Sign changing bump solutions for Schrödinger equations involving critical growth and indefinite potential wells

Yuxia Guo\textsuperscript{a,1}, Zhongwei Tang\textsuperscript{b,*,2}

\textsuperscript{a} Department of Mathematics, Tsinghua University, Beijing 100084, PR China
\textsuperscript{b} School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China

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Abstract

In this paper, we consider the following Schrödinger equations with critical growth

$$\Delta u + (\lambda a(x) - \delta)u = |u|^{2^* - 2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 4$, $2^*$ is the critical Sobolev exponent, $a(x) \geq 0$ and its zero sets are not empty, $\lambda > 0$ is a parameter, $\delta > 0$ is a constant such that the operator $(-\Delta + \lambda a(x) - \delta)$ might be indefinite for $\lambda$ large. We prove that if the zero sets of $a(x)$ have several isolated connected components $\Omega_1, \ldots, \Omega_k$ such that the interior of $\Omega_i$ ($i = 1, 2, \ldots, k$) is not empty and $\partial \Omega_i$ ($i = 1, 2, \ldots, k$) is smooth. Then for $\lambda$ sufficiently large, the equation admits, for any $i \in \{1, 2, \ldots, k\}$, a solution which is trapped in a neighborhood of $\Omega_i$. The key ingredients of the paper are using a flow argument and a combination of global linking and local linking.

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* Corresponding author.
E-mail addresses: yguo@math.tsinghua.edu.cn (Y. Guo), tangzw@bnu.edu.cn (Z. Tang).
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1. Introduction and main results

We consider the following equation of the form:

$$-\Delta u + (\lambda a(x) - \delta)u = |u|^{2^*-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 4$, $\lambda > 0$ is a parameter, $a(x)$ is a given potential, $2^*$ is the critical Sobolev exponent. Such equations arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. Part of the interest and significance is due to the fact that the solutions of this type equation are related to the existence of the solitary wave solutions to the Schrödinger equation of the form:

$$i \partial_t \Psi = -\Delta \Psi + V(x)\Psi - |\Psi|^{2^*-2}\Psi \quad \text{in} \quad \mathbb{R}^N,$$

where $\Psi : \mathbb{R} \times \mathbb{R}^N \to C$ is an unknown function, $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a given potential.

Due to its applications in physics, in mathematical literature, in recent years, much attention has been devoted to the study of the existence of one-bump or multi-bump bound states for the equation of the form (1.1) or

$$-\hbar^2 \Delta v(x) + V(x)v(x) = |v(x)|^{p-2}v(x) \quad \text{in} \quad \mathbb{R}^N.$$

There are enormous investigations on problem (1.3) (or (1.1)) under various assumptions on the potential function $V(x)$. For example, for the subcritical cases, i.e., $2 < p < 2^*$, under the assumptions that $V(x)$ is a bounded function having a non-degenerate critical point, for sufficiently small $\hbar > 0$, Floer and Weinstein [13] established the existence of a standing wave solutions for (1.3). Moreover, they showed that the solutions concentrate near the given non-degenerate critical point of $V(x)$ as $\hbar$ tends to $0$. Their results were later generalized by Oh [18,19] to the higher-dimensional case and the existence of multi-bump solutions concentrating near several non-degenerate critical points of $V(x)$ was obtained as $\hbar$ tends to $0$. For more results, we refer the readers to Ambrosetti, Badiale and Cingolani [1]; Ambrosetti, Malchiodi and Secchi [2]; Bartsch and Tang [3]; Bartsch and Wang [4]; Byeon and Wang [6,7], Cingolani and Lazzo [9]; Cingolani and Nolasco [10]; Del Pino and Felmer [11,12] and the references therein.

As far as the critical case is concerned, due to the lack of the compactness, the problem gets more challenges. There are some results under stronger assumptions on $V(x)$. We firstly refer to the work by Benci and Cerami [5], they considered the following problem:

$$\begin{align*}
&\begin{cases}
-\varepsilon^2 \Delta u(x) + V(x)u(x) = |u(x)|^{2^*-2}u(x) & \text{in} \quad \mathbb{R}^N, \\
&u(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases}
\end{align*}$$

For $\varepsilon = 1$, under the assumption that $V(x) \not\equiv 0$, $V(x) \in L^\infty_\text{loc}(\mathbb{R}^N)$ and $\|V(x)\|_{L^\infty_\text{loc}(\mathbb{R}^N)}$ is sufficiently small, they proved the existence of bound states for (1.4); For $\varepsilon > 0$ small, under the assumptions that the interior part of zero sets $\Omega := \text{int}V^{-1}(0)$ of $V(x)$ is non-empty and $\|V(x)\|_{L^\infty_\text{loc}(\mathbb{R}^N)}$ is sufficiently small, Chabrowski and Yang [8] proved that the problem (1.4) admits cat(\Omega) many solutions; We also refer the readers to paper by Zhang, Chen and Zou [22] for the critical growth problems. Recently, Tang [21] considered the problem (1.1) with critical
exponents and indefinite potential function, i.e., \( a(x) \geq 0, \lambda > 0 \) is a parameter and \( \delta > 0 \) is a constant which can be arbitrary large such that the operator \( -\Delta + \lambda a(x) - \delta \) is indefinite, under some suitable assumptions on \( a(x) \) and \( \delta \), the author proved the existence of the least energy solution which localized near the potential well \( \text{int} a^{-1}(0) \) for \( \lambda \) large.

When the zero sets \( \text{int} a^{-1}(0) \) admit more than one isolated connected components, then it is natural to ask whether (1.1) has a family of solutions \( u_{\lambda} \) which converges, as \( \lambda \to \infty \), to the least energy solution in some selected isolated zero sets of \( a(x) \) and to 0 elsewhere? In our previous paper (see Guo and Tang [16]), by using local mountain pass technique combining Contraction Image Principle, we answered this question for \( \delta > 0 \) small such that the operator \( -\Delta + \lambda a(x) - \delta \) is definite.

In this paper, we assume that \( \delta > 0 \) can be arbitrarily large. In this case, the operator \( -\Delta + \lambda a(x) - \delta \) might be indefinite, which implies that the least energy solution of the limit problem corresponding to (1.1) is not mountain pass type solution anymore but linking type solution instead, which makes the problem getting more challenge. Besides the lack of the compactness caused by the critical growth and unbounded domain, in order to overcome the difficulties cased by the indefinite potential, more other techniques and different approaches are needed.

Our key ingredient in the proof of the present paper is by using a flow argument together with a combination of a global linking applied in \( H^1_{0}(\Omega_{i}) \) for a prescribed \( i \in \{1, 2, \cdots, k\} \) and a local linking near \( 0 \in H^1_{0}(\Omega_{j}) \) for \( 1 \leq j \leq k \) and \( j \neq i \). This idea is originally due to Bartsch and Tang [3] in dealing with the problem with the subcritical growth.

We assume:

\[ (A_1) \ a(x) \in C(\mathbb{R}^N, \mathbb{R}) \text{ satisfies } a(x) \geq 0, \ Omega := \text{int} \{ a^{-1}(0) \} \text{ is nonempty and has smooth} \]
\[ \text{boundary satisfying } \Omega = a^{-1}(0). \]
\[ (A_2) \ \liminf_{|x| \to \infty} a(x) > 0. \]
\[ (A_3) \ \Omega \text{ consists of } k \text{ components: } \Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k \text{ and } \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset, \text{ for all } i \neq j. \]
\[ (A_4) \text{ For every } 1 \leq i \leq k, \text{ the operator } -\Delta - \delta \text{ defined on } H^1_{0}(\Omega_i) \text{ is non-degenerate.} \]

Remark 1.1. By the assumptions \( (A_1), (A_2) \), we can see that \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). Indeed, we can replace the assumption \( (A_2) \) by the following weaker one \( (\tilde{A}_2) \): There exists \( M > 0 \) such that the measure of the set \( \{ x \in \mathbb{R}^N : a(x) \leq M \} \) is finite. Then one can easily see that the assumption \( (A_2) \) is stronger than \( \tilde{A}_2 \), since we only need to take \( M = \frac{1}{2} \liminf_{|x| \to \infty} a(x) \). But this weaker assumption implies that the measure of the zero set \( \Omega \) of \( a(x) \) is finite.

Remark 1.2. By the assumption \( (A_3) \), there is a positive number \( \rho > 0 \) such that

\[ \Omega_i^\rho \cap \Omega_j^\rho = \emptyset, \quad \text{for } i \neq j, 1 \leq i, j \leq k, \]

where \( D^\rho := \{ x \in \mathbb{R}^N : \text{dist}(x, D) < \rho \} \) for any domain \( D \subset \mathbb{R}^N \).

Remark 1.3. Since the zero set \( \Omega \) of \( a(x) \) is bounded in \( \mathbb{R}^N \), the operator \( -\Delta - \delta \) has discrete spectrum in \( H^1_{0}(\Omega_i) \) \( (i = 1, 2, \cdots, k) \) and the assumption \( (A_4) \) implies that 0 is not an eigenvalue of the operator \( -\Delta - \delta \) in \( H^1_{0}(\Omega_i) \) \( (i = 1, 2, \cdots, k) \).
Before the statement of the main results, let us fix some notations. First of all, it follows from Remark 1.2, we have $\Omega = \bigcup_{j=1}^{k} \Omega_j$. By assumptions (A2) and (A3), we may choose $\lambda_0$ large enough such that for all $\lambda \geq \lambda_0$,

$$\lambda a(x) - \delta \geq \delta_0 > 0, \quad \text{for } x \in \mathbb{R}^N \setminus \Omega^0. \quad (1.5)$$

We denote, for every function $V(x)$, $V^+(x) = \max\{V(x), 0\}$ and $V^-(x) = \max\{-V(x), 0\}$. Let $V_\lambda(x) := \lambda a(x) - \delta$ and $V_{\lambda_0}(x) := \lambda_0 a(x) - \delta$, from (1.5), it is easy to see that for $\lambda \geq \lambda_0$, supp$V_\lambda^-(x) \subset \Omega^0$. Let $E_\lambda := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_\lambda^+(x)u^2 < \infty \}$ with the norm:

$$\|u\|_{\lambda}^2 = \int_{\mathbb{R}^N} [\|\nabla u\|^2 + V_\lambda^+ u(x)^2].$$

By (1.5), it is easy to see that for $\lambda \geq \lambda_0$, $(E_\lambda, \|\cdot\|_{\lambda})$ embed continuously in $H^1(\mathbb{R}^N)$.

We consider the functional $J_\lambda(u)$ defined on $E_\lambda$ by:

$$J_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^N} [\|\nabla u\|^2 + V_\lambda(x)u^2] - \frac{1}{2^n} \int_{\mathbb{R}^N} |u|^{2^*}, u \in E_\lambda. \quad (1.6)$$

Then $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and the critical points of $J_\lambda$ are solutions of (1.1).

Note that when $\lambda$ large, the zero set $\Omega$ of $a(x)$ plays an important role and the following problem in $\Omega_i$ ($i \in \{1, 2, \cdots, k\}$) appears as a “limit” problem of (1.1):

$$\left\{ \begin{array}{l}
-\Delta u - \delta u = |u|^{2^*-2}u, \\
\quad \text{ } \quad u = 0 \\
\quad \quad \text{ } \quad \text{ } \quad \text{x \in } \Omega_i, \\
\quad \quad \text{ } \quad \text{x \in } \partial \Omega_i,
\end{array} \right. \quad (1.7)$$

and the solutions of (1.7) can be characterized as the critical points of the following functional:

$$I_{\Omega_i}(u) := \frac{1}{2} \int_{\Omega_i} [\|\nabla u\|^2 - \delta u^2 - \frac{1}{2^n} |u|^{2^*}], u \in H^1_0(\Omega_i).$$

As mentioned above, the operator $L_0 = -\Delta - \delta$ has discrete spectrum in $H^1_0(\Omega_i)$ ($1 \leq i \leq k$), we denote the eigenvalues by $\mu_1^i, \mu_2^i, \cdots, \mu_m^i, \mu_{m+1}^i, \cdots$ and the corresponding eigenfunctions by $\psi_1^i, \psi_2^i, \cdots, \psi_m^i, \psi_{m+1}^i, \cdots$, which are orthogonal with each other. By assumption (A4), we may assume that there is an $m_i \geq 0$ such that,

$$\mu_{m_i}^i < 0 < \mu_{m_i+1}^i, \quad \text{for every } i = 1, 2, \cdots, k. \quad (1.8)$$

Let $E_i = H^1_0(\Omega_i)$ and $E_i^- := \text{span}\{\psi_1^i, \psi_2^i, \cdots, \psi_{m_i}^i\}$, in particular, if $m_i = 0$, then we take $E_i^- = \{0\}$. From (1.8), for any $u \in E_i^-$, $(L_0 u, u) < 0$, here $(\cdot, \cdot)$ denotes the standard product in $L^2(\Omega_i)$.

Let $E_i^{\perp}$ be the orthogonal compensation of $E_i^-$ in $E_i$ and $P_i^- : H^1_0(\Omega_i) \to E_i^-$ be the orthogonal projection. Let

$$\mathcal{N}_i := \{ u \in H^1_0(\Omega_i) \} \setminus \{0\} : I_{\Omega_i}^i(u) \cdot u = 0, P_i^- I_{\Omega_i}^i(u) = 0 \}.$$
and
\[ c_i = \inf_{u \in \mathcal{N}_i} I_{\Omega_i}(u). \]  

By Tang [21], we know that under our assumption on \( \delta \) and \( a(x) \), \( c_i \) is achieved by a nontrivial solution \( w_i \) of (1.7).

Our main results are:

**Theorem 1.4.** Suppose \((A_1)-(A_4)\) hold. Then for any \( i = 1, 2, \ldots, k \) and \( \varepsilon > 0 \), there exists \( \Lambda = \Lambda(\varepsilon) > 0 \) such that, for \( \lambda \geq \Lambda \), (1.1) has a solution \( u_\lambda \in E_\lambda \) satisfying
\[ |J_\lambda(u_\lambda) - c_i| \leq \varepsilon, \] \[ \int_{\mathbb{R}^N \setminus \Omega_i^p} (|\nabla u_\lambda|^2 + (\lambda a(x) - \delta)u_\lambda^2) \leq \varepsilon. \]

Moreover, for any sequence \( \lambda_n \to \infty \), we can extract a subsequence \( \lambda_{n_i} \) such that \( u_{\lambda_{n_i}} \) converges strongly in \( H^1(\mathbb{R}^N) \) to a function \( u(x) \) which solves (1.7) and \( I_{\Omega_i}(u) = c_i \).

The paper is organized as follows: In Section 2, we discuss the Nehari–Pankov manifold and study the properties of the least energy solutions. In Section 3, we consider the penalized functional of \( J_\lambda \) and study the compactness of the modified functional. In Section 4, we study the asymptotic behavior of the critical points for the modified functional. After this, we gave the \( L^\infty \) estimate of the critical points of the modified functional in Section 5. Based on the behavior of the eigenvalues of the operator \(-\Delta - \lambda a(x) - \delta \) as \( \lambda \to \infty \) and on an intersection properties, in Sections 6 and 7, we define a possible critical value of the original functional \( J_\lambda \) for \( \lambda > 0 \) large and construct a new linking by using a flow argument. The proof of the main theorem is given in Section 8. Section 9 is devoted to the proof of Proposition 7.3.

2. The Nehari–Pankov manifold

In this section, we introduce some properties of Nehari–Pankov manifold (see [3,14]).

Let \( \mathcal{O} \subset \mathbb{R}^N \) be an open subset, \( b \in L^\infty_{\text{loc}}(\mathcal{O}) \) be a potential which is bounded below. The functional
\[ J(u) = \frac{1}{2} \int_{\mathcal{O}} (|\nabla u|^2 + b(x)u^2) - \frac{1}{p} \int_{\mathcal{O}} |u|^p \]
is well defined for \( u \in H^1(\mathcal{O}) \) satisfying \( \int_{\mathcal{O}} |b|^2 < \infty \). We write \( E \) for either of the energy spaces \( \{ u \in H^1(\mathcal{O}) : \int_{\mathcal{O}} |b|^2 < \infty \} \) or \( \{ u \in H^1_0(\mathcal{O}) : \int_{\mathcal{O}} |b|^2 < \infty \} \). Note that in our case, the operator \(-\Delta + b(x)\) has finite Morse index and is nondegenerate on \( E \). Thus \( E \) can be split as an orthogonal sum \( E = E^- \oplus E^+ \) of the negative and positive eigenspaces of \(-\Delta + b(x)\), and \( \dim E^- = \infty \). Let \( P^- : E \to E^- \) denote the orthogonal projection. The Nehari–Pankov manifold is defined as:
\[ \mathcal{N} := \{ u \in E \setminus \{ 0 \} : P^- \nabla J(u) = 0, \; DJ(u)[u] = 0 \} \subset E \setminus E^- . \]
It has been introduced by Pankov [20] in a situation where \( \dim E^- = \infty \), and it coincides with the Nehari manifold if \( E^- = \{0\} \). In order to formulate certain geometric properties of \( \mathcal{N} \), we first introduce some notations. For \( w \in E \setminus E^- \) and \( R > r > 0 \), set

\[
H_w := \{ v + tw : v \in E^-, \ t > 0 \}.
\]

\[
A_{w,r,R} := \{ v + tw : v \in E^-, \ \|v\| < R, \ t \in (r, R) \} \subset H_w.
\]

Then we have

\[
\mathcal{N} = \{ w \in E \setminus E^- : \nabla(J|H_w) = 0 \}.
\]

**Proposition 2.1.**

a) For every \( w \in E^+ \setminus \{0\} \), there exist \( t_w > 0 \) and \( \varphi(w) \in E^- \) such that \( H_w \cap \mathcal{N} = \{ \varphi(w) + t_w \cdot w \} \).

b) For every \( w \in \mathcal{N} \) and every \( u \in H_w \setminus \{w\} \), there holds \( J(u) < J(w) \).

c) \( c_0 := \inf_{w \in \mathcal{N}} J(u) > 0 \)

d) For every \( w \in \mathcal{N} \), it holds \( \|P^+w\| > \max\{\|P^-w\|, \sqrt{2c_0}\} \).

e) For \( w \in \mathcal{N} \) and \( 0 < r < \|w\| < R \), the map

\[
f : H_w \to E^- \times \mathbb{R}, \quad \text{such that} \quad f(u) := (P^- \nabla J(u), DJ(u)[u])
\]

has degree \( \deg(f, A_{w,r,R}, 0) = 1 \). Here we identify \( H_w \subset E^- \oplus \mathbb{R}w \) and \( E^- \times \mathbb{R}^+ \subset E^- \times \mathbb{R} \).

**Proof.** The proofs of a)–d) can be found in [14].

For the proof of e), we observe that \( f \) is homotopic to \( \nabla(J|H_w) : H_w \to E^- \oplus \mathbb{R}w \cong E^- \times \mathbb{R} \). By a) and b), the constrained functional \( J|H_w \) has a unique critical point \( w \), which is the global maximum. Since the local degree of a global maximum is \(-1\) we deduce

\[
\deg(f, A_{w,r,R}, 0) = \deg(\nabla(J|H_w), A_{w,r,R}, 0) = (-1)^{\dim H_w}.
\]

**Remark 2.2.** Set \( d := \dim E^+ \) and let \( e_1, \ldots, e_d \) be an orthonormal basis of \( E^- \). Let \( A := \{(s, t) \in \mathbb{R}^d \times \mathbb{R} : |s| \leq 1, \ 0 \leq t \leq 1 \} \), \( B := \partial A \subset \mathbb{R}^{d+1} \). Then for given \( w \in \mathcal{N} \) and \( 0 < r < \|w\| < R \), the map

\[
h_{w,r,R} : (A, B) \to (E, E \setminus \mathcal{N}), \quad h_{w,r,R}(s, t) := R \sum_{i=1}^{d} s_i e_i + ((1 - t)r + tR)w
\]

is well defined. Moreover, it is not difficult to see that all maps \( h_{w,r,R} \) are homotopic. As a consequence of **Proposition 2.1**, we have

\[
c_0 = \inf_{u \in \mathcal{N}} J(u) = \inf_{w \in \mathcal{N}} \max_{0 < r < \|w\| < R} J(u) = \inf_{u \in A_{w,r,R}} \max_{\gamma \in \Gamma(s,t) \in A} J \circ \gamma(s,t),
\]
where
\[ \Gamma = \{ \gamma : (A, B) \rightarrow (E, E \setminus \mathcal{N}) \mid \gamma|_B \text{ is homotopic to some } h_{w,r,R} \}. \]

The proof of the following result is standard, we omit it.

**Proposition 2.3.** If \( J \) satisfies the Palais–Smale condition at the level \( c_0 = \inf_{u \in \mathcal{N}} J(u) \), then \( c_0 \) is achieved by a least energy solution \( u_0 \in \mathcal{N} \).

### 3. Penalization of the nonlinearity and the compactness of the modified functional

In order to overcome the difficulties caused by the critical growth of the nonlinearity and the unboundedness of the domain. In this section, we first modify the functional \( J_\lambda \) by penalizing the nonlinearity term of the equation, then we show that, under some energy level, the modified functional satisfies the Palais–Smale (P.S. for shortness) condition.

For any small constant
\[ 0 < \gamma_0 \leq \frac{1}{2}, \tag{3.1} \]
we define a function \( f(t) \) by:
\[
f(t) = \begin{cases} 
|t|^{2^* - 2}t, & |t| \leq \gamma_0^{\frac{1}{2^* - 2}}, \\
\gamma_0 t, & |t| \geq \gamma_0^{\frac{1}{2^* - 2}}.
\end{cases}
\]

Let us denote
\[ g(x, u) = \chi_{\Omega^\rho}(x)|u|^{2^* - 2}u + (1 - \chi_{\Omega^\rho}(x))f(u), \]
and
\[ G(x, t) = \int_0^t g(x, s)ds = \frac{1}{2^*} \chi_{\Omega^\rho}(x)|u|^{2^*} + (1 - \chi_{\Omega^\rho}(x))F(u), \]
where \( \chi_{\Omega}(x) \) is the characteristic function of \( \Omega \), \( F(s) = \int_0^t f(s)ds \). We define the modified functional by:
\[ \Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V_\lambda(x)|u|^2 \right) dx - \int_{\mathbb{R}^N} G(x, u)dx. \tag{3.2} \]

Then one can check that a critical point of \( \Phi_\lambda \) corresponds to a weak solution of the following equation:
\[ -\Delta u + V_\lambda(x)u = g(x, u) \quad \text{in } \mathbb{R}^N. \tag{3.3} \]
By the definition of \( g(x, t) \), we know that \( g(x, t) = |t|^{2^* - 2} t \), if \( |t| \leq \gamma_0^{-\frac{1}{2^*}} \). Thus a solution \( u \) of (3.3) is also a solution of the original problem (1.1), if \( |u(x)| \leq \gamma_0^{-\frac{1}{2^*}} \) for all \( x \in \mathbb{R}^N \setminus \Omega_0^\rho \).

In the following, we will show that the functional \( \Phi_\lambda \) satisfies the Palais–Smale condition under certain energy level \( c \).

**Lemma 3.1.** Suppose that \( \{u_n\} \) is a \((P.S.)_c\) sequence of the modified functional \( \Phi_\lambda \), that is a sequence satisfying

\[
\Phi_\lambda(u_n) \to c, \quad \Phi'_\lambda(u_n) \to 0. \tag{3.4}
\]

Then there exists a positive constant \( \lambda_0 > 0 \) such that for any \( \lambda \geq \lambda_0 \), \( \{u_n\} \) is bounded. That is there exists a constant \( C \), which is independent of \( \lambda \) and \( n \) such that

\[
\sup \|u_n\|_\lambda \leq C. \tag{3.5}
\]

**Proof.** Since \( \{u_n\} \) is a \((P.S.)_c\) sequence, we have

\[
c + o(1) + \epsilon_n \|u_n\| = \Phi_\lambda(u_n) - \frac{1}{2} \Phi'_\lambda(u_n)u_n
\]

\[
= \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\Omega_0^\rho} |u_n|^{2^*} dx + \int_{\mathbb{R}^N \setminus \Omega_0^\rho} \left( \frac{1}{2} f(u_n)u_n - F(u_n) \right) dx, \tag{3.6}
\]

where \( \epsilon_n \to 0 \) as \( n \to \infty \).

By the definition of \( f(t) \) and \( F(t) \), we can see that, for any \( t \in \mathbb{R} \), \( \frac{1}{2} f(t)t - F(t) \geq 0 \). It follows from (3.6) that there is a constant \( C > 0 \) such that

\[
\int_{\Omega_0^\rho} |u_n|^{2^*} dx \leq C(c + o(1) + \epsilon_n \|u_n\|). \tag{3.7}
\]

Similarly, we have

\[
c + o(1) + \epsilon_n \|u_n\| = \Phi_\lambda(u_n) - \frac{1}{2^*} \Phi'_\lambda(u_n)u_n
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_\lambda(x) u_n^2) dx - \int_{\mathbb{R}^N} G(x, u_n) dx
\]

\[
- \frac{1}{2^*} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_\lambda(x) u_n^2) dx + \frac{1}{2^*} \int_{\mathbb{R}^N} g(x, u_n) dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_\lambda^+(x) u_n^2) dx - \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} V_\lambda^-(x) u_n^2 dx
\]

\[
+ \int_{\mathbb{R}^N \setminus \Omega_0^\rho} \left( \frac{1}{2^*} f(u_n)u_n - F(u_n) \right) dx
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2^s} \right) \left( \|u_n\|_{\lambda}^2 - \int_{\Omega^e} V_{\lambda_0}(x)u_n^2 dx - \gamma_0 \int_{\mathbb{R}^N \setminus \Omega^e} u_n^2 dx \right).
\]

Combining with (3.7), we obtain that
\[
\left( \frac{1}{2} - \frac{1}{2^s} \right)(1 - \gamma_0)\|u_n\|_{\lambda}^2 \leq c + o(1) + \epsilon_n \|u_n\| + C(c + o(1) + \epsilon_n \|u_n\|_{\lambda}),
\]
which, by the definition of \(\gamma_0\) in (3.1), implies that the estimate (3.5) holds for some constant \(C > 0\) independent of \(\lambda \geq \lambda_0\) for some positive numbers \(\lambda_0\) and \(n\), this \(\lambda\) independence is implied by \(\|u\| \leq C\|u\|_{\lambda}\), where \(C\) is independent of \(\lambda\) (for \(\lambda\) large). The proof of Lemma 3.1 is completed. \(\Box\)

Now we are ready to give the following compactness results.

**Proposition 3.2.** Suppose that \(\{u_n\}\) is a \((P.S.)_c\) sequence for \(\Phi_\lambda\) with
\[
c < \frac{1}{N} S^N,
\]
where \(S\) is the best Sobolev constant. Then there exists a subsequence of \(\{u_n\}\), which converges strongly in \(E_\lambda\) to a critical point \(u\) of \(\Phi_\lambda\) such that \(\Phi_\lambda(u) = c\).

**Proof.** By Lemma 3.1, we know that \(\{u_n\}\) is bounded. Thus there exists a subsequence of \(\{u_n\}\) (still denoted by \(\{u_n\}\)) such that
\[
\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\mathbb{R}^N),
\]
\[
u_n \rightarrow u \quad \text{a.e. in } \mathbb{R}^N,
\]
\[
u_n \rightarrow u \quad \text{weakly in } L^{2^*}(\mathbb{R}^N),
\]
\[
u_n \rightarrow u \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N).
\]

Then by a standard argument, we can see that \(\Phi'_\lambda(u) = 0\) and \(\Phi'_\lambda(u) \geq 0\). Next we show that \(u_n \rightarrow u\) strongly in \(E_\lambda\). Let \(v_n = u_n - u\), it follows from Brezis–Lieb’s Lemma that \(\{v_n\}\) is also a Palais–Smale sequence of \(\Phi_\lambda\) satisfying \(\Phi'_\lambda(v_n) \rightarrow 0\) and
\[
\lim_{n \rightarrow \infty} \Phi_\lambda(v_n) = c - \Phi_\lambda(u) \leq c - \frac{1}{N} S^N.
\]

Hence it is sufficient to prove that \(v_n \rightarrow 0\) strongly in \(E_\lambda\). Without loss of generality, up to a subsequence, we assume, on the contrary, that \(\lim_{n \rightarrow \infty} \|v_n\|_{\lambda}^2 = b > 0\). Note that \(\{v_n\}\) is also bounded in \(E_\lambda\) and \(v_n \rightarrow 0\) strongly in \(L^2(\Omega^e)\). Thus we have
\[
o(1) = \Phi'_\lambda(v_n) \cdot v_n
\]
\[
= \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V_\lambda(x)v_n^2(x)) dx - \int_{\mathbb{R}^N} g(x, v_n)v_n dx
\]
\[
\geq \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_+^+(x) v_n^2(x) \right] dx - \int_{\Omega^\rho} V_{\lambda_0}^- v_n^2 dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx.
\]

The strong convergence of \( v_n \to 0 \) in \( L^2(\Omega^\rho) \) indicates that as \( n \to \infty \),

\[
\int_{\Omega^\rho} V_{\lambda_0}^- v_n^2 dx \to 0,
\]

which implies that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^{2^*} dx \geq \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_+^+(x) v_n^2(x) \right] dx = b > 0.
\]

On the other hand

\[
\Phi_\lambda(v_n) - \frac{1}{2^*} \Phi_\lambda'(v_n)v_n = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_\lambda(x) v_n^2(x) \right] dx
\]

\[
+ \int_{\mathbb{R}^N \setminus \Omega^\rho} \left( F(v_n) - \frac{1}{2^*} f(v_n) v_n \right) dx.
\]

By the definition of \( f(t) \) and \( F(t) \), we have

\[
\int_{\mathbb{R}^N \setminus \Omega^\rho} \left( F(v_n) - \frac{1}{2^*} f(v_n) v_n \right) dx \geq 0.
\]

Thus

\[
\left( \frac{1}{2} - \frac{1}{2^*} \right) b = \left( \frac{1}{2} - \frac{1}{2^*} \right) \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_\lambda(x) v_n^2(x) \right] dx
\]

\[
\leq \lim_{n \to \infty} \Phi_\lambda(v_n) \leq c < \frac{1}{N} S_{N^\rho}^N,
\]

which implies that

\[
b < S_{N^\rho}^N.
\]

Note that \( v_n \to 0 \) in \( L^2(\Omega^\rho) \) as \( n \to \infty \) and \( \lambda_0 a_0 \geq \delta \), where \( \lambda_0 \) is defined as in (1.5) and \( a_0 := \inf_{x \in \mathbb{R}^N \setminus \Omega^\rho} a(x) > 0 \). Thus we deduce that for \( \lambda \geq \lambda_0 \),
\[ b = \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V^+(x) v_n^2(x) \right] dx \]
\[ \geq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \]
\[ \geq \lim_{n \to \infty} S \left( \int_{\mathbb{R}^N} |v_n|^2 dx \right)^{\frac{2}{N}} \]
\[ \geq S b^{\frac{2}{N}} \text{ (by (3.11))}. \]

We deduce that \( b \geq S \frac{N}{2} \), which contradicts with (3.12) and hence \( v_n \to 0 \) strongly in \( E_\lambda \). This completes the proof of Proposition 3.2. \( \square \)

4. Asymptotic behavior of the critical points to the modified functional

In this section, we study the asymptotic behavior of critical points of \( \Phi_\lambda \) as \( \lambda \) large.

**Proposition 4.1.** Suppose \( u_{\lambda_n} \in E_{\lambda_n} \) and \( 0 < M < \frac{1}{N} S^{N/2} \) such that \( \lambda_n \to \infty \) as \( n \to \infty \) and

\[ \Phi_{\lambda_n}(u_{\lambda_n}) \leq M, \quad \|\nabla \Phi_{\lambda_n}(u_{\lambda_n})\|_{E_{\lambda_n}} \to 0. \]  

Then we have, as \( n \to \infty \), up to a subsequence

(i) \( u_{\lambda_n} \to u \) strongly in \( H^1(\mathbb{R}^N) \).
  
(ii) \( u_i := u|_{\Omega_i} \quad (1 \leq i \leq k) \) is a solution of (1.7) and \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \).

(iii) \( \lim_{n \to \infty} \Phi_{\lambda_n}(u_{\lambda_n}) = \sum_{i=1}^{k} I_{\Omega_i}(u_i) \).

**Proof.** Indeed, by a similar argument as in the proof of Lemma 3.1, we can show that

\[ \|u_{\lambda_n}\|_{H^1(\mathbb{R}^N)} \leq C \|u_{\lambda_n}\|_{L^2} \leq C. \]

Thus up to a subsequence we may assume that \( u_{\lambda_n} \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \).

We first show that \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \), where \( \Omega = \Omega_1 \cup \Omega_2 \cup \cdots \), \( \Omega_k \) are the zero sets \( a^{-1}(0) \).

Let \( C_m = \{ x \in \mathbb{R}^N : a(x) \geq \frac{1}{m} \} \), then

\[ \int_{C_m} u_{\lambda_n}^2 \leq \frac{1}{\lambda_n m} \int_{C_m} \lambda_n a(x) u_{\lambda_n}^2 \]
\[ \leq \frac{2}{\lambda_n m} \int_{C_m} (\lambda_n a(x) - \delta) u_{\lambda_n}^2 \]
\[ \leq \frac{2}{\lambda_n m} \|u_{\lambda_n}\|_{L^2}^2 \leq \frac{2}{\lambda_n m} C. \]
We have
\[
\int_{C_m} u^2 \leq \liminf_{n \to \infty} \int_{C_m} u_{\lambda_n}^2 = 0, \quad \text{as } n \to \infty,
\]
which indicates that \( u \equiv 0 \) in \( C_m \). We get \( u \equiv 0 \) in \( \bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \Omega \).

Next we show that \( u|_{\Omega_i} \) (\( 1 \leq i \leq k \)) is a solution of (1.7). Indeed, for any \( \psi \in H^1_0(\Omega_i) \), since
\[
\lim_{n \to \infty} /Phi_1^\prime \lambda_n (u_{\lambda_n}) \cdot u_{\lambda_n} = 0,
\]
we can deduce that \( u \) satisfies the following problem:

\[
\begin{cases}
- \Delta u - \delta u = |u|^{2^*-2} u, & x \in \Omega_i, \\
u = 0, & x \in \partial \Omega_i.
\end{cases}
\]

(4.2)

Thus (ii) is proved.

At last, we come to show (i), namely \( u_{\lambda_n} \to u \) (as \( n \to \infty \)) strongly in \( H^1(\mathbb{R}^N) \). We show this by a contradiction argument. Let us denote \( I(u) = \sum_{i=1}^{k} I_{\Omega_i}(u) \) and we take \( v_n = u_{\lambda_n} - u \), to show (i), it is sufficient to prove that \( \|v_n\|_{\lambda_n} \to 0 \) as \( n \to \infty \). Suppose, on the contrary that, up to a subsequence,
\[
\liminf_{n \to \infty} \|v_n\|_{\lambda_n} = \liminf_{n \to \infty} \int_{\mathbb{R}^N} [|
abla v_n|^2 + V_{\lambda_n}^+ v_n^2] = b' > 0.
\]

Since
\[
\Phi_{\lambda_n}(v_n) = \Phi_{\lambda_n}(u_{\lambda_n}) - I(u) + o(1)
\leq \Phi_{\lambda_n}(u_{\lambda_n}) + o(1) \quad (\text{since } I(u) \geq 0)
\leq M + o(1)
< \frac{1}{N} S^{N/2} \quad \text{(for } n \text{ large enough)},
\]
and \( u_{\lambda_n} \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \), similarly as the proof of (3.10), we have
\[
\int_{\Omega_o} |v_n|^2 \, dx \to 0.
\]

(4.4)

Note that, for \( n \) large, \( \text{supp} V_{\lambda_n}^- \subset \Omega_o \) and \( V_{\lambda_n}^- \leq V_{\lambda_0}^- \), we have
\[
0 = \Phi_{\lambda_n}^\prime (u_{\lambda_n}) \cdot u_{\lambda_n} - I'(u) \cdot u + o(1)
= \Phi_{\lambda_n}^\prime (v_n) \cdot v_n + o(1)
\]
\[
\int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_\lambda(x)v_n^2(x) \right] dx - \int_{\mathbb{R}^N} g(x, v_n)v_n dx + o(1) \\
\geq \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_\lambda^+(x)v_n^2(x) \right] dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o(1),
\]

which implies that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^{2^*} dx \geq \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V_\lambda^+(x)v_n^2(x) \right] dx = b' > 0.
\]

Combining with (4.3), we obtain that

\[
b' < S_N^{\frac{N}{2}}.
\]

On the other hand, by using the similar arguments as in the proof of Proposition 3.2, we can prove that \( b' \geq S_N^{\frac{N}{2}} \), which contradicts with (4.5) and we proved that \( \|v_n\|_{\Lambda_n} \to 0 \) as \( n \to \infty \).

(iii) is a direct result of (i) and (ii). The proof of Proposition 4.1 is completed. \( \square \)

5. \( L^\infty \) estimate of the critical points for modified functional

In this section, we will show that the critical points \( u_\lambda \) of \( \Phi_\lambda \) with bounded energy are indeed a solution of the original problem (1.1) for \( \lambda \) large. More precisely, we have the following proposition.

**Proposition 5.1.** Fix \( M > 0 \), there is a constant \( \tau > 0 \) such that for any critical points \( u_\lambda \) of \( \Phi_\lambda \) with \( \Phi_\lambda(u_\lambda) \leq M \), we have:

\[
|u_\lambda(x)| \leq \frac{\tau}{\sqrt{\lambda}}, \text{ for all } x \in \mathbb{R}^N \setminus \Omega^\rho,
\]

which implies that there is a \( \Lambda^* > 0 \) such that for \( \lambda \geq \Lambda^* \), \( |u_\lambda(x)| \leq \gamma_0 \frac{1}{\rho^{2-\lambda}} \) for any \( x \in \mathbb{R}^N \setminus \Omega^\rho \) and hence \( u_\lambda(x) \) is also a solution of the original problem (1.1).

Before giving the proof of Proposition 5.1, we first present an \( L^\infty \) estimate for the solutions \( u_\lambda \) outside of \( \Omega^\rho \). More precisely, we have

**Lemma 5.2.** Suppose \( u_\lambda \) are the critical points of \( \Phi_\lambda \) such that \( \Phi_\lambda(u_\lambda) \leq M \), where \( M \) is a constant independent of \( \lambda \). Then there exist two constants \( M_1 > 0 \), \( M_1 > 0 \) independent of \( \lambda \) such that for \( \lambda \) large, any \( y \in \mathbb{R}^N \setminus \Omega^\rho \) and any \( 0 < r < \frac{\rho}{4} \), it holds

\[
\sup_{x \in B_r(y)} u_\lambda(x) \leq \frac{M_1}{r^{m_1}}.
\]
Proof. We prove Lemma 5.2 by Moser’s iteration. The similar arguments can be found in the paper by Ni, Pan and Takagi [17] (see also Guo and Tang [16]), for the completeness, we give the details of the proof. Firstly, by Proposition 4.1, it is easy to see

\[ \lim_{\lambda \to \infty} \int_{\mathbb{R}^N \setminus \Omega} u^2_\lambda \, dx = 0. \]

Hence, for a small number \( \eta_0 > 0 \) (which we will be specified later), we have, for \( \lambda \) large enough,

\[ \int_{\mathbb{R}^N \setminus \Omega} u^2_\lambda \leq 2 \eta_0. \] (5.3)

Let \( \psi \) denote a smooth cut-off function and \( \beta > 1 \) be a number, both of them will be specified later. Multiply (3.3) by \( \psi^2 u^\beta_\lambda \), we have

\[ \int_{\mathbb{R}^N} \left[ \nabla (\psi^2 |u_\lambda|^\beta) \nabla u_\lambda + V_\lambda u_\lambda \psi^2 |u_\lambda|^\beta \right] = \int_{\mathbb{R}^N} g(x, u_\lambda) \psi^2 |u_\lambda|^\beta. \]

That is

\[ \beta \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^\beta - 2 u_\lambda |\nabla u_\lambda|^2 + 2 \int_{\mathbb{R}^N} \psi |u_\lambda|^\beta \nabla u_\lambda \nabla \psi + \int_{\mathbb{R}^N} V_\lambda u_\lambda \psi^2 |u_\lambda|^\beta = \int_{\mathbb{R}^N} g(x, u_\lambda) \psi^2 |u_\lambda|^\beta. \] (5.4)

On the other hand, by Hölder’s inequality and Young’s inequality, we have

\[ \left| \int_{\mathbb{R}^N} \psi |u_\lambda|^\beta \nabla u_\lambda \nabla \psi \right| \leq \frac{\beta}{2} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta-1} |\nabla u_\lambda|^2 + \frac{2}{\beta} \int_{\mathbb{R}^N} |\nabla \psi|^2 |u_\lambda|^{\beta+1}. \]

Note that for any \( x \in \mathbb{R}^N \) and \( u \geq 0 \), we have \( g(x, u) \leq u^{2^* - 1} \). Thus the inequality (5.4) leads to

\[ \frac{\beta}{2} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta-1} |\nabla u_\lambda|^2 + \int_{\mathbb{R}^N} V_\lambda u_\lambda \psi^2 |u_\lambda|^\beta \leq \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta+2^*-1} + \frac{2}{\beta} \int_{\mathbb{R}^N} |\nabla \psi|^2 |u_\lambda|^{\beta+1}. \]

Since for \( \lambda \geq \lambda_0 \), \( 0 \leq V^-_\lambda(x) \leq V^-_{\lambda_0}(x) \) and \( \text{supp} V^-_{\lambda_0}(x) \subset \Omega^\rho \), take \( M_0 := |V^-_{\lambda_0}(x)|_{L^\infty} \), it deduces that
\[
\int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta - 1} |\nabla u_\lambda|^2 \leq \frac{2}{\beta} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta + 2^* - 1} + \frac{4}{\beta^2} \int_{\mathbb{R}^N} |\nabla \psi|^2 |u_\lambda|^{\beta + 1} \\
+ \frac{2M_0}{\beta} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta + 1}.
\] (5.5)

By Sobolev imbedding theorem, we have

\[
S \left( \int_{\mathbb{R}^N} \left( \psi |u_\lambda|^{\frac{\beta + 1}{2}} \right)^{2^*} \right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla (\psi |u_\lambda|^{\frac{\beta + 1}{2}})|^2 \\
\leq (\beta + 1) \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta - 1} |\nabla u_\lambda|^2 + 2 \int_{\mathbb{R}^N} |u_\lambda|^{\beta + 1} |\nabla \psi|^2.
\] (5.6)

Combining with (5.6) and (5.5), we get

\[
S \left( \int_{\mathbb{R}^N} \left( \psi |u_\lambda|^{\frac{\beta + 1}{2}} \right)^{2^*} \right)^{\frac{2}{2^*}} \leq \frac{2(\beta + 1)}{\beta} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta + 2^* - 1} + \frac{2(\beta + 1)M_0}{\beta} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta + 1} \\
+ \left( \frac{4(\beta + 1)}{\beta^2} + 2 \right) \int_{\mathbb{R}^N} |u_\lambda|^{\beta + 1} |\nabla \psi|^2.
\] (5.7)

Now for \( y \in \mathbb{R}^N \setminus \Omega^0 \) and \( 0 < r < \frac{\rho}{8} \), we specify the cut-off function \( \psi \) by

\[
\psi = \begin{cases} 
1, & x \in B_{2r}(y), \\
0, & x \in \mathbb{R}^N \setminus B_{4r}(y),
\end{cases}
\]

with \(|\nabla \psi| \leq \frac{C}{r}\). By Hölder’s inequality, we have

\[
\int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\beta + 2^* - 1} \leq \left[ \int_{\mathbb{R}^N} \left( \psi^2 |u_\lambda|^{\beta + 1} \right)^{\frac{2^*}{2}} \right]^{\frac{2}{2^*}} \cdot \left[ \int_{\mathbb{R}^N \setminus \Omega} |u_\lambda|^{2^*} \right]^{\frac{2}{2^*}} \\
\leq [2\eta_0]^N \left[ \int_{\mathbb{R}^N} \left( \psi^2 |u_\lambda|^{\beta + 1} \right)^{\frac{2^*}{2}} \right]^{\frac{2}{2^*}}.
\]
Take $\beta = 2^* - 1$ and $\eta_0 > 0$ is such that $\frac{2(\beta + 1)}{\beta} [2\eta_0]^{\frac{2}{N}} = \frac{S}{7}$, then (5.7) becomes

$$S \left( \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^{\frac{(\beta+2)^2}{\beta}} \right)^{\frac{\beta}{2}} \leq \frac{8NM_0}{N + 2} \int_{\mathbb{R}^N} \psi^2 |u_\lambda|^2 + \left( \frac{16N(N-2)}{(N+2)^2} + 2 \right) \int_{\mathbb{R}^N} |u_\lambda|^2 |\nabla \psi|^2, \quad (5.8)$$

which implies that for any $y \in \mathbb{R}^N \setminus \Omega^\rho$,

$$\int_{B_{2r}(y)} |u_\lambda|^{\frac{(\beta+2)^2}{\beta}} \leq C(r) \int_{B_{4r}(y)} |u_\lambda|^2. \quad (5.9)$$

Now, we will use the above estimate combining with Moser’s iteration argument to complete the proof.

Let $Z_\lambda = |u_\lambda|^{\frac{\beta+1}{2}}$, where $\beta > 1$ will be chosen later, then (5.7) becomes

$$S \left( \int_{\mathbb{R}^N} (\psi Z_\lambda)^{2^*} \right)^{\frac{2}{2^*}} \leq \frac{2(\beta + 1)}{\beta} \int_{\mathbb{R}^N} \psi^2 Z_\lambda^2 |u_\lambda|^{2^*-2} + \frac{2M_0(\beta + 1)}{\beta} \int_{\mathbb{R}^N} \psi^2 Z_\lambda^2$$

$$+ \left( \frac{4(\beta + 1)}{\beta^2} + 2 \right) \int_{\mathbb{R}^N} Z_\lambda^2 |\nabla \psi|^2, \quad (5.10)$$

where $\psi$ is a cut-off function supported in $B_{2r}(y)$ with $y \in \mathbb{R}^N \setminus \Omega^\rho$ and $r \leq \frac{\rho}{4}$, which will be specified later in each step of the iteration process. Again by Hölder’s inequality, the first term in (5.10) can be estimated by

$$\int_{\mathbb{R}^N} \psi^2 Z_\lambda^2 |u_\lambda|^{2^*-2} \leq \left[ \int_{\mathbb{R}^N} (\psi Z_\lambda)^{\frac{2q^2}{q-2}} \right]^{\frac{q-2}{q}} \cdot \left[ \int_{B_{2r}(y)} |u_\lambda|^{\frac{(2^*-2)q^2}{2}} \right]^{\frac{2}{q}},$$

where $q = \frac{N^2}{N-2} > N$ and $2 < \frac{2q}{q-2} < 2^*$. Thus, for any $\varepsilon > 0$,

$$\|\psi Z_\lambda\|_{L^{\frac{2q}{q-2}}(\mathbb{R}^N)} \leq \varepsilon \|\psi Z_\lambda\|_{L^{2^*}(\mathbb{R}^N)} + \varepsilon^{-\sigma} \|\psi Z_\lambda\|_{L^2(\mathbb{R}^N)},$$

where $\sigma = \frac{N-2}{2}$. By (5.10), we have

$$S \left( \int_{\mathbb{R}^N} (\psi Z_\lambda)^{2^*} \right)^{\frac{2}{2^*}} \leq \frac{2M_0(\beta + 1)}{\beta} \int_{\mathbb{R}^N} \psi^2 Z_\lambda^2 + \left( \frac{4(\beta + 1)}{\beta^2} + 2 \right) \int_{\mathbb{R}^N} Z_\lambda^2 |\nabla \psi|^2$$

$$+ \frac{2(\beta + 1)}{\beta} C_1 \left( \varepsilon \|\psi Z_\lambda\|_{L^{2^*}(\mathbb{R}^N)} + \varepsilon^{-\sigma} \|\psi Z_\lambda\|_{L^2(\mathbb{R}^N)} \right). \quad (5.11)$$
where, using the fact that $y \in \mathbb{R}^N \setminus \Omega^\rho$ and (5.9), (5.3), $C_1$ can be estimated as follows

$$C_1 = \left[ \int_{B_{2\varepsilon}(y)} |u_\lambda|^{\frac{(\sigma-2)q}{2}} \right]^{\frac{2}{q}} \leq \left[ \int_{\mathbb{R}^N \setminus \Omega} |u_\lambda|^{\frac{(\sigma+2)q}{2}} \right]^{\frac{2}{q}} \leq (2\eta_0)^{\frac{2}{q}}.$$

Setting $\varepsilon = \frac{S\beta}{4(\beta + 1)C_1}$, we obtain from (5.11) that

$$\left( \int_{\mathbb{R}^N} (\psi Z_\lambda)^{2^*} \right)^{\frac{1}{2^*}} \leq \frac{2}{S} \left[ 4M_0 + 2C_2(\beta + 1)\sigma + 1 \right] \int_{\mathbb{R}^N} \psi^2 Z_\lambda^2 + \frac{2C_2}{S} \int_{\mathbb{R}^N} Z_\lambda^2 |\nabla \psi|^2,$$

(5.12)

where $C_2$ is a constant independent of $\beta$.

Now for $r \leq r_2 < r_1 \leq 2r$, we choose $\psi$ such that $\psi \equiv 1$ in $B_{r_2}(y)$ and $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_{r_1}(y)$. Then we deduce from (5.12) that

$$\|Z_\lambda\|_{L^{2^*}(B_{r_2}(y))} \leq \frac{C_3 R^2}{(r_1 - r_2)^2} (1 + \beta)^\sigma \|Z_\lambda\|_{L^2(B_{r_1}(y))},$$

(5.13)

i.e.,

$$\|Z_\lambda\|_{L^{2^*}(B_{r_2}(y))} \leq \frac{C_3 R}{(r_1 - r_2)} h^\frac{N}{2^*} \|Z_\lambda\|_{L^2(B_{r_1}(y))},$$

(5.14)

where $h = 1 + \beta$, $R = \max\{r, 1\}$ and $C_3$ is a constant independent of $r, \beta$. Set

$$N(p, r) = \left( \int_{B_r(y)} |Z_\lambda|^p \right)^{\frac{1}{p}}.$$

Then we can rewrite (5.14) in terms of $N(\cdot, \cdot)$:

$$N(kh, r_2) \leq \left( \frac{2C_3 R}{(r_1 - r_2)} \right)^{\frac{2}{k}} h^\frac{N}{2^*} N(h, r_1),$$

(5.15)

where $k = \frac{N}{N-2}$. Let $p = 2^*$ and $h = h_m = pk^m, r_m = r(1 + 2^{-m})$, for $m = 0, 1, 2, \cdots$. It follows from (5.15) that

$$N(pk^{m+1}, r_{m+1}) = N(kh_m, r_{m+1}) \leq \left( \frac{2C_3 R}{(r_m - r_{m+1})} \right)^{\frac{2}{k}} h_m^\frac{N}{2^*} N(h_m, r_m),$$

(5.16)
\[ \left( \frac{4C_3 R}{r} \right)^{2k-m} \left( \frac{2k}{r} \right)^{\frac{2}{p} mk-m} N(p k^m, r_m) \leq \cdots \leq \left( \frac{4C_3 R}{r} \right)^{2k-\sum_{j=0}^{\infty} j^{-i}} \left( \frac{2k}{r} \right)^{\frac{2}{p} \sum_{j=0}^{\infty} jk^{-j}} N(p, 2r). \] (5.17)

Let \( m \to \infty \), we have

\[ \sup_{x \in B_r(y)} u_{\lambda}(x) = \lim_{s \to \infty} N(s, r) \leq C_4(r) N(p, 2r) \] (5.18)

\[ = C_4(r) \| u_{\lambda} \|_{L^2(2r(2r, y))} \]
\[ \leq C_4(r) \left[ \int_{\mathbb{R}^N \setminus \Omega} \left| u_{\lambda} \right|^{2^*} \right]^1 \]
\[ \leq C_4(r)(2\eta_0)^{\frac{1}{2^*}}, \] (5.19)

where

\[ C_4(r) = \left( \frac{4C_3 R}{r} \right)^{2k-\sum_{j=0}^{\infty} j^{-i}} \left( \frac{2k}{r} \right)^{\frac{2}{p} \sum_{j=0}^{\infty} jk^{-j}}. \]

Thus we proved Lemma 5.2 with

\[ m_1 = \sum_{j=1}^{\infty} \left( \frac{N-2}{N} \right)^j \]

and

\[ M_1 = \left( 4C_3 R \left( \frac{2N}{N-2} \right)^{\frac{N}{4}} \right) \sum_{j=1}^{\infty} \left( \frac{N-2}{N} \right)^j \left( 2 \left( \frac{N}{N-2} \right)^{\frac{N}{4}} \right)^j \sum_{j=1}^{\infty} j \left( \frac{N-2}{N} \right)^j \left( 2\eta_0 \right)^{\frac{N-2}{2^*}}. \]

A direct result of the arguments in the proof of Lemma 5.2 is the following exterior Harnack-type inequality.

**Lemma 5.3.** Suppose all the assumptions of Lemma 5.2 are satisfied. Then for any \( 0 < r < \frac{1}{4} \rho \), there is a constant \( C > 0 \) such that for any \( y \in \mathbb{R}^N \setminus \Omega^\rho \), it holds

\[ \sup_{x \in B_r(y)} u_{\lambda}(x) \leq C \left[ \int_{B_{2r}(y)} \left| u_{\lambda} \right|^{2^*} \right]^1. \] (5.20)
Now we are ready to present the proof of Proposition 5.1.

**Proof of Proposition 5.1.** By the proof of Lemma 3.1, one can find a positive constant, still denoted by $M$ such that for $\lambda \geq \lambda_0$, we have

$$\|u_\lambda\|_\lambda \leq M. \quad (5.21)$$

On the other hand, since $a(x)$, $a(x) \geq a_0 > 0$ for all $x \in \mathbb{R}^N \setminus \Omega^\rho$. Thus for $\lambda$ large enough, it holds that $\lambda a(x) - \delta \geq \frac{1}{2} a_0$, for all $x \in \mathbb{R}^N \setminus \Omega^\rho$. As a consequence of (5.21), we have

$$\int_{\mathbb{R}^N \setminus \Omega^\rho} \frac{\lambda}{2} a_0 |u_\lambda|^2 \leq \int_{\mathbb{R}^N \setminus \Omega^\rho} \left[ |\nabla u_\lambda|^2 + \frac{\lambda}{2} a_0 |u_\lambda|^2 \right] \leq M,$$

which implies that

$$\int_{\mathbb{R}^N \setminus \Omega^\rho} |u_\lambda|^2 \leq \frac{1}{\lambda} \frac{2M}{a_0}. \quad (5.22)$$

We may assume that $\frac{2M}{a_0} \geq 1$. Otherwise, we can take $M$ is properly large.

Now take $r = \frac{\rho}{8}$ and $q > 2^*$ be fixed, by the interpolation inequality, we have, for any $y \in \mathbb{R}^N \setminus \Omega^\rho$,

$$\int_{B_{2r}(y)} |u_\lambda|^{2^*} \leq \left( \int_{B_{2r}(y)} |u_\lambda|^2 \right)^{\alpha} \left( \int_{B_{2r}(y)} |u_\lambda|^q \right)^{1-\alpha}, \quad (5.23)$$

where $\alpha \in (0, 1)$ is such that $2^* = 2\alpha + (1-\alpha)q$, i.e., $\alpha = \frac{q-2^*}{q-2}$. We may choose $q > 2^*$ such that $\alpha = \frac{1}{2}$. By Lemma 5.2, we have

$$\left( \int_{B_{2r}(y)} |u_\lambda|^q \right)^{1-\alpha} \leq \left( M_1^q (2r)^{N-m_1 q} \omega_N \right)^{(1-\alpha)} \leq \left( 4^{m_1 q-N} M_1^q \rho^{N-m_1 q} \omega_N \right)^{\frac{1}{2}}, \quad (5.24)$$

where $\omega_N$ is the volume of the unit ball $B_1(0)$. Combining (5.22)–(5.24), we obtain that for any $y \in \mathbb{R}^N \setminus \Omega^\rho$, 

\[ \int_{B_2(y)} |u^\lambda|^2 \leq \left( 4^m q - N \right)^{\frac{1}{2}} \rho^{N - m^1 q} \omega_N \left( \frac{1}{\lambda} \right)^{\frac{1}{2}} \left( \frac{1}{2} M a_0 \right) \alpha \leq \frac{C_5}{\sqrt{\lambda}}, \quad (5.25) \]

where \( C_5 = \frac{2M}{a_0} \left( 4^m q - N \right)^{\frac{1}{2}} \rho^{N - m^1 q} \omega_N \). As a consequence of Lemma 5.3, we have for any \( y \in \mathbb{R}^N \setminus \Omega^\rho \),

\[ \sup_{x \in B_r(y)} u^\lambda (x) \leq \frac{C_5}{\sqrt{\lambda}}, \quad (5.26) \]

which implies that

\[ \|u^\lambda(x)\|_{L^\infty(\mathbb{R}^N \setminus \Omega^\rho)} \leq \frac{C_5}{\sqrt{\lambda}}. \]

We complete the proof of Proposition 5.1 by taking \( \tau = C_5 \). \( \square \)

6. Definition of the critical value

As in Section 1, for \( i \in \{1, 2, \ldots, k\} \), we denote \( E_i := H_0^1(\Omega_i) \) and \( P_i^- : E_i \to E_i^- \) the orthogonal projection. Then by the results in A. Szulkin, T. Weth and M. Willem [15] (see also Tang [21]), we see that under the condition (A4),

\[ c_i := \inf_{u \in N_i} I_{\Omega_i} (u) > 0 \quad (6.1) \]

is achieved by some \( w_i \in N_i \) such that \( c_i = I_{\Omega_i} (w_i) \), where \( N_i \) is the Nehari–Pankov manifold corresponding to the functional \( I_{\Omega_i} \) and is defined by

\[ N_i = \{ u \in E_i \setminus \{0\} : P_i^- I_{\Omega_i}' (u) = 0, I_{\Omega_i}' (u) \cdot u = 0 \}. \]

Set \( d_i := \dim E_i^- \) for \( 1 \leq i \leq k \), and let \( e_{ji} \) (\( j = 1, \ldots, d_i \)) be an orthogonal basis of \( E_i^- \). Let

\[ A := \left\{ (s_1, \ldots, s_k, t) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \times \mathbb{R} : \|s_i\|_{\mathbb{R}^{d_i}} \leq 1, i = 1, \ldots, k, 0 \leq t \leq 1, \right\} \]

and \( B := \partial A \). In the following of the paper, we fix \( i \in \{1, 2, \ldots, k\} \) and take \( R > \|w_i\| \) large and \( 0 < r < \|w_i\| \) small, which are to be determined later, where \( w_i \) is the least energy solution of (1.7). We define the map \( \gamma_0 : A \to E \) by

\[ \gamma_0(s, t) := R \sum_{j=1}^{d_i} s_{ij} e_{ij} + ((1 - t)r + t R) w_i + \sum_{m \neq i}^r \sum_{j=1}^{d_m} s_{mj} e_{mj}. \]
Since $I_{\Omega_m}(u) \leq 0$ for $u \in E_m^-$, we have

$$
\sum_{m \neq i} I_{\Omega_m} \left( r \sum_{j=1}^{d_m} s_{mj} e_{mj} \right) \leq 0 \quad \text{for all } s_{mj}.
$$

Hence, if some $s_{ij} \neq 0$ or $t > 0$ then

$$
\Phi_\lambda(\gamma_0(s, t)) = I_{\Omega_i} \left( R \sum_{j=1}^{d_i} s_{ij} e_{ij} + ((1-t)r + tR)w_i \right) + \sum_{m \neq i} I_{\Omega_m} \left( r \sum_{j=1}^{d_m} s_{mj} e_{mj} \right)
$$

$$
\rightarrow -\infty, \quad \text{as } R \rightarrow \infty.
$$

If $t = 0$ and $r = 0$, then $\Phi_\lambda(\gamma_0(s, t)) \leq 0$. It follows that for $R > 0$ large and $r > 0$ small there holds

$$
\Phi_\lambda(\gamma_0(s, t)) < c_i \quad \text{for all } (s, t) \in B, \lambda \geq 0. \quad (6.2)
$$

Moreover, if $r$ is small enough, there exists $\alpha > 0$ such that for any $1 \leq i \leq k$,

$$
I_{\Omega_i}(u) \geq \alpha \|u\|^2_{E_j} \quad \text{for } u \in E_i^+, \|u\|_{E_i} \leq r. \quad (6.3)
$$

We fix $r, R$ satisfying (6.2) and (6.3) and define the sets

$$
\mathcal{H}_\lambda := \{ h : A \times [0, 1] \rightarrow E : h \in C^0, \ h(s, t, 0) = \gamma_0(s, t), \ J_\lambda(h(s, t, \tau)) \text{ is non-increasing with respect to } \tau \}
$$

and

$$
\Gamma_\lambda := \{ \gamma : A \rightarrow E \mid \exists h \in \mathcal{H}_\lambda \ \forall (s, t) \in A : \gamma(s, t) = h(s, t, 1) \}.
$$

Then we arrive at a minimax description of a possible critical value:

$$
c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{(s,t) \in A} J_\lambda(\gamma(s,t)). \quad (6.4)
$$

Since $\gamma_0 \in \Gamma_\lambda$ (by taking $h = id$), we have

$$
c_\lambda \leq c_i. \quad (6.5)
$$
7. Lower bound estimate for $c_\lambda$

In this section, we estimate the lower bound for $c_\lambda$. For simplicity, in the following we simplify the notation of $I_{\Omega_i}(u)$ be $I_i(u)$. The main arguments of this section is motivated by the work of Bartsch and Tang [3], where a subcritical problem was considered.

Consider the functional $I_\lambda^i : X_i = H^1(\Omega_i^\rho) \to \mathbb{R}$ defined by

$$I_\lambda^i(u) := \frac{1}{2} \int_{\Omega_i^\rho} \left( |\nabla u|^2 + (\lambda a(x) - \delta)u^2 \right) - \frac{1}{2^*} \int_{\Omega_i^\rho} |u|^{2^*}.$$ 

We define a modified Nehari–Pankov manifold associated with $I_\lambda^i$ by

$$N_\lambda^i := \{ u \in X_i \setminus \{0\} : P_i^- (\nabla I_\lambda^i(u)) = 0, \ D I_\lambda^i(u)[u] = 0 \},$$

where $P_i^- : X_i \to E_i^-$ is the projection on $E_i^-$. We denote the discrete spectrum of $L_\lambda$ by $\mu_{i,\lambda}^1, \ldots, \mu_{i,\lambda}^m, \ldots$ and set $V_{i,\lambda}^m$ be the corresponding eigenspace with respect to $\mu_{i,\lambda}^m$. As in Section 1, we denote $\mu_i^1, \ldots, \mu_i^m, \ldots$, be the eigenvalues of the operator $L_0 = -\Delta - \delta$ and set $V_i^m$ be the corresponding eigenspace with respect to $\mu_i^m$. Then with a similar argument as in the proof of Lemma 2.1 in [21], we have the following:

**Lemma 7.1.** $\mu_{i,\lambda}^m \to \mu_i^m$ and $V_{i,\lambda}^m \to V_i^m$ as $\lambda \to \infty$.

Here $V_{i,\lambda}^m \to V_i^m$ means that, for any given sequence of $\lambda_j \to \infty$ (as $j \to \infty$) and sequence of normalized eigenfunctions $\psi_j \in V_{i,\lambda}^m$, there exists a normalized eigenfunction $\psi \in V_i^m$ such that $\psi_j \to \psi$ strongly in $H^1(\mathbb{R}^N)$ along a subsequence (as $j \to \infty$). Let us define the infimum

$$c_\lambda^i := \inf_{u \in N_i^\lambda} I_\lambda^i(u).$$

Then, we have the following asymptotic behavior for $c_\lambda^i$ as $\lambda \to \infty$.

**Lemma 7.2.** $c_\lambda^i \to c_i$ as $\lambda \to \infty$.

**Proof.** Note that for every $u_i \in H^1_i(\Omega_i)$, $I_\lambda^i(u_i) = I_i(u_i)$ and thus by the definition of $N_i^\lambda$ we have $N_i \subset N_i^\lambda$ and which implies that

$$c_\lambda^i \leq c_i.$$ \hspace{1cm} (7.1)

On the other hand, it is easy to see that $c_\lambda^i$ is nondecreasing with respect to $\lambda$. Thus (7.1) implies that the limit $\lim_{\lambda \to \infty} c_\lambda^i$ exists and

$$\lim_{\lambda \to \infty} c_\lambda^i \leq c_i.$$ \hspace{1cm} (7.2)

Now we prove the inverse of (7.2).
In the following, we will show that \( c_0 > 0 \) and it can be achieved by a critical point of \( I_i^\lambda \) for \( \lambda \) large. To show this, we denote \( \tilde{X}^+_i := \{ u \in X_i : P_i^- u = 0 \} \). We first prove that there is \( \beta > 0 \) such that for \( \lambda \) large,

\[
I_i^\lambda (v) \geq \beta \quad \text{for all} \quad v \in N_i^{\lambda^2}.
\]  (7.3)

Indeed, for every \( v \in N_i^{\lambda^2} \), we have \( v \in \tilde{X}^+_i \) and

\[
\int_{\Omega_i^0} |\nabla v|^2 + (\lambda a(x) - \delta) v^2 = \int_{\Omega_i^0} |v|^2^*.
\]  (7.4)

Let us denote \( X_{\lambda}^+ := \{ u \in X_i : Q_{\lambda}^- u = 0 \} \), where \( Q_{\lambda}^- : X_i \rightarrow X_{\lambda}^- \) is the orthogonal projection on \( X_{\lambda}^- \), which is the negative eigenspace associated to the operator \( L_{\lambda} := -\Delta + \lambda a(x) - \delta \) in \( X_i \). I.e., \( X_{\lambda}^- = \text{span}\{\psi_{\lambda}^1, \ldots, \psi_{\lambda}^k\} \), where \( \psi_{\lambda}^j \) are the negative eigenfunctions to the operator \( L_{\lambda} \). By Lemma 7.1, we have, as \( \lambda \rightarrow \infty \), for every \( 1 \leq j \leq k \),

\[
\psi_{\lambda}^j \rightarrow \psi_j, \quad (7.5)
\]

where \( \psi_j \ (j = 1, 2, \ldots, k) \) are the negative eigenfunctions to the operator \( L_0 \) in \( E_i \) such that \( E_i^- = \text{span}\{\psi_1, \ldots, \psi_k\} \). Let us denote \((\cdot, \cdot)\) be the inner product in \( X_i \), we have, for any \( v \in \tilde{X}^+_i \),

\[
Q_{\lambda}^- v = (v, \psi_{\lambda}^1)\psi_{\lambda}^1 + \cdots + (v, \psi_{\lambda}^k)\psi_{\lambda}^k
= (v, \psi_1)\psi_1 + \cdots + (v, \psi_k)\psi_k
+ (v, \psi_{\lambda}^1 - \psi_1)\psi_{\lambda}^1 + \cdots + (v, \psi_{\lambda}^k - \psi_k)\psi_{\lambda}^k
+ (v, \psi_{\lambda}^1)(\psi_{\lambda}^1 - \psi_1) + \cdots + (v, \psi_{\lambda}^k)(\psi_{\lambda}^k - \psi_k)
= o(\lambda) \|v\|_{X_i} \quad \text{(by the definition of} \ \tilde{X}^+_i \text{and} \ (7.5)),
\]  (7.6)

where \( o(\lambda) \rightarrow 0 \) as \( \lambda \rightarrow \infty \).

Now we decompose \( v = v_1 + v_2 \) with \( v_1 \in X_{\lambda}^- \) and \( v_2 \in X_{\lambda}^+ \) such that

\[
\|v_1\| = o(\lambda) \|v\|_{X_i}, \quad \|v_2\| = (1 - o(\lambda)) \|v\|_{X_i}.
\]

By (7.4) we have

\[
I_i^\lambda (v) = \left( \frac{1}{2} - \frac{1}{2^n} \right) \int_{\Omega_i^0} \left( |\nabla v|^2 + (\lambda a(x) - \delta)v^2 \right)
= \left( \frac{1}{2} - \frac{1}{2^n} \right) \left[ \int_{\Omega_i^0} |\nabla v_1|^2 + (\lambda a(x) - \delta)v_1^2 + \int_{\Omega_i^0} |\nabla v_2|^2 + (\lambda a(x) - \delta)v_2^2 \right]
\geq C_1 (1 - o(\lambda)) \|v\|_{X_i}^2 - C_2 o(\lambda) \|v\|_{X_i}^2,
\]  (7.7)
where $C_j$ ($j = 1, 2$) are constants independent of $\lambda$. Thus for $\lambda$ large enough, (7.3) follows from (7.7).

On the other hand, by standard arguments, we see that $I_i^\lambda$ satisfies the $(P.S.)_{c_i^\lambda}$ condition since $c_i^\lambda \leq c_i < \frac{1}{N} S_{\lambda}$. Thus following a similar argument as in A. Szulkin, T. Weth [14], one can obtain that $c_i^\lambda$ is achieved by a critical point $w_i^\lambda$ of $I_i^\lambda$.

Therefore, for a given sequence $\lambda_n \to \infty$, we deduce from (7.2) that $w_i^{\lambda_n}$ is uniformly bounded in $H^1(\Omega_i^\rho)$. Without loss of generality, we may assume $w_i^{\lambda_n} \to w$ in $H^1(\Omega_i^\rho)$. As in the proof of Proposition 4.1, we deduce that $w_i^{\lambda_n}$ is achieved by a critical point $w_i\lambda_n$ of $I_i^\lambda$, in particular $w \neq 0$. Moreover,

$$DI_i^{\lambda_n}(w_i^{\lambda_n})[w_i^{\lambda_n}] \to DI_i(w)[w]$$

and

$$P_i^- \nabla I_i^{\lambda_n}(w_i^{\lambda_n}) \to P_i^- \nabla I_i(w);$$

Thus $w \in N_i$ and

$$c_i \leq I_i(w) = \lim_{\lambda \to \infty} c_i^\lambda.$$ (7.8)

The desired result follows from (7.2) and (7.8). $\square$

For fixed $i \in \{1, 2, \cdots, k\}$, let $\Omega_0 := \bigcup_{j \neq i} \Omega_j$ and $\Omega_0^\rho := \bigcup_{j \neq i} \Omega_j^\rho$. We denote $X_0 := H^1(\Omega_0^\rho) = \bigoplus_{j \neq i} X_j$ and $E_0 := H^1_0(\Omega_0^\rho) = \bigoplus_{j \neq i} E_j$. Let $X_0^{-}$ be the negative eigenspace associated to $-\Delta + \lambda a - \delta$ in $X_0$, and let $E_0^-$ be the negative eigenspace associated to $-\Delta - \delta$ in $E_0$. Clearly $X_0^{-} = \bigoplus_{j \neq i} X_j^{-}$ and $E_0^- = \bigoplus_{j \neq i} E_j^-$. Finally, let $Q_0^{-} : X_0 \to X_0^{-}$ and $P_0^- : E_0 \to E_0^-$ be the orthogonal projections.

The following linking property for $\gamma \in \Gamma_\lambda$ is the key to the proof of the lower bound of $c_\lambda$. We postpone its proof in the last section.

**Proposition 7.3.** If $\lambda$ is sufficiently large, then for any $\gamma \in \Gamma_\lambda$, there exists $(s, t) \in A$ such that $u := \gamma(s, t)$ satisfies

$$u_i := u|_{\Omega_i^\rho} \in N_i^\lambda,$$ (7.9)

and

$$u_0 \perp E_0^-, \|u_0\| < r.$$ (7.10)

By the aid of Proposition 7.3, we have the following:

**Lemma 7.4.** $c_\lambda \geq c_i^\lambda$. 

Proof. By Proposition 7.3, we have that for given $\gamma \in \Gamma_\lambda$, there exists $(s, t) \in A$ such that $u := \gamma(s, t)$ satisfies (7.9) and (7.10). It follows from (6.3) that $I_{\lambda}^0(u_0) \geq 0$, hence

$$\max_{A} \Phi_{\lambda} \circ \gamma \geq \Phi_{\lambda}(u) \geq I_{\lambda}^i(u_0) \geq c_i^\lambda.$$  

The desired result follows immediately. $\square$

As a consequence of (7.1) and Lemma 7.4 and Lemma 7.2, we have:

Proposition 7.5. There holds $\lim_{\lambda \to \infty} c_{\lambda} = c_i$ and for $\lambda$, large, $c_{\lambda}$ is achieved by a critical point $u_{\lambda}$ of $\Phi_{\lambda}$.

Proof. In fact, for $\lambda$ large enough, (6.2) implies

$$c_{\lambda} > \max_{(s, t) \in B} J_{\lambda} (\gamma_0(s, t)).$$

A standard argument yields that $c_{\lambda}$ is achieved by a critical point $u_{\lambda}$ of $\Phi_{\lambda}$ provided $\lambda \geq \Lambda_0$ thanks to Proposition 3.2. And a direct result of Proposition 5.1 is that $u_{\lambda}$ is a solution of $(S_{\lambda})$ for $\lambda$ large. $\square$

Remark 7.6. Take $u_{\lambda}$ be the critical point of $\Phi_{\lambda}$ obtained in Proposition 7.5. Then by Proposition 4.1, we conclude that $u_{\lambda} \to u$ strongly in $H^1(\mathbb{R}^N)$ as $\lambda \to \infty$, and $u_j = u|_{\Omega_j}$ $(1 \leq j \leq k)$ is a solution of (1.7) with $i$ replaced by $j$ and $u|_{\mathbb{R}^N \setminus \Omega} \equiv 0$. Furthermore for each $1 \leq j \leq k$, either $u_j \neq 0$ and $I_j(u_j) \geq c_j$ or $u_j = 0$. Moreover $\Phi_{\lambda}(u_{\lambda}) \to \sum_{j=1}^{k} I_j(u_j)$. On the other hand, as we showed in Proposition 7.5, $\Phi_{\lambda}(u_{\lambda}) = c_{\lambda} \to c_i$. Thus we have $\sum_{j=1}^{k} I_j(u_j) = c_i$. If we assume further that for any $1 \leq m, j \leq k$ and $m \neq j$ indicates $c_m \neq c_j$. Then we must have $u_i \neq 0$ such that $I_i(u_i) = c_i$ and for $j \neq i, u_j \equiv 0$, which implies that under this further condition, we indeed have finished the proof of our main result Theorem 1.4. However, in general, the above facts cannot assure that $I_i(u_i) = c_i$ and for $j \neq i, u_j \equiv 0$. Thus to complete the proof of our main result, we need more arguments and which is realized by the following flow argument.

8. Proof of Theorem 1.4

For $u \in E$ and $M \subset \mathbb{R}^N$ measurable, we use the notation

$$\|u\|_{\lambda, M} := \left( \int_M \left( |\nabla u|^2 + \left( \lambda a(x) - \delta \right) u^2 \right) \right)^{1/2}.$$  

Choose $\varepsilon > 0$ small so that $B_{\varepsilon}(0) := \{u|u \in H^1_0(\Omega_j), \|u\| < \varepsilon \}$ contains only $0 \in H^1_0(\Omega_j)$ as critical point of $I_j$ for all $j \neq i$. We also assume that $\varepsilon < \sqrt{Nc_i}$. Let

$$D_{\varepsilon} = \{u \in E_\lambda : \|u\|_{L^\infty, \Omega_j} \leq \varepsilon/3, \|u\|_{\Lambda, \Omega_j} \leq \sqrt{Nc_i} \leq \varepsilon/3 \}.$$
It is easy to check that $D^c_\Lambda \cap \Phi^c_\lambda$ contains all functions of the form:

$$w(x) = \begin{cases} w_i(x) & x \in \Omega_i, \\ 0 & x \in \mathbb{R}^N \setminus \Omega_i; \end{cases}$$

where $w_i$ minimizes $I_i$ in $\mathcal{N}_i$ (see Section 6).

**Lemma 8.1.** There exist $\sigma_0 > 0$ and $\Lambda_1 \geq \Lambda_0$ such that

$$\|\nabla \Phi_\lambda(u)\|_\lambda \geq \sigma_0 \quad \text{for } \lambda \geq \Lambda_1 \text{ and } u \in \left(D^2_\Lambda \setminus D^c_\lambda\right) \cap \Phi^c_\lambda. \quad (8.1)$$

**Proof.** We argue by contradiction. Suppose that there exist $\lambda_n \to \infty$ and $u_n \in \left(D^2_\lambda \setminus D^c_\lambda\right) \cap \Phi^c_\lambda$ such that $\|\nabla \Phi_\lambda(u)\|_{\lambda_n} \to 0$. Applying Proposition 4.1, we conclude that, up to a subsequence, $u_n \to u$ in $E$ and $u|\Omega_j$ ($j = 1, 2, \cdots, k$) is a critical point of $I_j$. Moreover

$$\lim_{n \to \infty} \|u_n\|_{\lambda_n, \Omega_j} = \int_{\Omega_j} (|\nabla u|^2 - \delta u^2) \quad \text{for } j = 1, 2, \cdots, k \quad (8.2)$$

and

$$\lim_{n \to \infty} \|u_n\|_{\lambda_n, \mathbb{R}^N \setminus \Omega} = 0, \quad \text{(8.3)}$$

which implies that $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. On the other hand, by (8.2) and the definition of $D^c_\lambda$, we have

$$\int_{\Omega_j} (|\nabla u|^2 - \delta u^2) \leq 2\varepsilon,$$

this implies that for $j \neq i$,

$$I_j(u|\Omega_j) \leq \frac{2}{N}\varepsilon.$$ 

By the choice of $\varepsilon$, we obtain that for $j \neq i$, $u|\Omega_j = 0$. Again, (8.2) and the choice of $\varepsilon$ imply that $u|\Omega_i \neq 0$, hence $I_i(u|\Omega_i) \geq c_i$. Notice that $\Phi_{\lambda_n}(u_n) \leq c_i$, we have $I_i(u|\Omega_i) = c_i$. It deduces that

$$\int_{\Omega_i} (|\nabla u|^2 - \delta u^2) = \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-1} c_i = N c_i.$$

Combining (8.2) with (8.3), we have $u_n \in D^c_\lambda$ for large $n$, which contradicts with that fact that $u_n \in D^2_\Lambda \setminus D^c_\lambda$. \quad $\Box$

The following proposition is the key of the proof of our main result.

**Proposition 8.2.** Let $\Lambda_1$ be the constant given in Lemma 8.1 and $\Lambda^*$ be given in Proposition 5.1. Then for $\lambda \geq \max\{\Lambda_1, \Lambda^*\}$, there exists a solution $u_\lambda$ of (1.1) satisfying $u_\lambda \in D^c_\lambda \cap \Phi^c_\lambda$. 

Proof. We argue indirectly and assume that $\Phi_\lambda$ has no critical points in $D^\varepsilon_\lambda \cap \Phi^{c_j}_\lambda$. By Proposition 3.2, we see that $\Phi_\lambda$ satisfies the (P.S.)\(_c\) condition for $c \leq c_i$. Thus there exists a constant $d_\lambda > 0$ such that

$$\|\nabla \Phi_\lambda(u)\|_{\lambda} \geq d_\lambda \text{ for all } u \in D^\varepsilon_\lambda \cap J^{c_i}_\lambda.$$  \hspace{1cm} (8.4)

By Lemma 8.1, it holds

$$\|\nabla \Phi_\lambda(u)\|_{\lambda} \geq \sigma_0 \text{ for all } u \in (D^{2\varepsilon}_\lambda \setminus D^\varepsilon_\lambda) \cap \Phi^{c_j}_\lambda.$$  

Let $\varphi : E \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\varphi(u) = \begin{cases} 1 & \text{for } u \in D^{3\varepsilon/2}_\lambda, \\ 0 & \text{for } u \notin D^{2\varepsilon}_\lambda, \end{cases}$$

and $0 \leq \varphi(u) \leq 1$ for every $u \in E$. Then the vector field

$$V : \Phi^{c_j}_\lambda \to E, \quad V(u) = -\varphi(u) \frac{\nabla \Phi_\lambda(u)}{\|\nabla \Phi_\lambda(u)\|_{\lambda}},$$

is well defined, Lipschitz continuous and satisfies

$$\|V(u)\|_{\lambda} \leq 1, \text{ for all } u.$$  \hspace{1cm} (8.5)

We consider the associated flow $\eta : [0, \infty) \times \Phi^{c_j}_\lambda \to \Phi^{c_j}_\lambda$ defined by

$$\dot{\eta}(\tau, u) = \frac{d\eta}{d\tau}(\tau, u) = V(\eta(\tau, u)), \quad \eta(0, u) = u.$$  

Obviously $\eta$ satisfies

$$\frac{d}{d\tau} \Phi_\lambda(\eta(\tau, u)) = -\varphi(u) \|\nabla \Phi_\lambda(u)\|_{\lambda} \leq 0,$$  \hspace{1cm} (8.6)

and

$$\eta(\tau, u) = u, \quad \text{for all } \tau \geq 0, \text{ } u \in \Phi^{c_j}_\lambda \setminus D^{2\varepsilon}_\lambda.$$  \hspace{1cm} (8.7)

Since $\gamma_0(s, t) \notin D^{2\varepsilon}_\lambda$ for $(s, t) \in B$, it follows from (8.7)

$$\eta(\tau, \gamma_0(s, t)) = \gamma_0(s, t), \quad \text{for } (s, t) \in B, \text{ } \tau \geq 0.$$  \hspace{1cm} (8.8)

Recall that $\text{supp } \gamma_0(s, t) \subset \bigcup_{j=1}^{k} \Omega_j$ for every $(s, t) \in A$, hence $\Phi_\lambda(\gamma_0(s, t))$ and $\|\gamma_0(s, t)\|_{\lambda, \Omega}$ do not depend on $\lambda \geq 0$. On the other hand

$$\Phi_\lambda(\gamma_0(s, t)) \leq c_i, \quad \text{for } (s, t) \in A,$$
and there exists a unique \((s^*, t^*) \in A\) (see (9.9)) such that \(\Phi_\lambda(\gamma_0(s^*, t^*)) = c_i\). That is, \(\gamma_0(s^*, t^*))|_{\Omega_i} = w_i\) and \(\gamma_0(s^*, t^*))|_{\Omega_j} = 0\) for \(j \neq i\). Thus we have
\[
m_0 := \max\{\Phi_\lambda(u) : u \in \gamma_0(A) \setminus D^\varepsilon_\lambda\} < c_i, \tag{8.9}
\]
which is independent of \(\lambda\).

Now we claim that for large \(\bar{\tau}\),
\[
\max_{(s, t) \in A} \Phi_\lambda(\eta(\bar{\tau}, \gamma_0(s, t))) \leq \max\{m_0, c_i - \sigma_0 \varepsilon / 6\} \tag{8.10}
\]
with \(\sigma_0, m_0\) from (8.1), (8.9), respectively. In fact, (8.9) yields \(\Phi_\lambda(\eta(\tau, \gamma_0(s, t))) \leq m_0\) if \(\gamma_0(s, t) \not\in D^\varepsilon_\lambda, \ \tau \geq 0\). In the case \(\gamma_0(s, t) \in D^\varepsilon_\lambda\), we consider the behavior of \(\tilde{\eta}(\tau) := \eta(\tau, \gamma_0(s, t))\).

Set \(\tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}\) and \(\bar{\tau} = \sigma_0 \varepsilon / 6 \tilde{d}_\lambda\), where \(d_\lambda\) is from (8.4). Then there are two cases:

1) \(\tilde{\eta}(\tau) \in D^{3\varepsilon/2}_\lambda\) for all \(\tau \in [0, \bar{\tau}]\).
2) \(\tilde{\eta}(\tau_0) \in \partial D^{3\varepsilon/2}_\lambda\) for some \(\tau_0 \in [0, \bar{\tau}]\).

In case 1), we have \(\varphi(\tilde{\eta}(\tau)) \equiv 1\) and \(\|\nabla \Phi_\lambda(\tilde{\eta}(\tau))\|_\lambda \geq \tilde{d}_\lambda\) for all \(\tau \in [0, \bar{\tau}]\). Then (8.1) implies
\[
\Phi_\lambda(\tilde{\eta}(\tau)) = \Phi_\lambda(\gamma_0(s, t)) + \int_0^\bar{\tau} \frac{d}{ds} \Phi_\lambda(\tilde{\eta}(s)) ds
= \Phi_\lambda(\gamma_0(s, t)) - \int_0^\bar{\tau} \varphi(\tilde{\eta}(s))\|\nabla \Phi_\lambda(\tilde{\eta}(s))\|_\lambda ds
\leq c_i - \int_0^\bar{\tau} \tilde{d}_\lambda ds = c_i - \tilde{d}_\lambda \bar{\tau} = c_i - \sigma_0 \varepsilon / 6.
\]

In case 2), there exist \(0 \leq \tau_1 < \tau_2 \leq \bar{\tau}\) such that
\[
\tilde{\eta}(\tau_1) \in \partial D^e_\lambda, \quad \tilde{\eta}(\tau_2) \in \partial D^{3\varepsilon/2}_\lambda, \tag{8.11}
\]
and
\[
\tilde{\eta}(\tau) \in D^{3\varepsilon/2}_\lambda \setminus D^e_\lambda \text{ for all } \tau \in [\tau_1, \tau_2]. \tag{8.12}
\]

It follows from (8.11) that
\[
\|\tilde{\eta}(\tau_1)\|_{\lambda, \mathbb{R}^N \setminus \Omega^\varepsilon_i} \leq \varepsilon / 3, \quad \|\tilde{\eta}(\tau_1)\|_{\lambda, \Omega^e_i} - \sqrt{N} c_i \leq \varepsilon / 3,
\]
and
\[
\|\tilde{\eta}(\tau_2)\|_{\lambda, \mathbb{R}^N \setminus \Omega^\varepsilon_i} = \frac{\varepsilon}{2} \quad \text{or} \quad \|\tilde{\eta}(\tau_2)\|_{\lambda, \Omega^e_j} - \sqrt{N} c_i = \frac{\varepsilon}{2}.
\]
This implies
\[ \| \tilde{\eta}(\tau_1) - \tilde{\eta}(\tau_2) \| \geq \varepsilon / 6. \] (8.13)

Now (8.5), (8.13) and the mean value theorem imply that \( \tau_2 - \tau_1 \geq \varepsilon / 6 \). By (8.1), we deduce that
\[
\Phi_\lambda(\tilde{\gamma}(s, t)) = \Phi_\lambda(\tilde{\eta}(s)) - \int_0^s \Phi_\lambda(\tilde{\eta}(s)) \| \nabla \Phi_\lambda(\tilde{\eta}(s)) \| ds \\
\leq c_i - \int_{\tau_1}^{\tau_2} \sigma_0 ds = c_i - \sigma_0(\tau_2 - \tau_1) \leq c_i - \sigma_0 \mu / 6,
\]
and (8.10) is proved.

Define \( \tilde{h}(s, t, r) := \eta(r \tilde{\tau}, \gamma_0(s, t)) \) and \( \tilde{\gamma}(s, t) := \tilde{h}(s, t, 1) = \eta(\tilde{\tau}, \gamma_0(s, t)) \). Observe that \( \tilde{h} \in H_\lambda \) due to (8.6), (8.8), hence \( \gamma \in \Gamma_\lambda \). Thus
\[
c_\lambda \leq \Phi_\lambda(\tilde{\gamma}(s, t)) \leq \max \{ m_0, c_i - \sigma_0 \mu / 6 \}. \] (8.14)

On the other hand, by Proposition 7.5, we have \( c_\lambda \to c_i \) as \( \lambda \to \infty \). This contradicts (8.10), and thus \( \Phi_\lambda \) has a critical point \( u_\lambda \in D_{\epsilon, i}^\lambda \). By Proposition 7.5, \( u_\lambda \) is a solution of the original problem (1.1).

Finally we are ready to give the proof of the main result.

**Proof of Theorem 1.4.** Let \( u_\lambda \) be a solution of (1.1) obtained in Proposition 8.2. Applying Proposition 4.1, for any given sequence \( \lambda_n \to \infty \), we can extract a subsequence, which satisfies the conclusion of Proposition 4.1. With the same argument as in the proof of Lemma 8.1, we can extract a subsequence of \( u_{\lambda_n} \) (still denoted by \( u_{\lambda_n} \)) such that \( u_{\lambda_n} \to u \) in \( E \), and \( u|_{\mathbb{R}^N \setminus \Omega_i} = 0 \). Furthermore
\[
\lim_{n \to \infty} \int_{\Omega_i} \left( \frac{1}{2} |\nabla u_{\lambda_n}|^2 - \delta u_{\lambda_n}^2 \right) = c_i \tag{8.15}
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega_i} \left( |\nabla u_{\lambda_n}|^2 + (\lambda_n a(x) - \delta) u_{\lambda_n}^2 \right) = 0. \tag{8.16}
\]

Since the limits in (8.15) and (8.16) do not depend on the choice of the sequence \( \lambda_n \to \infty \), Theorem 1.4 is proved.

\[ \square \]
9. Proof of Proposition 7.3

In this section, we give the proof of Proposition 7.3. For \( u \in E_\lambda \), we write \( u_i := u|_{\Omega_i^p} \) for \( i \in \{1, 2, \cdots, k\} \). Define the map \( f_\lambda : E_\lambda \to E_0^- \times (E_i^- \times \mathbb{R}) \) by

\[
f_\lambda(u) = (f_{\lambda,0}(u), f_{\lambda,i}(u))
\]

with

\[
f_{\lambda,0} := P_0^- : E_\lambda \to E_0^-.
\]

Moreover, we define the map \( f_{\lambda,i} : E_\lambda \to E_i^- \times \mathbb{R} \) by

\[
f_{\lambda,i}(u) := \left( P_i^- (\nabla I_\lambda^i(u_i)), D I_\lambda^i(u_i)[u_i] \right).
\]

Then one can see that

\[
f_\lambda(u) = 0 \iff u_0 \perp E_0^- \text{ and } u_i \in N_{\lambda}^i.
\]

(9.1)

Now we consider \( \gamma \in \Gamma_\lambda \), let \( h \in \mathcal{H}_\lambda \) be a homotopy from \( \gamma_0 \) to \( \gamma \). We will show that for \( \lambda \) large, there exists \( (s, t) \in A \) such that \( u = \gamma(s, t) \) satisfying \( f_\lambda(u) = 0 \) and \( \|u_0\| < r \). This will be done with a degree argument.

We first claim that for \( (s, t, \tau) \in A \times [0, 1] \), \( u := h(s, t, \tau) \) and \( \lambda \) large, the following holds:

\[
f_\lambda(u) = 0 \implies \|u_0\|_{X_0} \neq r.
\]

(9.2)

In order to see this, let us denote \( \tilde{X}_0^+ := \{ u \in X_0 : P_0^- u = 0 \} \). Then by Lemma 7.1 and (6.3), there is \( \beta > 0 \) such that for \( \lambda \) large,

\[
I_0^\lambda(v) \geq \beta \text{ for all } v \in \tilde{X}_0^+, \|v\|_{X_0} = r,
\]

(9.3)

and

\[
I_0^\lambda(v) \geq 0 \text{ for all } v \in \tilde{X}_0^+, \|v\|_{X_0} \leq r.
\]

(9.4)

Indeed, we recall that \( Q_0^\lambda^- : X_0 \to X_0^\lambda^- \) is the orthogonal projection on \( X_0^\lambda^- \) which is negative eigenspace associated to the operator \( L_\lambda := -\Delta + \lambda V - \delta \) in \( X_0 \). Thus for any \( v \in \tilde{X}_0^+ \), by Lemma 7.1, similar with the proof of (7.6), we have

\[
Q_0^\lambda^- v = o(\lambda) \|v\|_{X_0},
\]

where \( o(\lambda) \to 0 \) as \( \lambda \to \infty \). Now we decompose \( v = v_1 + v_2 \) with \( v_1 \in X_0^\lambda^- \) and \( v_2 \in \tilde{X}_0^+ \) such that

\[
\|v_1\| = o(\lambda) \|v\|_{X_0}, \quad \|v_2\| = (1 - o(\lambda)) \|v\|_{X_0}.
\]
Thus

\[ I_0^\lambda (v) = \int_{\Omega_0^\rho} |\nabla v|^2 + (\lambda a(x) - \delta) v^2 - \int_{\Omega_0^\rho} |v|^{2^*} \]

\[ = \int_{\Omega_0^\rho} |\nabla v_1|^2 + (\lambda a(x) - \delta) v_1^2 + \int_{\Omega_0^\rho} |\nabla v_2|^2 + (\lambda a(x) - \delta) v_2^2 - \int_{\Omega_0^\rho} |v|^{2^*} \]

\[ \geq C_1 (1 - o(\lambda)) \|v\|_{X_0}^2 - C_2 o(\lambda) \|v\|_{X_0}^2 - C_3 \|v\|_{X_0}^{2^*}, \] (9.5)

where \( C_i (i = 1, 2, 3) \) are constants independent of \( \lambda \). Thus for \( \lambda \) large enough, we deduce from (9.5) that (9.3) and (9.4).

Next we prove (9.2) by contrary. Assume that

\[ \|u_0\|_{X_0} = r. \] (9.6)

Then for \( \lambda \geq \Lambda_0 \),

\[ \Phi_\lambda (u) = \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega^\rho} \left( |\nabla u|^2 + (\lambda a - \delta) u^2 \right) - \int_{\mathbb{R}^N \setminus \Omega^\rho} G(x, u) \]

\[ + \sum_{j=1}^{k} \left( \frac{1}{2} \int_{\Omega_j^\rho} \left( |\nabla u|^2 + (\lambda a - \delta) u^2 \right) - \int_{\Omega_j^\rho} G(x, u) \right) \]

\[ \geq \sum_{j=1}^{k} \left( \frac{1}{2} \int_{\Omega_j^\rho} \left( |\nabla u|^2 + (\lambda a - \delta) u^2 \right) - \frac{1}{2^*} \int_{\Omega_j^\rho} |u|^{2^*} \right) \]

\[ = \sum_{j=1}^{k} I_j^\lambda (u|_{\Omega_j^\rho}). \]

Thus we get, for \( u = h(s, t, r) \) with \( f_\lambda (u) = 0 \),

\[ \Phi_\lambda (u) \geq \sum_{j=1}^{k} I_j^\lambda (u_j) \geq \beta + c_i^j > c_i. \] (9.7)

The last inequality follows from Lemma 7.2.

On the other hand, noticing that \( \Phi_\lambda (h(s, t, \tau)) \) is nonincreasing with respect to \( \tau \in [0, 1] \), we have

\[ \Phi_\lambda (u) = \Phi_\lambda (h(s, t, \tau)) \leq \Phi_\lambda (h(s, t, 0)) = \Phi_\lambda (\gamma_0(s, t)) \leq c_i, \]

which contradicts with (9.7). This contradiction implies that (9.6) is impossible, (9.2) is proved.
Now we consider the sets
\[ G_\lambda := \{(s, t, \tau) \in A \times [0, 1] : f_\lambda(h(s, t, \tau)) = 0\} \]
and
\[ G_\lambda^0 := \{(s, t, \tau) \in G_\lambda : u = h(s, t, \tau) \text{ satisfies } \|u_0\|_{X_0} < r\}. \]
By (9.2), for \( \lambda \) large, there exists a neighborhood \( U_\lambda \) of \( G_\lambda^0 \) in \( A \times [0, 1] \) such that \( U_\lambda \cap (G_\lambda \setminus G_\lambda^0) = \emptyset \). We define \( \bar{U}_\lambda := \{(s, t) \in A : (s, t, \tau) \in U_\lambda\} \). The lemma is proved if we can find \( (s, t) \in \bar{U}_\lambda \) such that \( f_\lambda(\gamma(s, t)) = 0 \). By the homotopy invariance of the degree, we have
\[ \deg(f_\lambda \circ \gamma, U_\lambda^1, 0) = \deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0). \quad (9.8) \]
Setting
\[ s^* = (0, \ldots, 0) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k} \quad \text{and} \quad t^* = \frac{1 - r}{R - r}, \quad (9.9) \]
we have
\[ G \cap (A \times \{0\}) = \{(s^*, t^*, 0)\}, \]
and therefore
\[ \deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0) = \deg(f_\lambda \circ \gamma_0, A, 0). \quad (9.10) \]
In the following, let us calculate the degree \( \deg(f_\lambda \circ \gamma_0, A, 0) \).
Note that \( \gamma_0 \) is linear in \((s, t)\) and it defines a homeomorphism from \( A \) to \( A' \):
\[ \gamma_0 : A \to A' := B_{0, r} \times A_{w_i, r, R} \subset E_0^- \times H_{w_j} \subset H_0^1(\Omega), \]
where \( A_{w_i, r, R} \subset H_{w_i} \subset E_i^- \oplus \mathbb{R}w_i \) is defined as in (2.1) and (2.2), and
\[ B_{0, r} := \left\{ u \in E_0^- : u = r \sum_{j=1}^{d_i} s_{ij} e_{ij}, \ |s_{ij}| \leq 1 \right\}. \]
It follows that
\[ \deg(f_\lambda \circ \gamma_0, A, 0) = \pm \deg(f_\lambda, A', 0). \quad (9.11) \]
Moreover, since \( A' \subset H_0^1(\Omega) \), for \( u \in A' \), we have \( u_j = u|_{\Omega_j} \in H_j^1(\Omega_j) \) \( (j = 1, 2, \ldots, k) \) and \( I_j^\lambda(u_j) = I_j(u_j) \). Which implies that for \( u \in A' \),
\[ f_\lambda(u) = (g_0(u_0), g_i(u_i)) \]
with
\[ g_0(u_0) = P_0^-(u_0), \quad g_i(u_i) = \left( P_i^-(\nabla I_i(u_i)), D I_i^+(u_i)[u_i] \right). \]

By Proposition 2.1e), we have
\[ \deg(f_{\lambda}, A', 0) = \deg(g_0, B_{0,r}, 0) \cdot \deg(g_i, A_{w_i,r,R}, 0). \tag{9.12} \]

But we see that
\[ \deg(g_0, B_{0,r}, 0) = \deg(P_0^-, B_{0,r}, 0) = 1, \tag{9.13} \]

thus
\[ \deg(g_i, A_{w_i,r,R}, 0) = (-1)^{\dim H_{w_i}}. \tag{9.14} \]

Combining with equations in (9.8)–(9.14), we obtain the existence of \((s, t) \in U_{\lambda}^1\) with \(f_{\lambda}(\gamma(s, t)) = 0\). Therefore, it follows that \(u = \gamma(s, t)\) satisfying \(\|u_0\|_{X_0} < r\) and \(f_{\lambda}(u) = 0\). This proves Proposition 7.3.\qed

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References