LEAST ENERGY SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS INVOLVING THE HALF LAPLACIAN AND POTENTIAL WELLS

MIAOMIAO NIU
School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875 P. R. China

ZHONGWEI TANG
School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875 P. R. China

(Communicated by the associate editor name)

Abstract. In this paper, we are concerned with the existence of least energy solutions of nonlinear Schrödinger equations involving the half Laplacian

$$(-\Delta)^{1/2}u(x) + \lambda V(x)u(x) = u(x)^{p-1}, \quad x \in \mathbb{R}^N,$$

for sufficiently large $\lambda$, $2 < p < \frac{2N}{N-1}$ for $N \geq 2$. $V(x)$ is a real continuous function on $\mathbb{R}^N$. Using variational methods we prove the existence of least energy solution $u(x)$ which localize near the potential well $\text{int}(V^{-1}(0))$ for $\lambda$ large. Moreover, if the zero sets $\text{int}(V^{-1}(0))$ of $V(x)$ include more than one isolated components, then $u_\lambda(x)$ will be trapped around all the isolated components. However, in Laplacian case, when the parameter $\lambda$ large, the corresponding least energy solution will be trapped around only one isolated component and become arbitrary small in other components of $\text{int}(V^{-1}(0))$. This is the essential difference with the Laplacian problems since the operator $(-\Delta)^{1/2}$ is nonlocal.

1. Introduction and main results. We are concerned with the following nonlinear Schrödinger equations involving the half Laplacian

$$\begin{cases} (-\Delta)^{1/2}u(x) + \lambda V(x)u(x) = u(x)^{p-1}, & x \in \mathbb{R}^N, \\ u(x) \geq 0, & u(x) \in H^{1/2}(\mathbb{R}^N), \end{cases}$$

here $2 < p < 2^\ast := \frac{2N}{N-1}$ for $N \geq 2$, $V(x)$ is the potential, which is a real valued function on $\mathbb{R}^N$.

In recent years, much attention has been devoted to the study of the fractional Laplacian. The fractional powers of the Laplacian, which are called fractional Laplacian and correspond to Lévy stable processes, appear in anomalous diffusion phenomena in physics, biology as well as other areas. They occur in flame propagation, chemical reaction in liquids, population dynamics. Lévy diffusion processes have discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponential decaying tails. These processes have been applied to American options in mathematical finance for modeling the jump processes of the financial derivatives such as futures, forwards, options, and swaps, see [2] and references therein. Moreover,

2000 Mathematics Subject Classification. Primary: 35Q55; Secondary: 35J65.

Key words and phrases. Nonlinear Schrödinger equation; Least energy solution; Half Laplacian; Variational methods.

The second author was supported by National Science Foundation of China(11571040).
they play important roles in the study of the quasi-geostrophic equations in geophysical fluid dynamics.

There are many results which are concerned with the problems involving the fractional Laplacian. Firstly, we refer the readers to the work by Caffarelli and Silvestre [6], in which a new formulation of the fractional Laplacian through Dirichlet-Neumann maps was introduced. By this formulation, they transferred the nonlocal problem to a local problem defined in a higher half space. After their pioneering work, there are many investigations to the fractional Laplacian problem by using variational methods. For example, using variational methods, Cabré and Tan [5] established the existence of positive solutions for fractional problems in a bounded domain with power-type nonlinearities. We also refer the work by Juan Dávila, Manuel del Pino and Juncheng Wei[9], where they considered the following fractional problem

\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u - u^q = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N), \]

where \(0 < \varepsilon < 1\), \(1 < q < \frac{N+2s}{N-2s}\) and \(V(x)\) is a sufficiently smooth potential with \(\inf V(x) > 0\), \(\varepsilon > 0\) is a small parameter. Via a Lyapunov-Schmidt variational reduction, they proved the existence of multiple spike solutions which as \(\varepsilon\) small concentrate at separate places in the case of stable critical points and the existence of multiple spikes which as \(\varepsilon\) small concentrate at the same points.

For the following related fractional Schrödinger equation

\[ (-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N, \]

(1.3)

with \(0 < s < 1\) and \(V : \mathbb{R}^N \rightarrow \mathbb{R}\) is the potential function, there are also many investigations. We firstly refer the reader to the most recent paper by Frank, Lenzmann and Silvestre [17], where the authors obtained the uniqueness results to fractional Laplacian problem related to (1.3). For other results, see also Bona and Li [4], Cheng[7], de Bouard and Saut [10], Dipierro, Palatucci and Valdinoci[13], Felmer et al[14], Frank and Lenzmann [16], Maris [20] and references therein.

We also refer the readers to the paper by Jin, Li and Xiong [18], where the authors considered the following fractional Laplacian equations with lower order terms

\[ (-\Delta)^s u = au + b, \quad x \in B_1, \]

(1.4)

where \(a, b \in C_0^{\alpha}(B_1)\) with \(0 < \alpha \notin \mathbb{N}\) and \(2s + \alpha\) is not an integer. They proved some priori estimates results for the solutions of the above equation (1.4), such as the local Schauder estimates for nonnegative solutions. We also refer the work by Tan and Xiong [23], where they established a Harnack inequality in the case of \(u \in C^2(B_1) \cap C(B_1)\).

The analogue problem to (1.1) for the Laplacian, for instance, the following problem

\[ -h^2 \Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N. \]

(1.5)

has been investigated widely in the last decades. Much attention has been devoted to the study of the existence and uniqueness for one-bump or multi-bump bound states of (1.5). In [15], using a Lyapunov-Schmidt reduction, Floer and Weinstein established the existence of a standing wave solution of (1.5) when \(N = 1, p = 3\) and \(V(x)\) is a bounded function which has a non-degenerate critical point for sufficiently small \(h > 0\). Moreover they showed that \(u\) concentrates near the given non-degenerate critical point of \(V\) when \(h\) tends to 0. Their method and results were later generalized by Oh[21], [22] to the higher-dimensional case with \(2 < p < \frac{2N}{N-2}\) and the existence of multi-bump solutions concentrating near several non-degenerate critical points of \(V\) as \(h\) tends to 0 was obtained. We also refer to Ambrosetti, Badiale and Cingolani [1], Cingolani and Nolasco [8], Pino and Felmer [11], [12] for the Laplacian problems.

Related to the equation (1.5), the second author [24] considered the following equation with eletra-magnetic field and potential wells

\[ - (\nabla + iA(x))^2 u(x) + (\lambda a(x) + 1)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N. \]

(1.6)

Under some proper conditions on \(a(x)\) and \(A(x)\), he proved the existence of least energy solutions to problem (1.6) which localize near the potential well int\((a^{-1}(0))\) for \(\lambda\) large. Similar investigation to equation (1.6) but there is no eletra-magnetic field, one can refer the work by Bartsch and Wang[3].

Now we are ready to present our main assumptions, we assume that:

\( (V_1) \) \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) such that \( V(x) \geq 0, \) \( \Omega := \text{int} V^{-1}(0) \) is non-empty with smooth boundary and \( \Omega = V^{-1}(0); \)
There exists $M_0 > 0$ such that

$$\mu \left\{ x \in \mathbb{R}^N : V(x) \leq M_0 \right\} < \infty,$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Before stating our main results, we firstly give some notations and remarks.

To treat the nonlocal problem (1.1), we will study a corresponding extension problem in one more dimension, which allows us to investigate problem (1.1) by studying a local problem via classical nonlinear variational methods.

Let us denote the closure of the set of smooth functions compactly supported in $\mathbb{R}^{N+1}_+$, by $D^{1,2}(\mathbb{R}^{N+1}_+)$, with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^{N+1}_+)} := \left( \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 \, dx \, dy \right)^{1/2}.$$

And we also introduce the fractional Sobolev space $H^{1/2}(\mathbb{R}^N)$ which is a Banach space with the norm

$$\|u\|_{H^{1/2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+1}} \, dx \, dy + \int_{\mathbb{R}^N} |u(x)|^2 \, dx \right)^{1/2}.$$

Let

$$E := \left\{ v \in D^{1,2}(\mathbb{R}^{N+1}_+) : v(-,0) \in L^2(\mathbb{R}^N) \right\}.$$

Then $E$ is the Hilbert space under the inner product

$$(u, v) = \int_{\mathbb{R}^{N+1}_+} \nabla u \nabla v \, dx \, dy + \int_{\mathbb{R}^N} u(x, 0)v(x, 0) \, dx,$$

and the norm induced by the inner product $(\cdot, \cdot)$ is

$$\|v\|_E = \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy + \int_{\mathbb{R}^N} v(x, 0)^2 \, dx \right)^{1/2}.$$

Indeed, for every $v(x, y) \in E$, we denote $v(x, 0)$ be the trace of $v(x, y)$ on $\mathbb{R}^N$ and take

$$tr_{\mathbb{R}^N} E := \{ v(x, 0) : v(x, y) \in E \}.$$

Then by the definition of $E$, we have

$$tr_{\mathbb{R}^N} E = H^{1/2}(\mathbb{R}^N). \quad (1.7)$$

We take

$$E_\lambda := \left\{ v \in D^{1,2}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} \lambda V(x)v(x, 0)^2 \, dx < \infty \right\}.$$

Then $E_\lambda$ is the Hilbert space under the inner product

$$(u, v)_\lambda = \int_{\mathbb{R}^{N+1}_+} \nabla u \nabla v \, dx \, dy + \int_{\mathbb{R}^N} \lambda V(x)u(x, 0)v(x, 0) \, dx,$$

and the norm induced by the inner product $(\cdot, \cdot)_\lambda$ is

$$\|v\|_\lambda = \left( \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy + \int_{\mathbb{R}^N} \lambda V(x)v(x, 0)^2 \, dx \right)^{1/2}.$$

We mention that the half Laplacian in the whole space is a well studied operator. Let $u \in C^\infty_c(\mathbb{R}^N)$ be a smooth function. Then there is a unique harmonic extension $v \in C^\infty(\mathbb{R}^{N+1}_+)$ of $u$ in a half space such that $D^k v(x, y) \to 0$ as $|x, y| \to \infty$, for all $k \geq 0$ and $v(x, 0) = u(x)$. It is the solution of the following Laplacian problem

$$\begin{align*}
\Delta v &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
v &= u \quad \text{on } \mathbb{R}^N \times \{0\} = \partial \mathbb{R}^{N+1}_+.
\end{align*}$$

Consider the operator $T : u \mapsto -\partial_y v(-, 0)$. Since $\partial_y v$ is still a harmonic function, if we apply the operator twice, we obtain

$$T \circ T u = -\partial_y v|_{y=0} = -\Delta_x v|_{y=0} = -\Delta u \text{ in } \mathbb{R}^N.$$
Thus the operator $T$ that maps the Dirichlet-type data $u$ to the Neumann-type data $-\partial_y v(\cdot, 0)$ is actually the half Laplacian. In this way we can study problem (1.1) by variational methods for a local problem. More precisely, we will study the following boundary value problem in a half space:

$$
\begin{aligned}
& -\Delta v = 0 \quad \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
& \frac{\partial v}{\partial \nu} = v^{p-1} - \lambda V(x)v \quad \text{on } \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\}, \\
& v \geq 0 \quad \text{in } \mathbb{R}^N = \mathbb{R}^N \times (0, \infty),
\end{aligned}
$$

where $\nu$ is the unit outer normal to $\mathbb{R}^N \times \{0\}$. If $v$ satisfies (1.8), then the trace $u$ on $\mathbb{R}^N \times \{0\}$ of the function $v$ will be a solution of problem (1.1). By studying (1.8), we establish the results for (1.1).

The energy functional associated with (1.8) is defined by

$$
J_\lambda(v) := \frac{1}{2} \int_{\mathbb{R}^{N+1}} |\nabla v|^2 \, dx \, dy + \int_{\mathbb{R}^N} V(x)v(x, 0)^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} v^+(x, 0)^p \, dx \quad \text{for } v \in E_\lambda,
$$

where $v^+$ denotes the positive part of $v$ for every function $v$, in other words, $v^+ = \max\{v, 0\}$.

We define the Nehari manifold

$$
\mathcal{M}_\lambda := \left\{ v \in E_\lambda \setminus \{0\} : \int_{\mathbb{R}^{N+1}} |\nabla v|^2 \, dx \, dy + \lambda \int_{\mathbb{R}^N} V(x)|v(x, 0)|^2 \, dx = \int_{\mathbb{R}^N} v^+(x, 0)^p \, dx \right\}
$$

and let

$$
c_\lambda := \inf \{ J_\lambda(v) : v \in \mathcal{M}_\lambda \}
$$

be the infimum of $J_\lambda$ on the Nehari manifold $\mathcal{M}_\lambda$.

For $\lambda$ large, the following problem

$$
\begin{aligned}
(-\Delta)^{1/2} u(x) &= u(x)^{p-1}, \quad x \in \Omega, \\
u(x) &\geq 0, \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\end{aligned}
$$

(1.9)

is some kind of limit problem of (1.1). We shall prove that there exists a least energy solution of (1.1) converging for $\lambda \to \infty$ to a least energy solution of (1.9).

Similarly, to consider the problem (1.9), we will study the following mixed boundary value problem in a half space:

$$
\begin{aligned}
& -\Delta v = 0 \quad \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
& v = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
& \frac{\partial v}{\partial \nu} = v^{p-1} \quad \text{on } \Omega \times \{0\}, \\
& v \geq 0 \quad \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty),
\end{aligned}
$$

(1.10)

where $\nu$ is the unit outer normal to $\Omega \times \{0\}$. If $v$ satisfies (1.10), then the trace $u$ on $\Omega \times \{0\}$ of the function $v$ will be a solution of (1.9).

To consider problem (1.10), we define a subspace $E_0$ of $E$ as follows

$$
E_0 := \left\{ v(x, y) \in E : v(x, 0) = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}
$$

(1.11)

Similarly, by the definition of $E_0$, we also have $\text{tr}_\Omega E_0 = H^{1/2}(\Omega)$.

The energy functional associated with (1.10) is defined by

$$
\Phi(v) := \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy - \frac{1}{p} \int_{\Omega} v^+(x, 0)^p \, dx \quad \text{for } v \in E_0.
$$

Comparing with the Nehari manifold $\mathcal{M}_\lambda$, we define the Nehari manifold

$$
\mathcal{N} := \left\{ v \in E_0 \setminus \{0\} : \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy = \int_{\Omega} v^+(x, 0)^p \, dx \right\}
$$

and let

$$
c(\Omega) := \inf \{ \Phi(v) : v \in \mathcal{N} \}
$$

be the infimum of $\Phi$ on the Nehari manifold $\mathcal{N}$.

**Definition 1.1.** We say that a function $u_\lambda(x) = v_\lambda(x, 0)$ is a least energy solution of (1.1) if $c_\lambda$ is achieved by $v_\lambda \in \mathcal{M}_\lambda$ which is a critical point of $J_\lambda$. Similarly we say that a function $u(x) = v(x, 0)$ is a least energy solution of (1.9) if $c(\Omega)$ is achieved by $v \in \mathcal{N}$ which is a critical point of $\Phi$. 

Now we give our main results and which are:

**Theorem 1.2.** Suppose \((V_1)\) and \((V_2)\) hold. Then for \(\lambda\) large, \((1.1)\) has a least energy solution \(u_\lambda(x) = \psi_\lambda(x,0)\). Furthermore, any sequence \(\lambda_n\) (\(\lambda_n \to \infty\) as \(n \to \infty\)), \(\{u_{\lambda_n}(x)\}\) has a subsequence such that \(u_{\lambda_n}\) converges in \(H^{1/2}(\mathbb{R}^N)\) along the subsequence to a least energy solution \(u\) of \((1.9)\).

As in the case of the least energy solution of \((1.1)\), any solutions of \((1.1)\) converges for \(\lambda \to \infty\) towards solutions of \((1.9)\). More precisely, we have the following result.

**Theorem 1.3.** Suppose \((V_1)\) and \((V_2)\) hold. Let \(u_n = v_n(\cdot,0), n \in \mathbb{N}\) be a sequence of solutions of \((1.1)\) with \(\lambda\) being replaced by \(\lambda_n\) (\(\lambda_n \to \infty\) as \(n \to \infty\)) such that \(\limsup_{n \to \infty} J_{\lambda_n}(v_n) < \infty\). Then \(u_n(x) = v_n(x,0)\) converges strongly along a subsequence in \(H^{1/2}(\mathbb{R}^N)\) to a solution \(u\) of \((1.9)\).

Our paper is organized as follows: In section 2, we give a compactness result, Section 3 is devoted to the “limit” problem and Section 4 contains the proofs of the main results.

We will use the same \(C\) to denote various generic positive constants, and we will use \(o(1)\) to denote quantities that tend to 0 as \(\lambda\) (or \(n\)) \(\to \infty\).

2. **Compactness result.** The main result in this section is the following compactness result. To begin with, we firstly give the definition of \((PS)_c\) condition and \((PS)_{c,\varepsilon}\) sequence.

**Definition 2.1.** Let \(X\) be a Banach space, \(\varphi \in C^1(X,\mathbb{R})\) and \(c \in \mathbb{R}\). The functional \(\varphi\) satisfies the \((PS)_c\) condition if any sequence \(\{u_n\} \subseteq X\) such that

\[
\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0,
\]

has a convergent subsequence. We call a sequence \(\{u_n\} \subseteq X\) is a \((PS)_c\) sequence of a functional \(\varphi\) if \((2.1)\) is satisfied.

Now we give the following compactness result.

**Proposition 1.** Suppose \((V_1)\) and \((V_2)\) hold. Then for any \(C_0 > 0\), there exists \(\Lambda_0 > 0\) such that \(J_\lambda\) satisfies the \((PS)_{c,\varepsilon}\)-condition for all \(\lambda \geq \Lambda_0\) and \(c \leq C_0\).

The proof consists of a series of lemmas which occupy the rest of this section.

**Lemma 2.2.** Let \(\lambda_0 > 0\) be any fixed constant. Then for \(\lambda \geq \lambda_0 > 0\), \(V(x)\) satisfies \((V_1)\) and \((V_2)\), \(E_\lambda\) is continuously embeded in \(E\).

**Proof.** From the definition of \(E\) and \(E_\lambda\), to show the lemma, we only need to prove the following estimate

\[
\int_{\mathbb{R}^N} v(x,0)^2 dx \leq C \left( \int_{\mathbb{R}^{N+1}} |\nabla v|^2 dx dy + \lambda \int_{\mathbb{R}^N} V(x)v(x,0)^2 dx \right).
\]

Let us denote

\[
D := \left\{ x \in \mathbb{R}^N : V(x) \leq M_0 \right\},
\]

and

\[
D^{\delta_0} := \left\{ x \in \mathbb{R}^N : \text{dist}(x, D) \leq \delta_0 \right\}.
\]

Take \(\zeta \in C^\infty(\mathbb{R}^N), 0 \leq \zeta \leq 1\) and for the above fixed small \(\delta_0\),

\[
\zeta(x) = \begin{cases} 
1 & x \in D, \\
0 & x \notin D^{\delta_0}, \\
|\nabla \zeta| \leq C/\delta_0.
\end{cases}
\]

Thus for any function \(v \in E_\lambda\), we obtain that

\[
\int_{\mathbb{R}^N} (1 - \zeta^2)v(x,0)^2 dx = \int_{\mathbb{R}^N \setminus D} (1 - \zeta^2)v(x,0)^2 dx + \int_D (1 - \zeta^2)v(x,0)^2 dx
\]

\[
\leq \frac{1}{\lambda_0 M_0} \lambda \int_{\mathbb{R}^N} V(x)v(x,0)^2 dx,
\]

\[ (2.3) \]
and

\[
\int_{\mathbb{R}^N} \zeta^2 v(x,0)^2 \, dx = \int_{D^N} \zeta^2 v(x,0)^2 \, dx \\
\leq \mu(D^N)^{1-\frac{2}{p}} \left( \int_{D^N} |v(x,0)|^2 \, dx \right)^{\frac{2}{p}} \\
\leq \mu(D^N)^{1-\frac{2}{p}} \left( \int_{\mathbb{R}^N} |v(x,0)|^2 \, dx \right)^{\frac{2}{p}} \\
\leq C \int_{\mathbb{R}^N} |\nabla v|^2 \, dx dy,
\]

(2.5)

where we have used the assumption \((V_2)\) and the well known Sobolev inequality. The inequality states for \(v \in D^{1,2}(\mathbb{R}^{N+1})\),

\[
\left( \int_{\mathbb{R}^N} |v(x,0)|^{2^*} \, dx \right)^{\frac{1}{2^*}} \leq C_0 \left( \int_{\mathbb{R}^{N+1}} |\nabla v|^2 \, dx dy \right)^{\frac{1}{2}},
\]

where \(C_0\) depends only on \(N\). Therefore, we indeed have proved (2.2) by adding up to the two inequalities (2.4) and (2.5) together. Thus the proof of the lemma is completed.

The following Lemma shows that 0 is an isolated critical point of \(J_\lambda\).

**Lemma 2.3.** Let \(K_\lambda\) denote the set of critical points of \(J_\lambda\), \(\lambda \geq \lambda_0 > 0\). Then there exists \(\sigma > 0\) independent of \(\lambda\) such that \(|v|\|_\lambda \geq \sigma\) for all \(v \in K_\lambda \setminus \{0\}\).

**Proof.** By lemma 2.2, for any \(v \in K_\lambda \setminus \{0\}\),

\[
0 = J'_\lambda(v) \cdot v = \int_{\mathbb{R}^{N+1}} |\nabla v|^2 \, dx dy + \lambda \int_{\mathbb{R}^N} V(x) |v(x,0)|^2 \, dx - \int_{\mathbb{R}^N} v^+(x,0)p \, dx \\
\geq |v|_\lambda^2 - C_0 |v|_E^p \\
\geq |v|_\lambda^2 - C_0 |v|_\lambda^p,
\]

where \(C_0 > 0\) is independent of \(\lambda \geq 0\). Thus we see that there exists \(\sigma > 0\) such that \(|v|\|_\lambda \geq \sigma\).

**Lemma 2.4.** There exists \(c_0 > 0\) independent of \(\lambda \geq \lambda_0 > 0\) such that if \(\{v_n\}\) is a \((PS)_c\)-sequence of \(J_\lambda\). Then

\[
\lim_{n \to \infty} \|v_n\|_{\lambda}^2 \leq \frac{2p}{p-2}
\]

and either \(c \geq c_0\) or \(c = 0\).

**Proof.** First we prove that any \((PS)_c\)-sequence must be bounded. In fact, for \(n\) large enough

\[
c = \lim_{n \to \infty} \left( J_\lambda(v_n) - \frac{1}{p} J'_\lambda(v_n) \right) \\
\geq \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^{N+1}} |\nabla v_n|^2 \, dx dy + \left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\mathbb{R}^N} V(x) v_n(x,0)^2 \, dx \\
\geq \frac{p-2}{2p} \lim_{n \to \infty} \|v_n\|_{\lambda}^2,
\]

which proves (2.6).

On the other hand, there is a constant \(C > 0\) independent of \(\lambda \geq \lambda_0 \geq 0\) such that

\[
J'_\lambda(v) \cdot v = \int_{\mathbb{R}^{N+1}} |\nabla v|^2 \, dx dy + \lambda \int_{\mathbb{R}^N} V(x) |v(x,0)|^2 \, dx - \int_{\mathbb{R}^N} v^+(x,0)p \, dx \\
\geq |v|_\lambda^2 - C_0 |v|_E^p.
\]

Thus there exists \(\sigma_1 > 0\) independent of \(\lambda\) such that

\[
\frac{1}{4} |v|_\lambda^2 \leq J'_\lambda(v) \cdot v \text{ for } |v|_\lambda < \sigma_1.
\]

(2.7)

Now, if \(c < \frac{\sigma_1^2(p-2)}{2p}\) and \(\{v_n\}\) is a \((PS)_c\)-sequence of \(J_\lambda\), then

\[
\limsup_{n \to \infty} \|v_n\|_{\lambda}^2 \leq \frac{2cp}{p-2} < \sigma_1^2.
\]
Hence, \( \|v_n\|_\lambda < \sigma_1 \) for \( n \) large, then by (2.7)
\[
\frac{1}{2} \|v_n\|_\lambda^2 \leq J'_\lambda(v_n) \cdot v_n = o(1) \|v_n\|_\lambda,
\]
which implies \( \|v_n\|_\lambda \rightarrow 0 \) as \( n \rightarrow \infty \). Therefore \( J_\lambda(v_n) \rightarrow 0 \), that is \( c = 0 \). Thus \( c_0 = \frac{\sigma^2(p - 2)}{2p} \) is as required.

**Lemma 2.5.** There exists \( \delta_0 > 0 \) such that any \((PS)_c\)-sequence \( \{v_n\} \) of \( J_\lambda \) with \( \lambda \geq 0 \) and \( c > 0 \) satisfies
\[
\liminf_{n \rightarrow \infty} \|v_n^+(\cdot, 0)\|_{L^p(\mathbb{R}^N)}^p \geq \delta_0 c. \tag{2.8}
\]

**Proof.** From the proof of Lemma 2.4 we know that \( \{v_n\} \) is bounded and hence
\[
c = \lim_{n \rightarrow \infty} \left( J_\lambda(v_n) - \frac{1}{2} J'_\lambda(v_n) \cdot v_n \right)
= \left( \frac{1}{2} - \frac{1}{p} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} v_n^+(x, 0)^p dx
= \frac{(p - 2)}{2p} \lim_{n \rightarrow \infty} \|v_n^+(\cdot, 0)\|_{L^p(\mathbb{R}^N)}^p,
\]
which implies (2.8) with \( \delta_0 = \frac{2p}{p - 2} \).

**Lemma 2.6.** Let \( C_1 \) be fixed. Then for any \( \varepsilon > 0 \) there exists \( \lambda_\varepsilon > 0 \) and \( R_\varepsilon > 0 \) such that if \( \{v_n\} \) is a \((PS)_c\)-sequence of \( J_\lambda \) with \( \lambda \geq \lambda_\varepsilon, c \leq C_1 \), then
\[
\limsup_{n \rightarrow \infty} \int_{B^c_{R_\varepsilon}} v_n^+(x, 0)^p dx \leq \varepsilon, \tag{2.9}
\]
where \( B^c_{R_\varepsilon} = \{ x \in \mathbb{R}^N : |x| \geq R_\varepsilon \} \).

**Proof.** For \( R > 0 \), we set
\[
A(R) := \{ x \in \mathbb{R}^N : |x| > R, V(x) \geq M_0 \}
\]
and
\[
B(R) := \{ x \in \mathbb{R}^N : |x| > R, V(x) < M_0 \}.
\]
Then by Lemma 2.4, we can obtain that
\[
\int_{A(R)} v_n(x, 0)^2 dx \leq \frac{1}{\lambda M_0} \int_{\mathbb{R}^N}\left( x \right) v_n(x, 0)^2 dx
\leq \frac{1}{\lambda M_0} \left( \int_{\mathbb{R}^N+1} |v_n(x)|^2 dx + \int_{\mathbb{R}^N} x V(x) v_n(x, 0)^2 dx \right)
\leq \frac{1}{\lambda M_0} \left( \frac{2p}{p - 2} C_1 + o(1) \right). \tag{2.10}
\]
Using the H"older inequality and (2.6), we obtain that for \( 1 < q < N/(N - 1) \),
\[
\int_{B(R)} v_n(x, 0)^2 dx \leq \left( \int_{\mathbb{R}^N} v_n(x, 0)^{2q} dx \right)^{1/q} \mu(B(R))^{1/q'}
\leq C \|v_n\|_{\lambda}^2 \cdot \mu(B(R))^{1/q'}
\leq C \frac{2p}{p - 2} C_1 \cdot \mu(B(R))^{1/q'}, \tag{2.11}
\]
where \( C = C(N, q) \) is a positive constant and \( q' \) is such that \( 1/q + 1/q' = 1 \). \( \mu(B(R)) \) denotes the Lebesgue measure of \( B(R) \). Setting \( \theta := \frac{N(p - 2)}{p} \), the interpolation inequality and the Sobolev
trace inequality yield
\[
\int_{B_R} v_n^p(x,0) dx \\
\leq \left( \int_{B_R} v_n(x,0)^2 dx \right)^{\frac{p(1-\theta)}{2}} \left( \int_{B_R} |v_n(x,0)|^{2^*} dx \right)^{\frac{p}{2^*}} \\
\leq C \left( \int_{B_R} v_n(x,0)^2 dx \right)^{\frac{p(1-\theta)}{2}} \left( \int_{\mathbb{R}^N} |v_n(x,0)|^{2^*} dx \right)^{\frac{p}{2^*}} \\
\leq C \left( \int_{A(R)} v_n(x,0)^2 dx + \int_{B(R)} v_n(x,0)^2 dx \right)^{\frac{p(1-\theta)}{2}} \|v_n\|_{p^\ast}^{p}.
\]

From (2.10), the first summand on the right can be made arbitrarily small if λ large. On the other hand, from (2.11), the second summand on the right will be arbitrarily small if R large since μ(B(R)) → 0 as R → ∞ by assumption (V2). This completes the proof.

The following lemma is well known and we only give the result without proof.

**Lemma 2.7.** (Brézis-Lieb Lemma, 1983) Let \{u_n\} ⊂ L^p(\mathbb{R}^N), 1 ≤ p < ∞. If
a) \{u_n\} is bounded in L^p(\mathbb{R}^N),
b) u_n → u almost everywhere on \mathbb{R}^N, then
\[
\lim_{n \to \infty} (|u_n|^p - |u|^p) = |u|^p.
\]  
(2.12)

**Lemma 2.8.** If 2 ≤ p < 2\ast and v_n → v in E_\lambda, v_n^\lambda := v_n - v then
\[
\lim_{n \to \infty} \sup_{\|\lambda\|_{\lambda} \leq 1} \int_{\mathbb{R}^N} \left( \frac{\nu_n^p + v_n(x,0)^{p-1} + v_n^p + v(x,0)^{p-1} - v_n^p - v(x,0)^{p-1}}{2} \right) dx = 0, \ v \in E_\lambda.
\]  
(2.13)

**Proof.** Let us define \( f(v) := v^p - v(x,0)^{p-1} \). By the mean value theorem, we have, almost everywhere on \mathbb{R}^N,
\[
|f(v_n) - f(v_n^\lambda)| \leq (p - 1) \left( |v_n(x,0)| + |v(x,0)| \right)^{p-2} |v(x,0)|.
\]

For R > 0 and w ∈ E_\lambda, we obtain from the Hölder’s inequality, we have
\[
\int_{|x| > R} |f(v_n) - f(v_n^\lambda)| w(x,0) dx \\
\leq c_1 \left( \int_{|x| > R} |v_n(x,0)|^{p-2} + \|v(x,0)|^{p-2} \right) \|w\|_p \cdot \left( \int_{|x| > R} |v(x,0)|^{p} dx \right)^{\frac{1}{p}} \\
\leq c_2 \||\lambda\|_\lambda \left( \int_{|x| > R} |v(x,0)|^{p} dx \right)^{\frac{1}{p}}.
\]

We also have that
\[
\int_{|x| > R} f(v)w(x,0) dx \leq \|w\|_p \left( \int_{|x| > R} |v(x,0)|^{p} dx \right)^{\frac{p-1}{p}}.
\]
Thus, for every ε > 0, there exists R > 0 such that, for every w ∈ E_\lambda,
\[
\int_{|x| > R} (f(v_n) - f(v_n^\lambda) - f(v)) w(x,0) dx \leq \varepsilon \||\lambda\|_\lambda.
\]  
(2.14)

It follows from the Rellich imbedding theorem, up to a subsequence,
v_n(\cdot,0) → v(\cdot,0) in L^p(\mathbb{R}^N) for any 2 ≤ p < 2\ast,
v_n,0(\cdot,0) → v(\cdot,0) in L^p(B_R) for any 2 ≤ p < 2\ast,
v_n(\cdot,0) → v(\cdot,0) almost everywhere on \mathbb{R}^N.

Then there exists a subsequence of v_n (we still denote v_n) and g ∈ L^p(B_R) such that
\[
|v_n(x,0)|, |v(x,0)| \leq g(x).
\]
It is easy to obtain
\[ |f(v_n) - f(v_n^1) - f(v)| \leq C|g|^p. \]
It follows from Lebesgue dominated convergence theorem that
\[ \lim_{n \to \infty} \int_{|x| \leq R} \left| f(v_n) - f(v_n^1) - f(v) \right| dx = 0. \tag{2.15} \]

Since
\[ \left| \int_{|x| \leq R} (f(v_n) - f(v_n^1) - f(v)) w(x, 0) dx \right| \leq \|w\|_p \|f(v_n) - f(v_n^1) - f(v)\|_{L^\infty(B_R)}. \tag{2.16} \]

Thus combining (2.14), (2.15) and (2.16) we obtain (2.13) and the proof of the lemma is finished.

**Lemma 2.9.** Let \( \lambda \geq \lambda_0 > 0 \) be fixed and let \( \{v_n\} \) be a \( (PS)_c \)-sequence of \( J_\lambda \). Then up to a subsequence, \( v_n \rightharpoonup v \) in \( E_\lambda \) with \( v \) being a weak solution of (1.8). Moreover, \( v_n^1 = v_n - v \) is \( (PS)_{c'} \)-sequence with \( c' = c - J_\lambda(v) \).

**Proof.** Firstly, by Lemma 2.4 we know that \( \{v_n\} \) is bounded in \( E_\lambda \) and hence \( \{v_n\} \) is bounded in \( E \). Then, up to a subsequence, \( v_n \rightharpoonup v \) in \( E \) as \( n \to \infty \). We recall (1.7) and obtain
\[ v_n(\cdot, 0) \rightharpoonup v(\cdot, 0) \text{ in } H^{1/2}(\mathbb{R}^N), \tag{2.17} \]
\[ v_n(\cdot, 0) \rightharpoonup v(\cdot, 0) \text{ in } L^p(\mathbb{R}^N) \text{ for any } 2 \leq p < 2^\star, \tag{2.18} \]
\[ v_n(\cdot, 0) \to v(\cdot, 0) \text{ in } L_{loc}^p(\mathbb{R}^N) \text{ for any } 2 \leq p < 2^\star, \tag{2.19} \]
\[ v_n(\cdot, 0) \to v(\cdot, 0) \text{ almost everywhere on } \mathbb{R}^N, \tag{2.20} \]
where \( 2^\star = \frac{2N}{N-1} \) is the critical Sobolev exponent. Thus for any \( w \in E_\lambda \) we have
\[ J_\lambda'(v_n) \cdot w = \int_{\mathbb{R}^{N+1}} \nabla v_n \nabla w dx + \lambda \int_{\mathbb{R}^N} V(x)v_n(x)w(x, 0) dx - \frac{1}{p} \int_{\mathbb{R}^N} v_n^p(x)w(x, 0) dx \]
\[ \to \int_{\mathbb{R}^{N+1}} \nabla v \nabla w dx + \lambda \int_{\mathbb{R}^N} V(x)v(x)w(x, 0) dx - \frac{1}{p} \int_{\mathbb{R}^N} v^p(x)w(x, 0) dx \]
\[ = J_\lambda'(v) \cdot w. \]
Therefore,
\[ \langle J_\lambda'(v), w \rangle = \lim_{n \to \infty} \langle J_\lambda'(v_n), w \rangle = 0, \tag{2.21} \]
which indicates that \( v \) is a critical point of \( J_\lambda \).

Let \( v_n^1 = v_n - v \), we will prove that as \( n \to \infty \),
\[ J_\lambda(v_n^1) \to c - J_\lambda(v) \tag{2.22} \]
and
\[ J_\lambda'(v_n^1) \to 0. \tag{2.23} \]
To show (2.22), we observe that
\[ J_\lambda(v_n^1) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} |\nabla v_n| dx + \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x)v_n^1(x, 0)^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} v_n^1(x, 0)^p dx \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|\nabla v_n|^2 + |v_n|^2 - 2\nabla v_n \nabla v) dx + \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x)(v_n(x, 0)^2 + v(x, 0)^2) \]
\[ - 2v_n(x, 0)v(x, 0) dx - \frac{1}{p} \int_{\mathbb{R}^N} v_n^1(x, 0)^p dx \]
\[ = \int_{\mathbb{R}^N} v_n(x) - J_\lambda(v) - (v_n^1, v_\lambda) - \frac{1}{p} \int_{\mathbb{R}^N} v_n^1(x, 0)^p dx + \frac{1}{p} \int_{\mathbb{R}^N} v_n^1(x, 0)^p dx \]
\[ - \frac{1}{p} \int_{\mathbb{R}^N} v_n^1(x, 0)^p dx. \tag{2.24} \]
From Lemma 2.7, \( \int_{\mathbb{R}^N} v_n^+ (x,0)^p \, dx - \int_{\mathbb{R}^N} v_0^+ (x,0)^p \, dx - \int_{\mathbb{R}^N} v_{n+1}^+ (x,0)^p \, dx \to 0 \) as \( n \to \infty \). On the other hand, we know that \( (v_n^+, v)_\lambda \to 0 \), as \( n \to \infty \). Thus from (2.24) we indeed have obtained (2.22). Now we come to show (2.23). From (2.21) we have for any \( w \in E_\lambda \)

\[
\langle J'_\lambda(v_n), w \rangle = \langle J'_\lambda(v_0), w \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v_{n+1}^+) \partial_v v_n^+ \, dx \, dw = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v_0^+) \partial_v v_n^+ \, dx \, dw - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v_0^+) \partial_v v_0^+ \, dx \, dw.
\]

By Lemma 2.8 and the fact \( J'_\lambda(v_0) \to 0 \), we have

\[
\lim_{n \to \infty} \sup_{\|w\|_{\lambda} \leq 1} |\langle J'_\lambda(v_n), w \rangle| = 0,
\]

which implies (2.23) and this completes the proof.

Now we come to prove Proposition 1.

**Proof of Proposition 1.** We choose \( 0 < \varepsilon < \delta_0 c_0 / 2 \), where \( c_0 > 0 \) and \( \delta_0 > 0 \) are same constants defined in Lemma 2.4 and Lemma 2.5 respectively. Then for the given constant \( C_0 > 0 \), we choose \( \Lambda \) and \( R_C > 0 \) as in Lemma 2.6. Thus we claim that \( \Lambda_0 := \Lambda_C \) is the constant as required in Proposition 1.

Take \( \{v_n\} \) be a \((PS)_c\)-sequence of \( J_\lambda \) with \( \lambda \geq \Lambda_0 \) and \( c \leq C_0 \). As in Lemma 2.9, we may assume that \( v_n \rightharpoonup v \) in \( E_\lambda \) and \( v_n^+ = v \) is a \((PS)_c\)-sequence of \( J_\lambda \) with \( c = \lambda J_\lambda(v) \). If \( c' > 0 \) then \( c' \geq c_0 \) by Lemma 2.4. As a consequence of Lemma 2.5,

\[
\liminf_{n \to \infty} \|v_n^+ (\cdot, 0)\|^p_{L^p(\mathbb{R}^N)} \geq \delta_0 c' \geq \delta_0 c_0.
\]

On the other hand, Lemma 2.6 implies

\[
\limsup_{n \to \infty} \int_{B(x_0)} v_n^+ (x,0)^p \leq \varepsilon < \delta_0 c_0 / 2.
\]

This implies \( v_n^+ \rightharpoonup v^+ \) in \( E_\lambda \) with \( v^+ \neq 0 \), this is a contradiction. Therefore \( c' = 0 \) hence, \( v_n^+ \to 0 \) in \( E_\lambda \) by Lemma 2.4. This completes the proof of Proposition 1.

Recalling the definition of \( c_\lambda \) in Section 1 and applying Proposition 1 to the functional \( J_\lambda \), we obtain the following corollary.

**Corollary 1.** For any \( p \in (2,2^*) \), there exists \( \Lambda_0 > 0 \) such that \( c_\lambda \) is achieved for all \( \lambda \geq \Lambda_0 \) at some \( v_\lambda \in E_\lambda \) which is a solution of (1.8).

3. Limit problem. Let us recall that the following problem is the “limit” problem of (1.8)

\[
\begin{cases}
-\Delta v = 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
\frac{\partial v}{\partial n} = v^{p-1} & \text{on } \Omega \times \{0\}, \\
v \geq 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty),
\end{cases}
\]

and the corresponding functional of (3.1) is defined by

\[
\Phi(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \left| \nabla v \right|^2 \, dx \, dy - \frac{1}{p} \int_{\Omega} v^+(x,0)^p \, dx
\]

for \( v \in E_0 \), where \( E_0 \) is defined as in Section 1. Again as defined in Section 1, the following energy

\[
c(\Omega) := \inf \{ \Phi(v) : v \in \mathcal{N} \}
\]

is the infimum of \( \Phi \) on the Nehari manifold \( \mathcal{N} \). We will see that \( c(\Omega) \) is achieved by a least energy \( v \in \mathcal{N} \). To show that, we firstly give an imbedding lemma which is standard.

**Lemma 3.1.** Let \( 2 < p < \frac{2N}{N-2} \) for \( N \geq 2 \). Then \( \text{tr}_{\Omega} E_0 \) is compactly embedded in \( L^p(\Omega) \).

**Proof.** Note that \( \text{tr}_{\Omega} E_0 \subset H^{1/2}(\Omega) \) and the fact that the embedding \( H^{1/2}(\Omega) \hookrightarrow L^p(\Omega) \) is compact for \( 2 < p < \frac{2N}{N-2} \) for \( N \geq 2 \) immediately implies the Lemma 3.1.

By a standard argument applying the the above compactness result Lemma 3.1, we can obtain the following existence lemma.

**Lemma 3.2.** The infimum \( c(\Omega) \) is achieved by a function \( v \in \mathcal{N} \) which is a least energy solution of (3.1).
Remark 2. By Remark 1 combining Hopf’s Lemma, one can easily check that the least energy solution of (3.1) satisfies $v \in \text{im} \Omega = \Omega$ immediately indicates that $\Delta u$ is an essential difference from the local operator Laplacian since in Laplacian case, $-\Delta u = 0$ in $\Omega$ for any domain $\Omega$.

Remark 1. When the zero set $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Then we have $v(x, 0) \geq 0$ both in $\Omega_1$ and in $\Omega_2$. Indeed, suppose on the contrary that $v \in \mathcal{N}$ is the least energy solution of (3.1) with $v(x, 0) = 0$ in $\Omega_1$ and $v(x, 0) \geq 0$ in $\Omega_2$. Then on one hand, $(-\Delta)^{1/2} v(x, 0) = \int_{\mathbb{R}^N} \frac{v(x, 0) - v(y, 0)}{|x - y|^{N+1}} dy < 0$ in $\Omega_1$.

On the other hand, $(-\Delta)^{1/2} v(x, 0) = v(x, 0)^{p-1} = 0$ for $x \in \Omega_1$. This contradiction shows that the least energy solution of (3.1) satisfies $v(x, 0) \geq 0$ both in $\Omega_1$ and in $\Omega_2$. The phenomenon is an essential difference from the local operator Laplacian since in Laplacian case, $u = 0$ in $\Omega$ immediately indicates that $\Delta u = 0$ in $\Omega$ for any domain $\Omega$.

Remark 2. By Remark 1 combining Hopf’s Lemma, One can easily check that $u(x) = v(x, 0) > 0$ for all $x \in \Omega$.

4. Proofs of main results. In this section we will give the proofs of our main results. To begin with, we firstly give an asymptotic behavior for $c_\lambda$ as $\lambda$ large. More precisely, we have the following lemma:

Lemma 4.1. $c_\lambda \to c(\Omega)$ as $\lambda \to \infty$.

Proof. It is easy to see that $c_\lambda \leq c(\Omega)$ for all $\lambda \geq 0$. It is not difficult to check that $c_\lambda$ is monotone increasing with respect to $\lambda$ large according to the definition of $c_\lambda$:

$$c_\lambda := \inf_{\lambda \in \mathcal{M}_\lambda} \left( \frac{1}{2} - \frac{1}{p} \right) \left( \int_{\mathbb{R}^{N+1}} |\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} V(x) v(x, 0)^2 dx \right).$$

Now, assume on the contrary that for a sequence $\lambda_n \to \infty$ such that

$$\lim_{n \to \infty} c_{\lambda_n} = k < c(\Omega).$$

First of all, Lemma 2.4 implies $k > 0$ and by Corollary 1, for $n$ large enough, there exists a sequence $v_n \in \mathcal{M}_{\lambda_n}$ which is a solution of (1.8) with $\lambda$ being replaced by $\lambda_n$ such that $J_{\lambda_n}(v_n) = c_{\lambda_n}$. Similar to the proof of Lemma 2.4, it is easy to verify that $\{v_n\}$ is bounded in $E$, thus we may assume that $v_n \to v$ in $E$ and

$$v_n(x, 0) \to v(x, 0) \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \leq q < 2^*. \quad (4.1)$$

We firstly claim that $v(\cdot, 0)_{|\Omega^c} = 0$ and hence $v \in E_0$, where $\Omega^c = \{x \in \mathbb{R}^N : x \notin \Omega\}$. In fact, if $v(\cdot, 0)_{|\Omega^c} \neq 0$, then there exists a compact subset $F \subset \Omega^c$ with $\text{dist}(F, \Omega) > 0$ such that $v(\cdot, 0)_{|F} \neq 0$. Then by (4.1), we have

$$\int_{F} v_n(x, 0)^2 dx \to \int_{F} v(x, 0)^2 dx > 0.$$ 

However, since $V(x) \geq \epsilon_0 > 0$ for all $x \in F$ and for some $\epsilon_0 > 0$, it follows that

$$J_{\lambda_n}(v_n) \geq \frac{p - 2}{2p} \lambda_n \int_{F} V(x) v_n(x, 0)^2 dx \geq \frac{p - 2}{2p} \lambda_n \epsilon_0 \int_{F} v_n(x, 0)^2 dx \to \infty \text{ as } n \to \infty,$$

this is a contradiction.

Next we show that $v_n(\cdot, 0) \to v(\cdot, 0)$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Indeed, if not, then by the Concentration Compactness Lemma of P.L. Lions[10], there exists $\delta > 0$, $\rho > 0$ and $x_n \in \mathbb{R}^N$ with $|x_n| \to \infty$ such that

$$\liminf_{n \to \infty} \int_{B_{\rho}(x_n)} |v_n(x, 0) - v(x, 0)|^2 dx \geq \delta > 0.$$
Then we have
\[
J_{\lambda_n}(v_n) = \frac{p-2}{2p} \int_{\mathbb{R}_+^{N+1}} |\nabla v_n|^2 dx dy + \frac{p-2}{2p} \int_{\mathbb{R}^N} \lambda_n V(x)v_n(x,0)^2 dx \\
\geq \frac{p-2}{2p} \lambda_n \int_{B_p(x_n)\cap \{x: V(x) \geq M_0\}} V(x)v_n(x,0)^2 dx \\
= \frac{p-2}{2p} \lambda_n \left( M_0 \int_{B_p(x_n)\cap \{x: V(x) \geq M_0\}} |v_n(x,0) - v(x,0)|^2 dx - M_0 \int_{B_p(x_n)\cap \{x: V(x) \leq M_0\}} v_n(x,0)^2 dx \right) \\
\geq \frac{p-2}{2p} \lambda_n \left( M_0 \int_{B_p(x_n)} |v_n(x,0) - v(x,0)|^2 dx - \alpha(1) \right) \\
\rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]

For the last inequality we have used the Hölder inequality and the fact
\[
\mu(B_p(x_n) \cap \{x: V(x) \leq M_0\}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

This contradiction implies \( v_n(\cdot,0) \rightarrow v(\cdot,0) \) in \( L^p(\mathbb{R}^N) \). By this strong convergence, one can easily check that \( v \geq 0 \) is a solution of the following problem
\[
\begin{cases}
-\Delta v = 0 & \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
\frac{\partial v}{\partial \nu} = v^{p-1}(x,0) & \text{on } \Omega, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\] (4.2)

Furthermore again from the strong convergence \( v_n(\cdot,0) \rightarrow v(\cdot,0) \) in \( L^p(\mathbb{R}^N) \), we have
\[
k = \lim_{n \rightarrow \infty} c_{\lambda_n} = \lim_{n \rightarrow \infty} J_{\lambda_n}(v_n) = \lim_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} v_n^+(x,0)^p dx = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} v^+(x,0)^p dx.
\]

Namely we obtain that
\[
\Phi(v) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} v^+(x,0)^p dx = k,
\]
which implies \( v \in \mathcal{N} \) and \( \lambda = c(\Omega) \). This is also a contradiction. Thus we proved that \( \lim_{\lambda \rightarrow \infty} c_{\lambda} = c(\Omega) \). Thus the proof of this lemma is completed.

Now we are ready to give the proofs of Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2:** It suffices to prove that any sequence of \( v_n \in E_{\lambda_n} \) with \( v_n \in \mathcal{M}_{\lambda_n}, J_{\lambda_n}(v_n) = c_{\lambda_n} \) (\( \lambda_n \rightarrow \infty \) as \( n \rightarrow \infty \)) converges in \( E \) along a subsequence to a least energy solution of (1.10). As in the proof of Lemma 2.4, we can obtain that such a sequence \( v_n \) must be bounded in \( E \). Thus without any loss of generality, we may assume that \( v_n \rightarrow v \) in \( E \) and \( v_n(\cdot,0) \rightarrow v(\cdot,0) \) in \( L^p_{loc}(\mathbb{R}^N) \) for \( 2 < q < 2^* \).

To complete the proof, it is sufficient to prove that \( v_n \rightarrow v \) strongly in \( E \) and \( v \in \mathcal{N} \) is a least energy solution of (1.10) such that \( \Phi(v) = c(\Omega) \). Firstly, as in the proof of Lemma 4.1, we can prove that \( v \geq 0 \) is a solution of the following problem
\[
\begin{cases}
-\Delta v = 0 & \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
\frac{\partial v}{\partial \nu} = v^{p-1}(x,0) & \text{on } \Omega, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\] (4.3)

and \( v_n^+(x,0) \rightarrow v^+(x,0) \) strongly in \( L^p(\mathbb{R}^N) \).

Now we claim that
\[
\lambda_n \int_{\mathbb{R}^N} V(x)v_n(x,0)^2 dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^{N+1}} |\nabla v_n|^2 dx dy \rightarrow \int_{\mathbb{R}^{N+1}} |\nabla v|^2 dx dy.
\]
Indeed, if either
\[ \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} V(x)v_n(x,0)^2 \, dx > 0 \]
or
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx > \int_{\mathbb{R}^N} |\nabla v|^2 \, dx. \]
Then we get that
\[ \int_{\mathbb{R}^N} |\nabla v|^2 \, dx < \int_\Omega u^+ (x,0)^p \, dx. \]
Thus there is \( \alpha \in (0,1) \) such that \( \alpha \varepsilon \in \mathcal{N} \) and
\[ c(\Omega) \leq \Phi(\alpha \varepsilon) \]
\[ = \frac{p-2}{2p} \int_{\mathbb{R}^N} |\nabla \alpha \varepsilon|^2 \, dx \]
\[ < \frac{p-2}{2p} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \]
\[ \leq \lim_{n \to \infty} \frac{p-2}{2p} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} \lambda_n V(x)v_n(x,0)^2 \, dx \right) \]
\[ = \lim_{n \to \infty} J_{\lambda_n}(v_n) \]
\[ = c(\Omega), \]
which is a contradiction. Thus we complete the proof of Theorem 1.2.

**Proof of Theorem 1.3:** Suppose \( \{u_n = v_n(\cdot,0)\} \in H^{1/2}(\mathbb{R}^N) \) is a solution of (1.1) with \( \lambda \) being replaced by \( \lambda_n \) (\( \lambda_n \to \infty \) as \( n \to \infty \)). It is easy to see that such a sequence \( v_n \) must be bounded in \( E \). We may assume that \( u_n \to u \) in \( E \) and \( v_n(\cdot,0) \to v(\cdot,0) \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \) for \( 2 < p < 2^* \). As in the proof of Lemma 4.1, we can prove that \( (\cdot,0)|_{\partial \Omega} = 0 \) and \( v \in E_0 \) is solution of (1.10). Moreover \( v_n(\cdot,0) \to v(\cdot,0) \) in \( L^p(\mathbb{R}^N) \) for \( 2 \leq p < 2^* \). As in the proof of Theorem 1.2, it suffices to show \( v_n \to v \) in \( E \). We observe that
\[ \int_{\mathbb{R}^N} |\nabla (v_n - v)|^2 \, dx + \int_{\mathbb{R}^N} \lambda_n V(x) |v_n(x,0) - v(x,0)|^2 \, dx \]
\[ = \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} \lambda_n V(x) v_n(x,0)^2 \, dx - \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \]
\[ - \int_{\mathbb{R}^N} \lambda_n V(x) v(x,0)^2 \, dx + o(1) \]
\[ = \int_{\mathbb{R}^N} v_n^+(x,0) \, dx - \int_\Omega v^+(x,0) \, dx + o(1) \]
\[ = o(1). \]
Here we used the fact that \( v_n \) and \( v \) lie on the Nehari manifold \( \mathcal{M}_{\lambda_n} \) and \( \mathcal{N} \) respectively. This completes the proof of Theorem 1.3.

**REFERENCES**


E-mail address: miaomiaoniu@mail.bnu.edu.cn
E-mail address: tangzw@bnu.edu.cn