Sign-changing solutions of critical growth nonlinear elliptic systems

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Abstract

In this paper, we are concerned with the existence of sign-changing solutions of a class of nonlinear elliptic systems with critical growth.

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, we are concerned with the existence of sign-changing solutions of the following elliptic systems:

\[
\begin{align*}
-\Delta u &= \frac{2\alpha}{\alpha + \beta} |u|^{2^*-2} |v|^{\beta} u + \lambda u, \quad x \in \Omega, \\
-\Delta v &= \frac{2\beta}{\alpha + \beta} |u|^2 |v|^{\beta - 2} v + \delta v, \quad x \in \Omega, \\
u &= 0, \quad v = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where $\lambda > 0$, $\delta > 0$ are parameters and $\alpha > 1$, $\beta > 1$ satisfying $\alpha + \beta = 2^*, \quad 2^* = \frac{2N}{N-2}$ is the critical Sobolev exponents.

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We call a pair of functions \((u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)\) a weak solution of (1.1) if
\[
\int_\Omega (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 - \lambda u \varphi_1 - \delta u \varphi_2) \, dx - \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^\alpha - 2|v|^\beta u \varphi_1 \, dx \\
- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^2|v|^{\beta - 2} v \varphi_2 \, dx, \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^1_0(\Omega) \times H^1_0(\Omega).
\]
The corresponding energy functional of problem (1.1) is
\[
I_{\lambda, \delta}(u, v) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2 - \lambda u^2 - \delta v^2) \, dx \\
- \frac{2}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^\beta \, dx, \quad (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega).
\] (1.2)

Notice that for any \((\varphi_1, \varphi_2) \in H^1_0(\Omega) \times H^1_0(\Omega),\)
\[
I'_{\lambda, \delta}(u, v)(\varphi_1, \varphi_2) \\
= \int_\Omega (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 - \lambda u \varphi_1 - \delta u \varphi_2) \, dx - \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^\alpha - 2|v|^\beta u \varphi_1 \, dx \\
- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^2|v|^{\beta - 2} v \varphi_2 \, dx.
\]

It is well known that the nontrivial solution of problem (1.1) are equivalent to the nonzero critical points of functional \(I_{\lambda, \delta}\) in \(H^1_0(\Omega) \times H^1_0(\Omega)\).

Let
\[
S_{\alpha, \beta} = \inf_{u \in H^1_0(\Omega) \times H^1_0(\Omega)/\{0\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2)}{\left(\int_\Omega |u|^\alpha |v|^\beta \, dx\right)^{\frac{2}{\alpha + \beta}}},
\] (1.3)

Then, by Alevs [1], we have
\[
S_{\alpha, \beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha + \beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha + \beta)}\right) S,
\]
where \(S\) is the best Sobolev constants defined by
\[
S = \inf_{u \in H^1_0(\Omega)/\{0\}} \frac{\int_\Omega (|\nabla u|^2)}{\left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*}}
\]
and \(S\) is achieved if and only if when \(\Omega = \mathbb{R}^N\) by the following function
\[
V(x) = \frac{(N(N - 2))(N - 2)^{N-2/4}}{(1 + |x|^{2})^{(N-2)/2}}.
\]
The function \(V\) called a instant satisfies
\[-\Delta V = V^{2^* - 1}, \quad x \in \mathbb{R}^N.\]
Moreover
\[
\int_{\mathbb{R}^N} |\nabla V|^2 = \int_{\mathbb{R}^N} V^{2^*} = S^{N/2}.
\]
Recently, for $\lambda \in (0, \lambda_1)$, $\delta \in (0, \lambda_1)$, Alevs proved that there exists a least energy positive solution $U_0 = (u_0, v_0) \in H^1_0(\Omega) \times H^1_0(\Omega)$ of (1.1) such that

$$I_{\lambda, \delta}(U_0) = \inf \{ I_{\lambda, \delta}(U) : U \in H^1_0(\Omega) \times H^1_0(\Omega), I'_{\lambda, \delta}(U)U = 0, \ U \neq 0 \},$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$. Later, Han studied the effect of domain topology on the number of positive solutions of (1.1) (see [5]), he also obtained the existence of high energy solutions of (1.1) (see [6]).

A nature problem is that whether problem (1.1) exists solutions $(u, v)$ such that $u$ or $v$ is change sign in $\Omega$?

In present paper, we aim to answer this question. Our main results are as follows:

**Theorem A.** Suppose $N \geq 7$, $\lambda \in (0, \lambda_1)$, $\delta \in (0, \lambda_1)$, then there exists at least four pairs of nontrivial solutions $(\pm u, \pm v)$ of (1.1) such that both $u$ and $v$ change sign in $\Omega$.

**Remark 1.1.** Notice that if $(u, v)$ is a solution of problem (1.1), then $(-u, v), (u, -v), (-u, -v)$ are all the solutions of (1.1). So the group of solution $(\pm u, \pm v)$ includes four pairs of solutions of (1.1), namely $(u, v), (-u, v), (u, -v)$ and $(-u, -v)$.

For single equation, in 1986, Cerami et al. [4] considered the existence of sign-changing solution of the following problem

$$\begin{aligned}
-\Delta u &= |u|^{2^*-2}u + \lambda u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}$$

(1.4)

where $\lambda \in (0, \lambda_1)$. Since then, there have been many papers studied the existence of sign-changing solutions of single equations or systems (see [2,3] and [8]).

Our results extend those of Cerami et al. [4] from which our idea comes. However, since we are concerned with elliptic systems, in order to obtain our results, we need more delicate estimates.

We point out that to ensure the existence of sign-changing solution of (1.1), $N \geq 7$ is necessary in some sense. The cases of $N = 3, 4, 5, 6$ are more delicate. In fact, for single equation when $\Omega$ is the unit ball, it has been proved that problem (1.4) has no radial sign-changing solutions in the case $N = 4, 5, 6$ if $\lambda > 0$ is small enough.

We will use the same $C$ to denote various generic positive constants, and we will use $O(t), o(t)$ to mean $|O(t)| \leq Ct, |o(t)|/t \to 0$ as $t \to 0$, and all integral are on $\Omega$ unless particularly point out.

### 2. Proof of the main results

We denote $U =: (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega), U = 0$ if and only if $u \equiv 0, v \equiv 0$, we denote $U^+ = (u^+, v^+)$ and $U^- = (u^-, v^-)$, where for any function $u, u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\}$. Moreover, we define $U \geq 0$ if and only if $u \geq 0, v \geq 0$.

Let $g(u, v)$ be the functional defined in $H^1_0(\Omega) \times H^1_0(\Omega)$ by

$$g(u, v) = \begin{cases}
2 \int |u|^2 |v|^\beta \\
\int |\nabla u|^2 + |\nabla v|^2 - \lambda u^2 - \delta v^2 - 2^*|u|^2 |v|^{2^*-2}v, & \text{if } U \neq 0,
0 & \text{if } U = 0.
\end{cases}$$
Denote by
\[ M = \{ U \in H^1_0(\Omega) \times H^1_0(\Omega) : g(U^+) = g(U^-) = 1 \}, \]
\[ N = \{ U \in H^1_0(\Omega) \times H^1_0(\Omega) : |g(U^+) - 1| < \frac{1}{2} \}. \]

It is not difficult to see that \( M \neq \emptyset \).

We claim that for any \( U = (u, v) \in N \), there exists a constant \( \alpha > 0 \) such that
\[ \|U^\pm\| = \|u^\pm\| + \|v^\pm\| \geq \alpha > 0. \]  
(2.1)

Indeed, since
\[ |g(U^\pm) - 1| < \frac{1}{2}, \]
a.e.
\[ \frac{1}{2} < g(U^\pm) < \frac{3}{2}. \]

Thus
\[ 2 \int |u^\pm|^2 |v^\pm|^\beta \geq \frac{1}{2} \int |\nabla u^\pm|^2 - \alpha (u^\pm)^2 + |\nabla v^\pm|^2 - \beta (v^\pm)^2, \]
by Hölder inequality and \( \lambda \in (0, \lambda_1), \delta \in (0, \lambda_1) \), we have
\[ \frac{\alpha}{\alpha + \beta} \int |u^\pm|^2 + \frac{\beta}{\alpha + \beta} \int |v^\pm|^2 \geq \frac{1}{4} c \int |\nabla u^\pm|^2 + |\nabla v^\pm|^2 \]
\[ = \frac{1}{4} c \|U^\pm\|^2, \]
where \( c \) is a positive constant. By Sobolev inequality, we easily obtain (2.1).

The following Lemma is some what well known (a similar statement for single equation is proved for instance in Zhu [8]) (see also [4]), for the sake of completeness, we give here the proof.

**Lemma 2.1.** There is a sequence \( \{U_n\} \subset \tilde{N} \) such that as \( n \to \infty \)
\[ I_{\lambda, \delta}(U_n) \to c_1, \]
\[ I'_{\lambda, \delta}(U_n) \to 0, \]  
(2.2)

where
\[ c_1 = \inf \{ I_{\lambda, \delta}(U) : U \in \tilde{M} \}. \]

**Proof.** Firstly, we denote \( P = \{ U = (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) : U \geq 0 \} \). Correspondingly we note \( -P = \{ U = (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) : -U \geq 0 \} \). Let \( \Sigma \) be the set of maps \( \sigma \) such that
\[
\begin{array}{ll}
(a) & \sigma \in C(D, H^1_0(\Omega) \times H^1_0(\Omega)) \text{ where } D = [0, 1] \times [0, 1] \\
(b) & \sigma(s, 0) \in P \\
(c) & \sigma(0, s) \in -P \\
(d) & \sigma(1, s) \in -P \\
(e) & (I_{\lambda, \delta} \cdot \sigma)(s, 1) \leq 0, \quad (g \cdot \sigma)(s, 1) \geq 2 \\
\end{array}
\]

We claim that \( \Sigma \neq \emptyset \).
In fact, for any \( U = (u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) \) with \( U^+, U^- \neq 0 \), we define
\[
\sigma = \sigma(t_1, t_2) = t_2(k(1 - t_1)A^+ - kt_1A^-),
\]
when \( k \) is large enough, it is easy to know that \( \sigma \in \Sigma \). Now we divide the proof of this lemma into three steps.

*Step 1:* we prove that
\[
\inf_{\sigma \in \Sigma} \sup_{U \in \sigma(D)} I_{\lambda, \delta}(U) = \inf_{U \in M} I_{\lambda, \delta}(U).
\]
(2.3)

We claim that
\[
\forall \sigma \in \Sigma, \quad \exists (\tilde{u}, \tilde{v}) \in \sigma(D) \cap \tilde{M}.
\]
(2.4)

In fact, for any \( \sigma \in \Sigma \)
\[
g(\sigma^+(x)) - g(\sigma^-(x)) \begin{cases} 
\geq 0, & \forall x \in [(0, s) : s [0, 1]) \\
\leq 0, & \forall x \in [(1, s) : s [0, 1])
\end{cases}
\]
so from Miranda’s Theorem [7] we deduce that \( \exists x \in D \) such that
\[
g(\sigma^+(x)) - g(\sigma^-(x)) = 0 = g(\sigma^+(\tilde{x})) + g(\sigma^-(\tilde{x})) - 2,
\]
then \( \tilde{U} = (\tilde{u}, \tilde{v}) = \sigma(\tilde{x}) \in \tilde{M} \).

Conversely for \( \tilde{U} = (\tilde{u}, \tilde{v}) \in \tilde{M} \), we denote by \( \tilde{\sigma} \) a map of \( \Sigma \) such that
\[
\tilde{\sigma}(D) \subset \{ \xi U^+ + \zeta U^- : \xi, -\zeta \in \mathbb{R}^+ \cup \{0\} \},
\]
from the definition of \( g \) and the behavior of \( I_{\lambda, \delta} \), it comes
\[
\max_{\tilde{\sigma}(D)} I_{\lambda, \delta}(U) = I_{\lambda, \delta}(\tilde{U}).
\]
(2.5)

So combining (2.4) and (2.5), we obtain (2.3).

*Step 2:* Consider a minimizing sequence \( \{ \tilde{U}_n \} \) of \( c_1 = \inf \{ I_{\lambda, \delta}(U) : U \in \tilde{M} \} \) and denote by \( \tilde{\sigma}_n \) be the corresponding sequence of maps in the class of \( \Sigma \) satisfying (2.5). Thus
\[
\lim_{n \to \infty} \max_{\tilde{\sigma}_n(D)} I_{\lambda, \delta}(U) = \lim_{n \to \infty} I_{\lambda, \delta}(\tilde{U}_n) = c_1.
\]
(2.6)

We claim that there exists \( \{ U_n \} \subset H^1_0(\Omega) \times H^1_0(\Omega) \) such that
\[
\lim_{n \to \infty} \text{dist}(U_n, \tilde{\sigma}_n(D)) = 0,
\]
\[
\lim_{n \to \infty} I_{\lambda, \delta}'(U_n) = 0, \quad \lim_{n \to \infty} (U_n) = c_1.
\]
(2.7)

We argue by the way of contradiction. Suppose that there exists a \( \gamma > 0 \), for \( n \) large enough, \( \tilde{\sigma}_n(D) \cap N_{\gamma} = \emptyset \), where
\[
N_{\gamma} = \left\{ U \in H^1_0(\Omega) \times H^1_0(\Omega) : \exists \tilde{U} \in H^1_0(\Omega) \times H^1_0(\Omega), \text{ such that } \|U - \tilde{U}\| \leq \gamma, \quad |I_{\lambda, \delta}(\tilde{U})| - c_1| \leq \gamma \right\}.
\]
Using Hofer Lemma, we can construct a continuous map \( \eta : [0, 1] \times H^1_0(\Omega) \times H^1_0(\Omega) \to H^1_0(\Omega) \times H^1_0(\Omega) \) which verifies for some \( \varepsilon \in (0, \frac{1}{2}) \),

\[
\eta(0, U) = U, \quad \eta(t, -U) = -\eta(t, U), \quad \forall t \in [0, 1];
\]

\[
\eta(t, U) = U, \quad \forall U \in I_{c_1-\varepsilon} \cup (H^1_0(\Omega) \times H^1_0(\Omega) \setminus I_{c_1+\varepsilon}), \quad t \in [0, 1];
\]

\[
\eta(1, I_{c_1+(\varepsilon/2)} \setminus N_{\gamma}) \subset I_{c_1-\varepsilon};
\]

\[
\eta(1, I_{c_1+(\varepsilon/2)} \cap P \setminus N_{\gamma}) \subset I_{c_1-\varepsilon} \cap P,
\]

where

\[
I_\varepsilon = \{ U \in H^1_0(\Omega) \times H^1_0(\Omega) : I_{\lambda, \delta}(U) \leq \varepsilon \}.
\]

Choose \( \tilde{n} \) such that

\[
\tilde{\sigma}_{\tilde{n}}(D) \subset I_{c_1+(\varepsilon/2)}, \quad \tilde{\sigma}_{\tilde{n}}(D) \cap N_{\gamma} = \emptyset.
\]

Then if we define

\[
\tilde{\sigma}(s, t) = \eta(1, \tilde{\sigma}_{\tilde{n}}(s, t)), \quad \forall (s, t) \in D.
\]

We get \( \tilde{\sigma} \in \Sigma \) and \( \tilde{\sigma}(D) \subset I_{c_1-(\varepsilon/2)} \). So

\[
c_1 = \inf_{\sigma \in \Sigma} \max_{U \in \sigma(D)} I_{\lambda, \delta}(U) = \max_{U \in \tilde{\sigma}(D)} I_{\lambda, \delta}(U) \leq c_1 \leq \varepsilon + \frac{\varepsilon}{2},
\]

which is absurd.

Step 3: Now, we show that \( \{U_n\} \) satisfying (2.7) belongs to \( \tilde{N} \).

For the sequence \( \{\tilde{U}_n\} \), \( \{\tilde{\sigma}_{\tilde{n}}\} \) in Step 2, by (2.7), there exists

\[
\tilde{U}_n = \xi_n \tilde{U}_n^+ + \zeta \tilde{U}_n^- \in \tilde{\sigma}_{\tilde{n}}(D),
\]

such that

\[
dist(U_n, \tilde{U}_n) \to 0 \quad \text{as} \quad n \to \infty. \quad (2.8)
\]

By (2.6), we deduce that \( \forall \varepsilon \geq 0 \), \( \exists n_0 \), such that for \( n \geq n_0 \)

\[
I_{\lambda, \delta}(\tilde{U}_n) = I_{\lambda, \delta}(\tilde{U}_n^+) + I_{\lambda, \delta}(\tilde{U}_n^-) < c_1 + \varepsilon
\]

\[
I_{\lambda, \delta}(\tilde{U}_n) = I_{\lambda, \delta}(\xi_n \tilde{U}_n^+) + I_{\lambda, \delta}(\zeta \tilde{U}_n^-) > c_1 - \varepsilon
\]

and

\[
I_{\lambda, \delta}(\tilde{U}_n^+) \geq I_{\lambda, \delta}(\xi_n \tilde{U}_n^+) \quad I_{\lambda, \delta}(\tilde{U}_n^-) \geq I_{\lambda, \delta}(\zeta \tilde{U}_n^-).
\]

So

\[
I_{\lambda, \delta}(\tilde{U}_n^+) = I_{\lambda, \delta}(\xi_n \tilde{U}_n^+) \geq I_{\lambda, \delta}(\tilde{U}_n^+) - 2\varepsilon \geq c_0 - 2\varepsilon,
\]

\[
I_{\lambda, \delta}(\tilde{U}_n^-) = I_{\lambda, \delta}(\zeta \tilde{U}_n^-) \geq I_{\lambda, \delta}(\tilde{U}_n^-) - 2\varepsilon \geq c_0 - 2\varepsilon.
\]

(2.9)

Thus by (2.8) and (2.9), there is a positive constant \( a > 0 \) such that for \( n \geq n_1 \)

\[
\|U_n^\pm\| \geq a > 0.
\]
Now since $I_{\lambda, \delta}^1(U_n)(U_n) \to 0$, namely

$$I_{\lambda, \delta}^1(u_n, v_n)(u_n, v_n) = \int (|\nabla u_n|^2 + |\nabla v_n|^2 - \lambda |u_n|^2 - \delta |v_n|^2) \, dx - 2 \int |u_n|^2 |v_n|^\beta \to 0.$$ 

Thus, we have $U_n = (u_n, v_n) \in \tilde{N}$ for $n$ large enough and this completes the proof. □

**Lemma 2.2.** Assume sequence $U_n = (u_n, v_n) \in \tilde{N}$ satisfying (2.2) and

$$c_1 < c_0 + \frac{2}{N} \left( \frac{S_{2, \beta}}{2} \right)^{N/2},$$

(2.10)

then $\{U_n = (u_n, v_n)\}$ is strongly relatively compact in $H^1_0(\Omega) \times H^1_0(\Omega)$.

**Proof.** Since $U_n = (u_n, v_n) \in \tilde{N}$ satisfying (2.2), a.e.

$$\frac{1}{2} \int (|\nabla u_n|^2 + |\nabla v_n|^2 - \lambda |u_n|^2 - \delta |v_n|^2) \, dx - \frac{2}{\alpha + \beta} \int |u_n|^2 |v_n|^\beta \to c_1 + o(1)$$

$$\left( -\Delta u_n - \lambda u_n - \frac{2\alpha}{\alpha + \beta} |u_n|^{2-\alpha} |v_n|^\beta u_n \right) \to 0 \quad \text{in} \quad [H^1_0(\Omega) \times H^1_0(\Omega)]^{-1}.$$

Thus it is easy to check that $\{U_n\} \subset \tilde{N}$ is bounded in $H^1_0(\Omega) \times H^1_0(\Omega)$ and

$$\int (|\nabla u_n|^2 + |\nabla v_n|^2 - \lambda |u_n|^2 - \delta |v_n|^2) \, dx - \int |u_n|^2 |v_n|^\beta \to 0,$$

then (by (2.1)) there exists $k_n, \tilde{k}_n$ with $\lim_{n \to \infty} k_n = 1, \lim_{n \to \infty} \tilde{k}_n = 1$ such that

$$k_n U^+_n - \tilde{k}_n U^-_n \in \tilde{M}.$$

Without loss of generality, we assume that

$$\lim_{n \to \infty} I_{\lambda, \delta}(U^+_n) = c^{(1)}_1, \quad \lim_{n \to \infty} I_{\lambda, \delta}(U^-_n) = c^{(2)}_1.$$

So $I_{\lambda, \delta}(k_n U^+_n) \geq c_0$ and

$$I_{\lambda, \delta}(U^+_n) = I_{\lambda, \delta}(k_n U^+_n) + (I_{\lambda, \delta}(U^+_n) - I_{\lambda, \delta}(k_n U^+_n))$$

$$\geq c_0 + \frac{1}{2} \left( 1 - k_n^2 \right) \int (|\nabla u_n^+|^2 + |\nabla v_n^+|^2 - \lambda |u_n^+|^2 - \delta |v_n^+|^2) \, dx$$

$$- \frac{2}{\alpha + \beta} \left( 1 - k_n^{\alpha + \beta} \right) \int |u_n^+|^2 |v_n^+|^\beta.$$

Then it follows that

$$c^{(1)}_1 \geq c_0.$$ 

(2.11)

Similarly, we have

$$c^{(2)}_1 \geq c_0.$$ 

(2.12)
On the other hand, we have

\[ c_1 = c_1^{(1)} + c_1^{(2)} < c_0 + \frac{2}{N} \left( \frac{S_{\lambda, \beta}}{2} \right)^{N/2}. \]  

(2.13)

Combining (2.11), (2.12) and (2.13) we have

\[ c_0 \leq c_1^{(1)} < \frac{2}{N} \left( \frac{S_{\lambda, \beta}}{2} \right)^{N/2}, \]

\[ c_0 \leq c_1^{(2)} < \frac{2}{N} \left( \frac{S_{\lambda, \beta}}{2} \right)^{N/2}. \]

(2.14)

Now without loss of generality, we only prove that \( \{U_n^+\} \) is strongly relatively compact in \( H_0^1(\Omega) \times H_0^1(\Omega) \).

Since

\[ I_{\lambda, \delta}(U_n^+) \rightarrow c_1^{(1)} < \frac{2}{N} \left( \frac{S_{\lambda, \beta}}{2} \right)^{N/2}, \]

\[ I'_{\lambda, \delta}(U_n^+) \rightarrow 0. \]  

(2.15)

It is easy to check that \( \{U_n^+ = (u_n^+, v_n^+)\} \) is bounded in \( H_0^1(\Omega) \times H_0^1(\Omega) \). Then by Sobolev embedding theorem, there exists a subsequence again denote \( \{U_n^+ = (u_n^+, v_n^+)\} \) such that \( u_n^+ \rightarrow u_0^+ \), \( v_n^+ \rightarrow v_0^+ \) in \( H_0^1(\Omega) \) and they converge strongly in \( L^p(\Omega) \) for \( 2 \leq p < 2^* = \alpha + \beta \). By (2.15), it is easy to see that \( U_0^+ = (u_0^+, v_0^+) \) is a solution of problem (1.1). Now we prove that \( u_n^+ \rightarrow u_0^+ \), \( v_n^+ \rightarrow v_0^+ \) strongly in \( H_0^1(\Omega) \) and thus \( (u_0^+, v_0^+) \neq (0, 0) \).

Notice that \( u_0^+ = 0 \) if and only if \( v_0^+ = 0 \). In fact, if \( u_0^+ = 0 \), then \( v_0^+ \) satisfies

\[
\begin{cases}
-\Delta v_0^+ = \delta v_0^+, & x \in \Omega, \\
v_0^+ = 0, & x \in \partial \Omega.
\end{cases}
\]

Since \( 0 < \delta < \lambda_1 \), where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( \Omega \), So we have \( v_0^+ = 0 \).

We denote \( U_0^+ = (u_0^+, v_0^+) \), by (2.15) we have

\[
o(1) = (I'_{\lambda, \delta}(U_n^+) - I'_{\lambda, \delta}(U_0^+), U_n^+ - U_0^+) \]

\[ = \int (|\nabla (u_n^+ - u_0^+)|^2 + |\nabla (v_n^+ - v_0^+)|^2 - \lambda |u_n^+ - u_0^+|^2 - \delta |v_n^+ - v_0^+|^2) \, dx \\
- 2 \int |u_n^+ - u_0^+|^2 |v_n^+ - v_0^+|^\beta + o(1) \\
\geq \|u_n^+ - u_0^+\|^2 + \|v_n^+ - v_0^+\|^2 \\
\times (1 - 2S_{\lambda, \beta}^{(2^*/2)}[\|u_n^+ - u_0^+\|^2 + \|v_n^+ - v_0^+\|^2]^{(2^*/2)-1}) + o(1).
\]

We obtain that either \( U_n^+ = (u_n^+, v_n^+) \rightarrow U_0^+ = (u_0^+, v_0^+) \) strongly and thus \( (u_0^+, v_0^+) \neq (0, 0) \) or

\[ I_{\lambda, \delta}(U_n^+) = I_{\lambda, \delta}(U_0^+ - U_0^+) + I_{\lambda, \delta}(U_0^+) + o(1) \]

\[ \geq \frac{1}{N} \|u_n^+ - u_0^+\|^2 + \|v_n^+ - v_0^+\|^2 + o(1) \]

\[ \geq \frac{2}{N} \left( \frac{S_{\lambda, \beta}}{2} \right)^{N/2} + o(1). \]
Let $n \to \infty$, we have
\[
\varepsilon_1^{(1)} \geq \frac{2}{N} \left( \frac{S_{x, \beta}}{2} \right)^{N/2},
\]
which is absurd, namely we prove that
\[
(u_n^+, v_n^+) \to (u_0^+, v_0^+) \neq (0, 0) \quad \text{in } H_0^1(\Omega) \times H_0^1(\Omega).
\]
Similarly, we can prove that
\[
(u_n^-, v_n^-) \to (u_0^-, v_0^-) \neq (0, 0) \quad \text{in } H_0^1(\Omega) \times H_0^1(\Omega).
\]
This completes the prove Lemma 2.2. □

Lemma 2.3. Under the assumption of Theorem A, we have
\[
c_1 < c_0 + \frac{2}{N} \left( \frac{S_{x, \beta}}{2} \right)^{N/2},
\]
(2.16)

Proof. It suffices to show that
\[
\sup_{\xi, \zeta \in \mathbb{R}} I_{\lambda, \delta}(\xi U_0 + \zeta \Psi) < c_0 + \frac{2}{N} \left( \frac{S_{x, \beta}}{2} \right)^{N/2},
\]
where $U_0 = (u_{0}, v_0)$ is the positive solution of (1.1) such that
\[
I_{\lambda, \delta}(U_0) = \inf \{ I_{\lambda, \delta}(U) : U \in H_0^1(\Omega) \times H_0^1(\Omega), I_{\lambda, \delta}'(U)U = 0, u \neq 0 \},
\]
$\Psi = (\phi BV_{\mu, x_0}, \Phi CV_{\mu, x_0})$, here $\phi \in C^\infty(B_{\rho/2}(x_0))$, $\phi$ identically 1 on $B_{\rho/2}(x_0)$, $x_0 \in \Omega$, $\rho \in \mathbb{R}^+$ to be specified later. $(B/C)^2 = (x/\beta)$. $V_{\mu, x_0}(x) = \mu^{-(N-2)/2} V((x-x_0)/\mu)$ and $V(x)$ is just narrated in Section 1.

Let
\[
\psi_{\mu} = \phi V_{\mu, x_0} = \phi \frac{[N(N-2)]^{(N-2)/4}}{[\mu^2 + |x-x_0|^2]^{N/2}},
\]
then by a direct calculation, it is easy to see that
\[
\begin{align*}
\int |\nabla \psi_{\mu}|^2 &= S^{N/2} + O(\mu^{N-2}), \\
\int |\psi_{\mu}|^2 &= S^{N/2} + O(\mu^N), \\
\int |\psi_{\mu}|^{2^*} &= S^{N/2} + O(\mu^{N-2}) \quad \text{for } N \geq 5, \\
\int |\psi_{\mu}|^q &= O(\mu^{N-((N-2)q/2)}) \quad \text{for } N \geq 5, \quad 1 \leq q < 2^*, \\
\int |\psi_{\mu}|^{2^*-1} &= O(\mu^{(N-2)/2}) \quad \text{for } N \geq 5.
\end{align*}
\]
In particular, for $q = 2^* - 1 = (N + 2)/(N - 2)$
\[
\int |\psi_{\mu}|^{2^*-1} = O(\mu^{(N-2)/2}) \quad \text{for } N \geq 5.
\]
It is well known that \((BV_{\mu,x_0}, CV_{\mu,x_0})\) achieves \(S_{x,\beta}\) for \((B/C)^2 = \alpha/\beta\) and one can choose \(B, C\) properly such that \((BV_{\mu,x_0}, CV_{\mu,x_0})\) satisfies the following systems:

\[
\begin{align*}
-\Delta u &= \frac{\alpha}{\alpha + \beta}|u|^{\alpha - 2}u, \quad x \in \mathbb{R}^N, \\
-\Delta v &= \frac{\beta}{\alpha + \beta}|u|^{\beta - 2}v, \quad x \in \mathbb{R}^N
\end{align*}
\]

and

\[
\int_{\mathbb{R}^N} |\nabla BV_{\mu,x_0}|^2 + |\nabla CV_{\mu,x_0}|^2 = \int_{\mathbb{R}^N} |BV_{\mu,x_0}|^2|CV_{\mu,x_0}|^\beta = S_{x,\beta}^{N/2}. \tag{2.18}
\]

Thus

\[
I_{\lambda,\delta}(\xi U_0 + \xi \Psi)
= \frac{1}{2} \int_{\Omega} |\nabla (\xi u_0 + \xi B\psi_\mu)|^2 - \varepsilon(\xi u_0 + \xi B\psi_\mu)^2 \\
+ \frac{1}{2} \int_{\Omega} |\nabla (\xi v_0 + \xi C\psi_\mu)|^2 - \varepsilon(\xi v_0 + \xi C\psi_\mu)^2 \\
- \frac{2}{\alpha + \beta} \int_{\Omega} |\xi u_0 + \xi B\psi_\mu|^2|\xi v_0 + \xi C\psi_\mu|^\beta
\leq \frac{\varepsilon^2}{2} \int_{\Omega} [|\nabla u_0|^2 - \varepsilon u_0^2] + [|\nabla v_0|^2 - \varepsilon v_0^2] \\
+ \frac{\varepsilon^2}{2} \int_{\Omega} [|\nabla B\psi_\mu|^2 - \varepsilon(B\psi_\mu)^2] + [|\nabla C\psi_\mu|^2 - \delta(C\psi_\mu)^2] \\
- \frac{2}{\alpha + \beta} \int_{\Omega} |u_0|^2|v_0|^\beta - \frac{2}{\alpha + \beta} \int_{\Omega} |B\psi_\mu|^2|C\psi_\mu|^\beta \\
+ K \int_{\Omega} |u_0||\psi_\mu|^\beta + K \int_{\Omega} |v_0||\psi_\mu|^\beta \\
+ K \int_{\Omega} |u_0|^2|\psi_\mu|^\beta + K \int_{\Omega} |v_0|^2|C\psi_\mu|^\beta \\
+ K \int_{\Omega} |u_0|^2|v_0|^\beta - |C\psi_\mu|^\beta \\
+ o(1), \tag{2.19}
\]

where \(K\) is a positive constant.

The following two estimates are somewhat well known, here we only give them without proof.

\[
\int_{\Omega} [|\nabla B\psi_\mu|^2 + |\nabla C\psi_\mu|^2] = \int_{\mathbb{R}^N} [|\nabla BV_{\mu,x_0}|^2 + |\nabla CV_{\mu,x_0}|^2] + O(\mu^{N-2}) \quad \text{for } N \geq 3
\]

\[
\int_{\Omega} |B\psi_\mu|^2|\psi_\mu|^\beta = \int_{\mathbb{R}^N} |BV_{\mu,x_0}|^2|CV_{\mu,x_0}|^\beta + O(\mu^{N-2}) \quad \text{for } N \geq 3. \tag{2.20}
\]

On the other hand,

\[
\int_{\Omega} |u_0||\psi_\mu|^{\alpha + \beta - 1} = \int_{B_{\rho}(x_0)} |u_0||\psi_\mu|^{\alpha + \beta - 1}
\leq \max_{B_{\rho}(x_0)} |u_0| \int_{B_{\rho}(x_0)} (|\psi_\mu|^{\alpha - 1}) \leq D\mu^{(N-2)/2} \max_{B_{\rho}(x_0)} |u_0|. \tag{2.21}
\]
Similarly
\[
\int_\Omega |v_0||\psi_\mu|^{\alpha+\beta-1} \leq D\mu^{(N-2)/2} \max_{B_\rho(x_0)} |v_0|,
\] (2.22)
\[
\int_\Omega |u_0|^2|\psi_\mu|^\beta \leq D\mu^{N-(\beta(N-2))/2} \max_{B_\rho(x_0)} |u_0|^2,
\] (2.23)
\[
\int_\Omega |v_0|^2|\psi_\mu|^\beta \leq D\mu^{N-(\beta(N-2))/2} \max_{B_\rho(x_0)} |v_0|^2,
\] (2.24)
\[
\int_\Omega |u_0|^2|v_0|^{\beta-1}|B\psi_\mu| \leq \max_{B_\rho(x_0)} \{|u_0|^2|v_0|^{\beta-1}\} \mu^{(N+2)/2},
\] (2.25)
\[
\int_\Omega |u_0|^{\beta-1}|v_0|^\beta |C\psi_\mu| \leq \max_{B_\rho(x_0)} \{|u_0|^{\beta-1}|v_0|^\beta\} \mu^{(N+2)/2}.
\] (2.26)

Combining (2.20)–(2.26) and (2.19), we have that
\[
\max_{\xi, \zeta, \mu} I_{\lambda, \delta} (\xi U_0 + \zeta \Psi_\mu)
\leq \max_{\xi, \zeta, \mu} \left\{ \frac{\varepsilon^2}{2} \int_\Omega [||\nabla u_0||^2 - \varepsilon u_0^2] + [||\nabla v_0||^2 - \delta v_0^2] - \frac{2}{\alpha + \beta} \varepsilon^{\alpha+\beta} \int_\Omega |u_0|^2|v_0|^{\beta} \right\}
+ \max_{\xi, \zeta, \mu} \left\{ \frac{\varepsilon^2}{2} \int_{\mathbb{R}^N} [||\nabla B V_{\mu, x_0}||^2] + [||\nabla C V_{\mu, x_0}||^2] - \frac{2}{\alpha + \beta} \varepsilon^{\alpha+\beta} \int_{\mathbb{R}^N} |B\psi_\mu|^2 |C\psi_\mu|^\beta \right\}
- \varepsilon \int_{B_\rho(x_0)} (B\psi_\mu)^2 + \delta \int_{B_\rho(x_0)} (C\psi_\mu)^2 + D\mu^{N-(\beta(N-2)/2)} \max_{B_\rho(x_0)} \{|v_0|^2 + |u_0|\}
+ D\mu^{(N-2)/2} \max_{B_\rho(x_0)} |v_0|.
\]
\[
\leq I_{\lambda, \delta} (u_0, v_0) + \max_{\xi, \zeta, \mu} \left\{ \frac{\varepsilon^2}{2} - \frac{2\varepsilon^{\alpha+\beta}}{\alpha + \beta} \right\} S_{S, \mu, \beta}^{N/2} - D\mu^{(N-2)/2} - D\mu^{N-(\beta(N-2)/2)}
- \lambda D\mu^2 - \delta D\mu^2
= c_0 + \frac{2}{N} \left( \frac{S_{S, \mu, \beta}}{2} \right)^{N/2} - D\mu^{(N-2)/2} - D\mu^{N-(\beta(N-2)/2)} - \lambda D\mu^2 - \delta D\mu^2,
\] (2.27)
where $D$ is different positive constant independent of $\mu$. Since in Theorem A, we assume that $N \geq 7$, thus $2^* = 2N/(N-2) \leq 3$. From $\alpha > 1$, $\beta > 1$, $\alpha + \beta = 2^*$, we know that $\alpha < 2$, $\beta < 2$, so $N - (\beta(N-2)/2) > 2$ and $(N-2)/2 > 2$, Let $\mu$ small enough we have that for $N \geq 7$
\[
\max_{\xi, \zeta, \mu} I_{\lambda, \delta} (\xi U_0 + \zeta \Psi_\mu) < c_0 + \frac{2}{N} \left( \frac{S_{S, \mu, \beta}}{2} \right)^{N/2}.
\] (2.28)

By the property of $U_0$ and $V_{\mu, x_0}$, it is easy to see that we can choose $x_0 \in \Omega$ and $\rho$ small enough such that for any sufficiently small $\mu$, there exists $\xi, \zeta \in \mathbb{R}$ such that $\xi U_0 + \zeta \Psi_\mu \in M$. Thus we complete the proof of Lemma 2.3. □

**End of the proof of Theorem A.** Combining Lemmas 2.2 and 2.3, we have proved that under the assumption of Theorem A, problem (1.1) has at least a solution $(u, v)$ such that both of $u$ and $v$ change sign in $\Omega$. 

Open problems. Whether there exists solutions \((u,v)\) of problem (1.1) such that only one of the functions \(u\) and \(v\) changes sign in \(\Omega\) and the other one keeps sign in \(?\)

References