Multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields and critical frequency

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A R T I C L E   I N F O

Article history:
Received 6 September 2007
Revised 25 July 2008
Available online 10 September 2008

MSC:
35J60
35B33

Keywords:
Nonlinear Schrödinger equation
Multi-bump standing waves
Potential well
Magnetic fields
Variational methods

A B S T R A C T

In this paper, we are concerned with the existence and asymptotic behavior of standing wave solutions \( \psi(x,t) = e^{-iEt} \) of nonlinear Schrödinger equations with electromagnetic fields \( i \frac{\partial \psi}{\partial t} = -(\nabla + iA(x))^2 \psi + \lambda W(x) \psi - f(|\psi|^2) \psi \), \((t,x) \in \mathbb{R} \times \mathbb{R}^N\), with \( E \) being a critical frequency in the sense that \( \inf_{x \in \mathbb{R}^N} W(x) = E \). We show that if the zero set of \( W - E \) has several isolated connected components \( \Omega_1, \ldots, \Omega_k \) such that the interior of \( \Omega_i \) is not empty and \( \partial \Omega_i \) is smooth, then for \( \lambda > 0 \) large there exists, for any non-empty subset \( J \subset \{1, 2, \ldots, k\} \), a standing wave solution which is trapped in a neighborhood of \( \bigcup_{j \in J} \Omega_j \).

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1. Introduction

We are concerned with nonlinear Schrödinger equations with an electromagnetic potential

\[
i \frac{\partial \psi}{\partial t} = -(\nabla + iA(x))^2 \psi + \lambda W(x) \psi - f(|\psi|^2) \psi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,
\]

here \( i \) is the imaginary unit, \( \lambda \) is the parameter corresponding to Plank constant, \( A(x) = (A_1(x), A_2(x), \ldots, A_N(x)) \) is a real vector (magnetic) potential with magnetic field \( B = \text{curl} A \) and \( W(x) \) is a scalar electric potential.
We are interested in standing wave solutions, i.e., solutions of type
\[
\psi(x, t) = \exp(-i\lambda Et)u(x)
\]  
(1.2)
to (1.1) when \( \lambda \) is sufficiently large, where \( E \) is a real number and \( u(x) \) is a complex-valued function which satisfies
\[
-(\nabla + iA(x))^2 u(x) + \lambda (W(x) - E)u(x) = f(|u(x)|^2)u(x), \quad x \in \mathbb{R}^N.
\]  
(1.3)

We say that a local complex-valued function \( u \) is \( k \)-bump, if \(|u|\) has exactly \( k \) local maxima in \( \mathbb{R}^N \). In this paper we consider the existence of multi-bump solutions to problem (1.3).

In recent years, much attention has been devoted to the study of the existence for one-bump or multi-bump bound states of (1.3) under the case \( A(x) \equiv 0 \). In particular, when \( f(t) = t^{\frac{p}{2}} - \frac{2}{N} \) for \( N \geq 3 \) and \( 2 < p < +\infty \) for \( N = 1, 2 \), which leads to investigate the positive solutions \( u : \mathbb{R}^N \to \mathbb{R} \) to the semilinear elliptic equation
\[
-\Delta u(x) + \lambda (W(x) - E)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N.
\]

Set \( v(x) = h^{-\frac{2}{p-2}}u(x) \) with \( h^{-2} = \lambda \), the above equation comes to
\[
-h^2 \Delta v(x) + (W(x) - E)v(x) = |v(x)|^{p-2}v(x), \quad x \in \mathbb{R}^N.
\]  
(1.4)

Under the condition \( \inf_{x \in \mathbb{R}^N} W(x) > E \), there have been enormous investigations on problem (1.4). In [19], using a Lyapunov–Schmidt reduction, Floer and Weinstein established the existence of a standing wave solutions of (1.4) and \( W(x) - E \) is a bounded function having a non-degenerate critical point for sufficiently small \( h > 0 \). Moreover they showed that \( u \) concentrates near the given non-degenerate critical point of \( W - E \) when \( h \) tends to 0. Their method and results were later generalized by Oh [22,23] to the higher-dimensional case and the existence of multi-bump solutions concentrating near several non-degenerate critical points of \( W - E \) as \( h \) tends to 0 was obtained. For more results, we refer to A. Ambrosetti, A. Malchiodi, S. Secchi [2], A. Ambrosetti, M. Badiale and S. Cingolani [1], S. Cingolani and M. Lazzo [11], S. Cingolani and M. Nolasco [14], M. Del Pino, P. Felmer [15,16].

It seems that Byeon and Wang [5,6] were the first to study energy level and the asymptotic behavior of positive solutions to problem (1.4) under the condition \( \inf_{x \in \mathbb{R}^N} W(x) = E \).

In [7], D. Cao and E.S. Noussair extended the results of Byeon and Wang [5] and [6]. They showed that if and the zero set of \( W - E \) has several isolated connected components \( \Omega_1, \ldots, \Omega_k \) such that the interior of \( \Omega_j \) is not empty and \( \partial \Omega_j \) is smooth, then for \( h > 0 \) small there exist, for any non-empty subset \( J \subset \{1, 2, \ldots, k\} \), solutions of problem (1.4) concentrating simultaneously at \( \bigcup_{j \in J} \Omega_j \), that is, to obtain solutions \( u_h \) such that \( h^{-\frac{2}{p-2}}u_h \to w \) as \( h \to 0 \) and \( w \) satisfies \( w = 0 \) for \( \mathbb{R}^N \setminus \bigcup_{j \in J} \Omega_j \) and \( w|_{\Omega_j} \) is the ground state solution of
\[
\begin{aligned}
-\Delta w &= w^{p-1}, \quad x \in \Omega_j, \\
w &> 0, \quad x \in \Omega_j, \\
w &= 0, \quad x \in \partial \Omega_j.
\end{aligned}
\]  
(1.5)

In their argument of the main results, the isolation of the ground state solution of (1.5) plays an important role.

There are also many works on the following similar nonlinear Schrödinger equations with an electromagnetic potential
\[
\left( \frac{h}{i} \nabla - A(x) \right)^2 u(x) + (W(x) - E)u(x) = f(|u(x)|^2)u(x), \quad x \in \mathbb{R}^N,
\]  
(1.6)
with \( A(x) \neq 0 \). The first result is due to Esteban and Lions [18]. For \( h > 0 \) fixed and for special classes of magnetic fields, they found existence of solutions for (1.9) by solving an appropriate minimization problems for the corresponding energy functional in the case of \( N = 2 \) and \( N = 3 \).

More recently, K. Kurata [21], S. Cingolani [9], S. Cingolani and S. Secchi [12], D. Cao and Z. Tang [8] have verified the existence of single-bump or multi-bump bound states of (1.6) under the condition \( \inf_{x \in \mathbb{R}^N} W(x) > E \).

In [13], S. Cingolani and S. Secchi proved the existence of standing wave solutions for (1.6) on \( \mathbb{R}^3 \), they dealt with the physically meaningful case of a constant magnetic field \( B = (s, 0, 0) \) having source in the potential \( A(x) = b/2(-x_2, x_1, 0) \) corresponding to the Lorentz gauge.

We refer to T. Bartsch, E.N. Dancer and S. Peng [4] in the case of \( W(x) - E \geq 0 \), they obtained the existence of multi-bump semi-classic bound states of (1.6) which concentrate simultaneously at the local minima of \( W(x) - E \) under the condition that \( W(x) - E \) is non-negative. Moreover, they obtained the asymptotic behavior of the bound states as \( h \) sufficiently small. In [10], S. Cingolani, L. Jeanjean and S. Secchi developed the result obtained in [4], they obtained the existence result of multi-peak solutions to (1.6) under more general nonlinearities which is nearly optimal. In particular, they dropped the isolatedness condition which required in [4] and they covered the case of nonlinearities, which are not monotone.

In the present paper, we consider the standing waves of (1.3) under the condition \( \inf_{x \in \mathbb{R}^N} W(x) = E \), we assume that the zero set of \( W - E \) has several isolated connected components \( \Omega_1, \ldots, \Omega_k \) such that the interior of \( \Omega_i \) is non-empty and \( \partial \Omega_i \) is smooth. We will obtain the similar results with D. Cao and E.S. Noussair [7]. However, our method has essential difference with the methods used in [7]. We mainly follow the idea of Y. Ding and K. Tanaka [17] to modify the nonlinearity and using the decay flow to obtaining our main results. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, since the appearance of electromagnetic potential \( A(x) \), we must consider our problem for complex-valued functions and so we need more delicate estimates.

Our paper is organized as follows: In Section 2, we describe our main results (Theorem 2.2). Section 3 is devoted to preliminary results. Section 4 contains the proofs of the main results.

We will use the same \( C \) to denote various generic positive constants, and we will use \( o(1) \) to denote quantities that tend to 0.

2. Main results

We set \( V(x) = W(x) - E \) and rewrite (1.3) in the following form

\[-(\nabla + iA(x))^2 u(x) + \lambda V(x) u(x) = f(|u(x)|^2) u(x), \quad x \in \mathbb{R}^N.\]  (2.1)

Our hypotheses on \( A(x) \) and \( V(x) \) are:

(\( A_1 \)) \( A_j(x) \in C^1(\mathbb{R}^N, \mathbb{R}) \) (\( j = 1, 2, \ldots, N \)).

(\( V_1 \)) \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( V(x) \geq 0 \) and \( \liminf_{|x| \to \infty} V(x) > 0 \).

(\( V_2 \)) The zero set of \( V(x), \Omega := \text{int} V^{-1}(0) \) is non-empty and has smooth boundary and \( \overline{\Omega} = V^{-1}(0) \).

(\( V_3 \)) \( \Omega \) consists of \( k \) components: \( \Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k \) and

\[ \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset \quad \text{for all } i \neq j. \]

We write

\[ \nabla_A u = (\nabla + iA)u. \]

Let

\[ H_A^1(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N) \}. \]
and hence $H^1_A(\mathbb{R}^N)$ is the Hilbert space under the scalar product
\[(u, v) = \text{Re} \int_{\mathbb{R}^N} ((\nabla u + iA(x)u)(\nabla v + iA(x)v) + u\bar{v}),\]
the norm induced by the product $(.,.)$ is
\[\|u\|_{H^1_A(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} (|\nabla A u|^2 + |u|^2) \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^N} (|\nabla u + iA(x)u|^2 + |u|^2) \right)^{\frac{1}{2}}
\[= \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + (|A(x)|^2 + 1)|u|^2) - 2 \text{Re} \int_{\mathbb{R}^N} iA(x)\bar{u}\nabla u \right)^{\frac{1}{2}}.\]

Let
\[E := \left\{ u \in H^1_A(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 < \infty \right\}\]
with the norms
\[\|u\|_{E}^2 = \int_{\mathbb{R}^N} (|\nabla A u|^2 + V(x)|u|^2).\]

We can easily see that $(E, \|\cdot\|_{E})$ is a Hilbert space and $E \subset H^1_A(\mathbb{R}^N)$. We define for open set $K \subset \mathbb{R}^N$,
\[H^1_A(K) = \left\{ u \in L^2(K) : \nabla A u \in L^2(K) \right\},\]
\[\|u\|_{H^1_A(K)} = \left( \int_{K} (|\nabla A u|^2 + |u|^2) \right)^{\frac{1}{2}},\]
\[E(K) = \left\{ u \in H^1_A(K) : \int_{K} V(x)|u|^2 < \infty \right\},\]
\[\|u\|_{E(K)} = \left( \int_{K} (|\nabla A u|^2 + V(x)|u|^2) \right)^{\frac{1}{2}}.\]

Let $H^0,1_A(K)$ be the Hilbert space defined by the closure of $C^\infty_0(K, \mathbb{C})$ under the scalar product $(.,.)$. Moreover, we have the following diamagnetic inequality (see [18] for example):
\[|\nabla A u(x)| \geq |\nabla |u(x)||, \text{ for } \forall u \in H^1_A(\mathbb{R}^N),\]
and this fact means that if $u \in H^1_A(\mathbb{R}^N)$, then $|u| \in H^1(\mathbb{R}^N)$. 


Remark 2.1. The spaces $H^1_A(\mathbb{R}^N)$ and the spaces $H^1(\mathbb{R}^N)$ are not comparable; more precisely, in general $H^1_A(\mathbb{R}^N) \not\subset H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \not\subset H^1_A(\mathbb{R}^N)$. However it is proved by G. Arioli and A. Szulkin [3] that if $K$ is bounded domain with regular boundary, then $H^1(K)$ and $H^1_A(K)$ are equivalent, where

$$H^1_A(K) = \{ u \in L^2(K): \nabla_A u \in L^2(K) \}$$

with the norm $\| u \|_{H^1_A(K)} = (\int_K (|\nabla_A u|^2 + |u|^2))^\frac{1}{2}$.

Our assumptions on the nonlinearity $f(t)$ are as follows:

(f1) $f : [0, \infty) \to \mathbb{R}$ is continuous and $\lim_{t \to 0} f(t) = 0$.

(f2) $\lim_{t \to \infty} \frac{f(t)}{t^s} = 0$ for some $s < 4/(N - 2)$ if $N \geq 3$, some $s > 0$ for $N = 1, 2$.

(f3) For some $2 < \theta < s + 2$ we have $0 < \frac{2}{s} f(t) < f(t)t$ for all $t > 0$ where $F(t) = \int_0^t f(\tau) d\tau$.

(f4) $f(t)$ is increasing on $(0, \infty)$.

Let

$$E_\lambda := \left\{ u \in H^1_A(\mathbb{R}^N): \int_{\mathbb{R}^N} V(x)|u|^2 < \infty \right\}$$

with the norms

$$\| u \|^2_\lambda = \int_{\mathbb{R}^N} (|\nabla_A u|^2 + \lambda V(x)|u|^2).$$

The energy functional associated with (2.1) is defined by

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u + iA(x)u|^2 + \lambda V(x)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} F(|u|^2) \quad \text{for } u \in E_\lambda,$$

where $F(t) = \int_0^t f(s) ds$.

We say that $u(x) \in E_\lambda$ is a least energy solution of (2.1) if and only if

$$J_\lambda(u) = c_\lambda := \inf \{ J_\lambda(u): u \in E_\lambda \setminus \{0\} \text{ is a solution of (2.1)} \}.$$

For $\lambda$ large, the potential well $\Omega = \text{int}(V^{-1}(0))$ plays an important role and the following problem

$$\begin{cases}
-(\nabla + iA(x))^2 u(x) = f(|u(x)|^2) u(x), & x \in \Omega, \\
u(x) \in H^0_A(\Omega)
\end{cases} \quad (D_\Omega)$$

is some kind of limit problem of (2.1) and the solutions are characterized as critical points of

$$I_\Omega(u) = \frac{1}{2} \int_\Omega |\nabla u + iA(x)u|^2 - \frac{1}{2} \int_\Omega F(|u|^2) \quad \text{for } u \in H^0_A(\Omega).$$

We also say that $u \in H^0_A(\Omega)$ is a least energy solution of $(D_\Omega)$ if and only if

$$I_\Omega(u) = c(\Omega) := \inf \{ I_\Omega(u): u \in H^0_A(\Omega) \setminus \{0\} \text{ is a solution of (D_\Omega)} \}.$$

For the following connected problem

$$-(\nabla + iA(x))^2 u(x) + (\lambda V(x) + 1)u(x) = |u(x)|^{p-2} u(x), \quad x \in \mathbb{R}^N,$$

(2.2)
where $V(x)$, $A(x)$ satisfy conditions (A$_1$) and (V$_2$), (V$_3$) and some other conditions. In [24], we proved the existence of least energy solutions of (2.2) which are concentrated on the one of subsets of zero set $\Omega$ of $V(x)$. In [25], we proved that for any non-empty subset $J \subset \{1, 2, \ldots, k\}$, (2.2) has a standing wave solution which is trapped in a neighborhood of $\bigcup_{j \in J} \Omega_j$ for $\lambda$ large.

In present paper, we consider problem (2.1) which has essential differences with (2.2) as a results of the potential of (2.1) is critical frequency case.

Our main results are:

**Theorem 2.2.** Suppose (A$_1$), (V$_1$)-(V$_3$) and (f$_1$)-(f$_4$) hold. Then for any $\varepsilon > 0$ and any non-empty subset $J$ of $\{1, 2, \ldots, k\}$, there exists $\Lambda = \Lambda(\varepsilon) > 0$ such that, for $\lambda \geq \Lambda$, (2.1) has a solution $u_\lambda \in E$ satisfying

$$\left| J\lambda(u_\lambda) - c(\Omega_j) \right| \leq \varepsilon \quad \text{for } j \in J, \quad \text{(2.3)}$$

$$\int_{\mathbb{R}^N, \Omega_j} \left( \left| \nabla u_\lambda + iA(x)u_\lambda \right|^2 + \lambda V(x)|u_\lambda|^2 \right) \leq \varepsilon, \quad \text{(2.4)}$$

where $\Omega_j = \bigcup_{j \in J} \Omega_j$. Moreover, for any sequence $\lambda_n \to \infty$, we can extract a subsequence $\lambda_{n_i}$ such that $u_{\lambda_{n_i}}$ converges strongly in $H^1_A(\mathbb{R}^N)$ to a function $u(x)$ which satisfies $u(x) = 0$ for $x \notin \Omega_j$, and the restriction $u|_{\Omega_j}$ is a least energy solution of

$$\begin{cases}
-(\nabla + iA(x))^2u(x) = f\left(|u(x)|^2\right)u(x), & x \in \Omega_j, \\
u(x) \in H^{0,1}_A(\Omega_j)
\end{cases}$$

for $j \in J$.

**Corollary 2.3.** Under the same assumption of Theorem 2.2, there exists $\Lambda > 0$ such that for $\lambda > \Lambda$, (2.1) has at least $2^k - 1$ bound states.

**Remark 2.4.** With the same argument in [10, Remark 1.2], under the condition of (A$_1$), (V$_1$)-(V$_3$) and (f$_1$)-(f$_4$), we can prove that the solution $u_\lambda \in E$, founded in Theorem 2.2 satisfies $|u_\lambda(x)| \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$ with $0 < \alpha < 0$. Indeed, set $u_\lambda = v + iw$, with $v, w$ real-valued, we have

$$-\Delta v + \lambda V(x)v = G := f\left(|u_\lambda|^2\right)v - 2A(x) \cdot \nabla w - |A(x)|^2v + (\text{div}A(x))w$$

and

$$-\Delta w + \lambda V(x)w = H := f\left(|u_\lambda|^2\right)w + 2A(x) \cdot \nabla v - |A(x)|^2w + (\text{div}A(x))v.$$  

Since $u_\lambda \in E$, it follows that $u_\lambda \in H^1(K)$ for each bounded set $K \subset \mathbb{R}^N$ and thus $v, w \in H^1(K, \mathbb{R}) \subset L^2(K, \mathbb{R})$. By (f$_2$) and (A$_1$), it is easy to check that $G, H \in L^s(K, \mathbb{R})$, where $s = \min\{2, \frac{2}{p-1}, 2\}$. Using standard bootstrap argument we can prove that $v, w \in C^{1,\alpha}(K, \mathbb{R})$ with $0 < \alpha < 1$.

3. Preliminaries

From the assumption (V$_3$) on $V(x)$, for $j \in \{1, 2, \ldots, k\}$, we can find bounded open subset $\Omega_j'$ with smooth boundary such that

(i) $\overline{\Omega}_j \subset \Omega_j'$ for all $j$,

(ii) $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for all $i \neq j$. 

In the following, we will prove the positivity of the operator $-(\nabla+iA(x))^2+\lambda V(x)$ acting on the space $E(K)$, where $K$ is one of the following sets:

$$K = \mathbb{R}^N, \quad \Omega_j' \quad (j = 1, 2, \ldots, k), \quad \text{or} \quad \mathbb{R}^N \setminus \bigcup_{j \in J} \Omega_j' \quad (J \subset \{1, 2, \ldots, k\}).$$

(3.1)

We can define a norm on $E(K)$ by

$$\|u\|_{\lambda,K}^2 = \int_K (|\nabla A u|^2 + \lambda V(x)|u|^2) \quad \text{for all } \lambda \geq 0.$$

For $K = \mathbb{R}^N$, $\|u\|_{\lambda,\mathbb{R}^N} = \|u\|_{\lambda}$. It is easy to see that $\|u\|_{\lambda,K}$ is equivalent to $\|u\|_{E(K)}$.

The following proposition is one of the keys of the our argument.

**Proposition 3.1.** There exist $\delta_0 > 0$, $\nu_0 > 0$ such that for the set given in (3.1) and for $\lambda \geq 0$

$$\delta_0 \|u\|_{\lambda,K}^2 \leq \|u\|_{\lambda,K}^2 - \nu_0 \|u\|_{L^2(K)}^2.$$

(3.2)

**Proof.** We firstly prove the case $K = \Omega_j'$ for some $j \in \{1, 2, \ldots, k\}$.

Let $d = \text{dist}(\Omega_j, \partial\Omega_j')$, since $\Omega_j \subset \Omega_j'$, we have $d > 0$. Let $\delta = \frac{d}{4}$ and we denote $\Omega_j^\delta = \{x \in \Omega_j': \text{dist}(x, \partial\Omega_j) < \delta\}$, thus there exists a positive number $\gamma_0 > 0$ such that $V(x) \geq \gamma_0$ on $\Omega_j' \setminus \Omega_j^\delta$. We choose a cut-off function $\xi \in C_0^\infty(\Omega_j')$ such that

$$\begin{cases} 
\xi(x) \equiv 1, & x \in \Omega_j^\delta, \\
0 \leq \xi(x) \leq 1, & x \in \Omega_j', \\
|\nabla \xi(x)| \leq C, & x \in \Omega_j', 
\end{cases}$$

where $C$ is a positive constant dependent only on the distance $d := \text{dist}(\Omega_j, \partial\Omega_j')$. By Poincaré inequality we have

$$\int_{\Omega_j'} \xi^2 |u|^2 \leq C \int_{\Omega_j'} |\nabla (\xi |u|)|^2.$$

From the definition of $\xi$, we know that $|\nabla \xi| \equiv 0$ on $\Omega_j^\delta$, since $V(x) \geq \gamma_0$ on $\Omega_j' \setminus \Omega_j^\delta$ and by the diamagnetic inequality we have that for $\lambda > 0$

$$\int_{\Omega_j'} \xi^2 |u|^2 \leq C \int_{\Omega_j'} |\nabla \xi|^2 |u|^2 + \xi^2 \lambda V(x)|u|^2 \leq C \int_{\Omega_j'} \lambda V(x)|u|^2 + \lambda |\nabla u|^2 = C \|u\|_{\lambda,\Omega_j'}^2.$$

Similarly, we have
\[
\int_{\Omega_j'} (1 - \zeta)^2 |u|^2 \leq C \int_{\Omega_j'} \lambda V(x) |u|^2 \leq C \int_{\Omega_j'} \lambda V(x) |u|^2 + |\nabla u|^2 = C \|u\|^2_{\lambda, \Omega_j'}. 
\]

Thus we have
\[
\int_{\Omega_j'} |u|^2 \leq C \|u\|^2_{\lambda, \Omega_j'}. \tag{3.3}
\]

where the constant \( C \) dependent only on \( d \) and \( \gamma_0 \). We choose \( \nu_0 < \frac{1}{C} \), from (3.3) it is easy to see that we can find \( \delta_0 > 0 \) such that inequality (3.2) holds.

Now we come to see the case of \( K = \mathbb{R}^N \). We choose \( R > 0 \) large enough such that \( \bigcup_{j=1}^k \overline{\Omega_j} \subset B_R(0) \), where \( B_R(0) \) denotes the ball centered at 0 with radius \( R \). We also define a cut-off function \( \zeta \in C_0^\infty(\mathbb{R}^N) \) such that
\[
\begin{aligned}
\zeta(x) &\equiv 1, & x &\in B_R(0), \\
0 &\leq \zeta(x) \leq 1, & x &\in \mathbb{R}^N, \\
|\nabla \zeta(x)| &\leq C, & x &\in \mathbb{R}^N.
\end{aligned}
\]

Since \( V(x) \) satisfies the condition \((V_2)\), thus we also can find a positive \( \gamma_0 > 0 \) such that \( V(x) \geq \gamma_0 \) on \( \mathbb{R}^N \setminus B_R(0) \). The following argument is similar the case of \( K = \Omega_j' \) and we omit it.

For the case \( K = \mathbb{R}^N \setminus \bigcup_{j \in J} \Omega_j' (J \subset \{1, 2, \ldots, k\}) \), the proof is similar with the case of \( K = \mathbb{R}^N \) and we also omit it here and thus we complete the proof of Proposition 3.1. \( \square \)

In what follows, we fix non-empty subset \( J \subset \{1, 2, \ldots, k\} \) and we set
\[
\Omega = \bigcup_{j \in J} \Omega_j, \quad \Omega_j' = \bigcup_{j \in J} \Omega_j', \quad \chi_{\Omega_j'}(x) = \begin{cases} 1 & \text{for } x \in \Omega_j', \\ 0 & \text{for } x \notin \Omega_j. \end{cases}
\]

From \((f_1)-(f_4)\) we see there exists an \( a > 0 \) such that
\[
f(\xi^2) \leq \nu_0 \text{ for } \xi \in [0, a], \quad \text{and } f(\xi^2) \geq \nu_0 \text{ for } \xi \geq a.
\]

Now we define
\[
\bar{f}(s^2) = \min\{f(s^2), \nu_0\} \tag{3.4}
\]

and
\[
g(x, s^2) = \chi_{\Omega_j'}(x)f(s^2) + (1 - \chi_{\Omega_j'}(x))\bar{f}(s^2).
\tag{3.5}
\]

Set \( G(x, t) = \int_0^t g(x, s) \, ds \), then it is easy to see that
\[
G(x, \xi^2) \leq g(x, \xi^2)\xi^2. \tag{3.6}
\]
We define
\[
\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla A u|^2 + \lambda V(x)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} G(x, |u|^2) : E \to \mathbb{R}.
\]

It is easy to check that \(\Phi_\lambda(u) \in C^2(E, \mathbb{R})\) and its critical points are solutions of
\[-(\nabla + iA(x))^2 u(x) + \lambda V(x)u(x) = g(x, |u|^2)u(x), \quad x \in \mathbb{R}^N.\]

We remark that \(\tilde{f}(t) = f(t)\) for \(t \in [0, a]\) and a critical point \(u(x)\) of \(\Phi_\lambda(u)\) is solution of (2.1) if and only if \(|u| \leq a\) in \(\mathbb{R}^N \setminus \Omega'_J\).

We have the following compactness results

**Proposition 3.2.** For \(\lambda \geq 0\), \(\Phi_\lambda(u)\) satisfies \((PS)_c\) condition for all \(c \in \mathbb{R}\). That is any sequence \((u_n) \subset E\) satisfying for \(c \in \mathbb{R}\)
\[
\Phi_\lambda(u_n) \to c, \quad (3.7)
\]
\[
\Phi'_\lambda(u_n) \to 0 \quad \text{strongly in } E^*, \quad (3.8)
\]
has a strongly convergent subsequence in \(E\), where \(E^*\) is the dual space of \(E\).

For giving the proof of Proposition 3.2, we need the following lemma firstly.

**Lemma 3.3.** Suppose that a sequence \((u_n) \subset E\) satisfies (3.7) and (3.8). Then there exists constant \(M(c)\) which is independent of \(\lambda \geq 0\) such that
\[
\limsup_{n \to \infty} \|u_n\|_\lambda^2 \leq M(c).
\]

**Proof.** It follows from (3.7) and (3.8) that
\[
\Phi_\lambda(u_n) - \frac{1}{\theta} \Phi'_\lambda(u_n)u_n = c + o(1) + \varepsilon_n \|u_n\|_\lambda,
\]
where \(\varepsilon_n \to 0\) as \(n \to \infty\). Thus
\[
\left(\frac{1}{2} - \frac{1}{\theta}\right) \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + \lambda V(x)|u_n|^2) - \frac{1}{2} \int_{\mathbb{R}^N} G(x, |u_n|^2) + \frac{1}{\theta} \int_{\mathbb{R}^N} g(x, |u_n|^2)|u_n|^2
\]
\[
= c + o(1) + \varepsilon_n \|u_n\|_\lambda.
\]

Since
\[
G(x, |u_n|^2) = \chi_{\Omega'_J}(x)F(|u_n|^2) + (1 - \chi_{\Omega'_J}(x))\tilde{F}(|u_n|^2),
\]
and
\[
g(x, |u_n|^2)|u_n|^2 = \chi_{\Omega'_J}(x)f(|u_n|^2)|u_n|^2 + (1 - \chi_{\Omega'_J}(x))\tilde{f}(|u_n|^2)|u_n|^2,
\]
where \(F(t) = \int_0^t f(s) \, ds\) and \(\tilde{F}(t) = \int_0^t \tilde{f}(s) \, ds\). Thus we have
\[
\frac{1}{2} \int_{\mathbb{R}^N} G(x, |u_n|^2) - \frac{1}{\theta} \int_{\mathbb{R}^N} g(x, |u_n|^2)|u_n|^2 \\
= \int_{\Omega_j} \left[ \frac{1}{2} F(|u_n|^2) - \frac{1}{\theta} f(|u_n|^2)|u_n|^2 \right] + \int_{\mathbb{R}^N \setminus \Omega_j} \left[ \frac{1}{2} \tilde{F}(|u_n|^2) - \frac{1}{\theta} \tilde{f}(|u_n|^2)|u_n|^2 \right].
\]  
(3.9)

We remark that for \( t \in [a, \infty) \),
\[
\frac{1}{2} \tilde{F}(t^2) - \frac{1}{\theta} \tilde{f}(t^2)t^2 = \frac{1}{2} \nu_0 t^2 - \frac{1}{\theta} \nu_0 t^2 = \left( \frac{1}{2} - \frac{1}{\theta} \right) \nu_0 t^2
\]
and for \( t \leq a \), by the assumption \((f_3)\) we have that
\[
\frac{1}{2} F(t^2) - \frac{1}{\theta} f(t^2)t^2 \leq 0.
\]
Thus from (3.9) we obtained that
\[
\left( \frac{1}{2} - \frac{1}{\theta} \right) \left( \|u_n\|_{L^\theta}^2 - \nu_0 \int_{\mathbb{R}^N} |u_n|^2 \right) \leq c + o(1) + \varepsilon_n \|u_n\|_{\lambda}.
\]

Hence from (3.2), we have
\[
\left( \frac{1}{2} - \frac{1}{\theta} \right) \delta_0 \|u_n\|_{L^\theta}^2 \leq c + o(1) + \varepsilon_n \|u_n\|_{\lambda}.
\]

Thus \( \|u_n\|_{\lambda} \) is bounded as \( n \to \infty \) and
\[
\limsup_{n \to \infty} \|u_n\|_{L^\theta}^2 \leq M(c) := \left( \frac{1}{2} - \frac{1}{\theta} \right)^{-1} \delta_0^{-1} c.
\]

This completes the proof of Lemma 3.3. \( \square \)

Now we give the proof of Proposition 3.2.

**Proof of Proposition 3.2.** From Proposition 3.1 and Lemma 3.3, we know that \( (u_n) \) is bounded in \( E_\lambda \) and thus is bounded in \( H_A^1(\mathbb{R}^N) \), so there exists a subsequence of \( (u_n) \) still denote \( (u_n) \) such that
\[
u_n \to u \quad \text{weakly in} \quad E_\lambda \left( H_A^1(\mathbb{R}^N) \right),
\]
\[
u_n \to u \quad \text{strongly in} \quad L_{\text{loc}}^p(\mathbb{R}^N).
\]

Now we prove that \( u_n \to u \) in \( E_\lambda \). First of all, it is easy to check that \( u \) is critical point of \( \Phi_\lambda(u) \), namely for any \( \psi \in E_\lambda \)
\[
\text{Re} \int_{\mathbb{R}^N} \left( \nabla A u \nabla A \psi + \lambda V(x) u \overline{\psi} \right) = \text{Re} \int_{\mathbb{R}^N} g(x, |u|^2)u \overline{\psi}.
\]
It follows from (3.7) and (3.8) that
\[
(\Phi_\lambda'(u_n) - \Phi_\lambda'(u))(u_n - u) \to 0.
\]
that is
\[
\int_{\mathbb{R}^N} \left( |\nabla_A(u_n - u)|^2 + \lambda V(x)|u_n - u|^2 \right) - \text{Re} \int_{\mathbb{R}^N} g(x, |u_n|^2) u_n (u_n - u) + \text{Re} \int_{\mathbb{R}^N} g(x, |u|^2) u (u_n - u) \ni
\]
\[
= \int_{\mathbb{R}^N} \left( |\nabla_A(u_n - u)|^2 + \lambda V(x)|u_n - u|^2 \right) - \text{Re} \int_{\Omega'_j} f(|u_n|^2) u_n (u_n - u) + \text{Re} \int_{\Omega'_j} f(|u|^2) u (u_n - u) + \text{Re} \int_{\mathbb{R}^N \setminus \Omega'_j} f(|u|^2) u (u_n - u),
\]
by the definition of \( f(t) \), we have
\[
\left| \text{Re} \int_{\mathbb{R}^N \setminus \Omega'_j} (f(|u_n|^2) u_n - f(|u|^2) u) (u_n - u) \right| \ni
\]
\[
= \left| \text{Re} \int_{\Omega'_j} (f(|u_n|^2) u_n - f(|u|^2) u) (u_n - u) \right| + \left| \text{Re} \int_{\mathbb{R}^N \setminus \Omega'_j} (f(|u_n|^2) u_n - f(|u|^2) u) (u_n - u) \right| \ni
\]
\[
\leq v_0 \|u_n - u\|_{L^2}^2 + v_0 \text{Re} \int_{\mathbb{R}^N \setminus \Omega'_j} u (u_n - u) \ni.
\]
Since \( u_n \to u \) in \( E_\lambda \), we have \( \text{Re} \int_{\mathbb{R}^N} u (u_n - u) \to 0 \), from \( u_n \to u \) in \( L^2(\Omega'_j) \), we know that \( \text{Re} \int_{\Omega'_j} u (u_n - u) \to 0 \). Thus, we have
\[
v_0 \left| \text{Re} \int_{\mathbb{R}^N \setminus \Omega'_j} u (u_n - u) \right| \to 0.
\]
We also remark that \( u_n \to u \) strongly in \( L^p(\Omega'_j) \) for \( 2 < p < 2^* \), thus by (3.2) and the assumption \((f_2)\), we have
\[
\delta_0 \|u_n - u\|_{L^2}^2 \leq \|u_n - u\|_{L^2}^2 - v_0 \|u_n - u\|_{L^2}^2 \ni
\]
\[
\leq \int_{\Omega'_j} f(|u_n|^2) u_n (u_n - u) - \int_{\Omega'_j} f(|u|^2) u (u_n - u) + O(1) \to 0
\]
as \( n \to \infty \). Therefore \( u_n \to u \) in \( E_\lambda \) and this completes the proof of Proposition 3.2. \( \square \)

**Proposition 3.4.** Assume sequence \((u_n) \subset E \) and \((\lambda_n) \subset [0, \infty) \) satisfying
\[
\lambda_n \to \infty, \quad \text{(3.10)}
\]
\[
\Phi_{\lambda_n}(u_n) \to \infty, \quad \text{(3.11)}
\]
\[
\|\Phi'_{\lambda_n}(u_n)\|_{\lambda_n}^2 \to 0. \quad \text{(3.12)}
\]
Then after extracting a sequence, still denoted by \( n \), we have

\[
\begin{align*}
\mathbf{u}_n &\rightharpoonup u \quad \text{weakly in } E \text{ and } H^1_A(\mathbb{R}^N)
\end{align*}
\]

for some \( u \in E \). Moreover

(i) \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \) and \( u(x) \) is a solution of

\[
\begin{cases}
- (\nabla + iA(x)) u(x) = f(\|u(x)\|^2)u(x), & x \in \Omega_j, \\
u(x) \in H^1_A(\Omega_j)
\end{cases}
\]

for \( j \in J \);

(ii) \( u_n \) converges to \( u(x) \) in a stronger sense, namely

\[
\|u_n - u\|_{\lambda_n} \to 0,
\]

\[
u_n \to u \quad \text{strongly in } E \text{ and } H^1_A(\mathbb{R}^N);
\]

(iii) \( u_n(x) \) also satisfies

\[
\begin{cases}
\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \to 0, \\
\Phi_{\lambda_n}(u_n) \to \sum_{j \in J} I_{\Omega_j}(u), \\
\|u_n - u\|_{\lambda_n, \mathbb{R}^N \setminus \Omega} \to 0, \\
\|u_n - u\|_{\lambda_n, \Omega_j} \to \int_{\Omega_j} |\nabla_A u|^2 \quad \text{for } j \in J.
\end{cases}
\]

**Proof.** As the similar proof with Lemma 3.3, we can prove that

\[
\limsup_{n \to \infty} \|u_n\|_{\lambda_n}^2 \leq M(c).
\]

Thus \((u_n)\) stays bounded as \( n \to \infty \) in \( E \) and \( H^1_A(\mathbb{R}^N) \), we may assume that for some \( u \in E \)

\[
\begin{align*}
\mathbf{u}_n &\rightharpoonup u \quad \text{weakly in } E \text{ and } H^1_A(\mathbb{R}^N), \\
u_n &\to u \quad \text{a.e. in } \mathbb{R}^N, \\
u_n &\to u \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{for } 2 \leq q < \frac{2N}{N-2}.
\end{align*}
\]

Now we come to show (i). Set \( C_m := \{x \in \mathbb{R}^N : V(x) \geq \frac{1}{m}\} \), for \( m \) large, we have

\[
\int_{C_m} |u_n|^2 \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^2 \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (\lambda_n V(x)|u_n|^2 + |\nabla_A u_n|^2) = \frac{m}{\lambda_n} \|u_n\|_{\lambda_n} \to 0.
\]

Thus \( u(x) = 0 \) on \( \bigcup_{m=1}^{\infty} \mathbb{R}^N \setminus \bigcup_{j=1}^{k} \Omega_j \). Next, for any \( \varphi \in C^\infty_0(\Omega_j, \mathbb{C}) \), \( j \in \{1, 2, \ldots, k\} \), we have

\[
|\Phi'_{\lambda_n}(u_n)\varphi| \leq \|\Phi'_{\lambda_n}(u_n)\|_{\lambda_n} \|\varphi\|_{\lambda_n} \to 0.
\]
here we use the fact that \( \| \varphi \|_{\lambda_n} \) indeed does not dependent on \( \lambda_n \). Thus we have

\[
\text{Re} \int_{\Omega_j} (\nabla_A u \nabla_A \varphi + u \varphi) = \text{Re} \int_{\Omega_j} g(x, |u|^2) u \varphi.
\]

By the definition of \( g(x, t) \), we know that for \( j \in J \), \( u(x) \) satisfies (3.13). For \( j \in \{1, 2, \ldots, k\} \setminus J \), setting \( \varphi = u(x) \) we have

\[
\int_{\Omega_j} |\nabla_A u|^2 - f(|u|^2) |u|^2 = 0,
\]

that is

\[
\|u\|_{0, \Omega_j'}^2 - \int_{\Omega_j'} f(|u|^2) |u|^2 = 0.
\]

On the other hand, since \( \Omega_j' \) is bounded with smooth boundary, by Remark 2.1 we have that \( \|u\|_{0, \Omega_j'}^2 \) is equivalent with \( \|u\|_{1, \Omega_j'}^2 \), by Proposition 3.1 we have that there exist two positive constants \( c_0 \) and \( d_0 \) such that

\[
0 = \|u\|_{0, \Omega_j'}^2 - \int_{\Omega_j'} f(|u|^2) |u|^2 \\
\begin{aligned}
g \geq c_0 \|u\|_{1, \Omega_j'}^2 - v_0 \int_{\Omega_j'} |u|^2 & \geq d_0 \|u\|_{0, \Omega_j'}^2.
\end{aligned}
\]

Thus \( u = 0 \) in \( \Omega_j \) for \( j \in \{1, 2, \ldots, k\} \setminus J \) and thus we get (i).

For (ii), we know that

\[
\Phi_{\lambda_n}'(u_n)(u_n - u) - \Phi_{\lambda_n}'(u)(u_n - u) \\
= \|u_n - u\|_{\lambda_n}^2 - \text{Re} \int_{\mathbb{R}^N \setminus \Omega_j'} f(|u_n|^2) u_n (u_n - u) + \text{Re} \int_{\mathbb{R}^N \setminus \Omega_j'} f(|u|^2) u (u_n - u) \\
- \text{Re} \int_{\Omega_j'} f(|u_n|^2) u_n (u_n - u) + \text{Re} \int_{\Omega_j'} f(|u|^2) u (u_n - u).
\]

Since \( u_n \rightarrow u \) in \( L^p(\Omega_j') \) and the assumption \( (f_2) \) we have

\[
\text{Re} \int_{\Omega_j'} f(|u_n|^2) u_n - f(|u|^2) u (u_n - u) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

On the other hand
\[ |\Phi'_{\lambda_n}(u_n)(u_n - u)| \leq \left\| \Phi'_{\lambda_n}(u_n) \right\|_{\lambda_n}^\ast \|u_n - u\|_{\lambda_n} \]
\[ \leq \left\| \Phi'_{\lambda_n}(u_n) \right\|_{\lambda_n}^\ast (\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) \to 0. \]

Thus we have

\[ \|u_n - u\|_{\lambda_n}^2 - \text{Re} \int_{\mathbb{R}^N\setminus \Omega_j} (\tilde{f}(|u_n|^2)u_n - \tilde{f}(|u|^2)u)(u_n - u) \to 0. \]

As a similar argument in the proof of Proposition 3.1, we obtain that

\[ \delta_0 \|u_n - u\|_{\lambda_n}^2 \leq \|u_n - u\|_{\lambda_n}^2 - \nu_0 \|u_n - u\|_{L^2(\mathbb{R}^N)} \]
\[ = \|u_n - u\|_{\lambda_n}^2 - \text{Re} \int_{\mathbb{R}^N\setminus \Omega_j} (\tilde{f}(|u_n|^2)u_n - \tilde{f}(|u|^2)u)(u_n - u) + o(1) \to 0 \]

and thus (ii) is obtained.

Now we show (iii). Indeed

\[ \frac{1}{2} \int_{\mathbb{R}^N}\lambda_n V(x)|u_n|^2 = \frac{1}{2} \int_{\mathbb{R}^N\setminus \Omega_j}\lambda_n V(x)|u_n|^2 \]
\[ = \frac{1}{2} \int_{\mathbb{R}^N\setminus \Omega_j}\lambda_n V(x)|u_n - u|^2 \leq \|u_n - u\|_{\lambda_n}^2 \to 0. \]

This completes the proof of Proposition 3.4. □

**Proposition 3.5.** There exists a constant \( \Lambda_0 > 0 \) such that if \( u_\lambda \) is a critical point of \( \Phi_\lambda(u) \) for \( \lambda \geq \Lambda_0 \), then \( |u_\lambda| \leq a \). In particular, \( u_\lambda \) solves the original problem \((S_\lambda)\).

**Proof.** We use notation \( B_r(x) = \{ y \in \mathbb{R}^N : |x - y| < r \} \). Since \( u_\lambda \in E \) is a critical point of \( \Phi_\lambda(u) \), namely \( u_\lambda \) satisfies the following equation

\[ -(\nabla + iA(x))^2 u_\lambda(x) + \lambda V(x) u_\lambda(x) = g(x, |u_\lambda|^2) u_\lambda(x), \quad x \in \mathbb{R}^N. \]

By Kato’s inequality

\[ \Delta |u_\lambda| \geq \text{Re} \left( \frac{\bar{u}_\lambda}{|u_\lambda|} \left( \nabla + iA(x) \right)^2 u_\lambda(x) \right). \]

there holds

\[ \Delta |u_\lambda(x)| - \lambda V(x)|u_\lambda(x)| - g(x, |u_\lambda|^2)|u_\lambda(x)| \geq 0, \quad x \in \mathbb{R}^N. \]

Since \( |u_\lambda| \geq 0 \) and \( V(x) \geq 0 \) we have

\[ \Delta |u_\lambda(x)| - (g(x, |u_\lambda|^2))|u_\lambda(x)| \geq 0, \quad x \in \mathbb{R}^N, \]
we use the subsolution estimate (see Theorem 8.17 in [20]) to get that there exists a constant $C(r)$ such that for any $1 < q < 2$

$$|u_\lambda(x)| \leq C(r) \int_{B_r(x)} |u_\lambda(x)|^q.$$ 

By Proposition 3.3, for any sequence $\lambda_n \to \infty$ we can extract a subsequence still denote $\lambda_n$ such that

$$u_{\lambda_n} \to u_0 \in H^{0,1}_A(\Omega_J) \quad \text{strongly in } H_0^1(\mathbb{R}^N).$$ 

In particular,

$$u_{\lambda_n} \to u_0 \in H^{0,1}_A(\Omega_J) \quad \text{strongly in } L^2_A(\mathbb{R}^N \setminus \overline{\Omega_J}).$$ 

Since $\lambda_n \to \infty$ is arbitrary, we have

$$u_\lambda \to u_0 \in H^{0,1}_A(\Omega_J) \quad \text{strongly in } L^2_A(\mathbb{R}^N \setminus \overline{\Omega_J}) \quad \text{as } \lambda \to \infty.$$ 

Thus, choosing $r \in (0, \text{dist}(\Omega_J, \mathbb{R}^N \setminus \Omega_J'))$, we have uniformly in $x \in \mathbb{R}^N \setminus \Omega_J'$ that

$$|u_\lambda(x)| \leq C(r) \int_{B_r(x)} |u_\lambda(x)|^q \leq C(r)(\text{meas } B_r(x))^{1-\frac{q}{2}}\|u_\lambda\|_{L^q(B_r(x))}^q \leq C(r)(\text{meas } B_r(x))^{1-\frac{q}{2}}\|u_\lambda\|_{L^q(\mathbb{R}^N \setminus \Omega_J)}^q \to 0.$$ 

This completes the proof of Proposition 3.5. □

4. Proof of main results

For $j \in J$ we consider the following two functionals

$$I_{\Omega_J}(u) = \frac{1}{2} \int_{\Omega_J} \left| \nabla u + iA(x)u \right|^2 - \frac{1}{2} \int_{\Omega_J} F(|u|^2) \quad \text{for } u \in H^{0,1}_A(\Omega_J),$$ 

and for $u \in E(\Omega'_j) = H^1_A(\Omega_j')$,

$$\Phi_{\lambda,\Omega'_j}(u) = \frac{1}{2} \int_{\Omega'_j} \left| \nabla u + iA(x)u \right|^2 + \lambda V(x)|u|^2 \quad \text{for } u \in H^1_A(\Omega_j').$$ 

By the assumptions $(f_1)$–$(f_4)$, one can easily see that both of $I_{\Omega_J}(u)$ and $\Phi_{\lambda,\Omega'_j}(u)$ have mountain pass geometry. That is,

(i) $I_{\Omega_J}(0) = \Phi_{\lambda,\Omega'_j}(0) = 0$. 

Lemma 4.1.

(ii) There exist $\rho_0 > 0$ and $\rho_1 > 0$ independent of $\lambda \geq 0$ such that

\[
\|u\|_{0,\Omega_j} \leq \rho_0 \rightarrow I_{\Omega_j}(u) \geq 0, \\
\|u\|_{0,\Omega_j} = \rho_0 \rightarrow I_{\Omega_j}(u) \geq \rho_1, \\
\|u\|_{0,\Omega_j} \leq \rho_0 \rightarrow \Phi_{\lambda,\Omega_j}(u) \geq 0, \\
\|u\|_{0,\Omega_j} = \rho_0 \rightarrow \Phi_{\lambda,\Omega_j}(u) \geq \rho_1.
\]

(4.2)

(4.3)

Here we use the notation:

\[
\|u\|_{0,\Omega_j} = \int_{\Omega_j} \left( |\nabla u + iA(x)u|^2 + |u|^2 \right) \quad \text{for} \quad u \in H^{1,0}(\Omega_j).
\]

(iii) There exists $\varphi_j(x) \in C_0^\infty(\Omega_j, \mathbb{C})$ such that

\[
\|\varphi_j(x)\|_{\lambda,\Omega_j} = \|\varphi_j(x)\|_{0,\Omega_j} \geq \rho_1, \\
\Phi_{\lambda,\Omega_j}(\varphi_j) = I_{\Omega_j}(\varphi_j) < 0.
\]

We define the following minimax values (mountain pass):

\[
c_j = \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} I_{\Omega_j}(\gamma(t)),
\]

\[
c_{\lambda,j} = \inf_{\gamma \in \Gamma_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,\Omega_j}(\gamma(t)),
\]

where

\[
\Gamma_j = \{ \gamma \in C([0,1], H^{1,0}(\Omega_j)): \gamma(0) = 0, \ I_{\Omega_j}(\gamma(1)) < 0 \},
\]

\[
\Gamma_{\lambda,j} = \{ \gamma \in C([0,1], H^{1,0}(\Omega_j)): \gamma(0) = 0, \ \Phi_{\lambda,\Omega_j}(\gamma(1)) < 0 \}.
\]

It is standard to verify the Palais–Smale condition for $I_{\Omega_j}(u)$ and $\Phi_{\lambda,\Omega_j}(u)$ and $c_j$, $c_{\lambda,j}$ are achieved by critical points. We denote the corresponding critical points by $\omega_j(x)$ and $\omega_{\lambda,j}(x)$ respectively.

We have the following lemma.

Lemma 4.1.

(i) $0 < \rho_1 \leq c_{\lambda,j} \leq c_j$ for all $\lambda \geq 0$.

(ii) $c_j$ ($c_{\lambda,j}$ respectively) is a least energy level for $I_{\Omega_j}(u)$ ($\Phi_{\lambda,\Omega_j}(u)$ respectively), that is

\[
c_j = \inf_{u \in H^{1,0}(\Omega_j) \setminus \{0\}} I_{\Omega_j}(u) \cdot u = 0,
\]

\[
c_{\lambda,j} = \inf_{u \in H^{1,0}(\Omega_j) \setminus \{0\}} \Phi_{\lambda,\Omega_j}(u) \cdot u = 0.
\]

(iii) $c_j = \max_{t \geq 0} I_{\Omega_j}(t\omega_j)$, $c_{\lambda,j} = \max_{t \geq 0} \Phi_{\lambda,\Omega_j}(t\omega_{\lambda,j})$.

(iv) $c_{\lambda,j} \rightarrow c_j$ as $\lambda \rightarrow \infty$. 
**Proof.** From (4.3), it is easy to see that $c_{\lambda, j} \geq \rho_1$. On the other hand, for any $u \in H^{0,1}_A(\Omega_j)$, we may extend $u$ to $\tilde{u} \in H^1_A(\Omega'_j)$ by

$$
\tilde{u} = \begin{cases} 
  u(x) & \text{in } \Omega_j, \\
  0 & \text{in } \Omega'_j \setminus \Omega_j, 
\end{cases}
$$

we regard $H^{0,1}_A(\Omega_j) \subset H^1_A(\Omega'_j)$. Thus we have $\Gamma_j \subset \Gamma_{\lambda, j}$ and

$$
c_{\lambda, j} = \inf_{\gamma \in \Gamma_{\lambda, j}} \max_{t \in [0, 1]} \Phi_{\lambda, \Omega_j}(\gamma(t)) \leq \inf_{\gamma \in \Gamma_j} \max_{t \in [0, 1]} \Phi_{\lambda, \Omega'_j}(\gamma(t)) = \inf_{\gamma \in \Gamma_j} \max_{t \in [0, 1]} I_{\Omega_j}(\gamma(t)) = c_j. \tag{4.4}
$$

Thus we have (i). Using the monotonicity of the term $f(|u|^2)$ with respect to $|u|$, the proof of (ii) and (iii) is standard.

Now we show (iv). Using Proposition 3.3, we may extract a subsequence $\lambda_n \to \infty$ such that $\omega_{\lambda, j} \to u_0$ strongly in $H^1_A(\Omega'_j)$, where $u_0 \in H^{0,1}_A(\Omega_j)$ is a solution of (3.7) and

$$
\Phi_{\lambda, \Omega_j}(\omega_{\lambda, j}) \to I_{\Omega_j}(u_0).
$$

By the definition of $c_j$, we have

$$
\limsup_{\lambda \to \infty} c_{\lambda, j} = \limsup_{\lambda \to \infty} \Phi_{\lambda, \Omega'_j}(\omega_{\lambda, j}) \geq I_{\Omega_j}(u_0) \geq c_j.
$$

Compare with (4.4), we get (iv) and this completes the proof of Lemma 4.1. \(\square\)

We choose $R \geq 2$ such that

$$
I_{\Omega_j}(R\omega_j) < 0. \tag{4.5}
$$

Without loss of generality, we assume that $J = \{1, 2, \ldots, l\}$ ($l \leq k$). We remark that the project $t \mapsto tR\omega_j$ belongs to $\Gamma_j$ and satisfies $\max_{t \in [0, 1]} I_{\Omega_j}(tR\omega_j) = c_j$ for any $j \in J$. Now we set

$$
\gamma_0(s_1, s_2, \ldots, s_l)(x) = \sum_{j=1}^l s_j R\omega_j(x) \quad \text{for all } (s_1, s_2, \ldots, s_l) \in [0, 1]^l, \tag{4.6}
$$

\[ \Gamma_j = \left\{ \gamma \in C([0, 1]^l, E) : \gamma(s_1, s_2, \ldots, s_l) = \gamma_0(s_1, s_2, \ldots, s_l) \right\} \] for all $(s_1, s_2, \ldots, s_l) \in \partial([0, 1]^l)$ and

$$
b_{\lambda, j} = \inf_{\gamma \in \Gamma_j} \max_{(s_1, s_2, \ldots, s_l) \in [0, 1]^l} \Phi_{\lambda}(\gamma(s_1, s_2, \ldots, s_l)).
$$

We remark that $\Gamma_j \neq \emptyset$ since $\gamma_0 \in \Gamma_j$ and thus $b_{\lambda, j}$ is well defined. We denote $c_j = \sum_{j=1}^l c_j$, we have the following lemmas.
Lemma 4.2.

(i) \( \sum_{j=1}^{l} c_{\lambda, j} \leq b_{\lambda, j} \leq c_{j} \) for all \( \lambda \geq 0 \).

(ii) \( \Phi_{\lambda}(\gamma(s_{1}, s_{2}, \ldots, s_{l})) \leq c_{j} - \rho_{1} \) for all \( \lambda \geq 0 \), \( \gamma \in \Gamma_{j} \) and \( (s_{1}, s_{2}, \ldots, s_{l}) \in \partial([0, 1]^{l}) \). Here \( \rho_{1} \) is given in (4.2), (4.3) and (1) in Lemma 4.1.

Proof. For any given \( \gamma \in \Gamma_{j} \), we denote for \( j = 1, 2, \ldots, l \)

\[
T_{j}(s_{1}, s_{2}, \ldots, s_{l}) = \frac{\int_{\Omega_{j}} f(|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2})|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2}}{\int_{\Omega_{j}} |\nabla \gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2} + \lambda V(x)|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2}}
\]

and we define a map \( T : [0, 1]^{l} \rightarrow \mathbb{R}^{l} \) as follows

\[
T(s_{1}, s_{2}, \ldots, s_{l}) = (T_{1}(s_{1}, s_{2}, \ldots, s_{l}), T_{2}(s_{1}, s_{2}, \ldots, s_{l}), \ldots, T_{1}(s_{1}, s_{2}, \ldots, s_{l})).
\]

We have for \( (s_{1}, s_{2}, \ldots, s_{l}) \in \partial([0, 1]^{l}) \)

\[
T_{j} = \frac{\int_{\Omega_{j}} f(|s_{j}R_{\omega_{j}}|^{2})|s_{j}R_{\omega_{j}}|^{2}}{\int_{\Omega_{j}} |\nabla A(s_{j}R_{\omega_{j}})|^{2} + \lambda V(x)|s_{j}R_{\omega_{j}}|^{2}} = \frac{\int_{\Omega_{j}} f(|s_{j}R_{\omega_{j}}|^{2})|\omega_{j}|^{2}}{\int_{\Omega_{j}} |\nabla A\omega_{j}|^{2} + \lambda V(x)|\omega_{j}|^{2}}.
\]

Thus for \( (s_{1}, s_{2}, \ldots, s_{l}) \in \partial([0, 1]^{l}) \),

\[
T(s_{1}, s_{2}, \ldots, s_{l}) = \left( \frac{\int_{\Omega_{j}} f(|s_{1}R_{\omega_{1}}|^{2})|\omega_{1}|^{2}}{\int_{\Omega_{j}} |\nabla A\omega_{1}|^{2} + \lambda V(x)|\omega_{1}|^{2}}, \ldots, \frac{\int_{\Omega_{j}} f(|s_{l}R_{\omega_{l}}|^{2})|\omega_{l}|^{2}}{\int_{\Omega_{j}} |\nabla A\omega_{l}|^{2} + \lambda V(x)|\omega_{l}|^{2}} \right).
\]

By the assumptions \( (f_{1}), (f_{3}) \) and \( (f_{4}) \) and (4.5) we have

\[
\deg(T, [0, 1]^{l}, (1, 1, \ldots, 1)) = 1. \tag{4.7}
\]

By the property of topological degree, there exists \( (s_{1}, s_{2}, \ldots, s_{l}) \in [0, 1]^{l} \) such that

\[
\frac{\int_{\Omega_{j}} f(|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2})|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2}}{\int_{\Omega_{j}} |\nabla A\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2} + \lambda V(x)|\gamma(s_{1}, s_{2}, \ldots, s_{l})(x)|^{2}} = 1 \quad \text{for all } j = 1, 2, \ldots, l. \tag{4.8}
\]

Now we come to show (i).

Since \( \gamma_{0} \in \Gamma_{j} \), we have

\[
b_{\lambda, j} \leq \max_{(s_{1}, s_{2}, \ldots, s_{l}) \in [0, 1]^{l}} \Phi_{\lambda}(\gamma_{0}(s_{1}, s_{2}, \ldots, s_{l}))
\]

\[
= \max_{(s_{1}, s_{2}, \ldots, s_{l}) \in [0, 1]^{l}} \sum_{j=1}^{l} l_{\Omega_{j}}(s_{j}R_{\omega_{j}})
\]

\[
= \sum_{j=1}^{l} c_{j} = c_{j}.
\]
On the other hand, remarking (4.8), we know that for any $\gamma \in \Gamma_j$, there exists $s_\gamma \in [0, 1]^l$ such that
\[
\int_{\Omega_j'} f \left( |\gamma'(s_\gamma)(x)|^2 \right) = 1 \quad \text{for all } j = 1, 2, \ldots, l,
\]
which means that $\Phi_{\gamma', \Omega_j'}(\gamma'(s_\gamma)(x)) = 0$ for all $j = 1, 2, \ldots, l$. Thus for $u(x) = \gamma(s_\gamma)(x)$, we have
\[
\Phi_\lambda(u) = \Phi_{\lambda, \mathbb{R}^N \setminus \Omega_j'}(u) + \sum_{j=1}^l \Phi_{\lambda, \Omega_j'}(u),
\]
where
\[
\Phi_{\lambda, \mathbb{R}^N \setminus \Omega_j'}(u) = \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega_j'} \left( |\nabla u + iA(x)u|^2 + \lambda V(x)|u|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega_j'} \tilde{F}(|u|^2).
\]
Since $F(|u|^2) \leq v_0|u|^2$, we have
\[
\Phi_{\lambda, \mathbb{R}^N \setminus \Omega_j'}(u) = \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega_j'} \left( |\nabla u + iA(x)u|^2 + \lambda V(x)|u|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega_j'} \tilde{F}(|u|^2)
\geq \frac{1}{2} \|u\|^2_{\lambda, \mathbb{R}^N \setminus \Omega_j'} - \frac{1}{2} \|u\|^2_{L^2(\mathbb{R}^N \setminus \Omega_j')}
\geq \frac{\delta_0}{2} \|u\|^2_{\lambda, \mathbb{R}^N \setminus \Omega_j'} \geq 0.
\]
Thus
\[
\Phi_\lambda(u) = \Phi_{\lambda, \mathbb{R}^N \setminus \Omega_j'}(u) + \sum_{j=1}^l \Phi_{\lambda, \Omega_j'}(u) \geq \sum_{j=1}^l \Phi_{\lambda, \Omega_j'}(u)
\geq \sum_{j=1}^l \inf\{\Phi_{\lambda, \Omega_j'}(v): v \in H^1_\lambda(\Omega_j) \setminus \{0\}, \Phi_{\lambda, \Omega_j'}'(v) \cdot v = 0\}
\geq \sum_{j=1}^l c_{\lambda, j}.
\]
Since $\gamma \in \Gamma_j$ is arbitrary, we have $b_{\lambda, j} \geq c_{\lambda, j}$.

For (ii), we remark that for any $\gamma \in \Gamma$ $\gamma(s_1, s_2, \ldots, s_l) = \gamma_0(s_1, s_2, \ldots, s_l)$ on $\partial([0, 1]^l)$, thus by the definition of $\gamma_0$, for $(s_1, s_2, \ldots, s_l) \in \partial([0, 1]^l)$ we have
\[
\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) = \sum_{j=1}^l l_{\partial j}(s_j R \omega_j)
\]
and \( I_{\Omega_j}(s_{j_0} R \omega_j) \leq c_j \) for all \( j = 1, 2, \ldots, l \). On the other hand, for some \( j_0 \in J \), \( s_{j_0} = 1 \) or \( s_{j_0} = 0 \) and thus \( I_{\Omega_{j_0}}(s_{j_0} R \omega_{j_0}) \leq 0 \). Therefore

\[
\Phi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)) \leq \sum_{j \neq j_0} I_{\Omega_j}(s_{j} R \omega_j) \leq c_j - \rho_1.
\]

This completes the proof of the whole lemma. \( \square \)

**Corollary 4.3.** \( b_\lambda, j \to c_j \) as \( \lambda \to \infty \), moreover \( b_\lambda, j \) is a critical point of \( \Phi_\lambda \) for large \( \lambda \).

**Proof.** From Lemma 4.1, we know that \( c_{\lambda, j} \to c_j \) as \( \lambda \to \infty \), thus from the above lemma, it is clear that \( b_\lambda, j \to c_j \) as \( \lambda \to \infty \). Thus, we may choose \( \lambda_0 \) large enough such that for all \( \lambda \geq \lambda_0 \), \( b_\lambda, j > c_j - \rho_1 \). Since \( \Phi_\lambda(u) \) satisfies Palais–Smale condition, by the standard deformation argument we can see that \( b_\lambda, j \) is a critical value of \( \Phi_\lambda(u) \) for \( \lambda \geq \lambda_0 \). This completes the proof of the corollary. \( \square \)

We use the following notation

\[
\Phi_{c_j}^\lambda = \{ u \in E: \Phi_\lambda(u) \leq c_j \}.
\]

We choose

\[
0 < \mu < \frac{1}{3} \min_{j \in \{1, 2, \ldots, l\}} c_j
\]

and define

\[
D^\mu_{\lambda} = \{ u \in E: \| u \|_{L^\infty(\Omega_j \setminus \Omega_j')} \leq \mu, \ |\Phi_\lambda R\omega_j|_j - c_j | \leq \mu \text{ for all } j \in J \}.
\]

We remark that \( \omega_j \) is the least energy solution of (3.7) and

\[
\Phi_{\lambda, R\omega_j} = \frac{1}{2} \int_{\Omega_j} (|\nabla \omega_j + i A(x) \omega_j|^2) - \frac{1}{2} \int_{\Omega_j} F(|\omega_j|^2) = c_j.
\]

Thus \( D^\mu_{\lambda} \cap \Phi_{c_j}^\lambda \) contains all the functions of the following form

\[
\omega(x) = \begin{cases} 
\omega_j(x), & x \in \Omega_j, \\
0, & x \in \mathbb{R}^N \setminus \Omega_j.
\end{cases}
\]

We have the following lemma.

**Lemma 4.4.** There exist \( \sigma_0 > 0 \) and \( \Lambda_0 > 0 \) independent of \( \lambda \) such that

\[
\| \Phi_\lambda'(u) \|^\ast_{\lambda} \geq \sigma_0 \text{ for } \lambda \geq \Lambda_0 \text{ and for all } u \in (D^\mu_{\lambda} \setminus \Phi_{c_j}^\lambda) \cap \Phi_{\lambda, R\omega_j}.
\]

**Proof.** We prove it by contradiction. Suppose that there exist \( \lambda_n \to \infty \) and \( u_n \in (D^\mu_{\lambda_n} \setminus \Phi_{c_j}^\lambda) \cap \Phi_{\lambda, R\omega_j} \) such that \( \| \Phi_\lambda'(u_n) \|_{\lambda_n}^\ast \to 0 \). Since \( u_n \in D^\mu_{\lambda_n} \), thus \( u_n \) is bounded in \( E \left( H^1_A(\mathbb{R}^N) \right) \) and \( \Phi_{\lambda_n}(u_n) \) stays bounded as \( n \to \infty \). We may assume that

\[
\Phi_{\lambda_n}(u_n) \to c \leq c_j
\]

up to a subsequence.
Applying Lemma 3.4, we can extract a subsequence of $u_n$ still denote $u_n$ such that $u_n \to u$ in $E (H_0^1(\mathbb{R}^N))$ and

$$
\lim_{n \to \infty} \Phi_{\lambda_n}(u_n) = \sum_{j=1}^I I_{\Omega_j}(u) \leq c_j,
$$

$$
\lim_{n \to \infty} \|u_n\|_{L^2(\Omega_j)}^2 = \int_{\Omega_j} (|\nabla u + iA(x)u|^2) \text{ for all } j \in J,
$$

$$
\lim_{n \to \infty} \int_{\Omega_j'} F(|u_n|^2) = \int_{\Omega_j} F(|u|^2),
$$

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega_j'} \left( |\nabla u_n + iA(x)u_n|^2 + \lambda_n V(x)|u_n|^2 \right) = 0.
$$

Since $c_j = \sum_{j=1}^J c_j$ and $c_j$ is the least energy level for $I_{\Omega_j}(u)$, thus we have two possibilities:

1. $I_{\Omega_j}(u|_{\Omega_j}) = c_j$ for all $j \in J$.
2. $I_{\Omega_j}(u|_{\Omega_j}) = 0$, that is $u|_{\Omega_j} = 0$ for some $j_0 \in J$.

If (1) occurs, we have

$$
\frac{1}{2} \int_{\Omega_j} |\nabla u + iA(x)u|^2 - \frac{1}{2} \int_{\Omega_j} F(|u|^2) = c_j \text{ for all } j \in J
$$

and it follows from (4.12)-(4.14) that $u_n \in D_{\lambda_n}^{\mu}$ for large $n$ which is a contradiction to $u_n \in (D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^{\mu})$.

If (2) occurs, from (4.12) and (4.13) that

$$
|\Phi_{\lambda_n.\Omega_j'}(u_n) - c_{j_0}| \to c_{j_0} \geq 3\mu.
$$

This is also a contradiction to $u_n \in (D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^{\mu})$ and we complete the proof. □

The following proposition is the key of the proof of our main result.

**Proposition 4.5.** Let $\mu$ satisfy (4.9) and let $\Lambda_0$ be the constant given in Lemma 4.4. Then for $\lambda \geq \Lambda_0$ there exists a solution $u_\lambda$ of $(S_\lambda)$ satisfying $u_\lambda \in D_\lambda^{\mu} \cap \Phi_{\lambda}^{c_j}$.

**Proof.** We argue indirectly and assume that $\Phi_\lambda(u)$ has no critical points in $D_\lambda^{\mu} \cap \Phi_{\lambda}^{c_j}$. Since $\Phi_\lambda(u)$ satisfy Palais–Smale condition, there exists a constant $d_\lambda > 0$ such that

$$
\|\Phi_\lambda'(u)\|_{\lambda} \geq d_\lambda \text{ for all } u \in D_\lambda^{\mu} \cap \Phi_{\lambda}^{c_j},
$$

and from Lemma 4.4 we have

$$
\|\Phi_\lambda'(u)\|_{\lambda} \geq \sigma_0 \text{ for all } u \in (D_{\lambda}^{2\mu} \setminus D_{\lambda}^{\mu}) \cap \Phi_{\lambda}^{c_j}.
$$
Let \( \varphi : E \rightarrow \mathbb{R} \) be a Lipschitz continuous function such that

\[
\varphi(u) = \begin{cases} 
1 & \text{for } u \in D^\lambda_{2\mu}, \\
0 & \text{for } u \notin D^\lambda_{2\mu}
\end{cases}
\]

and \( 0 \leq \varphi(u) \leq 1 \) for any \( u \in E \).

For any \( u \in \Phi^c_J \), we define

\[
W(u) = -\varphi(u) \frac{\Phi^c_J(u)}{\|\Phi^c_J(u)\|_\lambda^*} : \Phi^c_J \rightarrow E.
\]

Here we identify \( E^* \) and \( E \) by the Riesz representation theorem. We consider the following deformation flow \( \eta : [0, \infty) \times \Phi^c_J \rightarrow \Phi^c_J \) defined by

\[
d\eta = W(\eta(t, u)), \quad \eta(0, u) = u \in \Phi^c_J.
\]

\( \eta(t, u) \) has the following properties:

\[
\frac{d}{dt} \Phi^c_J(\eta(t, u)) = -\varphi(\eta(t, u)) \|\Phi^c_J(\eta(t, u))\|_\lambda^* \leq 0, \quad (4.15)
\]

\[
\left\| \frac{d\eta}{dt} \right\| \leq 1 \quad \text{for all } t, u, \quad (4.16)
\]

\[
\eta(t, u) = u \quad \text{for all } t \geq 0 \text{ and } u \in \Phi^c_J \setminus D^\lambda_{2\mu}. \quad (4.17)
\]

Let \( \gamma_0(s_1, s_2, \ldots, s_l) \in \Gamma_j \) be a path defined in (4.6) and we consider \( \eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) \) for large \( t \).

Since for all \( (s_1, s_2, \ldots, s_l) \in \partial([0, 1]^l), \gamma_0(s_1, s_2, \ldots, s_l) \notin D^\lambda_{2\mu} \), thus we have by (4.16) that

\[
\eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) = \gamma_0(s_1, s_2, \ldots, s_l) \quad \text{for all } (s_1, s_2, \ldots, s_l) \in \partial([0, 1]^l)
\]

and \( \eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) \in \Gamma_j \) for all \( t \geq 0 \).

Since \( \supp \gamma_0(s_1, s_2, \ldots, s_l)(x) \subset \Omega^j_\gamma \) for all \( (s_1, s_2, \ldots, s_l) \in \partial([0, 1]^l) \) and hence \( \Phi^c_J(\gamma_0(s_1, s_2, \ldots, s_l)(x)) \) and \( \|\gamma_0(s_1, s_2, \ldots, s_l)(x)\|_{\lambda, \Omega_j^\gamma} \), etc. do not depend on \( \lambda \geq 0 \). On the other hand

\[
\Phi^c_J(\gamma_0(s_1, s_2, \ldots, s_l)(x)) \leq c_J \quad \text{for all } (s_1, s_2, \ldots, s_l) \in [0, 1]^l
\]

and \( \Phi^c_J(\gamma_0(s_1, s_2, \ldots, s_l)(x)) = c_J \) if and only if \( s_j = \frac{1}{\pi} \), that is \( \gamma_0(s_1, s_2, \ldots, s_l)(x)|_{\Omega_j^\gamma} = \omega_j \) for all \( j \in J \).

Thus we have

\[
m_0 := \max\{\Phi^c_J(u) : u \in \gamma_0([0, 1]^l) \setminus D^\mu_{\lambda}\} \quad (4.18)
\]

is independent of \( \lambda \) and \( m_0 < c_J \).

From (4.16), it is easy to see that for any \( t > 0 \),

\[
\left\| \eta(0, \gamma_0(s_1, s_2, \ldots, s_l)) - \eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) \right\| \leq t.
\]
Since \( \Phi_{\lambda,j}(u) \in C^2(E_j) \) for all \( j = 1, 2, \ldots, l \), and the assumptions \((f_1)-(f_4)\), it is easy to see that for large number \( T \), there exists a positive number \( \gamma_0 > 0 \) which is independent of \( \lambda \) such that for all \( j = 1, 2, \ldots, l \) and \( t \in [0, T] \),

\[
\| \Phi'_{\lambda,j}(\eta(t, \gamma_0(s_1, s_2, \ldots, s_l))) \|_\lambda^* \leq \gamma_0.
\]  

(4.19)

We claimed that for large \( T \),

\[
\max_{(s_1, s_2, \ldots, s_l) \in \{0, 1\}^l} \Phi_{\lambda}(\eta(T, \gamma_0(s_1, s_2, \ldots, s_l))) \leq \max \left\{ m_0, c_j - \frac{1}{2} \tau_0 \mu \right\}.
\]

(4.20)

where \( m_0 \) is given in (4.18), \( \tau_0 = \max\{\sigma_0, \frac{\sigma_0}{\gamma_0} \} \) and \( \sigma_0 \) is given in (4.10).

In fact, if \( \gamma_0(s_1, s_2, \ldots, s_l)(x) \notin D_\lambda^{\mu_0} \), then by (4.18) we have \( \Phi_{\lambda}(\eta(T, \gamma_0(s_1, s_2, \ldots, s_l)(x))) \leq m_0 \) and thus (4.20) holds. Now we consider the case \( \gamma_0(s_1, s_2, \ldots, s_l)(x) \in D_\lambda^\mu \), we consider the behavior of \( \tilde{\eta}(t) := \eta(t, \gamma_0(s_1, s_2, \ldots, s_l)) \).

We set \( d_\lambda := \min\{d_s, \sigma_0\} \) and \( T = \frac{\sigma_0 \mu}{2_d_\lambda} \). We consider two cases:

1. \( \tilde{\eta}(t) \in D_\lambda^{\frac{3\mu}{2}d_\lambda} \) for all \( t \in [0, T] \).
2. \( \tilde{\eta}(t_0) \in \partial D_\lambda^{\frac{3\mu}{2}d_\lambda} \) for some \( t_0 \in [0, T] \).

When (1) holds, we have \( \psi(\tilde{\eta}(t)) \equiv 1 \) and \( \|\Phi'_{\lambda,j}(\tilde{\eta}(t))\|_\lambda^* \geq \tilde{d}_\lambda \) for all \( t \in [0, T] \). Thus by (4.15), we have

\[
\Phi_{\lambda,j}(\tilde{\eta}(t)) = \Phi_{\lambda,j}(\gamma_0(s_1, s_2, \ldots, s_l)) + \int_0^T \frac{d}{ds} \Phi_{\lambda,j}(\tilde{\eta}(t))
\]

\[
= \Phi_{\lambda,j}(\gamma_0(s_1, s_2, \ldots, s_l)) - \int_0^T \psi(\tilde{\eta}(s)) \| \Phi'_{\lambda,j}(\tilde{\eta}(s)) \|_\lambda^* \text{d}s
\]

\[
\leq c_j - \int_0^T \tilde{d}_\lambda \text{d}s = c_j - \tilde{d}_\lambda T
\]

\[
= c_j - \frac{1}{2} \sigma_0 \mu \leq c_j - \frac{1}{2} \tau_0 \mu.
\]

When (2) holds, there exists \( 0 \leq t_1 < t_2 \leq T \) such that

\[
\tilde{\eta}(t_1) \in \partial D_\lambda^{\mu},
\]

(4.21)

\[
\tilde{\eta}(t_2) \in \partial D_\lambda^{\frac{3\mu}{2}d_\lambda},
\]

(4.22)

\[
\tilde{\eta}(t) \in D_\lambda^{\frac{3\mu}{2}d_\lambda} \setminus D_\lambda^{\mu} \text{ for all } t \in [t_1, t_2].
\]

(4.23)

It follows from (4.22) that

\[
\| \tilde{\eta}(t_2) \|_{\lambda, \mathbb{R}^n \setminus \Omega_j} = \frac{3\mu}{2}
\]
or
\[ |\Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_2)) - c_{j_0}| = \frac{3\mu}{2} \]
for some \( j_0 \in J \).

We only see the later case, the former case can be dealt in a similar way. By (4.21),
\[ |\Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_1)) - c_{j_0}| \leq \mu. \]
Thus we have
\[ |\Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_2)) - \Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_1))| \geq |\Phi_{\lambda, \Omega^{j_0}}(\eta_2) - c_{j_0}| - |\Phi_{\lambda, \Omega^{j_0}}(\eta_1) - c_{j_0}| \geq \frac{1}{2}\mu. \]

On the other hand, by the mean value theorem, there exists \( t' \in (t_1, t_2) \) such that
\[ |\Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_2)) - \Phi_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t_1))| = \left| \Phi'_{\lambda, \Omega^{j_0}}(\tilde{\eta}(t')) \frac{d\tilde{\eta}}{dt}(t') \right| (t_2 - t_1). \]
From (4.16) and (4.19) we have that
\[ t_2 - t_1 \geq \frac{1}{2} \frac{\mu}{\gamma_0}. \]
Thus we have
\[ \Phi_{\lambda}(\tilde{\eta}(T)) = \Phi_{\lambda}(\gamma_0(s_1, s_2, \ldots, s_l)(x)) + \int_0^T \frac{d}{ds}\Phi_{\lambda}(\tilde{\eta}(s)) \, ds \]
\[ = \Phi_{\lambda}(\gamma_0(s_1, s_2, \ldots, s_l)(x)) + \int_0^T \psi(\tilde{\eta}(s)) \|\Phi'_{\lambda}(\tilde{\eta}(s))\|_A^\ast \, ds \]
\[ \leq c_J - \int_{t_1}^{t_2} \psi(\tilde{\eta}(s)) \|\Phi'_{\lambda}(\tilde{\eta}(s))\|_A^\ast \, ds \]
\[ = c_J - \sigma_0(t_2 - t_1) \]
\[ \leq c_J - \frac{1}{2} \tau_0 \mu \]
and thus (4.20) is proved. We recall that \( \tilde{\eta}(T) = \eta(T, \gamma_0(s_1, s_2, \ldots, s_l)) \in \Gamma_J \). Thus
\[ b_{\lambda, J} \leq \Phi_{\lambda}(\tilde{\eta}(T)) \leq \max\left\{ m_0, c_J - \frac{1}{2} \tau_0 \mu \right\}. \] (4.24)

However by Corollary 4.3, we have \( b_{\lambda, J} \to c_J \) as \( \lambda \to \infty \). This is a contradiction with (4.24), thus \( \Phi_{\lambda}(u) \) has a critical point \( u_\lambda(x) \in D_{\lambda}^{\mu} \) for large \( \lambda \) and by Proposition 3.4, \( u_\lambda(x) \) is a solution of the original problem \((S_\lambda)\). \( \square \)
Now we give the proof of main results.

**Proof of Theorem 2.2.** Let $u_{\lambda}(x)$ be a solution of the problem $(S_\lambda)$ obtained in Proposition 4.5, applying Lemma 3.3, for any given sequence $\lambda_n \to \infty$, we can extract a subsequence, still denote it by $\lambda_n$ which satisfies the conclusion of Proposition 3.4. With the same argument in the proof of Lemma 3.4, we can extract a subsequence of $u_{\lambda_n}$ still denote $u_{\lambda_n}$ such that $u_{\lambda_n} \to u$ in $E(H^1_A(\mathbb{R}^N))$ and

$$\lim_{n \to \infty} \Phi_{\lambda_n}(u_n) = c_j \quad \text{for all } j \in J,$$

(4.25)

$$\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega_j} (|\nabla u_{\lambda_n} + iA(x)u_{\lambda_n}|^2 + \lambda V(x)|u_{\lambda_n}|^2) = 0.$$

(4.26)

Since the limits in (4.25) and (4.26) do not depend on the choice of sequence $\lambda_n \to \infty$, thus we have (2.3) and (2.4) and the limit function $u(x)$ satisfies:

1. $u(x) \equiv 0$ for $x \in \mathbb{R}^N \setminus \Omega_j$.
2. $u(x)|_{\Omega_j}$ is a least energy solution of

$$\begin{cases}
-(\nabla + iA(x))^2 u(x) = f(|u(x)|^2)u(x), & x \in \Omega_j, \\
u(x) \in H^1_{0A}(\Omega_j)
\end{cases}$$

for $j \in J$.

This completes the proof of Theorem 2.2. □

**Acknowledgment**

The author would like to express his gratitude to the referee for his/her suggestions.

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