Multi-peak solutions to coupled Schrödinger systems with Neumann boundary conditions

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ABSTRACT

In this paper, we are concerned with the following two coupled Schrödinger systems in a bounded domain $\Omega \subset \mathbb{R}^N (N = 2, 3)$ with Neumann boundary conditions

$$
\begin{align*}
-\varepsilon^2 \Delta u + u &= \mu_1 u^3 + \beta uv^2, \\
-\varepsilon^2 \Delta v + v &= \mu_2 v^3 + \beta u^2 v, \\
\partial u/\partial n &= 0, \quad \partial v/\partial n = 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

(S$_\varepsilon$)

Suppose the mean curvature $H(P)$ of the boundary $\partial \Omega$ has several local minimums or local maximums, we obtain the existence of solutions with multi-peaks to the system with all peaks being on the boundary and all peaks locate either near the local maxima or near the local minima of the mean curvature at the boundary of the domain.

1. Introduction and main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N \leq 3)$, we are concerned with the following two coupled Schrödinger equations with Neumann boundary conditions in $\Omega$

$$
\begin{align*}
-\varepsilon^2 \Delta u + u &= \mu_1 u^3 + \beta uv^2, \\
-\varepsilon^2 \Delta v + v &= \mu_2 v^3 + \beta u^2 v, \\
\partial u/\partial n &= 0, \quad \partial v/\partial n = 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $\mu_1, \mu_2$ are positive constants, $n$ is the unit outer normal to $\partial \Omega$, $\beta \in \mathbb{R}$, $\varepsilon > 0$ is the parameter.

For $\Omega = \mathbb{R}^N$ and $\varepsilon = 1$, (S$_\varepsilon$) leads to investigate the following problems in $\mathbb{R}^N$

$$
\begin{align*}
-\Delta u + u &= \mu_1 u^3 + \beta uv^2, \\
-\Delta v + v &= \mu_2 v^3 + \beta u^2 v, \\
\partial u/\partial n &= 0, \quad \partial v/\partial n = 0, \quad \text{on } \partial \Omega, \\
\end{align*}
$$

(1.1)

Problem (1.1) arises in the Hartree–Fock theory for a double condensate i.e. a binary mixture of Bose–Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [7]). Physically, $u$ and $v$ are the corresponding condensate amplitudes, $\mu_j$
and $\beta$ are the intraspecies and interspecies scattering lengths. The sign of the scattering length $\beta$ determines whether the interactions of states [1] and [2] are repulsive or attractive. When $\beta > 0$, the interactions of states [1] and [2] are repulsive. In contrast, when $\beta < 0$, the interactions of states [1] and [2] are attractive.

There are a lot of papers which are concerned about the bound states of system (1.1). For example, in B. Sirakov [17], they analyzed for which $\beta$ problem (1.1) assures a least energy solution and for which $\beta$ problem (1.1) has no least energy solution. See also L.A. Maia and E. Nontefusco, B. Pellacci [14], T. Bartsch, Z.Q. Wang and J. Wei [4], M. Lucia and Z. Tang [13], T. Lin and J. Wei [11, 10, 12], J. Wei and T. Weth [19, 20] for the bound states of Schrödinger systems.

We want to point out that recently W. Yao and J. Wei [22] proved that for $0 < \beta < \min(\mu_1, \mu_2)$ and sufficiently small or $\beta > \max(\mu_1, \mu_2)$, the solution of (1.1) is unique. But for all $0 < \beta < \min(\mu_1, \mu_2)$, the uniqueness of solution of (1.1) is still open.

We also refer the readers to Ambrosetti, Antonio, Colraodo, Eduard [2, 1] for the bound states of Schrödinger equations. When the domain in (1.1) is replaced by a symmetric domain (possibly unbounded), T. Bartsch, N. Dancer and Z.Q. Wang [3] investigated the local and global bifurcation in terms of the parameter $\beta$ which provides a-priori bounds of solution branches.

A solution $(u, v)$ of $(S_\epsilon)$ which has a zero component $(u \equiv 0$ or $v \equiv 0$) will be called a standard solution. $(0, 0)$ is referred as the trivial solution of $(S_\epsilon)$. We are interested to study the nonstandard solutions of $(S_\epsilon)$.

The energy functional corresponding to $(S_\epsilon)$ is as follows:

$$J_\epsilon(u, v) := \frac{1}{2} \int_{\Omega} [\epsilon^2|\nabla u|^2 + u^2 + \epsilon^2|\nabla v|^2 + v^2]dx - \frac{1}{4} \int_{\Omega} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2)dx,$$

for every $(u, v) \in H^1(\Omega) \times H^1(\Omega)$.

As in [11], we consider the set

$$\mathcal{N}(\epsilon, \Omega) := \left\{(u, v) \in H^1(\Omega) \times H^1(\Omega), u \geq 0, v \geq 0 : \int_{\Omega} [\epsilon^2|\nabla u|^2 + u^2] = \int_{\Omega} [\mu_1 u^4 + \beta u^2 v^2] \right\}$$

and let

$$c_\epsilon = \inf_{(u, v) \in \mathcal{N}(\epsilon, \Omega)} J_\epsilon(u, v).$$

More recently, in Z. Tang [18], we proved that for any $\epsilon > 0$, when $-\infty < \beta < \min(\mu_1, \mu_2)$ or $\beta > \max(\mu_1, \mu_2)$, system $(S_\epsilon)$ exists a least energy solution $(u_\epsilon, v_\epsilon)$ which achieves $c_\epsilon$ and when $\min(\mu_1, \mu_2) < \beta < \max(\mu_1, \mu_2)$, system $(S_\epsilon)$ has no solution. In particular, we also discussed the asymptotic behavior of $(u_\epsilon, v_\epsilon)$ as $\epsilon$ goes to zero. More precisely, suppose $P_\epsilon, Q_\epsilon$ are the local maximum points of $u_\epsilon, v_\epsilon$ respectively. Let $H(P)$ denote the mean curvature at $P \in \partial \Omega$. Then we have proved that as $\epsilon$ small enough, both $P_\epsilon$ and $Q_\epsilon$ locate on the boundary of $\Omega$. Furthermore when $0 < \beta < \min(\mu_1, \mu_2)$ or $\beta > \max(\mu_1, \mu_2)$, as $\epsilon \to 0$, $(\frac{P_\epsilon - Q_\epsilon}{\epsilon}) \to 0$ and for $N = 2$ and $N = 3$

$$H(P_\epsilon) \to \max_{P \in \partial \Omega} H(P), \quad H(Q_\epsilon) \to \max_{P \in \partial \Omega} H(P).$$

Moreover, $(u_\epsilon, v_\epsilon) \to 0$ in $C^1_{\text{loc}}(\Omega \setminus \{P_\epsilon, Q_\epsilon\})$ and if we let $U_\epsilon(x) = u_\epsilon(x, +\epsilon y), V_\epsilon(x) = v_\epsilon(x, +\epsilon y)$, then as $\epsilon \to 0$, $(U_\epsilon, V_\epsilon) \to (U_0, V_0)$ such that $U_0, V_0 > 0$ and $(U_0, V_0)$ is a least energy solution (1.1) defined in half space $\mathbb{R}_+^N$ with Neumann boundary conditions.

It is therefore natural to ask if the mean curvature of the boundary $\partial \Omega$ admits several local maxima or local minima, can one construct solutions with multi-peak which concentrate at the local maximum points or local minimum points of the mean curvature of the boundary when $\epsilon > 0$ is sufficiently small?

In this paper we aim to answer these questions and our main results are as follows.

**Theorem 1.1.** Suppose $N = 2$ and $N = 3$. Let $\Gamma_i, i = 1, 2, \ldots, K$, be open sets in $\partial \Omega$ such that

$$\min_{P \in \Gamma_i} H(P) > \min_{P \in \Gamma_j} H(P) \quad (i = 1, 2, \ldots, K), \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for } 1 \leq i \neq j \leq K.$$ (1.3)

Then for $\epsilon$ small enough, when $0 \leq \beta < \min(\mu_1, \mu_2)$ or $\beta > \max(\mu_1, \mu_2)$ problem $(S_\epsilon)$ has a solution $(u_\epsilon, v_\epsilon)$ such that $u_\epsilon, v_\epsilon$ possess exactly $K$ local maximum points $P_{\epsilon, 1}, P_{\epsilon, 2}, \ldots, P_{\epsilon, K}$ and $Q_{\epsilon, 1}, Q_{\epsilon, 2}, \ldots, Q_{\epsilon, K}$ respectively such that $P_{\epsilon} := (P_{\epsilon, 1}, P_{\epsilon, 2}, \ldots, P_{\epsilon, K}) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_K$ and $Q_{\epsilon} := (Q_{\epsilon, 1}, Q_{\epsilon, 2}, \ldots, Q_{\epsilon, K}) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_K$. Moreover as $\epsilon \to 0$,

$$\frac{|P_{\epsilon, i} - Q_{\epsilon, i}|}{\epsilon} \to 0, \quad i = 1, 2, \ldots, K.$$ (1.4)

Furthermore

$$H(P_{\epsilon, i}) \to \min_{P \in \Gamma_i} H(P), \quad H(Q_{\epsilon, i}) \to \min_{P \in \Gamma_i} H(P), \quad i = 1, 2, \ldots, K.$$ (1.5)
Theorem 1.2. Suppose $N = 2$ and $N = 3$. Let $A_i$, $i = 1, 2, \ldots, K$, be open sets in $\partial \Omega$ such that

$$\max_{P \in A_i} H(P) < \max_{P \in A_j} H(P) \quad (i = 1, 2, \ldots, K), \quad \Lambda_i \cap \Lambda_j = \emptyset \quad \text{for} \quad 1 \leq i \neq j \leq K. \quad (1.4)$$

Then for $\varepsilon$ small enough, when $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$ problem $(S_\varepsilon)$ has a solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon$, $v_\varepsilon$ possess exactly $K$ local maximum points $P_1, P_2, \ldots, P_K$ and $Q_1, Q_2, \ldots, Q_K$ respectively such that $P_1, P_2, \ldots, P_K \in \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_K$ and $Q_1, Q_2, \ldots, Q_K \in \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_K$. Moreover as $\varepsilon \to 0$,

$$\frac{|P_i - Q_j|}{\varepsilon} \to 0, \quad i = 1, 2, \ldots, K.$$

Furthermore

$$H(P_i) \to \max_{P \in A_i} H(P), \quad H(Q_j) \to \max_{P \in A_j} H(P), \quad i = 1, 2, \ldots, K.$$

Remark 1.3. For scalar cases, the similar problem has been considered by Gui [8]. In that paper the author considered the case when the boundary of the domain admits several local maxima of the mean curvature, they first penalized the nonlinearity and then with a gradient flow argument combining delicate estimates obtained the main results. In our paper, we consider both cases when the boundary of the domain admits several local maxima and several local minima of the mean curvature. Our method of establishing the existence of multi-peak solutions consists first in reducing the problem a finite-dimensional one by a Lyapunov–Schmidt method. Then we use a maximizing and minimizing procedure to obtain multiple boundary spikes.

Remark 1.4. As was shown by P. Pucci and J. Serrin [16], if the mean curvature $H(P)$ admits two local minimum points. Then $H(P)$ should have one more critical point, which is a saddle point of $H(P)$. However our methods to prove Theorems 1.1 and 1.2 does not work for constructing a solution of $(S_\varepsilon)$ which, as $\varepsilon$ small concentrate at the saddle point. So constructing solutions of $(S_\varepsilon)$ which, as $\varepsilon$ small concentrate at saddle points of the mean curvature $H(P)$ is worthy to consider in the future.

At the end of this section, let us point out that although the idea was used before for the scalar equations, the adaptation to the procedure to our problem we are dealing is not trivial at all. Since we are dealing with the strong coupled systems, many delicate estimates are needed because of the interaction between two functions.

2. Technical analysis

Let $U(x)$ be the unique least energy solution of the following problem

$$\begin{cases}
-\Delta u(x) + u(x) = u(x)^3, & u(x) > 0, \quad x \in \mathbb{R}^N, \\
u(0) = \max_{x \in \mathbb{R}^N} u(x), & u(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases} \quad (2.1)$$

Then it is also well known that $U(x)$ is radially symmetric and

$$U(r) = Ar^{-\frac{N-1}{2}}e^{-r} \left(1 + O \left( \frac{1}{r} \right) \right), \quad U'(r) = -Ar^{-\frac{N+1}{2}}e^{-r} \left(1 + O \left( \frac{1}{r} \right) \right) \quad (2.2)$$

for $r$ large, and $A$ is a positive constant. Moreover

$$c_* := I(U(x)) = \inf_{v \in H^1(\mathbb{R}^N), v \neq 0} \sup_{t > 0} I(tv), \quad (2.3)$$

where

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^N} u^4 \, dx. \quad (2.4)$$

Let

$$k = \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}, \quad l = \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}, \quad (2.5)$$

when $0 \leq \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, by Sirakov [17],

$$(U_1, V_1) := (\sqrt{k}u, \sqrt{l}u) \quad (2.6)$$

is a least energy solution of (1.1).
Without loss of generality, we may assume that \( 0 \in \Omega \). By the following rescaling:
\[
\begin{align*}
x = \varepsilon z, \quad \Omega_\varepsilon := \{ z : \varepsilon z \in \Omega \},
\end{align*}
\]
problem (S\(_{\varepsilon}\)) becomes
\[
\begin{align*}
\Delta u - u + \mu_1 u^3 + \beta u v^2 &= 0, \quad \text{in } \Omega_\varepsilon, \\
\Delta v - v + \mu_2 v^3 + \beta u^2 v &= 0, \quad \text{in } \Omega_\varepsilon, \\
u > 0, \quad &v > 0, \quad \text{in } \Omega_\varepsilon, \\
\partial u/\partial n = 0, \quad &\partial v/\partial n = 0, \quad \text{on } \partial \Omega_\varepsilon.
\end{align*}
\]
\[\tag{2.7}\]

Let \( Q \in \partial \Omega \), since we assume that \( \partial \Omega \) is smooth, without loss of generality, we can find a constant \( R_0 > 0 \) and a smooth function \( g : \mathbb{R}^{N-1} \to \mathbb{R} \) such that \( g(0) = 0 \), \( V g(0) = 0 \). Moreover
\[
\begin{align*}
\Omega \cup B(Q, R_0) := \{ (x', x_N) \in B(Q, R_0) : x_N - Q_N > g(x' - Q') \},
\end{align*}
\]
where \( B(Q, R_0) = \{ x : |x - Q| < R_0 \} \) and \( x = (x', x_N) \), \( Q = (Q', Q_N) \) with \( x', Q' \in \mathbb{R}^{N-1} \). The mean curvature of \( \partial \Omega \) at \( Q \) is
\[
H(Q) = \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{\partial^2 g}{\partial x_i^2}(0).
\]
By the Taylor expansion we also have
\[
\begin{align*}
g(x' - Q') &= \frac{1}{2} \sum_{i,j=1}^{N-1} \frac{\partial^2 g}{\partial x_i \partial x_j}(0)(x_i - Q_i)(x_j - Q_j) + \frac{1}{6} \sum_{i,j,k=1}^{N-1} \frac{\partial^3 g}{\partial x_i \partial x_j \partial x_k}(0)(x_i - Q_i)(x_j - Q_j)(x_k - Q_k) \\
&\quad + O(|x' - Q'|^4).
\end{align*}
\]
For any \( x \in \Omega_0 := \Omega \cup B(Q, R_0) \), we introduce the transformation \( T \) by
\[
\begin{align*}
\begin{cases}
T_i(x) = x_i, & i = 1, 2, \ldots, N - 1, \\
T_N(x) = x_N - Q_N - g(x' - Q').
\end{cases}
\end{align*}
\[\tag{2.8}\]
Set
\[
y = \frac{T(x)}{\varepsilon}
\]
and thus \( x = T^{-1}(xy) \).
We set \( U_{Q,x} \) be the unique solution of
\[
\begin{align*}
\Delta u - u + \mu_1 U_1^3 \left( x - \frac{Q}{\varepsilon} \right) + \beta U_1 \left( x - \frac{Q}{\varepsilon} \right) V_1^2 \left( x - \frac{Q}{\varepsilon} \right) &= 0, \quad x \in \Omega_\varepsilon, \\
\frac{\partial u}{\partial n} = 0, \quad &x \in \partial \Omega_\varepsilon
\end{align*}
\]
\[\tag{2.9}\]
and \( V_{Q,x} \) be the unique solution of
\[
\begin{align*}
\Delta v - v + \mu_2 V_1^3 \left( x - \frac{Q}{\varepsilon} \right) + \beta U_1^2 \left( x - \frac{Q}{\varepsilon} \right) V_1 \left( x - \frac{Q}{\varepsilon} \right) &= 0, \quad x \in \Omega_\varepsilon, \\
\frac{\partial v}{\partial n} = 0, \quad &x \in \partial \Omega_\varepsilon.
\end{align*}
\]
\[\tag{2.10}\]
We define for \( x \in \Omega \)
\[
\psi_{Q,x} = U_1 \left( \frac{x - Q}{\varepsilon} \right) - U_{Q,x} \left( \frac{x}{\varepsilon} \right), \quad \phi_{Q,x} = V_1 \left( \frac{x - Q}{\varepsilon} \right) - V_{Q,x} \left( \frac{x}{\varepsilon} \right).
\]
\[\tag{2.11}\]
Then \( \psi_{Q,x} \) satisfies
\[
\begin{align*}
\varepsilon^2 \Delta u - u = 0, & \quad x \in \Omega, \\
\frac{\partial u}{\partial n_x} = \frac{\partial U_1 \left( \frac{x - Q}{\varepsilon} \right)}{\partial n_x}, & \quad x \in \partial \Omega
\end{align*}
\]
\[\tag{2.12}\]
and \( \phi_{Q,x} \) satisfies
\[
\begin{align*}
\varepsilon^2 \Delta v - v = 0, & \quad x \in \Omega, \\
\frac{\partial v}{\partial n_x} = \frac{\partial V_1 \left( \frac{x - Q}{\varepsilon} \right)}{\partial n_x}, & \quad x \in \partial \Omega.
\end{align*}
\]
\[\tag{2.13}\]
In fact for any \( x \in \partial \Omega \) we have
\[
\frac{\partial U_1}{\partial n_x} ( \frac{x - Q}{\varepsilon} ) \frac{\varepsilon |x - Q|}{|x - Q|} = U_1'( \frac{x - Q}{\varepsilon} ) \frac{\varepsilon |x - Q|}{|x - Q|} .
\] (2.14)

Thus for \( x \in \partial \Omega_0 \) we have
\[
\sqrt{1 + |\nabla g(x')|^2} \frac{\partial U_1}{\partial n_x} ( \frac{x - Q}{\varepsilon} ) = U' \left( \frac{x - Q}{\varepsilon} \right) \frac{\varepsilon |x - Q|^2}{|x - Q|} \sqrt{1 + |\nabla g(x')|^2}
\]
\[
= \frac{U_1'(|y|)}{|y|} \left\{ \frac{1}{2} \sum_{i,j=1}^{N-1} g_{ij}(0)y_iy_j + \frac{\varepsilon}{3} \sum_{i,j=1}^{N-1} g_{ij}(0)y_iy_j \right\} + O(\exp(-\mu|z|)),
\] (2.15)

where \( z = \frac{x - Q}{\varepsilon} \).

Let \( u_1 \) be the unique solution of
\[
\begin{cases}
\Delta u - v = 0, & \text{in } \mathbb{R}^N_+,
\end{cases}
\]
\[
\frac{\partial u}{\partial y_N} = - \frac{U_1'}{|y|} \frac{1}{2} \sum_{i,j=1}^{N-1} g_{ij}(0)y_iy_j, \quad \text{on } \partial \mathbb{R}^N_+.
\] (2.16)

and \( v_1 \) be the unique solution of
\[
\begin{cases}
\Delta v - v = 0, & \text{in } \mathbb{R}^N_+,
\end{cases}
\]
\[
\frac{\partial v}{\partial y_N} = - \frac{V_1'}{|y|} \frac{1}{2} \sum_{i,j=1}^{N-1} g_{ij}(0)y_iy_j, \quad \text{on } \partial \mathbb{R}^N_+.
\] (2.17)

where \( U_1' \), \( V_1' \) is the radial derivative of \( U_1 \) and \( V_1 \) respectively. By the definition of \( U_1 \) and \( V_1 \) in (2.6) and the exponential decay of both of \( U(r) U'(r) \), it is easy to check that \( u_1, v_1 \) are even functions in \( y' = (y_1, \ldots, y_{N-1}) \) and \( |u_1| \leq C e^{-\delta |y'|}, |v_1| \leq C e^{-\delta |y'|} \) for some \( \delta > 0 \).

Let \( \Phi \) be a smooth cut-off function such that \( \Phi(x) = 1, x \in B(0, R_0) \) and \( \Phi(x) = 0, x \in \mathbb{R}^N / B(0, R_0) \). The following two lemmas are due to Gui, Wei and Winter [9, Proposition 2.1] or Wei and Winter [21, Proposition 2.1], we here only give the results without proofs.

**Lemma 2.1.**
\[
\psi_{Q,e} = \varepsilon u_1(y) \Phi(x - Q) + \varepsilon^2 \psi_{Q,e}(x), \quad x \in \Omega
\]
and
\[
\phi_{Q,e} = \varepsilon v_1(y) \Phi(x - Q) + \varepsilon^2 \phi_{Q,e}(x), \quad x \in \Omega,
\]
where \( \varepsilon y = T(x) \). Moreover there is a constant \( C > 0 \) such that
\[
\int_{\Omega} \left[ \varepsilon^2 |\nabla \psi_{Q,e}|^2 + \psi_{Q,e}^2 \right] dx \leq C \varepsilon^N \quad \text{and} \quad \int_{\Omega} \left[ \varepsilon^2 |\nabla \phi_{Q,e}|^2 + \phi_{Q,e}^2 \right] dx \leq C \varepsilon^N.
\] (2.18)

**Lemma 2.2.**
\[
\begin{cases}
\frac{\partial U_1}{\partial \tau_q} - \frac{\partial U_{Q,e}}{\partial \tau_q} ( \frac{x - Q}{\varepsilon} ) = \omega_{U,j,1}(y) \Phi(x - Q_j) + \varepsilon \omega_{U,j,2}(x), & x \in \Omega,
\end{cases}
\]
and
\[
\begin{cases}
\frac{\partial V_1}{\partial \tau_q} - \frac{\partial V_{Q,e}}{\partial \tau_q} ( \frac{x - Q}{\varepsilon} ) = \omega_{V,j,1}(y) \Phi(x - Q_j) + \varepsilon \omega_{V,j,2}(x), & x \in \Omega,
\end{cases}
\]
where \( \varepsilon y = T(x) \), \( \tau_q \) denotes the tangential derivatives at \( Q \). Moreover \( \omega_{U,j,1} \) satisfies
\[
\begin{cases}
\Delta u - u = 0, & x \in \mathbb{R}^N_+,
\end{cases}
\]
\[
\frac{\partial u}{\partial y_N} = - \frac{1}{2} \left( \frac{U''}{|y|^2} - \frac{U'}{|y|^3} \right) \sum_{i=1}^{N-1} \frac{\partial^2 g}{\partial x_i \partial x_i} (0)y_iy_i, \quad y \in \partial \mathbb{R}^N_+.
\] (2.19)
and $\omega_{V,j,1}$ satisfies

$$\begin{cases}
\Delta u - u = 0, \\
\frac{\partial u}{\partial y_N} = -\frac{1}{2} \left( \frac{V''_y}{|y|^2} - \frac{V'_y}{|y|^3} \right) \sum_{i,j=1}^{N-1} \frac{\partial^2 g}{\partial x_i \partial x_j}(0) y_i y_j, \\
\end{cases} \quad x \in \mathbb{R}^N_+,$$

(2.20)

where $\omega_{U,j,1}$, $\omega_{V,j,1}$ are odd functions in $y' = (y_1, \ldots, y_{N-1})$ and $|\omega_{U,j,1}| \leq C e^{-\delta|y|}$, $|\omega_{V,j,1}| \leq C e^{-\delta|y|}$ for some $\delta > 0$. Furthermore, there is a constant $C > 0$ such that

$$\int_{\Omega} |\nabla \omega_{U,j,1}|^2 + \omega_{U,j,2} dx \leq C e^N \quad \text{and} \quad \int_{\Omega} |\nabla \omega_{V,j,1}|^2 + \omega_{V,j,2} dx \leq C e^N.$$

(2.21)

At the end of this section, we give a nondegeneracy property of $(U_1, V_1) = (\sqrt{U}, \sqrt{V})$, namely the following lemma which is a direct result of Dancer and Wei [6, Lemma 2.2 and Theorem 3.1] and Ni and Takagi [15, Lemma 4.2].

**Lemma 2.3.** The solution $(U_1, V_1)$ is nondegenerate, namely suppose $(\psi, \phi) \in H^2(\mathbb{R}^N_+) \times H^2(\mathbb{R}^N_+)$ satisfies the following eigenvalue problem:

$$\begin{cases}
\Delta \psi - \psi + 3 \mu_1 U_1^2 \psi + \beta V_1^2 \psi + 2 \beta U_1 V_1 \phi = 0, \\
\Delta \phi - \phi + 3 \mu_2 V_1^2 \phi + \beta U_1^2 \phi + 2 \beta U_1 V_1 \psi = 0, \\
\frac{\partial \psi}{\partial y_N} = \frac{\partial \phi}{\partial y_N} = 0, \\
\end{cases} \quad y \in \partial \mathbb{R}^N_+,$$

(2.22)

where $\beta \notin [\min(\mu_1, \mu_2), \max(\mu_1, \mu_2)]$ and $\mathbb{R}^N_+ = \{y' : y_N > 0\}$. Then

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \in \text{span} \begin{pmatrix} \partial U_1 \\ \partial V_1 \\ \partial U_1 \\ \partial V_1 \\ \partial y_N \\ \partial y_N \\ \partial y_{N-1} \end{pmatrix}.$$


### 3. Lyapunov–Schmidt reduction

In this section, we reduce problem (2.7) to finite dimensions by the Lyapunov–Schmidt reduction method. Let

$$H^2_N(\Omega_\epsilon) = \left\{ u \in H^2(\Omega_\epsilon) : \frac{\partial u}{\partial n} = 0 \right\},$$

where $n$ denotes the external unit normal vector at $x \in \Omega_\epsilon$.

For any $(u, v) \in H^2_N(\Omega_\epsilon) \times H^2_N(\Omega_\epsilon)$, we set

$$S_\epsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} S_1(u, v) \\ S_2(u, v) \end{pmatrix},$$

where

$$S_1(u, v) = \Delta u - u + \mu_1 u^3 + \beta vv^2, \quad S_2(u, v) = \Delta v - v + \mu_2 v^3 + \beta u^2 v.$$

Then solving system (2.7) is equivalent to solve

$$S_\epsilon \begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad (u, v) \in H^2_N(\Omega_\epsilon) \times H^2_N(\Omega_\epsilon).$$

(3.1)

The energy functional corresponding to (2.7) is as follows:

$$\Phi_\epsilon(u, v) := \frac{1}{2} \int_{\Omega_\epsilon} \left[ |\nabla u|^2 + u^2 + |\nabla v|^2 + v^2 \right] dx - \frac{1}{4} \int_{\Omega_\epsilon} \left( \mu_1 u^4 + \mu_2 v^4 + 2 \beta u^2 v^2 \right) dx,$$

(3.2)

for every $(u, v) \in H^1(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$.

We define

$$\Gamma := \{Q = (Q_1, Q_2, \ldots, Q_k) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k\}$$

and

$$\Lambda := \{Q = (Q_1, Q_2, \ldots, Q_k) \in \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_k\}.$$
For any $Q = (Q_1, Q_2, \ldots, Q_k) \in \Gamma$ (or $\Lambda$), we first consider the following linearized operator

$$
\tilde{L}_e(\psi, \phi) := \left(\frac{\partial^2}{\partial \tau_{Q_i}} \psi, \phi \right)_{H^2(\Omega_{\epsilon})} + 2H^2(\Omega_{\epsilon}) \times L^2(\Omega_{\epsilon}),
$$

where

$$
\tilde{L}_e(\psi, \phi) = \Delta \psi - \psi + 3\mu_1 \left( \sum_{i=1}^{K} U_{Q_i, \epsilon} \right)^2 \psi + \beta \left( \sum_{i=1}^{K} V_{Q_i, \epsilon} \right)^2 \psi + 2\beta \left( \sum_{i=1}^{K} U_{Q_i, \epsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i, \epsilon} \right) \phi
$$

and

$$
\tilde{L}_e(\psi, \phi) = \Delta \phi - \phi + 3\mu_2 \left( \sum_{i=1}^{K} V_{Q_i, \epsilon} \right)^2 \phi + \beta \left( \sum_{i=1}^{K} U_{Q_i, \epsilon} \right)^2 \phi + 2\beta \left( \sum_{i=1}^{K} U_{Q_i, \epsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i, \epsilon} \right) \psi.
$$

$U_{Q_i, \epsilon}, V_{Q_i, \epsilon}$ are the unique solutions of (2.9) and (2.10) respectively with $Q$ is replaced by $Q_i$.

We denote

$$
U_{Q_i, \epsilon} = \left( \frac{U_{Q_i, \epsilon}, V_{Q_i, \epsilon}}{V_{Q_i, \epsilon}} \right).
$$

It is easy to see that cokernel of $\tilde{L}_e$ coincides with its kernel and we choose cokernel and kernel as

$$
\mathcal{C}_{e, Q} = \mathcal{K}_{e, Q} = \text{span} \left\{ \frac{\partial U_{Q_i, \epsilon}}{\partial \tau_{Q_i}} : i = 1, 2, \ldots, K, j = 1, 2, \ldots, N - 1 \right\}.
$$

Let $P_{e, Q}$ denote the projection from $L^2(\Omega_{\epsilon}) \times L^2(\Omega_{\epsilon})$ onto $\mathcal{C}_{e, Q}$. Our goal in this section is to show that the system

$$
P_{e, Q} \circ S_i \left( \sum_{i=1}^{K} U_{Q_i, \epsilon} + \omega_{e, Q} \right) = 0
$$

has a unique solution $\omega_{e, Q} \in \mathcal{K}_{e, Q}$ for $e$ sufficiently small and $Q = (Q_1, Q_2, \ldots, Q_k) \in \tilde{\Gamma}$ (or $\tilde{\Lambda}$). We have the following results.

**Proposition 3.1.** Let $L_{e, Q} = P_{e, Q} \circ \tilde{L}_e$. There exist positive constants $\epsilon_0, \delta_0$ such that for all $e \in (0, \epsilon_0)$ and $Q = (Q_1, Q_2, \ldots, Q_k) \in \Gamma$ (or $\Lambda$)

$$
\|L_{e, Q} h\|_{L^2(\Omega_{\epsilon}) \times L^2(\Omega_{\epsilon})} \geq \delta_0 \|h\|_{H^2(\Omega_{\epsilon}) \times H^2(\Omega_{\epsilon})}
$$

(3.3)

for all $h \in \mathcal{K}_{e, Q}$.

**Proof.** We will follow the method used by Wei and Winter [21]. Suppose that (3.3) is false. Then there exist sequences $\{\epsilon_m\}, \{Q_{\epsilon_m}\}$ and $\{h_m\}(m = 1, 2, \ldots)$ with $\epsilon_m \to 0$, $Q_{\epsilon_m} \to \Gamma$ (or $\Lambda$) and $h_m = (h^1_m, h^2_m)^T \in \mathcal{K}_{e, Q}$ such that as $m \to \infty$

$$
\epsilon_m \to 0,
$$

(3.4)

$$
Q_{\epsilon_m} \to Q \in \Gamma \text{ (or $\Lambda$)},
$$

(3.5)

$$
\|L_{\epsilon_m, Q_{\epsilon_m}} h_m\|_{L^2(\Omega_{\epsilon_m}) \times L^2(\Omega_{\epsilon_m})} \to 0,
$$

(3.6)

$$
\|h_m\|_{H^2(\Omega_{\epsilon_m}) \times H^2(\Omega_{\epsilon_m})} = 1.
$$

(3.7)

For $j = 1, 2, \ldots, N - 1$ we denote

$$
e_{\epsilon, Q_{\epsilon_m}} = \frac{\partial U_{Q_{\epsilon_m}, \epsilon}}{\partial \tau_{Q_{\epsilon_m}, j}} \left( y - \frac{Q_{\epsilon_m}}{\epsilon_m} \right).
$$

We define an inner product

$$
\left( \begin{array}{c}
(u_1) \\
v_1
\end{array} \right)_e : = \int_{\Omega_{\epsilon}} (u_1 u_2 + v_1 v_2) dx, \text{ for } \left( \begin{array}{c}
(u_1) \\
v_1
\end{array} \right) \in L^2(\Omega_{\epsilon}) \times L^2(\Omega_{\epsilon}), \text{ } i = 1, 2.
$$

(3.8)

We denote

$$
\left( \begin{array}{c}
u_1 \\
v_1
\end{array} \right). \left( \begin{array}{c}
u_2 \\
v_2
\end{array} \right) = u_1 u_2 + v_1 v_2.
$$
By Lemma 2.2 and the fact $Q \in \tilde{\Gamma}$ (or $\tilde{\Lambda}$) with the exponential decay of $U(x)$ we have
\[
(e_{i_1,j_1,m}, e_{i_2,j_2,m})_{\Omega_m} = \delta_{i_1i_2} \delta_{j_1j_2} + O(e_m),
\]
where $\delta_{ij}$ is the Kronecker symbol. Furthermore, from (3.6) we have as $m \to \infty$
\[
\|\tilde{L}_{\tau_m} h_m\|_{L^2(\Omega_m^c) \times L^2(\Omega_m)} - \sum_{i=1}^N \sum_{j=1}^N \int_{\Omega_m} \tilde{L}_{\tau_m} h_m \cdot e_{i,j,m} \to 0. \tag{3.9}
\]
Let $\Omega_0, \mathcal{X}, g$ and $T$ be as defined in Section 2 (note that we allow $R_0 \to 0$ but $R_0 / \epsilon \to \infty$ as $\epsilon \to 0$). Then $T$ has an inverse $T^{-1}$ such that
\[
T^{-1} : T(B(Q, R_0) \cap \tilde{\Omega}) \to B(Q, R_0) \cap \tilde{\Omega}.
\]
We use the notation $T^j$ if $Q$ is replaced by $Q_i$ and we define $H_{i,m}(y)$ for $y \in \mathbb{R}^N_+: (y', y_N) : y_N > 0$ by
\[
H_{i,m}(y) = \mathcal{X} \left( \frac{(T^j)^{-1}(\epsilon_m y) - Q_{i,m}}{\epsilon_m} \right) h_{m} \left( \frac{(T^j)^{-1}(\epsilon_m y) - Q_{i,m}}{\epsilon_m} \right).
\]
Since $T^j$, $(T^j)^{-1}$ have bounded derivatives, it follows from (3.7) and the smoothness of $\mathcal{X}$ that
\[
\|H_{i,m}\|_{H^2(\mathbb{R}^N_+) \times H^2(\mathbb{R}^N_+) \rightarrow 0} \leq C
\]
for all $m$ sufficiently large. It is also easy to see that
\[
\|H_{i,m}\|_{H^2(\mathbb{R}^N_{+}) \times H^2(\mathbb{R}^N_{+}) \rightarrow 0} \to 0 \quad \text{as} \quad R \to \infty
\]
uniformly for all $k$ large. Thus up to a subsequence $H_{i,m}$ converges weakly in $H^2(\mathbb{R}^N_+) \times H^2(\mathbb{R}^N_+)$ to a limit $H_{i,\infty}$. We now aim to prove that $H_{i,\infty} \equiv (0, 0)$. First we prove that
\[
\int_{\mathbb{R}^N_+} H_{i,m} \cdot \frac{\partial U}{\partial y_j} dy = 0, \quad j = 1, 2, \ldots, N - 1, \tag{3.11}
\]
where $U = (U_1, V_1)$.

Indeed, we have
\[
\int_{\mathbb{R}^N_+} H_{i,m}(y) \cdot \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( (T^j)^{-1}(\epsilon_m y) - Q_{i,m} \right) dy
\]
\[
= \epsilon_m^{-N} \int_{\partial_0} \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) dx
\]
\[
= \epsilon_m^{-N} \int_{\Omega} h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) dx - \epsilon_m^{-N} \int_{\partial_0} \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) dx
\]
\[
= 0 + \epsilon_m^{-N} \int_{\partial_0} \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \left[ \frac{\partial U}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) - \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \right] dx
\]
\[
+ \epsilon_m^{-N} \int_{\partial_0} \left[ 1 - \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \right] h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \left[ \frac{\partial U}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) - \frac{\partial U_{Q_{i,m},e_m}}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \right] dx
\]
\[
- \epsilon_m^{-N} \int_{\partial_0} \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \frac{\partial U}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) dx
\]
\[
- \epsilon_m^{-N} \int_{\partial_0} \left[ 1 - \mathcal{X} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \right] h_{m} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) \cdot \frac{\partial U}{\partial \tau_{Q_{i,m}}} \left( \frac{x - Q_{i,m}}{\epsilon_m} \right) dx
\]
\[
\to 0 \quad \text{as} \quad m \to \infty.
\]
In the last expression of the above formula, the first two terms tend to zero due to Lemma 2.2. The last two terms tend to zero because of the exponential decay of $\frac{\partial U}{\partial \tau_{Q_{i,m}}}$ at infinity. Thus we indeed proved (3.11).
Let $\mathcal{K}_0$, $\mathcal{C}_0$ be the kernel and cokernel, respectively, of the linearized operator

$$L_0(\psi, \phi) := \begin{pmatrix} \Delta \psi - \psi + 3 \mu_1 U_2^\ast \psi + \beta V_2 \psi + 2 \beta U_1 V_1 \phi \\ \Delta \phi - \phi + 3 \mu_2 V_1^2 \phi + \beta U_1 \phi + 2 \beta U_1 V_1 \psi \end{pmatrix}$$

$$H_0^1(\mathbb{R}^N) \times H_0^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N).$$

Then by Lemma 2.3

$$\mathcal{K}_0 = \mathcal{C}_0 = \text{span} \left\{ \frac{\partial U}{\partial y_j} : j = 1, 2, \ldots, N - 1 \right\}.$$

Thus Eq. (3.11) implies that $H_{i, \infty} \in \mathcal{K}_0^\perp$. On the other hand by (3.6) and the exponential decay of $U$, after a further subsequence if necessary we have

$$L_0(H_{i, \infty}) = 0,$$

which indicates that $H_{i, \infty} \in \mathcal{K}_0$. Therefore $H_{i, \infty} = (0, 0)$. Hence we have

$$H_{i, m} \to (0, 0) \quad \text{weakly in } H_0^1(\mathbb{R}^N) \times H_0^1(\mathbb{R}^N) \quad \text{as } m \to \infty. \quad (3.12)$$

By the definition of $H_{i, m}$, we know that $h_m = (h_{i, m}^1, h_{i, m}^2) \to (0, 0)$ in $H_0^1(\Omega_m) \times H_0^1(\Omega_m)$ and thus by Lemma 2.1, we have

$$\int_{\Omega_m} \left[ 3 \mu_1 \left( \sum_{i=1}^{K} U_{Q_i, m, \epsilon} \right)^2 + \beta \left( \sum_{i=1}^{K} V_{Q_i, m, \epsilon} \right)^2 \right] dx \to 0 \quad \text{as } m \to \infty \quad (3.13)$$

and

$$\int_{\Omega_m} \left[ 3 \mu_2 \left( \sum_{i=1}^{K} V_{Q_i, m, \epsilon} \right)^2 + \beta \left( \sum_{i=1}^{K} U_{Q_i, m, \epsilon} \right)^2 \right] dx \to 0 \quad \text{as } m \to \infty. \quad (3.14)$$

On the other hand, by (3.7) we know that $\|h_{i, m}^1, h_{i, m}^2\|_{L^2(\Omega_m) \times L^2(\Omega_m)} \leq C$ for some constant $C > 0$ and uniformly for $m \geq 1$, where $2^*$ is the critical Sobolev exponent, more precisely, $2^* = +\infty$ for $N = 1, 2$ and $2^* = \frac{2N}{N-2}$ for $N = 3$. Thus combining (3.13) and (3.14), after the interpolation inequality, we indeed have

$$\left\| \left( 3 \mu_1 \left( \sum_{i=1}^{K} U_{Q_i, m, \epsilon} \right)^2 + \beta \left( \sum_{i=1}^{K} V_{Q_i, m, \epsilon} \right)^2 \right) h_{i, m}^1 \right\|_{L^2(\Omega_m)} \to 0 \quad \text{as } m \to \infty \quad (3.15)$$

and

$$\left\| \left( 3 \mu_2 \left( \sum_{i=1}^{K} V_{Q_i, m, \epsilon} \right)^2 + \beta \left( \sum_{i=1}^{K} U_{Q_i, m, \epsilon} \right)^2 \right) h_{i, m}^2 \right\|_{L^2(\Omega_m)} \to 0 \quad \text{as } m \to \infty. \quad (3.16)$$

(3.15) and (3.16) implies that

$$\|(\Delta - 1)h_m\|_{L^2(\Omega_m) \times L^2(\Omega_m)} \to 0 \quad \text{as } m \to \infty. \quad (3.17)$$

Combining (3.15)-(3.17) we have

$$\|h_m\|_{H^1(\Omega_m) \times H^1(\Omega_m)} \to 0 \quad \text{as } m \to \infty. \quad (3.18)$$

Again using (3.15)-(3.17) and combining (3.18) we show that

$$\|h_m\|_{H^2(\Omega_m) \times H^2(\Omega_m)} \to 0 \quad \text{as } m \to \infty \quad (3.19)$$

which contradicts with (7) and thus we complete the proof of Proposition 3.1.
By Proposition 3.1 we immediately have the following proposition.

**Proposition 3.2.** For all $\varepsilon \in (0, \varepsilon_0)$ and $Q = (Q_1, Q_2, \ldots, Q_k) \in \tilde{F}$ (or $\tilde{A}$), the map

$$L_{\varepsilon,Q} = P_{\varepsilon,Q} \circ \tilde{L}_\varepsilon : \mathcal{K}_{\varepsilon,Q}^\perp \to \mathcal{E}_{\varepsilon,Q}^\perp$$

is surjective.

**Proof.** We define a linear operator $\mathcal{T}$ from $L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ to itself as follows:

$$\mathcal{T} = P_{\varepsilon,Q} \circ \tilde{L}_\varepsilon.$$  

Its domain of definition is $H^2(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$. By the theory of elliptic systems it is easy to see that $\mathcal{T}$ is a self-adjoint operator on $L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ and also is a closed operator which deduces that the range of $\mathcal{T}$ is closed in $H^2(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$. Then by the Closed Range Theorem we know that the range of $\mathcal{T}$ is the orthogonal complement of its kernel which is, by Proposition 3.1, $\mathcal{K}_{\varepsilon,Q}$ and hence the proof is completed.

Now we are in a position to solve the system

$$P_{\varepsilon,Q} \circ \tilde{L}_\varepsilon \left( \sum_{i=1}^K U_{Q_i,m} + h_{Q_i,m} \right) = 0. \quad (3.20)$$

By Propositions 3.1 and 3.2, we conclude that the operator $L_{\varepsilon,Q}|_{\mathcal{K}_{\varepsilon,Q}^\perp}$ is invertible (call its inverse as $L_{\varepsilon,Q}^{-1}$) and thus we can rewrite (3.20) as

$$h = -(L_{\varepsilon,Q}^{-1} \circ P_{\varepsilon,Q}) \left( S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} + h \right) \right) - (L_{\varepsilon,Q}^{-1} \circ P_{\varepsilon,Q})N_{\varepsilon,Q}(h)$$

$$= G_{\varepsilon,Q}(h), \quad (3.21)$$

where

$$N_{\varepsilon,Q}(h) = S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} + h \right) - \left[ S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} \right) + S_\varepsilon \left( \sum_{i=1}^K V_{Q_i,\varepsilon} \right) \right] \cdot h. \quad (3.22)$$

We are going to show that the operator $G_{\varepsilon,Q}$ is a contraction on

$$B_{\varepsilon,\delta} := \{ h \in H^2(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon) : \| h \|_{H^2(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)} < \delta \}$$

for $\delta$ sufficiently small.

First we have the following technical lemma.

**Lemma 3.3.** For $\varepsilon$ sufficiently small, we have

$$|N_{\varepsilon,Q}(h)| \leq C \left( \| h \|^2 + \| h \|^3 \right), \quad (3.23)$$

$$\left\| S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} \right) \right\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon, \quad (3.24)$$

where $h = (h^1, h^2)^\top$ and $U_{Q_i,\varepsilon} = (U_{Q_i,\varepsilon}, V_{Q_i,\varepsilon})^\top$.

**Proof.** From the definition of $S_\varepsilon$,

$$S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} + h \right) = \begin{pmatrix} S_1 \left( \sum_{i=1}^K U_{Q_i,\varepsilon} + h^1, \sum_{i=1}^K V_{Q_i,\varepsilon} + h^2 \right) \\ S_2 \left( \sum_{i=1}^K U_{Q_i,\varepsilon} + h^1, \sum_{i=1}^K V_{Q_i,\varepsilon} + h^2 \right) \end{pmatrix},$$

$$S_\varepsilon \left( \sum_{i=1}^K U_{Q_i,\varepsilon} \right) = \begin{pmatrix} S_1 \left( \sum_{i=1}^K U_{Q_i,\varepsilon}, \sum_{i=1}^K V_{Q_i,\varepsilon} \right) \\ S_2 \left( \sum_{i=1}^K U_{Q_i,\varepsilon}, \sum_{i=1}^K V_{Q_i,\varepsilon} \right) \end{pmatrix},$$

and so on.
where
\[ S_1(u, v) = \Delta u - u + \mu_1 u^3 + \beta uv^2, \quad S_2(u, v) = \Delta v - v + \mu_2 v^3 + \beta u^2 v. \]

On the other hand
\[ S' \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) \cdot h = \left( \begin{array}{c} \tilde{I}_1^1(\hspace{1mm} h^1, h^2) \\ \tilde{I}_2^2(\hspace{1mm} h^1, h^2) \end{array} \right), \]

where
\[ \tilde{I}_1^1(h^1, h^2) = (\Delta - 1)h^1 + 3\mu_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) h^1 + \beta \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) h^1 + 2\beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) h^2 \]

and
\[ \tilde{I}_2^2(h^1, h^2) = (\Delta - 1)h^2 + 3\mu_2 \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) h^2 + \beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) h^2 + 2\beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) h^1. \]

After a direct calculation we have
\[ S_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} + h \right) - S_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) - \tilde{I}_1^1(h^1, h^2) \]
\[ = 3\mu_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) (h^1)^2 + \mu_1(h^1)^3 + \beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) (h^2)^2 + 2 \beta \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) h^1 h^2 + (h^1) h^2 \]

and
\[ S_2 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} + h \right) - S_2 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) - \tilde{I}_2^2(h^1, h^2) \]
\[ = 3\mu_2 \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) (h^2)^2 + \mu_2(h^2)^3 + \beta \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right) (h^1)^2 + 2 \beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) h^1 h^2 + (h^1)^2 h^2 \]

By \textbf{Lemma 2.1} we know that the $L^\infty$ norm of $U_{Q_i,\varepsilon}$, $V_{Q_i,\varepsilon}$ are bounded with respect to $\varepsilon$ and hence (3.23) follows.

Now we come to show (3.24). By the definition of $U_{Q_i,\varepsilon}$ and $V_{Q_i,\varepsilon}$ we have

\[ S_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) = \mu_1 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right)^3 + \beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right)^2 \]
\[ - \mu_1 \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^3 - \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \] \hspace{1cm} (3.25)

and

\[ S_2 \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) = \mu_2 \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right)^3 + \beta \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right) \left( \sum_{i=1}^{K} V_{Q_i,\varepsilon} \right)^2 \]
\[ - \mu_2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^3 - \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \] \hspace{1cm} (3.26)

Let
\[ \Omega_i = \left\{ x \in \Omega : |x - Q_i| \leq \frac{1 - \delta}{2} \min_{k \neq i} |Q_k - Q_i| \right\}, \quad i = 1, 2, \ldots, K, \]
\[ \Omega_{K+1} = \Omega \setminus \bigcup_{i=1}^{K} \Omega_i. \]

Now we estimate $S_\varepsilon \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \right)$ in each domain.
For $x \in \Omega_{K+1}$, we have $|x - Q_i| \geq \frac{1-\delta}{2} \min_{k \neq l} |Q_k - Q_l|$. Again by Lemma 2.1 and combining (3.25) and (3.26), we have for any $x \in (\Omega_{K+1})_\varepsilon$

$$\left| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \left( \frac{x}{\varepsilon} \right) \right) \right| \leq C e^{-\left(\frac{1-\delta}{2}\right) \min_{k \neq l} |Q_k - Q_l|}. $$

Thus we have

$$\left\| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} (x) \right) \right\|_{L^2((\Omega_{K+1})_\varepsilon) \times L^2((\Omega_{K+1})_\varepsilon)} \leq C \varepsilon^{-\left(\frac{1-\delta}{2}\right)} \min_{k \neq l} |Q_k - Q_l|. \tag{3.27}$$

In $\Omega_1$, by the exponential decay of $U_1, V_1$, it is easy to check that

$$\left| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} \left( \frac{x}{\varepsilon} \right) \right) \right| \leq C \left( \left\| \phi_{Q_i,\varepsilon} (x) \right\| + C e^{-\left(\frac{1-\delta}{2}\right) \min_{k \neq l} |Q_k - Q_l|},$$

where $\phi_{Q_i,\varepsilon}(x), \phi_{Q_i,\varepsilon}(x)$ as defined in (2.11) in the above section. Thus we have

$$\left\| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} (x) \right) \right\|_{L^2((\Omega_{1})_\varepsilon) \times L^2((\Omega_{1})_\varepsilon)} \leq C \varepsilon. \tag{3.28}$$

Combining (3.27) and (3.28) we proved (3.24). Thus the proof of the Lemma 3.3 is completed. $\square$

Now we come to see the operator $G_{\varepsilon, \varepsilon}(h)$. Indeed by Lemma 3.3, for $h \in B_{\varepsilon, \delta}$

$$\left\| G_{\varepsilon, \varepsilon}(h) \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)} \leq \delta_0^{-1} \left( \left\| P_{Q_1} \circ N_{Q_1,\varepsilon} \right\|_{L^2(\Omega_1) \times L^2(\Omega_1)} + \left\| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} (x) \right) \right\| \right)_{L^2(\Omega_1) \times L^2(\Omega_1)} \leq \delta \delta_0^{-1} C (\delta^2 + \varepsilon).$$

Similarly we also have

$$\left\| G_{\varepsilon, \varepsilon}(h^1) - G_{\varepsilon, \varepsilon}(h^2) \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)} \leq \delta_0^{-1} C \delta \left\| h^1 - h^2 \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)}.$$

Therefore $G_{\varepsilon, \varepsilon}$ is a contradiction on $B_{\varepsilon, \delta}$. By the Contraction Mapping Principle, there exists a unique fixed point $h_{\varepsilon, \varepsilon}$ which is a solution of (3.21). On the other hand since

$$\left\| G_{\varepsilon, \varepsilon}(h_{\varepsilon, \varepsilon}) \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)} \leq \delta_0^{-1} \left( \left\| N_{Q_1,\varepsilon} (h_{\varepsilon, \varepsilon}) \right\|_{L^2(\Omega_1) \times L^2(\Omega_1)} + \left\| S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} (x) \right) \right\| \right)_{L^2(\Omega_1) \times L^2(\Omega_1)} \leq \delta_0^{-1} C \left( \varepsilon + \delta \right) \left\| h_{\varepsilon, \varepsilon} \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)},$$

we have

$$\left\| h_{\varepsilon, \varepsilon} \right\|_{H^2(\Omega_1) \times H^2(\Omega_1)} \leq C \varepsilon. \tag{3.29}$$

Thus we indeed have proved the following lemma.

**Lemma 3.4.** There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $Q = (Q_1, Q_2, \ldots, Q_K) \in \Gamma$ (or $\Lambda$), there is a unique $h_{\varepsilon, \varepsilon} \in K_{\varepsilon, \varepsilon}$ satisfying (3.29) and $S_x \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} + h_{\varepsilon, \varepsilon} \right) \in E_{\varepsilon, \varepsilon}$.

### 4. Estimates for the reduced problem

In this section, we give a precise estimate of the energy for the reduced problem. Let $\Phi_\varepsilon (u, v)$ be the energy functional of (2.7) as defined in Section 2. We have the following proposition.

**Proposition 4.1.** Suppose $h_{\varepsilon, \varepsilon} \in K_{\varepsilon, \varepsilon}$ is obtained in Lemma 3.4. Then we have

$$\Phi_\varepsilon \left( \sum_{i=1}^{K} U_{Q_i,\varepsilon} (x) + h_{\varepsilon, \varepsilon} \right) = \left[ \frac{K}{2} \left( k^2 + H^2 \right) H(U) - \gamma_1 \sum_{i=1}^{K} H(Q_i) + O \left( \sum_{i,j=1}^{K} e^{-\frac{|Q_i - Q_j|}{\varepsilon}} \right) + o(\varepsilon) \right]. \tag{4.1}$$
where \( k = \sqrt{\frac{\mu_2 - \mu_1}{\mu_1 \mu_2 - \beta^2}} \) and \( l = \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}} \), as defined in (2.5) and \( I(u) \) is defined by (2.4), \( H(Q) \) denotes the mean curvature of the boundary at \( Q \), where

\[
\gamma_1 = \frac{k^2 + l^2}{N + 1} \int_{\partial Q} |\nabla U|^2 |\gamma|^2 dy.
\] (4.2)

To prove Proposition 4.1, we first give the following technical lemmas.

**Lemma 4.2.** For any \( Q = (Q_1, Q_2, \ldots, Q_K) \in I' \) (or \( A \)), when \( \epsilon \) small enough

\[
\phi_\epsilon \left( \sum_{i=1}^{K} U_{Q_i}(x) \right) = \frac{1}{2} \int_{\Omega_\epsilon} \left[ \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 + \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 \right] dx
\]

\[
+ \frac{1}{4} \int_{\Omega_\epsilon} \left[ \epsilon \left( \sum_{i=1}^{K} U_{Q_i}(x) \right) + \mu_1 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right) \right] dx
\]

\[
- \frac{1}{2} \int_{\Omega_\epsilon} \beta \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 dx.
\] (4.3)

**Proof.** First,

\[
\phi_\epsilon \left( \sum_{i=1}^{K} U_{Q_i}(x) \right) = \frac{1}{2} \int_{\Omega_\epsilon} \left[ \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 + \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 \right] dx
\]

\[
+ \frac{1}{4} \int_{\Omega_\epsilon} \left[ \epsilon \left( \sum_{i=1}^{K} U_{Q_i}(x) \right) + \mu_1 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right) \right] dx
\]

\[
- \frac{1}{2} \int_{\Omega_\epsilon} \beta \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 dx.
\] (4.4)

By the definition of \( U_{Q_i}(x) = (U_{Q_i}(x), V_{Q_i}(x)) \) we have

\[
\frac{1}{2} \int_{\Omega_\epsilon} \left[ \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 + \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 \right] dx
\]

\[
= \frac{1}{2} \int_{\Omega_\epsilon} \left[ \epsilon \left( \sum_{i=1}^{K} U_{Q_i}(x) \right) + \mu_1 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right) \right] dx
\]

\[
+ \frac{1}{2} \int_{\Omega_\epsilon} \beta \left( \sum_{i=1}^{K} U_{Q_i}(x) \right)^2 \left( \sum_{i=1}^{K} V_{Q_i}(x) \right)^2 dx.
\] (4.5)
\[
\frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \mu_1 \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 + \mu_2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 \right] dx \\
= \frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \mu_1 \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 + \mu_2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 \right] dx \\
+ \int_{\Omega_{\varepsilon}} \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 dx + I_1 + I_2. 
\]

(4.6)

Since \( U_1(x) = kU(x), V_1(x) = lU(x), \) thus by a standard argument (see Gui–Wei–Winter [9, Lemma 2.6] and Cao–Küpper [5, Appendix C]), we have

\[
\frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \mu_1 \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 + \mu_2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^4 \right] dx \\
+ \int_{\Omega_{\varepsilon}} \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 dx \\
= \frac{1}{2} \left( k^2 + l^2 \right) \int_{\partial B_{\varepsilon}} U^4 dy - \gamma_2 \varepsilon^2 \sum_{i=1}^{K} H(Q_i) + O \left( \sum_{i,j=1}^{K} e^{-|Q_i - Q_j|/\varepsilon} \right). 
\]

(4.7)

where

\[
\gamma_2 = \frac{1}{2} \left( k^2 + l^2 \right) \int_{\partial B_{\varepsilon}} U^4 |y|^2 dy.
\]

Now we estimate \( I_1 \) and \( I_2. \) Indeed by the exponential decay of \( U_1, V_1 \) we have

\[
I_1 = \frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \mu_1 \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^3 \left( U_{Q_i,\varepsilon}(x) - U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) \right] dx \\
+ \frac{1}{2} \int_{\Omega_{\varepsilon}} \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \left( U_{Q_i,\varepsilon}(x) - U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) dx \\
= \sum_{i=1}^{K} \frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \mu_1 \left( U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^3 \left( U_{Q_i,\varepsilon}(x) - U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) \right] dx \\
+ \sum_{i=1}^{K} \frac{1}{2} \int_{\Omega_{\varepsilon}} \beta \left( \sum_{i=1}^{K} U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) \left( \sum_{i=1}^{K} V_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right)^2 \left( U_{Q_i,\varepsilon}(x) - U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) dx \\
+ O \left( \sum_{i,j=1}^{K} e^{-|Q_i - Q_j|/\varepsilon} \right) \\
= \frac{1}{2} \sum_{i=1}^{K} \int_{\Omega_{\varepsilon}} \left[ \Delta U_{Q_i,\varepsilon}(x) - U_{Q_i,\varepsilon}(x) \right] \left( U_{Q_i,\varepsilon}(x) - U_1 \left( x - \frac{Q_i}{\varepsilon} \right) \right) dx + O \left( \sum_{i,j=1}^{K} e^{-|Q_i - Q_j|/\varepsilon} \right). 
\]

(4.8)

Now for every fixed \( i \in \{1, 2, \ldots, K\} \) we have

\[
= \frac{1}{2} e^{-N} \int_{\Omega} \left[ \varepsilon^2 \Delta U_{Q_i,\varepsilon}(x) - U_{Q_i,\varepsilon}(x) \right] \psi_{Q_i,\varepsilon} dx + O \left( e^{-|Q_i - Q_j|/\varepsilon} \right) \\
= \frac{1}{2} e^{-N} \int_{\Omega} \left[ \varepsilon^2 \Delta \psi_{Q_i,\varepsilon} - \psi_{Q_i,\varepsilon} \right] U_1 \left( x - \frac{Q_i}{\varepsilon} \right) dx \\
- \frac{1}{2} e^{-N} \int_{\Omega} \left[ \nabla \psi_{Q_i,\varepsilon} \cdot \nabla U_{Q_i,\varepsilon}(x) - \frac{\partial U_{Q_i,\varepsilon}(x)}{\partial n_x} \psi_{Q_i,\varepsilon} \right] dx \\
= - \frac{1}{2} e^{-N} \int_{\Omega} \left[ \nabla \psi_{Q_i,\varepsilon} \cdot \nabla U_{Q_i,\varepsilon}(x) - \frac{\partial U_{Q_i,\varepsilon}(x)}{\partial n_x} \psi_{Q_i,\varepsilon} \right] dx 
\]
On the other hand, since
\[ -\frac{1}{2} \int_{\partial(\Omega_0)} \left[ \frac{1}{\sqrt{1 + |V_\varepsilon(x')|^2}} \sum_{j=1}^{N-1} \frac{\partial g_j(x')}{\partial z_j} \frac{\partial U_1}{\partial z_j} \right] U_1(z) \, dz + O(\varepsilon^{N-2}) \]
and
\[ -\frac{1}{2} \int_{\partial(\Omega_0)} \left[ \frac{\varepsilon}{2\sqrt{1 + |V_\varepsilon(x')|^2}} \frac{U_1'(z)}{|z|} U_1(z) \left( \sum_{j=1}^{N-1} \frac{\partial^2 g_j(0)}{\partial z_j \partial z_s} + O(\varepsilon |z'|^3) \right) \right] \, dz + O(\varepsilon^{N-2}) \]

Thus by the radial symmetry of $U$ we have
\[ I_1 = -\frac{1}{4} \sum_{i=1}^{K} \int_{\Omega} \frac{U_1'(y', 0)}{|y', 0|} U_1(y', 0) \sum_{j, s=1}^{N-1} \frac{\partial^2 g_j(0)}{\partial z_j \partial z_s} y_j y_s \, dy + O(\varepsilon^2) + O(\varepsilon^{N-2}) \]
and
\[ I_2 = -\frac{(N-1)^2}{8} \sum_{i=1}^{K} \epsilon H(Q_i) \int_{\Omega} U^2(\varepsilon y') \, dy + O(\varepsilon^2). \]

Similarly we also have
\[ I_2 = -\frac{(N-1)^2}{8} \sum_{i=1}^{K} \epsilon H(Q_i) \int_{\Omega} U^2(\varepsilon y') \, dy + O(\varepsilon^2). \]

Thus we have
\[ \frac{1}{2} \int_{\Omega^N} \left( \frac{\sqrt{\sum_{i=1}^{K} U_{Q_i, \varepsilon}(x)}^2 + \sqrt{\sum_{i=1}^{K} V_{Q_i, \varepsilon}(x)}^2 + \left( \sum_{i=1}^{K} U_{Q_i, \varepsilon}(x) \right)^2 + \left( \sum_{i=1}^{K} V_{Q_i, \varepsilon}(x) \right)^2 \right) \, dx \]
\[ = \frac{1}{2} \left( K(k^2 + \beta^2) \int_{\Omega} U^4 \, dy - \gamma_3 \epsilon \sum_{i=1}^{K} H(Q_i) + O\left( \sum_{i=1}^{K} e^{-\frac{|\Omega - Q_i|}{\beta}} \right) \right), \]
Using the above estimate (4.4)–(4.11) we have

\[
-\frac{1}{4} \int_{\Omega_\varepsilon} \left[ \mu_1 \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right)^4 + \mu_2 \left( \sum_{i=1}^{K} V_{iQ,K}(x) \right)^4 \right] dx - \frac{1}{2} \int_{\Omega_\varepsilon} \beta \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right)^2 \left( \sum_{i=1}^{K} V_{iQ,K}(x) \right)^2 dx.
\]

Thus by (4.12)–(4.15) we have

\[
\phi_{\varepsilon} \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right) = \frac{1}{4} \left( k^2 + \beta^2 \right) \int_{\Omega_\varepsilon} U^4(y) dy - \gamma_4 \varepsilon \sum_{i=1}^{K} H(Q_i) + O \left( \sum_{i,j=1}^{K} e^{-\frac{|Q_i-Q_j|}{\varepsilon}} \right),
\]

where

\[
\gamma_4 = (k^2 + \beta^2) \left( \frac{1}{2} \int_{\mathbb{R}^{N-1}} U^4(y) |y|^2 dy - (N - 1) \int_{\mathbb{R}^{N-1}} U^2 dy \right).
\]

The last equality is due to the following two facts by Gui–Wei–Winter [9, Lemma 2.5]

\[
\frac{N - 3}{2} \gamma_1 = \frac{1}{4} \left( k^2 + \beta^2 \right) \int_{\mathbb{R}^{N-1}} U^4(y) |y|^2 dy - \frac{1}{2} \left( k^2 + \beta^2 \right) \int_{\mathbb{R}^{N-1}} U^2(y) |y|^2 dy
\]

and

\[
(N + 1) \gamma_1 = (k^2 + \beta^2) \left[ \frac{N - 1}{2} \int_{\mathbb{R}^{N-1}} U^2 dy - \int_{\mathbb{R}^{N-1}} U^2(y) |y|^2 dy + \int_{\mathbb{R}^{N-1}} U^4(y) |y|^2 dy \right].
\]

Hence we complete the proof of Lemma 4.2.

**Proof of Proposition 4.1.** For any \( Q \in \tilde{\Gamma} \) (or in \( \tilde{A} \)), we have

\[
\phi_{\varepsilon} \left( \sum_{i=1}^{K} U_{iQ,K}(x) + h_{\varepsilon,Q} \right) = \phi_{\varepsilon} \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right) + f_{Q,K}(h_{\varepsilon,Q}) + O(\| h_{\varepsilon,Q} \|_{H^2(\Omega_\varepsilon)}),
\]

where

\[
f_{Q,K}(h_{\varepsilon,Q}) = \int_{\Omega_\varepsilon} \left[ \nabla \sum_{i=1}^{K} U_{iQ,K}(x) \nabla h_{\varepsilon,Q} + \nabla \sum_{i=1}^{K} V_{iQ,K}(x) \nabla h_{\varepsilon,Q} \right.
\]

\[
+ \left( \sum_{i=1}^{K} U_{iQ,K}(x) h_{\varepsilon,Q}^1 \right) + \left( \sum_{i=1}^{K} V_{iQ,K}(x) h_{\varepsilon,Q}^2 \right) \right] dx
\]

\[
- \int_{\Omega_\varepsilon} \mu_1 \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right)^3 h_{\varepsilon,Q} + \mu_2 \left( \sum_{i=1}^{K} V_{iQ,K}(x) \right)^3 h_{\varepsilon,Q}^2 \right] dx
\]

\[
- \int_{\Omega_\varepsilon} \beta \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right)^2 \left( \sum_{i=1}^{K} V_{iQ,K}(x) \right) h_{\varepsilon,Q}^3 dx
\]

\[
- \int_{\Omega_\varepsilon} \beta \left( \sum_{i=1}^{K} U_{iQ,K}(x) \right)^2 \left( \sum_{i=1}^{K} V_{iQ,K}(x) \right) h_{\varepsilon,Q}^3 dx
\]
Proposition 5.1. We shall prove the following two propositions.

First, with similar arguments as the proofs of Propositions 3.1 and 3.2 one can prove that the operator

\[ K \text{ is invertible from } H^1(\Omega, \mathbb{R}^d) \]

Hence we complete the proof of Proposition 4.1.

At the end of this section, we show that \( h_{\varepsilon, Q} \) is indeed smooth in \( Q \).

Lemma 4.3. Let \( h_{\varepsilon, Q} \) be defined by Lemma 3.4. Then \( h_{\varepsilon, Q} \) is \( C^1 \) continuous in \( Q \).

Proof. Since the similar proof can be found in the proof of Lemma 3.5 by Gui–Wei–Winter [9], we here only give the sketch of proof.

Recall that \( h_{\varepsilon, Q} \) is a solution of

\[ P_{\varepsilon, Q} \circ S_{\varepsilon} \left( \sum_{i=1}^{K} U_{Q, i} + h_{\varepsilon, Q} \right) = 0 \]  

such that

\[ h_{\varepsilon, Q} \in \mathcal{K}^1_{\varepsilon, Q}. \]  

First, with similar arguments as the proofs of Propositions 3.1 and 3.2 one can prove that the operator

\[ P_{\varepsilon, Q} \circ S_{\varepsilon} \left( \sum_{i=1}^{K} U_{Q, i} + h_{\varepsilon, Q} \right) \]

is invertible from \( \mathcal{K}^1_{\varepsilon, Q} \) to \( C^1_{\varepsilon, Q} \). Second, it is not difficult to check that \( \frac{\partial P_{\varepsilon, Q}}{\partial Q_{ij}} \) are continuous in \( Q \) and thus we indeed have proved that \( h_{\varepsilon, Q} \) is indeed \( C^1 \) smooth in \( Q \).

5. Maximizing and minimizing procedure for reduced problem

In the section, we study a maximizing and minimizing problem. Let \( h_{\varepsilon, Q} \) be defined by Lemma 3.4, we define two functionals

\[ M_{\varepsilon}(Q) = \Phi_{\varepsilon} \left( \sum_{i=1}^{K} U_{Q, i}(x) + h_{\varepsilon, Q} \right) : \tilde{\Gamma} \to \mathbb{R} \]  

and

\[ N_{\varepsilon}(Q) = \Phi_{\varepsilon} \left( \sum_{i=1}^{K} U_{Q, i}(x) + h_{\varepsilon, Q} \right) : \tilde{\Lambda} \to \mathbb{R}. \]

We shall prove the following two propositions.

Proposition 5.1. For \( \varepsilon \) small, the following maximizing problem

\[ \max \{ M_{\varepsilon}(Q) : Q \in \Gamma \} \]

has a solution \( Q^* \in \Gamma \).
Proposition 5.2. For $\varepsilon$ small, the following minimizing problem
\[ \min\{N_\varepsilon(Q) : Q \in \Lambda\} \]
has a solution $Q^\varepsilon \in \Lambda$.

We first give the proof of Proposition 5.1.

Proof of Proposition 5.1. Since $\Phi_\varepsilon(\sum_{i=1}^K U_{\partial, \varepsilon}(x) + h, Q)$ is continuous in $Q$, the maximizing problem (5.3) has a solution $Q^\varepsilon \in \Gamma$. Now we show that $Q^\varepsilon \in \Gamma$. In fact for any $Q \in \Gamma$, by Proposition 4.1, we have
\[ M_\varepsilon(Q) = \left[ \frac{K}{2} (\kappa^2 + \hat{\kappa}) I(U) - \gamma_1 \varepsilon \sum_{i=1}^K H(Q_i) + O \left( \sum_{i,j=1}^K \varepsilon^{-\frac{|Q_i - Q_j|}{\varepsilon}} \right) + o(\varepsilon) \right]. \]
Since $M_\varepsilon(Q^\varepsilon)$ is the maximum we have for any $Q \in \Gamma$ and $\frac{|Q^\varepsilon - Q_i|}{\varepsilon} \to \infty$ as $\varepsilon \to 0$ we have
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \leq \gamma_1 \sum_{i=1}^K H(Q_i) + o(1). \]
Thus we choose $Q_i$ such that $H(Q_i)$ closes to $\min_{Q \in \Gamma_1} H(Q)$. This implies that for any $\delta > 0$, for $\varepsilon$ small we have
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \geq \gamma_1 \sum_{Q \in \Gamma_1} \min_{Q \in \Gamma_1} H(Q) + \gamma_1 \eta_0. \]
We note that if $Q \in \partial \Gamma$, then for some $i = 1, 2, \ldots, K$
\[ \min_{Q \in \partial \Gamma_1} H(Q) \geq \min_{Q \in \Gamma_1} H(Q) + \eta_0. \]
Here $\eta_0 > 0$ is a constant. Thus if $Q^\varepsilon \in \partial \Gamma$. Then
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \geq \gamma_1 \sum_{Q \in \Gamma_1} \min_{Q \in \Gamma_1} H(Q) + \gamma_1 \eta_0, \]
which contradicts with (5.4) if we choose $\delta$ small enough. Hence we complete the proof of Proposition 5.1.

Proof of Proposition 5.2. Similar with the proof of above proposition, we have
\[ N_\varepsilon(Q) = \left[ \frac{K}{2} (\kappa^2 + \hat{\kappa}) I(U) - \gamma_1 \varepsilon \sum_{i=1}^K H(Q_i) + O \left( \sum_{i,j=1}^K \varepsilon^{-\frac{|Q_i - Q_j|}{\varepsilon}} \right) + o(\varepsilon) \right]. \]
Suppose $N_\varepsilon(Q^\varepsilon)$ is the minimum for all $Q \in \bar{\Lambda}$. Since we also have for any $Q \in \Lambda$
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \geq \gamma_1 \sum_{i=1}^K H(Q_i) + o(1). \]
Thus we still choose $Q_i$ such that $H(Q_i)$ closes to $\max_{Q \in \Lambda_1} H(Q)$. This implies that for any $\delta > 0$, for $\varepsilon$ small we have
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \geq \gamma_1 \sum_{Q \in \Lambda_1} \max_{Q \in \Lambda_1} H(Q) - \delta. \]
We note that $Q \in \partial \Lambda$ implies that for some $i = 1, 2, \ldots, K$, $Q_i \in \partial \Lambda_i$ and hence
\[ H(Q^\varepsilon_i) \leq \max_{Q \in \partial \Lambda_i} H(Q) \leq \max_{Q \in \Lambda_i} H(Q) - \eta_0. \]
Thus if $Q^\varepsilon \in \partial \Lambda$. Then
\[ \gamma_1 \sum_{i=1}^K H(Q^\varepsilon_i) \leq \gamma_1 \sum_{Q \in \Lambda_1} \max_{Q \in \Lambda} H(Q) - \gamma_1 \eta_0, \]
which contradicts with (5.5) if we choose $\delta$ small enough. Hence we complete the proof of Proposition 5.2.
6. Proof of the main results

In this section, we give the proofs of the main results, we only give the proof of Theorem 1.1, the proof of Theorem 1.2 can be done in the same way and we omit it.

**Proof of Theorem 1.1.** By Lemmas 3.4 and 4.3, there exists \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \), we have a \( C^1 \) map

\[
h_{\varepsilon, \Omega^i} : \tilde{\Gamma} \to \mathcal{K}^z_{\varepsilon, \Omega^i}
\]
such that

\[
S_{\varepsilon} \left( \sum_{i=1}^{K} U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) = \sum_{k=1}^{N-1} \sum_{i=1}^{K} \frac{\partial U_{\varepsilon, i} \varepsilon}{\partial \tau_{Q_k}}
\]

for some \( (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{K(N-1)}) \in \mathbb{R}^{K(N-1)} \). By Proposition 5.1, we have \( Q \varepsilon \in \Gamma \) which achieves the maximum. Let \( u_\varepsilon = \sum_{i=1}^{K} U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \) and \( v_\varepsilon = \sum_{i=1}^{K} V_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \). Then we have

\[
\frac{\partial}{\partial \tau_{Q_k}} M(Q) = 0, \quad i = 1, 2, \ldots, K, \quad s = 1, \ldots, N - 1.
\]

Namely we have

\[
\int_{\Omega^i} \left[ \frac{\partial}{\partial \tau_{Q_k}} \left( \sum_{i=1}^{K} U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx + \int_{\Omega^i} \left[ \frac{\partial}{\partial \tau_{Q_k}} \left( \sum_{i=1}^{K} V_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx
\]

\[
- \int_{\Omega^i} \left[ \mu_1 u_\varepsilon^2 \frac{\partial}{\partial \tau_{Q_k}} \left( \sum_{i=1}^{K} U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx - \int_{\Omega^i} \beta u_\varepsilon (x)^2 v_\varepsilon (x) \frac{\partial}{\partial \tau_{Q_k}} \left( \sum_{i=1}^{K} U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) dx = 0.
\]

Thus we indeed have

\[
\int_{\Omega^i} \left[ \frac{\partial}{\partial \tau_{Q_k}} \left( U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx + \int_{\Omega^i} \left[ \frac{\partial}{\partial \tau_{Q_k}} \left( V_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx
\]

\[
- \int_{\Omega^i} \left[ \mu_1 u_\varepsilon^2 \frac{\partial}{\partial \tau_{Q_k}} \left( U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx - \int_{\Omega^i} \beta u_\varepsilon (x)^2 v_\varepsilon (x) \frac{\partial}{\partial \tau_{Q_k}} \left( V_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) dx = 0.
\]

By (6.1), we have for \( i = 1, 2, \ldots, K, \quad s = 1, \ldots, N - 1 \)

\[
\sum_{k=1}^{N-1} \sum_{i=1}^{K} \alpha_{k,i} \int_{\Omega^i} \left[ \frac{\partial}{\partial \tau_{Q_k}} \left( U_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) \right] dx + \frac{\partial}{\partial \tau_{Q_k}} \left( V_{\varepsilon, i} + h_{\varepsilon, \Omega^i} \right) dx = 0.
\]
Namely
\[
\sum_{k=1}^{K} \sum_{i=1}^{N-1} \alpha_{ki} \int_{\Omega} \frac{\partial(U_{Q,k,i})}{\partial \tau_{Q,i}} \cdot \frac{\partial(U_{Q,k,i} + h, q)}{\partial \tau_{Q,i}} \, dx = 0.
\] (6.3)

Since $h, q \in \mathcal{X}_{\epsilon, \gamma}$, we have
\[
\left| \int_{\Omega} \frac{\partial^2(U_{Q,k,i})}{\partial \tau_{Q,i} \partial \tau_{Q,i}} h, q \, dx \right| \leq \| \frac{\partial^2(U_{Q,k,i})}{\partial \tau_{Q,i} \partial \tau_{Q,i}} \|_{L^2(\Omega)} \| h, q \|_{L^2(\Omega \times \Omega)}^2 = 0 \left( \frac{1}{\epsilon} \right).
\] (6.4)

On the other hand
\[
\int_{\Omega} \frac{\partial(U_{Q,k,i})}{\partial \tau_{Q,i}} \cdot \frac{\partial(U_{Q,k,i} + h, q)}{\partial \tau_{Q,i}} \, dx = \frac{\epsilon}{2} \delta_{ij} \delta_{kl} (B + O(1)),
\] (6.5)

where
\[
B = \frac{(\mu_2 - \beta)(\mu_1 - \beta)}{\mu_1 - \beta^2} \int_{\mathbb{R}^m} \left( \frac{\partial(U(y))}{\partial y_1} \right)^2 \, dy > 0.
\]

Combining (6.4) and (6.5), (6.3) becomes a system of homogeneous equations for $\alpha_{ki}$ and the matrix of system is nonsingular since it is diagonally dominant. So we have $\alpha_{ki} = 0$ for $i = 1, 2, \ldots, K$, $s = 1, \ldots, N - 1$ and thus $u_\epsilon := (u_1, v_1)$ is a solution of (3.1).

By the construction of $u_\epsilon$, it is easy to see that by maximum principle, $u_\epsilon > 0$ and $v_\epsilon > 0$. Moreover,
\[
\Phi_{\epsilon}(u_\epsilon) \rightarrow \frac{K}{2} (k^2 + p^2) I(U)
\]
and both of $u_\epsilon$ and $v_\epsilon$ have exactly $K$ local maximum points $P_{1,1}^\epsilon, \ldots, P_{1,K}^\epsilon$ and $Q_{1,1}^\epsilon, \ldots, Q_{1,K}^\epsilon$ respectively such that $P_{1,i}^\epsilon, Q_{1,i}^\epsilon \in (\Gamma_i)$.

By our construction of the solution $u_\epsilon$, we have
\[
H(P_{1,i}^\epsilon) \rightarrow \min_{P \in \Gamma_i} H(P), \quad H(Q_{1,i}^\epsilon) \rightarrow \min_{Q \in \Gamma_i} H(P), \quad i = 1, 2, \ldots, K.
\]

On the other hand, by a similar argument as the proof of Tang [18, Theorem 1.2], we indeed have for any $i = 1, 2, \ldots, K$,
\[
\frac{Q_{1,i}^\epsilon - P_{1,i}^\epsilon}{\epsilon} \rightarrow 0 \quad \text{as} \ \epsilon \rightarrow 0.
\]

Thus we complete the proof of Theorem 1.1.

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