_\lambda(u_n) u_n = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u_n|^p dx. \tag{4.3}\]

By Hölder’s inequality, we have
\[
\int_{\mathbb{R}^N} V_\lambda^- |u_n|^2 dx \leq \|\delta(x)\|_{L^\infty} \int_{B_R} |u_n|^2 dx \leq \|\delta(x)\|_{L^\infty} |\Omega|^{1 - \frac{2}{p}} \left(\int_{B_R} |u_n|^p dx\right)^{\frac{2}{p}}. \tag{4.4}\]

Thus by (4.2), (4.3) and (4.4), we can easily obtain that $u_n$ is bounded in $X_\lambda$ for $\lambda > \Lambda_2$. Furthermore, if $u_n \to 0$ in $X_\lambda$ as $n \to +\infty$, then up to a subsequence, by Lebesgue Dominated theorem, we have $\int_{\mathbb{R}^N} V_\lambda^- |u_n|^2 dx \to 0$ as $n \to +\infty$. Thus (4.1) holds directly from (4.2).

Lemma 4.2 For $2 < p \leq 2^*$, $\lambda > \Lambda_2$, $\{u_n\}$ is a $(PS)_c$ sequence of $J_\lambda$, if $u_n \to 0$ in $X_\lambda$ as $n \to +\infty$. Then there exists a subsequence such that one of the following statements holds:

(i) $\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^p dx = 0$;

(ii) There exists $\sigma > 0$ which is independent of $\lambda$ such that

$$\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^p dx \geq \sigma.$$

Proof: Since $\{u_n\}$ is a $(PS)_c$ sequence of $J_\lambda$ and $u_n \to 0$ in $X_\lambda$ as $n \to +\infty$, then up to a subsequence, by Lebesgue Dominated theorem, we have

$$\int_{\mathbb{R}^N} |u_n|^p dx = o(1) = \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_\lambda |u_n|^2) dx = \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_\lambda^+ |u_n|^2) dx + o(1).$$

By Sobolev embedding theorem, for $\lambda > \Lambda_2$ we have

$$\int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_\lambda^+ |u_n|^2) dx \geq \Lambda \left(\int_{\mathbb{R}^N} |u_n|^p dx\right)^{\frac{2}{p}},$$

where $\Lambda$ does not depend on $\lambda$. Thus if $\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^p dx \neq 0$, then

$$\liminf_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^p dx \geq \Lambda^{\frac{p}{p-2}}.$$

We complete the proof of this lemma by taking $\sigma = \Lambda^{\frac{p}{p-2}}$.

Lemma 4.3 Let $2 < p < 2^*$, $N \geq 5$ and $M > 0$ be a constant which does not depend on $\lambda$, then for any $\varepsilon > 0$, there exist $\Lambda_\varepsilon > \Lambda_2$ such that for any $\lambda > \Lambda_\varepsilon$, $c < M$, $\{u_n\}$ is a $(PS)_c$ sequence of $J_\lambda$ and $u_n \to 0$ in $X_\lambda$ as $n \to +\infty$, up to a subsequence, we have

$$\limsup_{n \to +\infty} \int_{B_R^c} |u_n|^p dx \leq \varepsilon,$$

where $B_R^c = \{ x \in \mathbb{R}^N : |x| \geq R \}$. Especially, there exists $\Lambda_3 > \Lambda_2$ such that

$$\limsup_{n \to +\infty} \int_{B_R^c} |u_n|^p dx \leq \frac{\sigma}{2}.$$
Proof: For \( \lambda > \Lambda_2 \), by (2.2), we have
\[
\int_{B_R} u_n^2 dx \leq \frac{1}{\lambda M_0 - \| \delta(x) \|_{L^\infty}} \int_{B_R} (\lambda V(x) - \delta(x)) u_n^2 dx
\]
\[
\leq \frac{1}{\lambda M_0 - \| \delta(x) \|_{L^\infty}} \int_{B_R} (|\Delta u_n|^2 + V_\lambda |u_n|^2) dx
\]
\[
\leq \frac{1}{\lambda M_0 - \| \delta(x) \|_{L^\infty}} \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_\lambda^+ |u_n|^2) dx
\]
\[
\leq \frac{2 p M}{(p - 2)(\lambda M_0 - \| \delta(x) \|_{L^\infty})} \to 0 \quad \text{as} \quad \lambda \to \infty.
\]
By using Hölder’s inequality and Sobolev imbedding theorem, as \( \lambda \to +\infty \) we have
\[
\int_{B_R} |u_n|^p dx \leq C \left( \int_{B_R} |u_n|^{2^*} dx \right)^{\frac{p}{2} - 1} \left( \int_{B_R} |u_n|^2 dx \right)^{\frac{p(1 - \theta)}{2}}
\]
\[
\leq C \| u_n \|_\lambda^{\theta} \left( \int_{B_R} |u_n|^2 dx \right)^{\frac{p(1 - \theta)}{2}}
\]
\[
\leq C \left( \frac{2 p M}{p - 2} \right)^{\frac{p(1 - \theta)}{2}} \left( \int_{B_R} |u_n|^2 dx \right) \to 0,
\]
where \( \frac{1}{p} = \frac{\theta}{2^*} + \frac{1 - \theta}{2} \). Thus the proof of the lemma is completed.

We complete this subsection by showing the following lemma which compare \( c_\lambda \) and \( c(\Omega) \).

Lemma 4.4 For \( \lambda > \Lambda_2 \), \( 2 < p \leq 2^* \), the following estimate holds:
\[
0 < \tau < c_\lambda \leq c(\Omega).
\]
Proof: Since \( \mathcal{N}_\Omega \subset \mathcal{N}_\lambda \), then \( c_\lambda \leq c(\Omega) \). According to \((iv)\) in Lemma 2.9, we know that \( c_\lambda > \tau > 0 \). Thus we complete the proof of this lemma.

4.2 Existence of least energy solutions in subcritical case

In this subsection, we are concerned with the existence of least energy solutions for subcritical case. To begin with, we give the following proposition.

Proposition 4.5 For any \( \lambda > \Lambda_3 \), \( 2 < p < 2^* \), \( c_\lambda := \inf_{\mathcal{N}_\lambda} J_\lambda(u) \) is achieved by some \( u \neq 0 \).

Proof: For any \( \lambda > \Lambda_3 \), \( 2 < p < 2^* \), by the definition of \( c_\lambda \) and Ekeland Variational Principle, there exits a \((PS)_{c_\lambda} \) sequence \( \{ u_n \} \) of \( J_\lambda(u) \). By Lemma 4.1, we know that \( \{ u_n \} \) is bounded in
In this subsection, we consider the existence of least energy solutions for (1.1) in the critical case

\[ p = 2^\ast. \]

We have the following proposition.

**Proposition 4.6** For \( p = 2^\ast, \lambda > \Lambda_3 \), then \( c_\lambda := \inf_{\mathcal{N}_\lambda} J_\lambda(u) \) is achieved by some \( u \neq 0 \).

**Proof:** For any \( \lambda > \Lambda_3 \), by the definition of \( c_\lambda \) and Ekeland Variational Principle, there exists a \((PS)_{c_\lambda}\) sequence \( \{u_n\} \) of \( J_\lambda(u) \). According to Lemma 4.1, we know that \( \{u_n\} \) is bounded in \( X_\lambda \). Then up to a subsequence, we have

\[
\begin{cases}
    u_n \to u & \text{in } X_\lambda, \\
    u_n \to u & \text{in } L^p(\mathbb{R}^N), \\
    u_n \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \\
    u_n \to u & \text{a.e. in } \mathbb{R}^N.
\end{cases}
\]

as \( n \to \infty \). Thus \( J'_\lambda(u_n) = 0 \) and

\[
J_\lambda(u) = J_\lambda(u) - \frac{1}{2} J'_\lambda(u) u = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^p dx \geq 0.
\]

Let \( v_n = u_n - u \), by Brezis-Lieb’s Lemma (see [6]), we obtain that

\[
\|u_n\|_\lambda^2 = \|u\|_\lambda^2 + \|v_n\|_\lambda^2, \quad \|u_n\|_{L^p(\mathbb{R}^N)}^p = \|u\|_{L^p(\mathbb{R}^N)}^p + \|v_n\|_{L^p(\mathbb{R}^N)}^p.
\]

It is easy to obtain that

\[
J_\lambda(u_n) = J_\lambda(u) + J_\lambda(v_n) + o(1), \quad J'_\lambda(u_n) u_n = J'_\lambda(u) u + J'_\lambda(v_n) v_n + o(1).
\]

According to Lemma 8.1 and Lemma 8.2 in [33], we know that \( \{v_n\} \) is a \((PS)_d\) sequence of \( J_\lambda \) where \( d = c_\lambda - J_\lambda(u) \).

We may assume \( \lim_{n \to +\infty} \|v_n\|_{L^p(\mathbb{R}^N)}^p = b \). If \( b = 0 \), we easily obtain that \( v_n \to 0 \) in \( X_\lambda \), which implies \( u_n \to u \) in \( X_\lambda \). If \( b > 0 \), then by Lemma 4.2, we have \( b \geq \sigma \). On the other hand, if we take \( M = c(\Omega) \), then by Lemma 4.3 we immediately have

\[
b = \lim_{n \to +\infty} \|v_n\|_{L^p(\mathbb{R}^N)}^p = \lim_{n \to +\infty} \int_{B_R} |v_n|^p dx \leq \frac{\sigma}{2},
\]

which leads to a contradiction. Thus \( u_n \to u \) in \( X_\lambda \) and \( J_\lambda(u) = c_\lambda > 0 \). This implies \( u \in \mathcal{N}_\lambda \).

Therefore, \( c_\lambda \) is achieved by some \( u \in \mathcal{N}_\lambda \) and \( u \) is a nontrivial least energy solution to (1.1) for any \( \lambda > \Lambda_3 \). \( \square \)

### 4.3 Existence of least energy solutions in critical case

In this subsection, we consider the existence of least energy solutions for (1.1) in the critical case

\[ p = 2^\ast. \]

We have the following proposition.

**Proposition 4.6** For \( p = 2^\ast, \lambda > \Lambda_3 \), then \( c_\lambda := \inf_{\mathcal{N}_\lambda} J_\lambda(u) \) is achieved by some \( u \neq 0 \).

**Proof:** For any \( \lambda > \Lambda_3 \), by the definition of \( c_\lambda \) and Ekeland Variational Principle, there exists a \((PS)_{c_\lambda}\) sequence \( \{u_n\} \) of \( J_\lambda(u) \). According to Lemma 4.1, we know that \( \{u_n\} \) is bounded in \( X_\lambda \). Then up to a subsequence, we have

\[
\begin{cases}
u_n \to \nu & \text{in } X_\lambda, \\
    u_n \to u & \text{in } L^2\ast(\mathbb{R}^N), \\
    u_n \to u & \text{a.e. in } \mathbb{R}^N.
\end{cases}
\]

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Thus $J'_\lambda(u) = 0$ and
\[
J_\lambda(u) = J_\lambda(u) - \frac{1}{2} J'_\lambda(u) u = \left(\frac{1}{2} - \frac{1}{2^{**}}\right) \int_{\mathbb{R}^N} |u|^{2^{**}} dx \geq 0.
\]
Let $v_n = u_n - u$, by Brézis Lieb’s lemma, we have
\[
\|u_n\|_\lambda^2 = \|u\|_\lambda^2 + \|v_n\|_\lambda^2 + o(1),
\]
\[
\|u_n\|_{L^{2^{**}}(\mathbb{R}^N)}^2 = \|u\|_{L^{2^{**}}(\mathbb{R}^N)}^2 + \|v_n\|_{L^{2^{**}}(\mathbb{R}^N)}^2 + o(1).
\]
It is easy to obtain that
\[
J_\lambda(u_n) = J_\lambda(u) + J_\lambda(v_n) + o(1),
\]
and
\[
J'_\lambda(u_n) u_n = J'_\lambda(u) u + J'_\lambda(v_n) v_n + o(1).
\]
According to Lemma 8.1 and Lemma 8.2 in [33], we know that $\{v_n\}$ is a $(PS)_d$ sequence of $J_\lambda$ where $d = c_\lambda - J_\lambda(u)$. We may assume that
\[
\lim_{n \to +\infty} \|v_n\|_\lambda^2 = \lim_{n \to +\infty} \|v_n\|_{L^{2^{**}}(\mathbb{R}^N)}^2 = b > 0.
\]
On the one hand, we have
\[
b = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^{2^{**}} dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\Delta v_n|^2 + V_\lambda^+ v_n^2) dx \\
\geq \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\Delta v_n|^2 dx \\
\geq S \lim_{n \to +\infty} \left( \int_{\mathbb{R}^N} |v_n|^{2^{**}} dx \right)^{\frac{2}{2^{**}}} = S b^{\frac{2}{2^{**}}},
\]
Thus $b \geq S^{\frac{2}{2^{**}}}$. By Lemma 2.7, Lemma 3.3 and Lemma 3.4, we know that $0 < c_\lambda \leq c(\Omega) < \frac{2}{N} S^{\frac{2}{2^{**}}}$. Then we have
\[
\frac{2}{N} S^{\frac{2}{2^{**}}} > c_\lambda \geq \lim_{n \to +\infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta v_n|^2 + V_\lambda^+ v_n^2) dx - \frac{1}{2^{**}} \int_{\mathbb{R}^N} v_n^{2^{**}} dx \right) = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) b,
\]
Thus $b < S^{\frac{2}{2^{**}}}$ which leads to a contradiction. This implies that $u_n \to u$ strongly in $X_\lambda$ and $c_\lambda$ is achieved by $u$ in $\mathcal{N}_\lambda$. Thus $u \in \mathcal{N}_\lambda$ is a least energy solution of (1.1).
5 Asymptotic behavior of least energy solutions

In this section, we study the asymptotic behavior of \( c_\lambda \) as \( \lambda \to +\infty \).

We firstly give the asymptotic behavior of \( c_\lambda \) in the subcritical case, we have the following lemma.

**Lemma 5.1** Let \( 2 < p < 2^* \), \( N \geq 5 \), then for any \( \lambda_n \to +\infty \), up to a subsequence (still denoted by \( \lambda_n \)), we have

\[
\lim_{\lambda_n \to +\infty} c_{\lambda_n} = c(\Omega).
\]

**Proof:** Since \( 0 < \tau \leq c_\lambda \leq c(\Omega) < +\infty \) for \( \lambda > \Lambda_\delta \), then up to a subsequence, we may assume

\[
0 < \tau \leq \lim_{\lambda_n \to +\infty} c_{\lambda_n} = k \leq c(\Omega).
\]

For \( n = 1, 2, \cdots \), let \( u_n \in X_{\lambda_n} \) satisfies \( J_{\lambda_n}(u_n) = c_{\lambda_n} \) and \( J'_{\lambda_n}(u_n) = 0 \). According to Lemma 4.1, \( \{\|u_n\|_{\lambda_n}\} \) is bounded. By Lemma 2.1, \( \{u_n\} \) is also bounded in \( H^2(\mathbb{R}^N) \). Up to a subsequence, we have

\[
\begin{cases}
 u_n \to u & \text{in } H^2(\mathbb{R}^N), \\
 u_n \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \\
 u_n \to u & \text{in } L^p(\mathbb{R}^N), \\
 u_n \to u & \text{a.e. in } \mathbb{R}^N.
\end{cases}
\]

Firstly, we claim that \( u|_{\Omega^c} = 0 \), where \( \Omega^c := \{x : x \in \mathbb{R}^N \setminus \Omega\} \).

If not, we have \( u|_{\Omega^c} \neq 0 \). Then there exists a compact subset \( F \subset \Omega^c \) with dist \( \{F, \partial \Omega\} > 0 \) such that \( u|_F \neq 0 \) and

\[
\int_F u_n^2 dx \to \int_F u^2 dx > 0, \text{ as } n \to \infty.
\]

Moreover, by assumption \((V_2)\), there exists \( \varepsilon_0 > 0 \) such that \( V(x) \geq \varepsilon_0 \) for any \( x \in F \). Since \( \{u_n\} \) is bounded in \( H^2(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_{\lambda_n} u_n^2) dx = \int_{\mathbb{R}^N} |u_n|^p dx,
\]

then

\[
J_{\lambda_n}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_{\lambda_n} u_n^2) dx - \frac{1}{p} \int_{\mathbb{R}^N} |u_n|^p dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_{\lambda_n} u_n^2) dx
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{\mathbb{R}^N} \lambda_n V(x) u_n^2 dx - \|\delta(x)\|_{L^\infty} \int_{\mathbb{R}^N} u_n^2 dx \right)
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{F} \lambda_n \varepsilon_0 u_n^2 dx - \|\delta(x)\|_{L^\infty} \int_{\mathbb{R}^N} (|\Delta u_n|^2 + u_n^2) dx \right)
\]

\[
\to +\infty \text{ as } n \to +\infty.
\]

This contradiction shows that \( u|_{\Omega^c} = 0 \), by the smooth assumption on \( \partial \Omega \) we have \( u \in H(\Omega) \).
Now we are going to show that
\[ u_n \to u \quad \text{in } L^p(\mathbb{R}^N). \] (5.1)

Suppose (5.1) is not true, then by the Concentration Compactness Principle of P. L. Lions (see [24]), there exist \( \delta > 0, \rho > 0 \) and \( x_n \in \mathbb{R}^N \) with \( |x_n| \to +\infty \) such that
\[ \lim_{n \to \infty} \sup_{B_{\rho}(x_n)} |u_n - u|^2 dx \geq \delta > 0. \] (5.2)

By the choice of \( \{u_n\} \) and the facts that \( u|_{\Omega^c} = 0 \), we have
\[ J_{\lambda_n}(u_n) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |\Delta u_n|^2 + V_{\lambda_n} u_n^2 \, dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{B_{\rho}(x_n) \cap B_{\rho}(0)} \lambda_n V(x) u_n^2 \, dx - \|\delta(x)\|_{L^\infty} \int_{\mathbb{R}^N} u_n^2 \, dx \right) \]
\[ \geq \left( \frac{1}{2} - \frac{1}{p} \right) \left( \lambda_n M_0 \int_{B_{\rho}(x_n)} |u_n - u|^2 \, dx - \|\delta\|_{L^\infty} \int_{\mathbb{R}^N} u_n^2 \, dx \right) \]
\[ \to +\infty. \]

This contradiction induce that \( u_n \to u \) in \( L^p(\mathbb{R}^N) \).

Since \( J'_{\lambda_n}(u_n) = 0 \), then for any \( \psi \in H(\Omega) \), we have
\[ \int_{\mathbb{R}^N} (\Delta u_n \Delta \psi + V_{\lambda_n} u_n \psi) \, dx = \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \psi \, dx. \]

Let \( n \to +\infty \), we have
\[ \int_{\Omega} (\Delta u \Delta \psi - \delta u \psi) \, dx = \int_{\Omega} |u|^{p-2} u \psi \, dx. \]

Thus \( J'_{\Omega}(u) = 0 \). Since
\[ J_{\lambda_n}(u_n) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u_n|^p \, dx = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u|^p \, dx + o(1). \]

Then \( k = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^p \, dx > 0 \) which implies \( u \neq 0 \). Thus \( u \in \mathcal{N}_{\Omega} \) and
\[ J_{\Omega}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^p \, dx = k \geq c(\Omega). \]

This implies that \( k = c(\Omega) \). Furthermore, by Brézis-Lieb’s Lemma, we obtain that \( \|u_n - u\|_{\lambda_n}^2 \to 0 \) as \( n \to +\infty \). Thus according to Lemma 2.1, we have \( u_n \to u \) in \( H^2(\mathbb{R}^N) \). \( \square \)

Now we give the asymptotic behavior of \( c_{\lambda} \) in the critical case and which is
Lemma 5.2 Let $N \geq 8$, $p = 2^{**}$, then for any $\lambda_n \to +\infty$, up to a subsequence (still denoted by $\lambda_n$), we have
\[
\lim_{\lambda_n \to +\infty} c_{\lambda_n} = c(\Omega).
\]

Proof: Since $0 < \tau \leq c_{\lambda} \leq c(\Omega) < +\infty$ for $\lambda > \Lambda_3$, then up to a subsequence, we may assume
\[
0 < \tau \leq \lim_{\lambda_n \to +\infty} c_{\lambda_n} = k \leq c(\Omega).
\]

For $n = 1, 2, \cdots$, let $u_n \in X_{\lambda_n}$ satisfies $J_{\lambda_n}(u_n) = c_{\lambda_n}$ and $J'_{\lambda_n}(u_n) = 0$. As proved in Lemma 4.1, we can easily get that $\{u_n\}$ is bounded in $X_{\lambda_n}$, namely $\|u_n\|_{\lambda_n} \leq C$ for some $C > 0$. According to Lemma 2.1, $\{u_n\}$ is also bounded in $H^2(\mathbb{R}^N)$. Then up to a subsequence, we have
\[
\begin{aligned}
&u_n \rightharpoonup u \text{ in } H^2(\mathbb{R}^N), \\
&u_n \rightharpoonup u \text{ in } L^{2^{**}}(\mathbb{R}^N), \\
&u_n \to u \text{ in } L^2_{\text{loc}}(\mathbb{R}^N), \\
&u_n \to u \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
\]

Similar to the proof of Lemma 5.1, we have $u = 0$ on $\mathbb{R}^N \setminus \Omega$. Thus for each $\phi \in H(\Omega)$, as $n \to +\infty$, we have
\[
0 = J'_{\Omega}(u_n)\phi = \int_{\mathbb{R}^N} (\Delta u_n \Delta \phi + V_{\lambda_n} u_n \phi) dx - \int_{\mathbb{R}^N} u_n^{2^{**}} \phi dx
\]
\[
\to \int_{\Omega} (\Delta u \Delta \phi - \delta u^2 \phi) dx - \int_{\Omega} u^{2^{**}} \phi dx
\]
\[
= J'_{\Omega}(u)\phi.
\]

Thus $J'_{\Omega}(u) = 0$. Furthermore, we have
\[
J_{\Omega}(u) = J_{\Omega}(u) - \frac{1}{2} J'_{\Omega}(u) u = \left(\frac{1}{2} - \frac{1}{2^{**}}\right) \int_{\Omega} |u|^{2^{**}} dx \geq 0.
\]

Let $v_n = u_n - u$, by Brézis-Lieb’s Lemma, we have
\[
\int_{\mathbb{R}^N} |\Delta u_n|^2 dx = \int_{\Omega} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\Delta v_n|^2 dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |u_n|^{2^{**}} dx = \int_{\Omega} |u|^{2^{**}} dx + \int_{\mathbb{R}^N} |v_n|^{2^{**}} dx + o(1)
\]
and
\[
\int_{\mathbb{R}^N} V_{\lambda_n} u_n^2 dx = \int_{\mathbb{R}^N} V_{\lambda_n} u^2 dx + \int_{\mathbb{R}^N} V_{\lambda_n} v_n^2 dx + \int_{\mathbb{R}^N} 2V_{\lambda_n} uv_n dx
\]
\[
= - \int_{\Omega} \delta u^2 dx + \int_{\mathbb{R}^N} V_{\lambda_n} v_n^2 dx - 2 \int_{\Omega} \delta uv_n dx
\]
\[
= - \int_{\Omega} \delta u^2 dx + \int_{\mathbb{R}^N} V_{\lambda_n} v_n^2 dx + o(1).
\]
Thus we can easily get that

\[ J_{\lambda_n}(u_n) = J_{\Omega}(u) + J_{\lambda_n}(v_n) + o(1), \]

\[ J'_{\lambda_n}(u_n)u_n = J'_{\Omega}(u)u_n + J'_{\lambda_n}(v_n)v_n + o(1). \]

We may assume that

\[ b = \lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\Delta v_n|^2 + V_{\lambda_n} v_n^2)\,dx = \lim_{n \to +\infty} |v_n|^{2^*} \,dx > 0. \]

On the one hand, by Sobolev inequality, we have

\[ b = \lim_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} \,dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\Delta v_n|^2 + V_{\lambda_n} v_n^2)\,dx \geq \lim_{n \to +\infty} \int_{\mathbb{R}^N} |\Delta v_n|^2 \,dx \geq \lim_{n \to +\infty} S \left( \int_{\mathbb{R}^N} |v_n|^{2^*} \,dx \right)^{\frac{2}{2^*}}. \]

Thus \( b \geq S^*_N \). Recall that

\[ J_{\lambda_n}(v_n) = J_{\lambda_n}(v_n) - \frac{1}{2} J_{\lambda_n}(v_n)v_n + o(1) = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) \int_{\mathbb{R}^N} |v_n|^{2^*} \,dx + o(1), \]

then

\[ \frac{2}{N} S^*_N > c(\Omega) \geq \lim_{n \to +\infty} J_{\lambda_n}(v_n) = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) \lim_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} \,dx = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) b. \]

Thus \( b < S^*_N \) which leads to a contradiction. This implies that \( u_n \to u \) in \( L^{2^{**}}(\mathbb{R}^N) \). According to Lemma 2.1, we known that \( u_n \to u \) in \( H^2(\mathbb{R}^N) \). Furthermore,

\[ J_{\Omega}(u) = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) \int_{\Omega} |u|^{2^*} \,dx = \left( \frac{1}{2} - \frac{1}{2^{**}} \right) \lim_{n \to +\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \,dx \]

\[ = \lim_{n \to +\infty} J_{\lambda_n}(u_n) = k > 0, \]

which implies \( u \neq 0 \). Hence, \( u \in N_{\Omega} \) and

\[ c(\Omega) \leq J_{\Omega}(u) = k \leq c(\Omega) \]

which implies \( J_{\Omega}(u) = c(\Omega) \). \( \square \)
Finally, we complete our paper by proving our main result Theorem 1.4.

**Proof of Theorem 1.4:** The existence of least energy solutions to (1.1) is proved by Proposition 4.5 and Proposition 4.6 for $\lambda > \Lambda_2$. The asymptotic behavior of least energy solutions follows from Lemma 5.1 and Lemma 5.2 for $\lambda \to +\infty$. Thus we complete the proof of our main result Theorem 1.4.

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