Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields

Daomin Cao\textsuperscript{a,}* \textsuperscript{1}, Zhongwei Tang\textsuperscript{b}

\textsuperscript{a} Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100080, PR China
\textsuperscript{b} School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, PR China

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Abstract

In this paper, we are concerned with the existence and uniqueness of multi-bump bound states of the nonlinear Schrödinger equations with electromagnetic potential

\[
i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \psi + V(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N
\]

for sufficiently small \( \hbar > 0 \), where \( i \) is the imaginary unit, \( 2 < p < \frac{2N}{N-2} \) for \( N \geq 3 \) and \( 2 < p < +\infty \) for \( N = 1, 2 \). \( V(x) \) is a bounded real function on \( \mathbb{R}^N \), and \( A(x) = (A_1(x), A_2(x), \ldots, A_N(x)) \) is such that \( A_j(x) \) is a bounded real function on \( \mathbb{R}^N \) for \( j = 1, 2, \ldots, N \). For any finite collection of \( \{a^1, a^2, \ldots, a^k\} \) of non-degenerate critical points of \( V(x) \), we show that there exists a solution \( \psi(x, t) = \exp(-iEt/\hbar)u(x) \) which is a small perturbation of a sum of

\* Corresponding author. Fax: +86 10 62541689.
E-mail address: dmcao@mail.amt.ac.cn (D. Cao).

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one-bump solutions concentrated at $a^1, a^2, \ldots, a^k$ and $u(x)$ is unique up to rotations for small $\hbar > 0$.

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1. Introduction

We are concerned with nonlinear Schrödinger equations with a bounded electromagnetic potential

$$i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \psi + V(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N, \quad (1.1)$$

here, $i$ is the imaginary unit, $\hbar$ is the Planck constant, $2 < p < \frac{2N}{N-2}$ for $N \geq 3$ and $2 < p < +\infty$ for $N = 1, 2$. $L_A = \left( \frac{\hbar}{i} \nabla - A(x) \right)^2$ denotes a Schrödinger operator with a real-valued magnetic vector potential $A(x) = (A_1(x), A_2(x), \ldots, A_N(x))$, where $A_j(x)$ is a bounded real-valued function on $\mathbb{R}^N$ for $j = 1, 2, \ldots, N$. Actually, the magnetic field $B$ is nothing but $B = \text{curl} A$ if $N = 3$; in general dimension, $B$ should be thought of as a 2-form where $B_{j,k} = \partial_j A_k - \partial_k A_j$ and $V(x)$ is a bounded real-valued electric potential function on $\mathbb{R}^N$.

We are interested in standing wave solutions, i.e., solutions of type

$$\psi(x, t) = \exp(-iEt/\hbar)u(x) \quad (1.2)$$

to (1.1) when $\hbar$ is sufficiently small, where $E$ is a real number and $u(x)$ is a complex-valued function which satisfies

$$\left( \frac{\hbar}{i} \nabla - A(x) \right)^2 u(x) + (V(x) - E)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N. \quad (1.3)$$

The transition from quantum mechanics to classical mechanics can be formally described by letting $\hbar \to 0$ and thus the existence of solutions for $\hbar$ small has physical interest. Standing waves for $\hbar$ small are usually referred as semiclassical bound states.
When $A(x) \equiv 0$, (1.3) becomes
\[
-\hbar^2 \Delta u(x) + (V(x) - E)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N. \tag{1.4}
\]

In recent years, much attention has been devoted to the study of the existence and uniqueness for one- or multi-bump bound states of (1.4). In [17], using a Lyapunov–Schmidt reduction, Floer and Weinstein established the existence of a standing wave solutions of (1.4) when $N = 1$, $p = 3$ and $V(x)$ is a bounded function having a non-degenerate critical point for sufficiently small $\hbar > 0$. Moreover, they showed that $u$ concentrates near the given non-degenerate critical point of $V$ when $\hbar$ tends to 0. Their method and results were later generalized by Oh [28,29] to the higher-dimensional case with $2 < p < \frac{2N}{N-2}$ and existence of multi-bump solutions concentrating near several non-degenerate critical points of $V$ as $\hbar$ tends to 0 was obtained.

On the other hand, Rabinowitz in [30] used a global variational method to show the existence of “least energy” solutions of (1.4) (and some generalization) when $\hbar$ is small, and the condition imposed on $V$ is a global one, namely
\[
\liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) \geq E. \tag{1.5}
\]

These solutions concentrate near the global minima of $V$ as $\hbar$ tends to 0, as shown by Wang [32]. See Wang and Zeng [33], for more general case.

Existence of multi-bump solutions of (1.4) (even for more general forms) were obtained by Gui [21] when $\hbar$ is sufficiently small by using different variational methods under local conditions.

In [8], Cao and Heinz considered (1.4) for $\hbar$ small and proved uniqueness of multi-bump bound states of (1.4) as $\hbar$ is small and which can be roughly described as follows: if $V(x)$ has $k$ different non-degenerate critical points $a^1, a^2, \ldots, a^k$ and $u_h(x), v_h(x)$ are two families of multi-bump bound states of (1.4) which concentrate at $a^1, a^2, \ldots, a^k$, then as $\hbar$ small enough, they proved that $u_h(x) \equiv v_h(x)$.

Many results on the existence of multi-bump solutions for problems similar to (1.4) have been obtained in recent years. For equations with zero Dirichlet boundary condition on a bounded domain $\Omega$, solutions concentrating at one or several points were obtained by Rey [31] and by Bahri et al. [5] for the case of a critical nonlinearity and by Lu and Wei [27], Cao et al. [7], Li [24], Li and Nirenberg [25] for the case of a subcritical nonlinearity. Uniqueness of solutions concentrating at one point was obtained by Glangetas [19] for Dirichlet problems with critical nonlinearity on bounded domains.

We also refer to Ambrosetti et al. [2], Ambrosetti et al. [1], Cingolani and Lazzo [11], Cingolani and Nolasco [12], Del Pino and Felmer [14,15] for the case that $A(x) \equiv 0$.

When $A(x) \not\equiv 0$, existence of standing waves to (1.1) has been proved by Lions–Esteban [16] for $\hbar > 0$ fixed and for special classes of magnetic fields. They found existence by solving a appropriate minimization problems for the corresponding energy functional in the case of $N = 2$ and 3.

More recently, Kurata [22] has proved the existence of least energy solution of (1.3) for $h > 0$ under a condition relating $V(x)$ and $A(x)$. Cingolani [10] obtained...
multiplicity results of solutions of (1.3) concentrating at a single point for small $\bar{h} > 0$ by using topological argument, and showed that the magnetic fields $A(x)$ only contributes to the phase factor of the solitary solutions of (1.1) as $\bar{h}$ small enough. Moreover, Cingolani and Secchi [13] also proved the existence of the one-bump bound states of (1.4) which concentrates at a non-degenerate critical point of $V(x)$ as $\bar{h}$ goes to zero.

We also refer to Arioli and Szulkin [3] for the case that $A(x) \not\equiv 0$, $A_j \ (j = 1, \ldots, N)$ and $V(x)$ are periodic functions.

It is therefore natural to ask if problem (1.3) admits multi-bump solutions which concentrate at several distinct non-degenerate critical points of $V(x)$ when $\bar{h} > 0$ is sufficiently small. If such solutions do exist, can uniqueness be established?

In this paper, we aim to answer these questions. First, we prove the existence of multi-bump bound states of problem (1.3) and then prove that such solutions are unique up to a rotation.

Our method of establishing existence and uniqueness of multi-bump solutions consists first in reducing the problem a finite-dimensional one by a Lyapunov–Schmidt reduction and then to derive our conclusions by an application of classical degree theory. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem we are dealing is not trivial at all. Many delicate estimates are needed because of the interaction between bumps.

Our paper is organized as follows: In Section 2, our main results (Theorems 2.2–2.4) and some remarks are presented. In Section 3, we first describe our framework, then give equivalent statements of Theorems 2.2–2.4 (Theorems 3.1–3.3). In Section 4, we prove existence of multi-bump bound states of (1.3) and in Section 5, we prove uniqueness of one- and multi-bump solutions. Some propositions and estimates needed in Sections 4, 5 are listed in Appendices A–C.

Throughout Sections 2–5, we always assume that $V(x)$, $A_j(x)$, $j = 1, 2, \ldots, N$ are in $C^2(\mathbb{R}^N)$, and bounded even if it is not explicitly stated. Throughout this paper, all integrals are over $\mathbb{R}^N$ if not specifically indicated otherwise. We will use the same $C$ to denote various generic positive constants, and we will use $O(t), o(t)$ to mean $|O(t)| \leq C|t|$, $o(t)/t \to 0$ as $t \to 0$. Finally, $o(1)$ denotes quantities that tend to 0 as $\bar{h} \to 0$.

2. Main results

Let $A(x) = (A_1(x), A_2(x), \ldots, A_N(x))$ be a bounded real-valued vector function on $\mathbb{R}^N$. Without loss of generality, we will assume $E = 0$ and that $V$ is a positive bounded function satisfying $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$. We consider the following complex problem:

$$
\begin{cases}
\left( \frac{\bar{h}}{i} \nabla - A(x) \right)^2 v + V(x)v = |v|^{p-2}v, & x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} v(x) = 0.
\end{cases}
$$

(2.1)
Let $E^h$ be the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product

$$(u, v)_h = \text{Re} \int \left( \frac{\hbar}{i} \nabla u - A(x)u \right) \left( \frac{\hbar}{i} \nabla v - A(x)v \right) + V(x)u\bar{v},$$

the norm induced by the product $(., .)_h$ is

$$|u|_h = \left( \int \left| \frac{\hbar}{i} \nabla u - A(x)u \right|^2 + V(x)|u|^2 \right)^{1/2}$$

$$= \left( \int \hbar^2 |\nabla u|^2 + [|A(x)|^2 + V(x)]|u|^2 - 2 \text{Re} \int i A(x)u \nabla \bar{u} \right)^{1/2}.$$

The energy functional associated with (2.1)$_h$ is defined by

$$I_h(v) = \frac{1}{2} \int \left( \left| \frac{\hbar}{i} \nabla v - A(x)v \right|^2 + V(x)|v|^2 \right) - \frac{1}{p} \int |v|^p \quad \text{for } v \in E^h.$$

Let $a$ be a given point in $\mathbb{R}^N$, then the following problem:

$$\begin{cases} -\Delta u + V(a)u & = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \quad u(0) = \max_{x \in \mathbb{R}^N} u(x), \\ \lim_{|x| \to \infty} u(x) = 0 \end{cases} \quad (2.2)$$

has a unique solution $U_a$ (see [26, 23]), where $U_a$ is radially symmetric and satisfying for $|x| \leq 1$

$$|D^2 U_a(x)| \exp \left( \sqrt{V(a)} |x| \right) |x|^{N-1} \leq C,$$

where $C > 0$ is some constant (see [18]).

Define for any $a \in \mathbb{R}^N$, $\sigma \in \mathbb{R}$, $y \in \mathbb{R}^N$ and for any fixed $\sigma \in [0, 2\pi]$

$$E_{h,a,\sigma,y} = \left\{ u \in E_h^h : \begin{array}{l} u, U_a \left( \frac{x-a-y}{h} \right) e^{i\sigma + i \left( \frac{x-a-y}{h} \cdot A(a+y) \right)} = 0; \\ u \frac{\partial}{\partial y_j} \left( U_a \left( \frac{x-a-y}{h} \right) e^{i\sigma + i \left( \frac{x-a-y}{h} \cdot A(a+y) \right)} \right) = 0, j = 1, \ldots, N \end{array} \right\}.$$

(2.3)
Definition 2.1. We say that a family of functions \( \{u_h\}_{h>0} \) concentrates at a set of points \( \{a^1, \ldots, a^k\} \subset \mathbb{R}^N \) if there exist \( (\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k \), \( \{x^m_h\}_{h>0} \subset \mathbb{R} \), \( x^m_h \geq 0 \), \( \{y^m_h\}_{h>0} \subset \mathbb{R}^N \), \( |y^m_h| = o(1) \) for \( m = 1, \ldots, k \) and \( k \) non-negative functions \( \tilde{U}_m \in E^h \) (\( 1 \leq m \leq k \)) satisfying \( \tilde{U}_m(x) \neq 0 \) and \( \tilde{U}_m(0) = \max_{x \in \mathbb{R}^N} \tilde{U}_m(x) \) such that

\[
h^{-\frac{N}{2}} \left| u_h - \sum_{m=1}^{k} x^{m}_h \tilde{U}_m \left( \frac{x - a^m - y^m_h}{h} \right) e^{i\sigma_{m,h} + i \left( \frac{x - a^m - y^m_h}{h} \right) \cdot A(a^m + y^m_h)} \right| = o(1). \tag{2.4}
\]

We call a solution \( u_h \) of (2.1) \( h \)-bump solution for small \( h \) if \( \{u_h\}_{h>0} \) is a family of functions which concentrate at a set of \( k \) distinct points.

Our main results are the following.

Theorem 2.2. Suppose that \( V(x), A(x) \) are bounded real-valued scalar function and vector functions, respectively, \( \inf_{x \in \mathbb{R}^N} V(x) \geq V^* > 0 \) and \( V(x) \) has \( k \) distinct non-degenerate critical points \( a^1, \ldots, a^k \). Then there exists a positive number \( h_0 \) such that for any \( (\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k \) and \( 0 < h < h_0 \), (2.1) admits a solutions of the form

\[
u_h(x) = \sum_{m=1}^{k} x^{m}_h u^{m}_h \left( \frac{x - a^m - y^m_h}{h} \right) e^{i\sigma_{m,h} + i \left( \frac{x - a^m - y^m_h}{h} \right) \cdot A(a^m + y^m_h)} + \omega_h, \tag{2.5}\]

with \( x^{m}_h \in \mathbb{R}, y^{m}_h \in \mathbb{R}^N \) and \( \omega_h \in \bigcap_{m=1}^{k} E_{h, a^m, \sigma_m, y^m_h} \) satisfying for \( m = 1, \ldots, k \), as \( h \to 0 \)

\[
\begin{cases}
  x^{m}_h = 1 + o(1), \\
  |y^{m}_h| = o(1), \\
  |\omega_h|_h = o(h^{\frac{N}{2}}).
\end{cases} \tag{2.6}
\]

Theorem 2.3. Suppose that \( \{u^m_h\}_{h>0} \) is a family of solutions of (2.1) \( h \) concentrating at a set of \( k \) distinct points \( \{a^1, \ldots, a^k\} \subset \mathbb{R}^N \). Then each \( a^m \) (\( 1 \leq m \leq k \)) must be a critical point of \( V \) and \( u_h \) must be of the form

\[
u_h(x) = \sum_{m=1}^{k} x^{m}_h u^{m}_h \left( \frac{x - a^m - y^m_h}{h} \right) e^{i\sigma_{m,h} + i \left( \frac{x - a^m - y^m_h}{h} \right) \cdot A(a^m + y^m_h)} + \omega_h, \tag{2.7}\]

with \( x^{m}_h, y^{m}_h \) and \( \omega_h \in \bigcap_{m=1}^{k} E_{h, a^m, \sigma_m, h, y^m_h} \) satisfying (2.6).

Theorem 2.4. Suppose that \( \{v^1_h\}_{h>0}, \{v^2_h\}_{h>0} \) are two families of solutions of (2.1) \( h \) concentrating at a set of \( k \) distinct non-degenerate critical points \( \{a^1, \ldots, a^k\} \subset \mathbb{R}^N \) and hence \( \{v^1_h\}_{h>0}, \{v^2_h\}_{h>0} \) of form (2.7) correspond to \( a^{1}_m, x^{m}_h, y^{m}_h, \omega^{1}_h \), \( m= \ldots, k \).
1, \ldots, k and \ \sigma_{m, h}^2, \ \varphi_{m, h}, \ \omega_{m, h}, m = 1, \ldots, k, \ \text{respectively. Then, for } h \ \text{small enough, if } \ \sigma_{m, h}^1 = \sigma_{m, h}^2, \ \text{we have } v_h^{1} = v_h^{2}, \ \text{namely}
\varphi_{h, 1}^m = \varphi_{h, 2}^m, \ \ \omega_{h, 1} = \omega_{h, 2}.

Furthermore, \ \varphi_{h, 1}^m = \varphi_{h, 2}^m, \ \omega_{h, 1} = \omega_{h, 2} \ \text{satisfying for } m = 1, \ldots, k,
\left\{ \begin{array}{l}
\varphi_{h}^m = 1 + O(h), \\
|\omega_{h}| = O(h), \\
|\omega_{h}| = O(h^{\frac{N}{2}+1}).
\end{array} \right.
(2.8)

\textbf{Remark 2.5.} The solutions obtained by Cingolani in [10], Cingolani and Secchi in [13] satisfy the definition of concentration at one critical point of } V(x) \ \text{as } h \to 0.

\textbf{3. Technical framework and proofs of the main results}

In this section, we give the equivalent results of Theorems 2.2–2.4, that is, Theorems 3.1–3.3. As in [8], by the change of variables \(x \to x/h\) we see that \(v_h\) is a solution of \((2.1)_h\) if and only if \(u_h(x) = v_h(hx)\) is a solution of
\[
\left\{ \begin{array}{l}
\left( \frac{1}{i} \nabla - A(hx) \right)^2 u(x) + V(hx)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0.
\end{array} \right.
(3.1)_h
\]

Corresponding to the definitions of \((\cdot, \cdot)_h\) and \(E_h\) defined in Section 2 we denote by \(E_h\) the Hilbert space defined as the closure of \(C_0^\infty(\mathbb{R}^N, \mathbb{C})\) under the scalar product
\[
\langle u, v \rangle_h = Re \int \left( \frac{1}{i} \nabla u - A(hx)u \right) \left( \frac{1}{i} \nabla v - A(hx)v \right) + V(hx)u \bar{v}.
(3.2)
\]

Define
\[
F_{h, a, \sigma, y} = \left\{ u \in E_h : \begin{array}{l}
\left. u, U_a \left( x - \frac{a-y}{h} \right) e^{i\sigma+i\left( x - \frac{a-y}{h} \right) \cdot A(a+y)} \right|_{h} = 0; \\
\left. \frac{\partial}{\partial y_j} U_a \left( x - \frac{a-y}{h} \right) e^{i\sigma+i\left( x - \frac{a-y}{h} \right) \cdot A(a+y)} \right|_{h} = 0, \quad j = 1, \ldots, N.
\end{array} \right\}
(3.3)
\]
where \(U_a\) is the unique solution of \((2.2)\) as defined in Section 2.
Let $|| \cdot ||_h$ be the norm introduced by the scalar product defined by (3.2), by the boundedness of $V(x)$ and $A(x)$, it is easy to see that $|| \cdot ||_h$ is equivalent to the usual norm $|| \cdot ||$ of $H^1(\mathbb{R}^N, \mathbb{C})$.

Energy functional associated with (3.1)$_h$ is defined by

$$I_h(u) = \frac{1}{2} \int \left( \frac{1}{i} \nabla u - A(hx)u \right)^2 + V(hx)u^2 - \frac{1}{p} \int |u|^p \quad \text{for } u \in E_h.$$  \hspace{1cm} (3.4)

Note that the equivalence of norm $|| \cdot ||_h$ with the standard one in $H^1(\mathbb{R}^N, \mathbb{C})$, the functional $I_h(u)$ is well defined.

Theorems 2.2 is equivalent to the following:

**Theorem 3.1.** Suppose that $V(x), A(x)$ are bounded real-valued scalar function and vector functions, respectively, $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0$ and $V(x)$ has $k$ distinct non-degenerate critical points $a^1, \ldots, a^k$. Then there exists positive number $h_0$ such that for any $(\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k$ and $0 < h < h_0$, (3.1)$_h$ admits a solution of the form

$$u_h(x) = \sum_{m=1}^k \zeta_h^m U_{a^m} \left( x - \frac{a^m + y_h^m}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y_h^m}{h} \right) A(a^m + y_h^m) + \omega_h},$$ \hspace{1cm} (3.5)

with $\zeta_h^m \in \mathbb{R}, y_h^m \in \mathbb{R}^N$ and $\omega_h \in \bigcap_{m=1}^k F_{h,a^m,\sigma_m,y_h^m}$ satisfying for $m = 1, \ldots, k$, as $h \to 0$

$$\begin{align*}
\zeta_h^m &= 1 + o(1), \\
|y_h^m| &= o(1), \\
||\omega_h|| &= o(1). \hspace{1cm} (3.6)
\end{align*}$$

$u_h$ is a solution of (3.1)$_h$ if and only if $v_h(x) = u_h(x/h)$ is a solution of (2.1)$_h$.

We have the following equivalent statement of Theorems 2.3–2.4.

**Theorem 3.2.** Suppose that $\{u_h\}_{h>0}$ is a family of solutions of (3.1)$_h$ and $\{v_h(x) = u_h(x/h)\}_{h>0}$ concentrating at a set of $k$ distinct points $\{a^1, \ldots, a^k\} \subset \mathbb{R}^N$. Then each $a^m$ ($1 \leq m \leq k$) must be a critical point of $V$ and $u_h$ must be of the form

$$u_h(x) = \sum_{m=1}^k \zeta_h^m U_{a^m} \left( x - \frac{a^m + y_h^m}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y_h^m}{h} \right) A(a^m + y_h^m) + \omega_h},$$ \hspace{1cm} (3.7)

with $\zeta_h^m \in \mathbb{R}, y_h^m \in \mathbb{R}^N$ and $\omega_h \in \bigcap_{m=1}^k F_{h,a^m,\sigma_m,y_h^m}$ satisfying (3.6), where $\sigma_m \in [0, 2\pi]$ for each $m = 1, \ldots, k$. 
Theorem 3.3. Suppose that \( \{u^1_h\}_{h>0}, \{u^2_h\}_{h>0} \) are two families of solutions of (3.1) and \( v^1_h(x) = u^1_h(x) \), \( v^2_h(x) = u^2_h(x) \) concentrating at a set of \( k \) distinct non-degenerate critical points \( \{a^1, \ldots, a^k\} \subset \mathbb{R}^N \) and hence \( \{v^1_h\}_{h>0}, \{v^2_h\}_{h>0} \) has the form of (3.7) corresponding to \( \sigma^1_{m,h}, x^m_{h,1}, \omega_{h,1}, \) and \( \sigma^2_{m,h}, x^m_{h,2}, \omega_{h,2} \), \( m = 1, \ldots, k \), respectively. Then, for \( h \) small enough, if \( \sigma^1_{m,h} = \sigma^2_{m,h}, m = 1, \ldots, k \), we have \( v^1_h \equiv v^2_h \).

Namely

\[
x^m_{h,1} = x^m_{h,2}, \quad y^m_{h,1} = y^m_{h,2}, \quad m = 1, \ldots, k, \quad \omega_{h,1} \equiv \omega_{h,2}.
\]

Furthermore, \( x^m_h = x^m_{h,1} = x^m_{h,2}, \quad y^m_{h} = y^m_{h,1} = y^m_{h,2} \) and \( \omega_h = \omega_{h,1} (\equiv \omega_{h,2}) \) satisfying for \( m = 1, \ldots, k \),

\[
\begin{aligned}
x^m_h &= 1 + O(h), \\
|y^m_h| &= O(h), \\
\|\omega_h\|_h &= O(h).
\end{aligned}
\]

For \( x_0 \in \mathbb{R}^k, \delta > 0 \), set

\[
\Sigma_{\delta, x_0} = \left\{ (x, y) : x = (x^1, \ldots, x^k), y = (y^1, \ldots, y^k) \text{ satisfying } |x^m - x^m_0| < \delta, y^m \in B_\delta, 1 \leq m \leq k \right\},
\]

where \( B_\delta = B_\delta(0) = \{x \in \mathbb{R}^N : |x| < \delta \} \) and for any \( \sigma = (\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k \), we denote

\[
\psi(x, y) = \sum_{m=1}^k \chi^m U_{\sigma_m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left( x - \frac{a^m + y^m}{h} \right) A(a^m + y^m)},
\]

where \( a^1, a^2, \ldots, a^k \) are given points in \( \mathbb{R}^N \).

For small \( \eta > 0 \), we define

\[
W(\eta, h) = \left\{ u \in E_h : \|u - \psi(x, y)\|_h < \eta, \text{ for some } (x, y) \in \Sigma_{\delta, x_0} \right\}.
\]

We will choose \( \delta \) small enough so that \( B_{5\delta}(a^m) \cap B_{5\delta}(a^j) = \emptyset \) if \( j \neq m, j = 1, \ldots, k, m = 1, \ldots, k \). We have the following proposition, which can be proved by the same arguments as in [9] (see also [4]).

Proposition 3.4. There exists \( \delta_0 > 0, h_0 > 0 \) such that for any \( \sigma = (\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k \) and \( u \in W(\delta, h) \) with \( \delta \in (0, \delta_0), h \in (0, h_0] \), the following minimization problem:

\[
\inf \left\{ \|u - \psi(x, y)\|_h : (x, y) \in \Sigma_{4\delta, x_0} \right\}
\]
has a unique solution which lies in $\Sigma_{2\delta, \sigma_0}$ and $u$ can be written as

$$
\sum_{m=1}^{k} \mathcal{Z}_m^m U_{am} \left( x - \frac{a^m + y^m}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) A(a^m + y^m) + \omega_h},
$$

with $(x_h, y_h) \in \Sigma_{2\delta, \sigma_0}$ and $\omega_h \in \bigcap_{m=1}^{k} F_{h, a^m, \sigma_m, y^m}$.  

Define

$$
K_h(u) = \frac{\|u\|_{h}^2}{\left( \int |u|^{p} \right)^{2/p}}, \quad \ell_h(u) = \frac{\|u\|_{h}^2}{\int |u|^{p}}.
$$

Note that $w \in E_h \setminus \{0\}$ is a critical point of $K_h$ in $E_h$ if and only if $u = \ell_{h}^{\frac{1}{p-1}}(w)w$ is a solution of (3.1)$_h$.

For any given $\delta > 0$, $(a^1, a^2, \ldots, a^k) \in \mathbb{R}^{Nk}$, let $\mathcal{N}_{\delta}(a^1, a^2, \ldots, a^k) = (-\delta, \delta) \times B_{\delta}(a^1) \times \cdots \times B_{\delta}(a^k)$, where $B_{\delta}(a^m) = \{ x \in \mathbb{R}^{N} : |x - a^m| < \delta \}$ for $m = 1, \ldots, k$. For simplicity, we will denote $\mathcal{N}_{\delta}(a^1, a^2, \ldots, a^k)$ by $\mathcal{N}_{\delta}$. In our discussion, we will choose $a^1, a^2, \ldots, a^k$ to be critical points of $V$ and $\delta > 0$ is a small number to be determined.

For any given $(\sigma_1, \ldots, \sigma_k) \in [0, 2\pi]^k$, let

$$
M_{h, \eta}^k = \left\{ (x, y, \omega) : (x, y) \in \mathcal{N}_{\delta}, \omega \in \bigcap_{m=1}^{k} F_{h, a^m, \sigma_m, y^m}, \|\omega\|_{h} < \eta \right\}
$$

and let us denote

$$
\phi(x^1, \ldots, x^k, y^1, \ldots, y^k) = \sum_{m=1}^{k} (1 + x^m) U_{am} \left( x - \frac{a^m + y^m}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) A(a^m + y^m)},
$$

and for convenience, in the following, we denote $\bigcap_{m=1}^{k} F_{h, a^m, \sigma_m, y^m}$ by $F_1, \ldots, y_k$ in the following.

Define

$$
J_h(x^1, \ldots, x^k, y^1, \ldots, y^k, \omega) = K_h(\phi(x^1, \ldots, x^k, y^1, \ldots, y^k) + \omega)
$$

for $(x^1, \ldots, x^k, y^1, \ldots, y^k, \omega) \in M_{h, \eta}^k$.
From Proposition 3.4 we derive the following result, which proof is standard and thus is omitted (see for example, [8]).

**Proposition 3.5.** There are $\eta_0, h_0$ and $\delta$ small enough such that if $h \in (0, h_0], \eta \in (0, \eta_0]$, $(x, y, \omega)$ is a critical point of $J_h$ in $M^k_{h,\eta}$ if and only if

$$u(x) := \sum_{m=1}^{k} (1 + \alpha^m) U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} + \omega$$

is a critical point of $K_h$ in $E_h$.

We note that $(x, y, w) \in M^k_{h,\eta}$ is a critical point of $J_h$ if and only if there are numbers $\xi_m, \gamma_{j,m}, m = 1, \ldots, k$, $j = 1, \ldots, k$ such that

$$\frac{\partial J_h(x, y, \omega)}{\partial \alpha^m} = 0, \quad m = 1, \ldots, k, \quad (3.14)$$

$$\frac{\partial J_h(x, y, \omega)}{\partial y^m_j} = \sum_{s=1}^{N} \gamma_{s,m} \left( \omega, \frac{\partial^2}{\partial y^m_s \partial y^m_j} \left\{ U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} \right\} \right)_{\bar{h}},$$

$$j = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, k, \quad (3.15)$$

$$\left\{ \frac{\partial J_h(x, y, \omega)}{\partial \omega}, \varphi \right\}_{\bar{h}} = \sum_{m=1}^{k} \tilde{\xi}_m \left( \varphi, U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} \right)_{\bar{h}}$$

$$+ \sum_{j=1}^{N} \sum_{m=1}^{k} \gamma_{j,m} \left( \varphi, \frac{\partial}{\partial y^m_j} \left\{ U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} \right\} \right)_{\bar{h}}.$$  

(3.16)

In order to prove Theorem 3.1, following the idea of Rey [31], we show first that for $(x, y)$ given, for $h$ small enough, there exists $\omega_h(x, y) \in F_{y_1, \ldots, y_k}$ and $\xi_m, \gamma_{m,j}, m =
1, \ldots, k, j = 1, \ldots, N \) such that Eq. (3.16) is satisfied and \( \omega_h \to 0 \) strongly in \( E_h \) as \( h \to 0 \), and then we employ topological degree theory to find \( z_m, y_m, m = 1, \ldots, k \) such that Eqs. (3.14) and (3.15) are satisfied. Furthermore, by calculating the topological degree of every critical point of \( L_h \) (which will be defined later), we can prove Theorem 3.3. The proof of Theorem 3.2 is standard which can be seen in Section 5.

To simplify our presentation, we will only consider the case \( k = 2 \), the case \( k \geq 3 \) follows similarly.

### 4. Existence of multi-bump bound states

In this section, we assume \( V(x) \) has multiple critical points. To simplify our presentation, we consider the existence of two-bump bound states. The crucial estimates needed in this section are given in Appendices A–C.

Assuming that \( a^1, a^2 \) are two critical points of \( V(x) \), consider the following problem:

\[
\begin{cases}
\left( -\frac{1}{i} \nabla - A(hx) \right)^2 u(x) + V(hx)u(x) = |u(x)|^{p-2}u(x), & x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0.
\end{cases}
\] (4.1)

We will establish the following result:

**Theorem 4.1.** Assume \( a^1, a^2 \) are two non-degenerate critical points of \( V(x) \). Then there is a positive real number \( h_0 > 0 \) such that for any \( (\sigma_1, \sigma_2) \in [0, 2\pi]^2 \) and \( h \in (0, h_0] \), (4.1) has a solution of the form

\[
u_h(x) = U_{a^1} \left( x - \frac{a^1 + y^1_h}{h} \right) e^{i\sigma_1 + i\left( x - \frac{a^1 + y^1_h}{h} \right)} A(a^1 + y^1_h) + (1 + \alpha_h)U_{a^2} \left( x - \frac{a^2 + y^2_h}{h} \right) e^{i\sigma_2 + i\left( x - \frac{a^2 + y^2_h}{h} \right)} A(a^2 + y^2_h) + \omega_h,
\]

where as \( h \to 0 \), \( \alpha_h \to 0 \), \( y^m_h \to 0 \), \( m = 1, 2 \), \( \omega_h \to 0 \) in \( E_h \) and \( \omega_h \in \cap_{m=1}^2 F_{h,a^m,\sigma_m,y^m_h} \).

The proof will be accomplished via the two propositions in the sequel.

Note that if \( u(x) \) is a critical point of \( K_h \), then for any constants \( c, cu(x) \) is also a critical point of \( K_h \). So for simplicity, we will find a critical point of \( K_h \) of the form

\[
u_h(x) = U_{a^1} \left( x - \frac{a^1 + y^1_h}{h} \right) e^{i\sigma_1 + i\left( x - \frac{a^1 + y^1_h}{h} \right)} A(a^1 + y^1_h) + (1 + \alpha_h)U_{a^2} \left( x - \frac{a^2 + y^2_h}{h} \right) e^{i\sigma_2 + i\left( x - \frac{a^2 + y^2_h}{h} \right)} A(a^2 + y^2_h) + \omega_h.
\]
We denote

\[ A^m = U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) A(a^m + y^m)}, \]

\[ U_{a^m, y^m} = U_{a^m} \left( x - \frac{a^m + y^m}{h} \right), \]

\[ Y^m = \sigma_m + \left( x - \frac{a^m + y^m}{h} \right) \cdot A(a^m + y^m). \]

Let \( M_{\bar{h}}, K_{\bar{h}}, N_{\bar{h}} \) and \( J_{\bar{h}} \) be as defined in Section 3 for \( k=2 \) and denote \( \cap_{m=1}^{2} F_{h,a^m,\sigma_m,y^m} \) by \( F_{y^1,y^2} \).

By Proposition 3.5, Theorem 4.1 will follow provided we can establish the existence of a critical point \( (0, \sigma, y^1, y^2, \omega) \in M_{h,\eta}^2 \) for \( J_{\bar{h}} \). Corresponding to the definition of \( N_{\bar{h}} \) in Section 3, we denote \( N_{\bar{h}} = \{ (\sigma, y^1, y^2) : \sigma \in (\delta, \bar{\delta}), y^m \in B_\delta(a^m), m = 1, 2 \} \).

Define

\[ M_{h,\eta}^* = \left\{ (\sigma, y^1, y^2, \omega) : (\sigma, y^1, y^2) \in N_{\bar{h}}, \omega \in \bigcap_{m=1}^{2} F_{h,a^m,\sigma_m,y^m}, \| \omega \|_h < \eta \right\}, \]

\[ J_{h}^*(\sigma, y^1, y^2, \omega) = J_{h}(0, \sigma, y^1, y^2, \omega). \]

Thus, we need to prove that there are, for \( \bar{h} \) and \( \eta \) small enough, \( \xi_1, \xi_2, \gamma_1^1, \gamma_2^1, \ldots, \gamma_1^N, \gamma_2^1, \gamma_2^2, \ldots, \gamma_2^N \) such that

\[ \frac{\partial J_{h}^*}{\partial \sigma} = 0, \quad (4.2) \]

\[ \frac{\partial J_{h}^*}{\partial y_j^s} = \sum_{j=1}^{N} \gamma_j^s \left\langle \omega, \frac{\partial^2 A_{m, \sigma_m, y^m}}{\partial y_j^s} \right\rangle_{\bar{h}}, \quad s = 1, \ldots, N, \quad m = 1, 2, \quad (4.3) \]

\[ \left\langle \frac{\partial J_{h}^*}{\partial \omega}, \phi \right\rangle_{\bar{h}} = \xi_1 (A_1, \phi)_{\bar{h}} + \xi_2 (A_2, \phi)_{\bar{h}} + \sum_{j=1}^{N} \gamma_j^1 \left\langle \frac{\partial A_1}{\partial y_j^1}, \phi \right\rangle_{\bar{h}} + \sum_{j=1}^{N} \gamma_j^2 \left\langle \frac{\partial A_2}{\partial y_j^2}, \phi \right\rangle_{\bar{h}}. \quad (4.4) \]

We have the following proposition.
Proposition 4.2. Suppose that \( a^1, a^2 \) are two given critical points of \( V \). Then there exist \( h_1 > 0, \delta_1 > 0 \) such that for \( h \in (0, h_1], \delta \in (0, \delta_1], \zeta \in (-\delta, \delta), y \in B_{\delta}(a^m), \) \( m = 1, 2 \), there is a unique \( C^1 \)-map:

\[
(\zeta, y^1, y^2) \in N_\delta \rightarrow \omega_{h, \zeta, y^1, y^2} \in F_{y^1, y^2}
\]
such that (4.4) is satisfied for some \((\xi_1, \xi_2, \eta_1, \eta_2, \ldots, \eta_N, \eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{2N+2}\). Moreover,

\[
\|\omega_{h, \zeta, y^1, y^2}\|_h = O(|y^1|^2 + |y^2|^2 + h).
\] (4.5)

**Proof.** We will use arguments similar to those in [9]. Set

\[
\psi_h = A^1 + (1 + \zeta)A^2.
\]

Expanding \( J_h^*(\zeta, y^1, y^2, \omega) \) with respect to \( \omega \) at \( \omega = 0 \), we get

\[
J_h^*(\zeta, y^1, y^2, \omega) = K_h(\psi_h + \omega)
= J_h^*(\zeta, y^1, y^2, 0) + f_{h, \zeta, y^1, y^2}(\omega)
+ Q_{h, \zeta, y^1, y^2}(\omega) + R_{h, \zeta, y^1, y^2}(\omega),
\]

where \( f_{h, \zeta, y^1, y^2}(\omega) \) is the linear term, \( Q_{h, \zeta, y^1, y^2}(\omega) \) is the quadratic term and \( R_{h, \zeta, y^1, y^2}(\omega) \) is the higher-order term. Let \( \bar{p} = \min\{p, 3\} \). Since,

\[
J_h^*(\zeta, y^1, y^2, \omega) = \frac{\|\psi_h + \omega\|^2}{(\int |\psi_h + \omega|^p)^{\frac{2}{p} + 1}},
\]

by direct calculations we get

\[
f_{h, \zeta, y^1, y^2}(\omega) = -\frac{2\|\psi_h\|_{\bar{h}}^2}{(\int |\psi_h|^p)^{\frac{2}{p} + 1}} Re \int |\psi_h|^{p-2}\psi_h\bar{\psi},
\] (4.6)

\[
Q_{h, \zeta, y^1, y^2}(\omega)
= \frac{2}{(\int |\psi_h|^p)^{\frac{2}{p}}} \left\{ \|\omega\|^2_{\bar{h}} - (p - 2)\ell_h(\psi_h) \int |\psi_h|^{p-4} Re(\psi_h\bar{\psi}) Re(\psi_h\bar{\psi})
- \ell_h(\psi_h) \int |\psi_h|^{p-2}|\omega|^2
+ (p + 2) \frac{\|\psi_h\|_{\bar{h}}^2}{(\int |\psi_h|^p)^{\frac{2}{p}}} \left( Re \int |\psi_h|^{p-2}\psi_h\bar{\psi} \right)^2 \right\},
\] (4.7)
while $R_{h,x,y^1,y^2}(\omega)$ collects the higher-order terms satisfying

\[
\begin{align*}
R_{h,x,y^1,y^2}(\omega) &= O\left(\|\omega\|_h^p\right), \\
R'_{h,x,y^1,y^2}(\omega) &= O\left(\|\omega\|_h^{p-1}\right), \\
R''_{h,x,y^1,y^2}(\omega) &= O\left(\|\omega_h\|_h^{p-2}\right).
\end{align*}
\] (4.8)

For $\omega \in F_{y^1,y^2}$, by the estimates of (A.6) in Appendix C, we get

\[
Re \int |\psi_h|^{p-2} \psi_h \tilde{\omega} = Re \int |A^1|^{p-2} A^1 \tilde{\omega} + (1 + x)^{p-1} Re \int |A^2|^{p-2} A^2 \tilde{\omega} + O(h^2)\|\omega\|_h,
\]

We claim that there is a $\rho' > 0$ such that

\[
Q_{h,x,y^1,y^2}(\omega) \geq \rho'\|\omega\|_h^2, \quad \omega \in F_{y^1,y^2}.
\] (4.9)

In fact,

\[
Re \int |A^1|^{p-2} A^1 \tilde{\omega} = Re \int |U_{a^1,y^1}|^{p-2} U_{a^1,y^1} e^{iy^1} \tilde{\omega} = Re \int \left(\frac{1}{i} \nabla A^1 - A(a^1 + y^1)A^1\right) \left(\frac{1}{i} \nabla \omega - A(a^1 + y^1)\omega\right) + V(a^1)A^1 \tilde{\omega}
\]

\[
= \langle A^1, \omega \rangle_h + Re \int [\{A(hx) - A(a^1 + y^1)\}^2 + (V(hx) - V(a^1))]A^1 \tilde{\omega}
\]

\[
- Re \int i[A(hx) - A(a^1 + y^1)][A^1 \nabla \tilde{\omega} - \nabla A^1 \tilde{\omega}]
\]

\[
= Re \int \left[|A(hx + a^1 + y^1) - A(a + y^1)|^2 + (V(hx + a^1 + y^1) - V(a))\right]
\]

\[
\times U_{a^1}(x) \omega \left(x + \frac{a^1 + y^1}{h}\right) - Re \int i[A(hx + a^1 + y^1) - A(a^1 + y^1)]
\]

\[
\times \left[U_{a^1}(x) \nabla \omega \left(x + \frac{a^1 + y^1}{h}\right) - \nabla U_{a^1}(x) \omega \left(x + \frac{a^1 + y^1}{h}\right)\right]
\]

\[
= O(|y^1|^2 + h)\|\omega\|_h.
\] (4.10)
here, \( \langle A^1, \omega \rangle_h = 0 \) for all \( \omega \in F_{y^1,y^2} \) and \( a^1 \) is a critical point of \( V(x) \) have been used. Similarly,

\[
Re \int |A^2|^p - 2 A^2 \tilde{\omega} = O(|y^2|^2 + h) \| \omega \|_h. \tag{4.11}
\]

Thus,

\[
Q_{h,x,y^1,y^2}(\omega) \geq \frac{1}{(\int |\psi_h|^p)^{2/p}} \left\{ \| \omega \|^2_h - (p - 2)\ell_h(\psi_h) \int |\psi_h|^p - 4 Re(\psi_h \tilde{\omega}) Re(\psi_h \tilde{\omega}) - \ell_h(\psi_h) \int |\psi_h|^p |\omega|^2 + O(|y^1|^2 + |y^2|^2 + h) \right\}. \]

We now use Proposition B.1 in Appendix B to obtain

\[
Q_{h,x,y^1,y^2}(\omega) \geq \frac{\rho \| \omega \|^2_h + O(|y^1|^2 + |y^2|^2 + h) \| \omega \|^2_h}{(\int |U_{a^1}(x)|^p + (1 + \alpha)^p \int |U_{a^2}(x)|^p)^{2/p}}.
\]

which implies (4.9).

We note that \( f_{h,x,y^1,y^2}(\omega) \) is a continuous linear form over \( F_{y^1,y^2} \), equipped with the scalar product \( \langle , \rangle_h \) in \( E_h \). Therefore, there exists a unique element of \( F_{y^1,y^2} \), still denoted by \( f_{h,x,y^1,y^2} \), such that

\[
f_{h,x,y^1,y^2}(\omega) = \langle f_{h,x,y^1,y^2}, \omega \rangle_h.
\]

We claim that

\[
\| f_{h,x,y^1,y^2} \|_h = O(|y^1|^2 + |y^2|^2 + h). \tag{4.12}
\]

Indeed, from (4.10) and (4.11), we easily get (4.12).

We are now in the position to use the argument of Proposition 4 in Rey [31] to establish the existence of an element \( \omega_{h,x,y^1,y^2} \) such that (4.4) is satisfied for some
\[ \xi^1, \xi^2 \text{ and } \gamma_j^1 (j = 1, \ldots, N), \gamma_j^2 (j = 1, \ldots, N). \] Furthermore, there is a constant \( C \) such that
\[ \| \omega_{\bar{h}, x, y^1, y^2} \| h \leq C \| f_{\bar{h}, x, y^1, y^2} \| h. \]

This completes the proof of Proposition 4.2. \( \square \)

By Proposition 4.2 we can define
\[ L_{\bar{h}}(x, y^1, y^2) = J_{\bar{h}}^*(x, y^1, y^2, \omega_{\bar{h}, x, y^1, y^2}) = J_{\bar{h}}(0, x, y^1, y^2, \omega_{\bar{h}, x, y^1, y^2}), \]
where \( \omega_{\bar{h}, x, y^1, y^2} \) is obtained in Proposition 4.2 for each \( (x, y^1, y^2) \in N_\delta \). Here, and henceforth \( \delta > 0 \) is assumed to be small enough so that the corresponding results in Appendices A–C and Proposition 4.2 hold.

It is easy to see that \( (0, x, y^1, y^2, \omega_{\bar{h}, x, y^1, y^2}) \) is a critical point of \( J_{\bar{h}} \) on \( M_{\bar{h}, \eta}^2 \) if and only if \( (x, y^1, y^2) \) is a critical point of \( L_{\bar{h}} \) on \( N_\delta \) when \( \bar{h} \) and \( \delta \) are sufficiently small (similar arguments can be seen in Cao and Heinz [8]). Therefore, we can reduce the problem to a finite-dimensional one. We will use classical degree theory to establish the existence of the solutions of (4.1) via proving the existence of critical points of \( L_{\bar{h}} \) on \( N_\delta \).

To use degree theory, we state the following proposition giving the degree of \( \nabla L_{\bar{h}} \) with respect to 0.

**Proposition 4.3.** Suppose that \( a^1, a^2 \) are two non-degenerate critical points of \( V \). Then for any fixed small \( \delta_3 > 0 \) and \( \delta \in (0, \delta_3] \) there exists \( h_3^\delta > 0 \) such that for \( h \in (0, h_3^\delta] \)
\[ \deg(0, \nabla L_{\bar{h}}, N_\delta) = (-1)^{n_1 + n_2 + 1}, \] (4.13)

where \( n_1, n_2 \) are the number of negative eigenvalues of \( D^2 V(a^1) \) and \( D^2 V(a^2) \).

**Proof.** For simplicity, we will denote \( \omega_{\bar{h}, x, y^1, y^2} \) by \( \omega_{\bar{h}} \). We first find an approximation of \( \nabla L_{\bar{h}}(x, y^1, y^2) \). To this end, differentiate \( L_{\bar{h}}(x, y^1, y^2) \) with respect to \( y^1 \) and use the following decomposition of \( \frac{\partial \omega_{\bar{h}}}{\partial y_j^1} \), given in Appendix C,
\[ \frac{\partial \omega_{\bar{h}}}{\partial y_j^1} = \omega_{j, \bar{h}}^1 + \xi_{1, j}^1 A^1 + \xi_{1, j}^2 A^2 + \sum_{n=1}^N \gamma_{j, n, \bar{h}}^1 \frac{\partial A^1}{\partial y_n^1} + \sum_{n=1}^N \gamma_{j, n, \bar{h}}^2 \frac{\partial A^2}{\partial y_n^2}, \] (4.14)
where \( \omega_{j,h}^{1} \in F_{y^1,y^2} \),

\[
\frac{\partial L_{h}(x, y^1, y^2)}{\partial y^1_j} = DK_{h}(\psi_{h} + \omega_{h}) \left( \frac{\partial A_{1}^{1}}{\partial y^1_{j}} + \frac{\partial \omega_{h}}{\partial y^1_{j}} \right)
\]

\[
= DK_{h}(\psi_{h} + \omega_{h}) \left( \frac{\partial A_{1}^{1}}{\partial y^1_{j}} + \xi_{1,j} A_{1}^{1} + \xi_{2} A_{2}^{2} + \sum_{n=1}^{N} \gamma_{j,n,h}^{1} \frac{\partial A_{1}^{1}}{\partial y^1_{j}} + \sum_{n=1}^{N} \gamma_{j,n,h}^{2} \frac{\partial A_{2}^{2}}{\partial y^2_{n}} \right). \tag{4.15}
\]

We will show that \( DK_{h}(\psi_{h} + \omega_{h}) \left( \frac{\partial A_{1}^{1}}{\partial y^1_{j}} \right) \) is the main term and the other terms in (4.15) are of higher order.

Let us first estimate

\[
DK_{h}(\psi_{h} + \omega_{h}) \left( \frac{\partial A_{1}^{1}}{\partial y^1_{j}} \right) = -2 \int |\psi_{h} + \omega_{h}|^{p-2} (\psi_{h} + \omega_{h}) \frac{\partial A_{1}^{1}}{\partial y^1_{j}} + O(\bar{h}^2)
\]

where

\[
I_{1} = \left( A_{1}^{1}, \frac{\partial A_{1}^{1}}{\partial y^1_{j}} \right)_{h} + O(h^2)
\]

\[
= Re \int \nabla A_{1}^{1} \cdot \nabla \frac{\partial A_{1}^{1}}{\partial y^1_{j}} + Re \int \left[ |A(hx)|^{2} + V(hx) \right] A_{1}^{1} \cdot \frac{\partial A_{1}^{1}}{\partial y^1_{j}}
\]

\[-Re \int iA(hx) \left[ A_{1}^{1} \cdot \nabla \frac{\partial A_{1}^{1}}{\partial y^1_{j}} - \nabla A_{1} \frac{\partial A_{1}^{1}}{\partial y^1_{j}} \right] + O(h^2).
\]
By the definition of $A^1, U_{a^1, y^1}, Y^1$, we have

$$
\frac{\partial A^1}{\partial y_j^1} = \frac{\partial U_{a^1, y^1}}{\partial y_j^1} e^{iY^1} - \frac{i}{\hbar} A(a^1 + y^1) U_{a^1, y^1} e^{iY^1} + i \frac{\partial A(a^1 + y^1)}{\partial y_j^1} \cdot \left( x - \frac{a^1 + y^1}{\hbar} \right) U_{a^1, y^1} e^{iY^1},
$$

$$
\nabla \frac{\partial A^1}{\partial y_j^1} = \nabla \frac{\partial U_{a^1, y^1}}{\partial y_j^1} e^{iY^1} + i A(a^1 + y^1) U_{a^1, y^1} e^{iY^1} - \frac{i}{\hbar} A_j(a^1 + y^1) \nabla U_{a^1, y^1} e^{iY^1} + \frac{1}{\hbar} A(a^1 + y^1) A_j(a^1 + y^1) U_{a^1, y^1} e^{iY^1} + i \frac{\partial A(a^1 + y^1)}{\partial y_j^1} U_{a^1, y_j^1} e^{iY^1} + i \frac{\partial A(a^1 + y^1)}{\partial y_j^1} \cdot \left( x - \frac{a^1 + y^1}{\hbar} \right) U_{a^1, y_j^1} e^{iY^1} - A(a^1 + y^1) \frac{\partial A(a^1 + y^1)}{\partial y_j^1} \cdot \left( x - \frac{a^1 + y^1}{\hbar} \right) U_{a^1, y^1} e^{iY^1},
$$

$$
\nabla A^1 = \nabla U_{a^1, y^1} e^{iY^1} + i A(a^1 + y^1) U_{a^1, y^1} e^{iY^1}.
$$

Direct calculation shows that

$$
\text{Re} \int \nabla A^1 \cdot \nabla \frac{\partial A^1}{\partial y_j^1} = \int A(a^1 + y^1) \frac{\partial A(a^1 + y^1)}{\partial y_j^1} U_{a^1, y^1}^2 + |A(a^1 + y^1)|^2 U_{a^1, y^1} \frac{\partial U_{a^1, y^1}}{\partial y_j^1},
$$

where we have used the equality $\int \nabla U_{a^1, y^1} \nabla \frac{\partial U_{a^1, y^1}}{\partial y_j^1} = 0$ ($j = 1, \ldots, N$) which follows the symmetry of $U_{a^1}$. Next, we note that

$$
\text{Re} \int \left[ |A(hx)|^2 + V(hx) \right] A^1 \cdot \frac{\partial A^1}{\partial y_j^1} = \int \left[ |A(hx)|^2 + V(hx) \right] U_{a^1, y^1} \frac{\partial U_{a^1, y^1}}{\partial y_j^1},
$$
\[ \begin{align*}
\text{Re} \int iA(hx) \left[ A^1 \cdot \nabla \frac{\partial \Abar^1}{\partial y_j^1} - \nabla A^1 \frac{\partial A^1}{\partial y_j^1} \right] \\
= 2 \int A(hx)A(a^1 + y^1)U_{a^1,y^1} \frac{\partial U_{a^1,y^1}}{\partial y_j^1} - \int A(hx) \frac{\partial A(a^1 + y^1)}{\partial y_j^1} U_{a^1,y^1}^2. 
\end{align*} \]

Thus,

\[ \begin{align*}
I_1 &= \int \left[ |A(hx) - A(a^1 + y^1)|^2 + V(hx) \right] U_{a^1,y^1} \frac{\partial U_{a^1,y^1}}{\partial y_j^1} \\
&\quad - \int \left[ A(hx) - A(a^1 + y^1) \right] \frac{\partial A(a^1 + y^1)}{\partial y_j^1} U_{a^1,y^1}^2 \\
&= \frac{1}{2} \frac{\partial}{\partial y_j^1} \int \left[ |A(hx) - A(a^1 + y^1)|^2 + V(hx) \right] U_{a^1,y^1}^2.
\end{align*} \]

\[ I_2 = -\ell_h(\psi_h + \omega_h) \text{Re} \int |\psi_h + \omega_h|^{p-2}(\psi_h + \omega_h) \frac{\partial A^1}{\partial y_j^1}. \]

Direct calculation shows that

\[ \ell_h(\psi_h + \omega_h) = \frac{\|U_{a^1}\|^2 + (1 + \xi)^2\|U_{a^2}\|^2}{\int U_a^p + (1 + \xi)\int U_a^p} + O(\|\omega_h\|). \] (4.16)

By (4.5) we have

\[ \begin{align*}
\text{Re} \int |\psi_h + \omega_h|^{p-2}(\psi_h + \omega_h) \frac{\partial A^1}{\partial y_j^1} \\
= \int U_{a^1,y^1}^{p-1} \frac{\partial U_{a^1,y^1}}{\partial y_j^1} + (p - 1)\text{Re} \int |A^1|^{p-2} \omega_h \frac{\partial A^1}{\partial y_j^1} \\
&\quad - 2\text{Re} \int |A^1|^{p-2}iA^1 \omega_h + O(h^2) \\
= O(\|\omega_h\|) = O(|y^1|^2 + |y^2|^2 + h), \quad (4.17)\end{align*} \]
where we used
\[\int U_{a^i-y^i}^{p-1} \frac{\partial U_{a^i,y^i}}{\partial y_j} = 0 \quad (j = 1, \ldots, N)\]
and \(\left\langle \omega_h, \frac{\partial A^1}{\partial y_j} \right\rangle = 0\). Thus, (4.16) and (4.17) imply that
\[I_2 = O(|y^1|^2 + |y^2|^2 + h).\]
Furthermore,
\[\left(\int (|\psi_h + \omega_h|^p)^{2/p} \right) = \left(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^1}^p \right)^{2/p} + O(\|\omega_h\|).\]
Thus, we have
\[DK_h(\psi_h + \omega_h) \left( \frac{\partial A^1}{\partial y_j} \right)\]
\[= \frac{2}{(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^2}^p)^{2/p} \int \left[ |A(hx) - A(a^1 + y^1)|^2 + V(hx) \right] U_{a^1,y^i} \frac{\partial U_{a^1,y^i}}{\partial y_j} \]
\[- \frac{2}{(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^2}^p)^{2/p} \int \left[ A(hx) - A(a^1 + y^1) \right] \frac{\partial A(a^1 + y^1)}{\partial y_j} U_{a^1,y^i} \]
\[= \frac{2}{(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^2}^p)^{2/p} \frac{1}{2} \frac{\partial}{\partial y_j} \int \left[ |A(hx + a^1 + y^1)|^2 + V(hx) \right] U_{a^1,y^i}^2(x) + O(|y^1|^2 + |y^2|^2 + h)\]
\[= \frac{1}{(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^2}^p)^{2/p} \frac{\hat{\partial}}{\hat{\partial} y_j} \int \left[ (A(hx + a^1 + y^1) - A(a^1 + y^1)) \]
\[\times \left( \frac{\partial A(hx + a^1 + y^1)}{\partial y_j} - \frac{\partial A(a^1 + y^1)}{\partial y_j} \right) \right] U_{a^1}^2(x) \]
\[+ \frac{1}{2} \frac{1}{(\int U_{a^i}^p + (1 + \varepsilon)^p \int U_{a^2}^p)^{2/p} \frac{\partial}{\partial y_j} \int \frac{\partial V(hx + a^1 + y^1)}{\partial y_j} U_{a^1}^2(x) + O(|y^1|^2 + |y^2|^2 + h).\]
We hence deduce

\[
DK_h(\psi_h + \omega_h) \left( \frac{\partial A^1}{\partial y^1_j} \right)
= \frac{1}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}} \int \frac{\partial V(hx + a^1 + y^1)}{\partial y^1_j} U^2_{a^1}(x)
+ O(|y^1|^2 + |y^2|^2 + \bar{h})
= \frac{\int U^2_{a^1}(x)}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}} \sum_{l=1}^{N} \frac{\partial^2 V(a^1)}{\partial a_l \partial a_j} y^1_l
+ O(|y^1|^2 + |y^2|^2 + \bar{h}).
\]

(4.18)

By the estimates of \( \xi^1_{1,j}, \xi^2_{1,j} \) and \( \gamma^1_{j.n,h}, \gamma^2_{j.n,h} \) for \( n = 1, \ldots, N \) in Appendix C we get

\[
DK_h(\psi_h + \omega_h) \left( \frac{\xi^1_{1,j} A^1 + \xi^2_{1,j} A^2 + \sum_{n=1}^{N} \gamma^1_{j.n,h} \frac{\partial A^1}{\partial y^1_n} + \sum_{n=1}^{N} \gamma^2_{j.n,h} \frac{\partial A^2}{\partial y^2_n} }{\partial \bar{h}} \right)
= O(\|\omega\|_h) = O(|y^1|^2 + |y^2|^2 + \bar{h}).
\]

(4.19)

Combining (4.15)–(4.19), we obtain that

\[
\frac{\partial L_h(x, y^1, y^2)}{\partial y^1_j} = \frac{\int U^2_{a^1}}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}} \sum_{l=1}^{N} \frac{\partial^2 V(a^1)}{\partial a_l \partial a_j} y^1_l
+ O(|y^1|^2 + |y^2|^2 + \bar{h}).
\]

(4.20)

Similarly, we have

\[
\frac{\partial L_h(x, y^1, y^2)}{\partial y^2_j} = \frac{(1 + x)^2 \int U^2_{a^2}}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}} \sum_{l=1}^{N} \frac{\partial^2 V(a^2)}{\partial a_l \partial a_j} y^2_l
+ O(|y^1|^2 + |y^2|^2 + \bar{h}).
\]

(4.21)
We now derive estimates for $\frac{\partial L_h(x, y^1, y^2)}{\partial \alpha}$. Since $\frac{\partial \omega_h}{\partial \alpha} \in F_{y^1, y^2}$, we have

$$\frac{\partial L_h(x, y^1, y^2)}{\partial \alpha} = D K_h(\psi_h + \omega_h) \left( A^2 + \frac{\partial \omega_h}{\partial \alpha} \right) = D K_h(\psi_h + \omega_h)(A^2)$$

$$= \frac{2 \int U^p_{a^2} \int U^p_{a^1}}{\left( \int (U^p_{a^1} + (1 + x)^p U^p_{a^2}) \right)^{\frac{2}{p} + 1}} \times \{1 + x - (1 + x)^{p-1}\} + O(|y^1|^2 + |y^2|^2 + h^2). \tag{4.22}$$

Thus, choosing $\delta$ and $\hbar$ small enough, for $(x, y^1, y^2) \in \partial N_\delta$, there exists a positive constant $C$ independent of $\hbar$ such that

$$|\nabla L_h(x, y^1, y^2)| > C|(x, y^1, y^2)| \neq 0.$$

Consider an homotopy given by

$$G(t) = (1 - t)\nabla L_h(x, y^1, y^2) + t F(x, y^1, y^2), \quad t \in [0, 1],$$

with

$$F(x, y^1, y^2) = \left( \lambda_1(1 - (1 + x)^{p-2}), \lambda_2 B_1 y^1, \lambda_3 B_2 y^2 \right),$$

where

$$B_j = \left( \frac{\partial^2 V(a_i)}{\partial a_i^j \partial a_i^j} \right)_{N \times N}, \quad j = 1, 2,$$

$$\lambda_1 = \frac{2(1 + x) \int U^p_{a^2} \int U^p_{a^1}}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{\frac{2}{p} + 1}},$$

$$\lambda_2 = \frac{\int U^2_{a^1}}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}}, \quad \lambda_3 = \frac{(1 + x)^2 \int U^2_{a^2}}{\left( \int U^p_{a^1} + (1 + x)^p \int U^p_{a^2} \right)^{2/p}}.$$
G(t) \neq 0 \text{ for all } (x, y^1, y^2) \in \partial N_\delta.

By the classical property of the degree, we have

\[ \deg(0, \nabla L_\bar{h}, N_\delta) = \deg(0, F, N_\delta) = (-1)^{n_1+n_2+1}, \]

which completes the proof of Proposition 4.3. □

**Proof of Theorem 4.1.** By Proposition 4.3, we obtain from the property of topological degree that there exists \((\bar{0}, \bar{\omega}_h, \bar{\omega}_h, \bar{\omega}_h, \bar{\omega}_h, \bar{\omega}_h) \in M_{\bar{h}, \eta}^2\) such that (4.2), (4.3) and (4.4) are satisfied and hence we complete the proof of Theorem 4.1. □

### 5. Uniqueness of multi-bump bound states

In this section, we prove Theorems 3.2 and 3.3.

As in Section 4, we denote

\[ A^m_\bar{h} = U^m_\bar{u} \left( x - \frac{a^m + y^m_h}{\bar{h}} \right) e^{i\sigma_m,\bar{h} + ix \cdot A(a^m + y^m_h)}, \]

\[ U^m_\bar{u}, Y^m_\bar{v} = U^m_\bar{u} \left( x - \frac{a^m + y^m_h}{\bar{h}} \right), \]

\[ Y^m_\bar{v} = \sigma_m,\bar{h} + \left( x - \frac{a^m + y^m_h}{\bar{h}} \right) \cdot A(a^m + y^m_h). \]

**Proof of Theorem 3.2.** First of all, we claim that if \((v_\bar{h})_{\bar{h}>0}\) is a family of solutions of (2.1)_\bar{h} satisfying (2.4) then \(\lim_{\bar{h} \to 0} x^m_\bar{h} = 0\) for \(m = 1, \ldots, k\) and \(x^m_0 \bar{U}_m = U^m_\bar{u}\) for \(m = 1, \ldots, k\), where \(x^m_0 = \lim_{\bar{h} \to 0} x^m_\bar{h}\).

Indeed, using the equation \(\int \left( \frac{\hbar}{2} \nabla v_\bar{h} - A(x)v_\bar{h} + V|v_\bar{h}|^2 \right) = \int |v_\bar{h}|^p\), we obtain

\[
\sum_{m=1}^k (x^m_\bar{h})^2 \int \left( \left| \frac{1}{i} \nabla \bar{U}_m(x) e^{i\sigma_m+h+ix \cdot A(a^m+y^m_h)} \right| - A(hx + a^m + y^m_h) \bar{U}_m(x) e^{i\sigma_m+h+ix \cdot A(a^m+y^m_h)} \right)^2
+ V(hx + a^m + y^m_h) \left| \bar{U}_m(x) \right|^2
= \sum_{m=1}^k (x^m_\bar{h})^p \left( \int \left| \bar{U}_m(x) \right|^p + o(1) \right),
\]

which implies that \((x^m_\bar{h})_{\bar{h}>0}\) is uniformly bounded with respect to \(h > 0\) for \(m = 1, \ldots, k\).
Multiplying
\[
\left( \frac{\hbar}{i} \nabla - A(x) \right)^2 v_\hbar + V(x) v_\hbar = |v_\hbar|^{p-2} v_\hbar,
\]
by \( \chi^m_{\hbar} \tilde{U}_m \left( \frac{x-a_0^m-y_0^m}{\hbar} \right) e^{i \sigma_m + i A(a_0^m+y_0^m)} \) and integrating by parts on \( \mathbb{R}^N \) we obtain
\[
(\chi^m_{\hbar})^2 \int \left( \frac{1}{i} \nabla \left\{ \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m+y_0^m)} \right\} \right. \\
\left. - A(\hbar x + a_0^m + y_0^m) \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m+y_0^m)} \right)^2 \\
+ V(\hbar x + a_0^m + y_0^m) \left| \tilde{U}_m (x) \right|^2 \\
= (\chi^m_{\hbar})^p \left( \int \left| \tilde{U}_m (x) \right|^p + o(1) \right),
\]
and consequently
\[
\lim_{\hbar \to 0^+} \chi^m_{\hbar} < +\infty.
\]
Since \( \sigma_m, h \in [0, 2\pi] \) for every \( \hbar > 0 \), we may assume that \( \lim_{\hbar \to 0^+} \sigma_m, h = \sigma_m \) up to a subsequence. Let \( \tilde{\chi}^m_{\hbar} (x) = v_\hbar (\hbar x + a^m + y_0^m) \). By (2.4) and passing to a subsequence if necessary \( \tilde{\chi}^m_{\hbar} \to \chi^m_0 \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m)} \) weakly in \( H^1 (\mathbb{R}^N, \mathbb{C}) \) as \( \hbar \to 0 \), and
\[
(\frac{1}{i} \nabla - A(\hbar x + a_0^m + y_0^m))^2 \tilde{\chi}^m_{\hbar} + V(\hbar x + a_0^m + y_0^m) \tilde{\chi}^m_{\hbar} = |\tilde{\chi}^m_{\hbar}|^{p-2} \tilde{\chi}^m_{\hbar}. \tag{5.1}
\]
Suppose \( \phi \) is an arbitrarily fixed function in \( H^1 (\mathbb{R}^N, \mathbb{C}) \). By (5.1) we get
\[
Re \int \left( \frac{1}{i} \nabla \tilde{\chi}^m_{\hbar} - A(\hbar x + a_0^m + y_0^m) \tilde{\chi}^m_{\hbar} \right) \left( \frac{1}{i} \nabla \phi - A(\hbar x + a_0^m + y_0^m) \nabla \phi \right) \\
+ V(\hbar x + a_0^m + y_0^m) \tilde{\chi}^m_{\hbar} \phi = Re \int |\tilde{\chi}^m_{\hbar}|^{p-2} \tilde{\chi}^m_{\hbar} \phi. \tag{5.2}
\]
Taking \( \hbar \to 0 \) in (5.2) we obtain
\[
Re \int \left( \frac{1}{i} \nabla \chi^m_0 \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m)} \right) - A(a_0^m) \chi^m_0 \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m)} = \int \left( \frac{1}{i} \nabla \chi^m_0 \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m)} \right) - A(a_0^m) \chi^m_0 \tilde{U}_m (x) e^{i \sigma_m + i A(a_0^m)}
\]
\[
\times \left( \frac{1}{i} \nabla \varphi - A(a^m) \nabla \varphi \right) + V(a^m) \tilde{U}_m(x)e^{i\sigma_m + ix \cdot A(a^m) \varphi} = Re \int |x_0 \tilde{U}_m(x)e^{i\sigma_m + ix \cdot A(a^m)}|^{p-2} x_0 \tilde{U}_m(x)e^{i\sigma_m + ix \cdot A(a^m) \varphi}.
\]

Hence, \(x_0 \tilde{U}_m(x)e^{i\sigma_m}\) satisfies the complex equation

\[
\begin{cases}
-\Delta u + V(a^m)u = |u|^{p-2}u, & x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} |u(x)| = 0.
\end{cases}
\]

By Kurata [22], we know that \(x_0 \tilde{U}_m(x)\) is a non-negative solution of

\[
\begin{cases}
-\Delta u + V(a^m)u = u^{p-1}, & x \in \mathbb{R}^N, \\
u(0) = \max_{x \in \mathbb{R}^N} u(x), \\
\lim_{|x| \to \infty} u(x) = 0.
\end{cases}
\]

By standard regularity theory and the maximum principle, it follows that \(\tilde{U}_m > 0\) on \(\mathbb{R}^N\). The uniqueness result of Kwong [23] yields \(x_0 \tilde{U}_m(x) \equiv U_{a^m}\) since as in [18], one can show that \(\tilde{U}_m\) decays exponentially at infinity.

Thus, \(u_{\bar{h}}\) satisfies

\[
\left\| u_{\bar{h}} - \sum_{m=1}^{k} (1 + \tilde{z}_m^{a_{\bar{h}}})U_{a^m} \left( x - \frac{a^m + \tilde{y}_m^{a_{\bar{h}}}}{\bar{h}} \right)e^{i\sigma_{m,\bar{h}} + i\left(x - \frac{a^m + \tilde{y}_m^{a_{\bar{h}}}}{\bar{h}}\right)A(a^m + \tilde{y}_m^{a_{\bar{h}}})} \right\|_{\bar{h}} = o(1), \quad (5.3)
\]

where \(\tilde{z}_m^{a_{\bar{h}}} = o(1), \tilde{y}_m^{a_{\bar{h}}} = o(1), m = 1, \ldots, k\).

As a consequence, \(u_{\bar{h}}\) can be written uniquely as

\[
u_{\bar{h}} = \sum_{j=1}^{k} (1 + \tilde{z}_j^{a_{\bar{h}}})U_{a^m} \left( x - \frac{a^m + y_j^{a_{\bar{h}}}}{\bar{h}} \right)e^{i\sigma_{m,\bar{h}} + i\left(x - \frac{a^m + y_j^{a_{\bar{h}}}}{\bar{h}}\right)A(a^m + y_j^{a_{\bar{h}}})} + \omega_{\bar{h}}, \quad (5.4)
\]

with \(\tilde{z}_j^{a_{\bar{h}}} \in \mathbb{R}, y_j^{a_{\bar{h}}} \in \mathbb{R}^N, \omega_{\bar{h}} \in \cap_{m=1}^{k} F_{\bar{h}, a^m, \sigma_{m,\bar{h}}, y_j^{a_{\bar{h}}}}\) satisfying as \(\bar{h} \to 0\)

\[
\begin{cases}
\tilde{z}_m^{a_{\bar{h}}} = o(1), & m = 1, 2, \ldots, k, \\
|y_j^{a_{\bar{h}}}| = o(1), & m = 1, 2, \ldots, k, \\
\|\omega_{\bar{h}}\|_{\bar{h}} = o(1).
\end{cases}
\]

(5.5)
We will show that $a^m (m = 1, \ldots, k)$ is a critical point of $V$. Similarly, to proof of Proposition 4.2, we can obtain a better estimate for $\|\omega_\bar{h}\|_\bar{h}$, namely

$$\|\omega_\bar{h}\|_\bar{h} = O(\bar{h} + |y^1_\bar{h}| + \cdots + |y^k_\bar{h}|).$$

Multiplying

$$\left(\frac{1}{i} \nabla - A(hx)\right)^2 u_\bar{h} + V(hx) u_\bar{h} = |u_\bar{h}|^{p-2} u_\bar{h}$$

by $\frac{\partial}{\partial y^m_\ell} \left\{ U_{am} \left(x - \frac{a^m + y^m_\bar{h}}{h}\right) e^{i\sigma_{m,h} + i\left(x - \frac{a^m + y^m_\bar{h}}{h}\right) \cdot A(a^m + y^m_\bar{h})} \right\}$ and integrating by parts on $\mathbb{R}^N$, using (A.6) in Appendix A, the exponential decay of $U_{am}$, $\frac{\partial U_{am}}{\partial x_\ell}$ and the above estimate on $\omega_\bar{h}$, we obtain

$$\frac{\partial}{\partial y^m_\ell} \int V(hx + a^m + y^m_\bar{h}) U^2_{am} = O(\bar{h} + |y^1_\bar{h}| + \cdots + |y^k_\bar{h}|).$$

Consequently,

$$\frac{\partial V(a^m + y^m_\bar{h})}{\partial a_\ell} \int U^2_{am} = O(\bar{h} + |y^1_\bar{h}| + \cdots + |y^k_\bar{h}|), \quad \ell = 1, \ldots, N.$$

Hence, $\frac{\partial V(a^m)}{\partial a_\ell} = 0$ for $m = 1, 2, \ldots, k$, $\ell = 1, \ldots, N$ and the proof of Theorem 3.2 is complete. $\square$

Now, it only remains to prove Theorem 3.3. Without loss of generality, we only give the proof when $k = 2$. Before doing it, we first state a proposition which is crucial for the proof of Theorem 3.3.

**Proposition 5.1.** Suppose that $a^1, a^2$ are two non-degenerate critical points of $V$. Then for any fixed small $\delta_4 > 0$ and $\delta \in (0, \delta_4]$ there exists $h^4_\delta > 0$ such that for $h \in (0, h^4_\delta]$

$$S_h = \left\{(x_h, y^1_h, y^2_h) \mid (x_h, y^1_h, y^2_h) \in N_\delta, \nabla L_h(x_h, y^1_h, y^2_h) = 0\right\}$$

is finite. Furthermore, for each $(x_h, y^1_h, y^2_h) \in S_h$

$$\text{sign} \left(\text{Jac}(\nabla L_h(x_h, y^1_h, y^2_h))\right) = (-1)^{n_1 + n_2 + 1}, \quad (5.6)$$
where $n_1, n_2$ are the number of negative eigenvalues of $D^2V(a^1)$ and $D^2V(a^2)$, $\text{Jac}\left(\nabla L_h(x_h, y^1_h, y^2_h)\right)$ denotes the Jacobian of $\nabla L_h$ at $(x_h, y^1_h, y^2_h)$.

The proof of this Proposition 5.1 will be given at the end of this section. Now, we turn to the proof of Theorem 3.3.

**Proof of Theorem 3.3.** Suppose $\{u^1_h\}_{h>0}, \{u^2_h\}_{h>0}$ are two families of solutions satisfying the assumptions of Theorem 3.3. Then, by the proof of Theorem 3.2, we know that $u^1_h, u^2_h$ are of the form

$$u_h(x) = U_{a^1} \left( x - \frac{a^1 + y^1_h}{h} \right) e^{i\sigma_{1,h} + i \left( x - \frac{a^1 + y^1_h}{h} \right) \cdot \Lambda(a^1 + y^1_h)}$$

$$+ (1 + x_h) U_{a^2} \left( x - \frac{a^2 + y^2_h}{h} \right) e^{i\sigma_{2,h} + i \left( x - \frac{a^2 + y^2_h}{h} \right) \cdot \Lambda(a^2 + y^2_h)} + \omega_h,$$

with $x_h \in \mathbb{R}$, $y^m_h \in \mathbb{R}^N$, $\omega_h \in F_{h,a^1,\sigma_{1,h},y^1_h} \cap F_{h,a^2,\sigma_{2,h},y^2_h}$ satisfying (5.5). Here, we use the same notations for $u^1_h$ and $u^2_h$ although at this moment they may be different functions. As in the proof of (ii) of Theorem 3.1 in [8], (5.5) can be improved to

$$\begin{cases} 
  x_h = O(h), \\
  |y^m_h| = O(h), \quad m = 1, 2, \\
  \|\omega_h\|_h = O(h). 
\end{cases} \tag{5.7}$$

So, if we let $u_h = u^1_h$ or $u_h = u^2_h$, then

$$u_h = A^1_h + (1 + x_h) A^2_h + \omega_h,$$

where $x_h, y^m_h, m = 1, 2$ and $\omega_h$ satisfy (5.7).

As before setting $w_h = A^1_h + (1 + x_h) A^2_h + \omega_h$, then $w_h$ is a critical point of $K_h$ and therefore $(x_h, y^1_h, y^2_h)$ is a critical point of $L_h$ in view of (5.7). To complete the proof we only need to show that $L_h$ has only one critical point in $N_\delta$ if $\delta$ is sufficiently small.

Suppose that $S_h$ has $k_0$ points $(x^j_h, y^1_h, y^2_h)$, $1 \leq j \leq k_0$. Then by Proposition 5.1 there holds

$$\text{deg}(0, \nabla L_h, N_\delta) = \sum_{j=1}^{k_0} \text{sign} \left( \text{Jac} \left( \nabla L_h \left( x^j_h, y^1_h, y^2_h \right) \right) \right) = (-1)^{n_1 + n_2 + 1} k_0. \tag{5.8}$$
But it follows from Proposition 5.3 that
\[
\deg(0, \nabla L_{\hat{h}}, N_{\hat{h}}) = (-1)^{n_1+n_2+1}.
\]
Thus, \(k_0 = 1\). Therefore, \(u_1^{\hat{h}} \equiv u_2^{\hat{h}}\). □

Now, we return to the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Let \((x, y_1^{\hat{h}}, y_2^{\hat{h}})\) be a fixed point in \(S_{\hat{h}}\). We will get estimates on \(D^2L_{\hat{h}}(x, y_1^{\hat{h}}, y_2^{\hat{h}})\). Let
\[
\phi_{\hat{h}} = U_{a_1} \left( x - \frac{a_1^1 + y_1^{\hat{h}}}{\hat{h}} \right) e^{i\sigma_{1,h} + i \left( x - \frac{a_1^1 + y_1^{\hat{h}}}{\hat{h}} \right) A(a_1^1 + y_1^{\hat{h}})}
\]
\[
+ (1 + \alpha) U_{a_2} \left( x - \frac{a_2^2 + y_2^{\hat{h}}}{\hat{h}} \right) e^{i\sigma_{2,h} + i \left( x - \frac{a_2^2 + y_2^{\hat{h}}}{\hat{h}} \right) A(a_2^2 + y_2^{\hat{h}})}.
\]
As in Section 4 we derive
\[
L_{\hat{h}}(x, y_1^{\hat{h}}, y_2^{\hat{h}})
\]
\[
= DK_{\hat{h}}(\phi_{\hat{h}} + \omega_{\hat{h}}) \left( (1 + \alpha)^{j-1} \frac{\partial A_j^{\hat{h}}}{\partial y_j^{\hat{h}}} + \sum_{t=1}^{2} \left\{ x_{j,t}^{\hat{h}} A_t^{\hat{h}} + \sum_{n=1}^{N} \gamma_{n,t}^{j,h} \frac{\partial A_n^{\hat{h}}}{\partial y_n^{\hat{h}}} \right\} \right).
\]
Since \(\phi_{\hat{h}} + \omega_{\hat{h}}\) is a critical point of \(K_{\hat{h}}\), we have
\[
\frac{\partial^2 L_{\hat{h}}(x, y_1^{\hat{h}}, y_2^{\hat{h}})}{\partial y_\ell^{\hat{h}} \partial y_m^{\hat{h}}}
\]
\[
= D^2K_{\hat{h}}(\phi_{\hat{h}} + \omega_{\hat{h}}) \left( (1 + \alpha)^{j-1} \frac{\partial A_j^{\hat{h}}}{\partial y_j^{\hat{h}}} + \omega_{\hat{h}, \ell} + (1 + \alpha)^{m-1} \frac{\partial A_n^{\hat{h}}}{\partial y_n^{\hat{h}}}
\]
\[
+ \sum_{t=1}^{2} \left\{ \eta_{m,t}^{\hat{h}} A_t^{\hat{h}} + \sum_{n=1}^{N} \beta_{n,t}^{m,h} \frac{\partial A_n^{\hat{h}}}{\partial y_n^{\hat{h}}} \right\} \right).
\]
where

\[ D^2 K_h(\phi_h + \omega_h) \left( \sum_{i=1}^{2} \left\{ \beta_{\ell,i} A_{h}^{m} + \sum_{n=1}^{N} \gamma_{n,i} \frac{\partial A_{h}^{m}}{\partial y_n^{i}} \right\} \right) + D^2 K_h(\phi_h + \omega_h) \left( \sum_{i=1}^{2} \left\{ \beta_{s,i} A_{h}^{m} + \sum_{n=1}^{N} \gamma_{n,i} \frac{\partial A_{h}^{m}}{\partial y_n^{i}} \right\} \right), \]

(5.9)

Indeed,

\[ D^2 K_h(u)(\varphi, \psi) = \frac{2}{(\int |u|^p)^{2/p}} \left\{ \langle \varphi, \psi \rangle_h - \frac{2(u, \varphi)_h}{\int |u|^p} \text{Re} \int |u|^{p-2} u \overline{\psi} - \frac{2(u, \psi)_h}{\int |u|^p} \right\} \times \text{Re} \int |u|^{p-2} u \overline{\varphi} + (p + 2) \frac{||u||^{2}_h}{(\int |u|^p)^{2}} \text{Re} \int |u|^{p-2} u \overline{\varphi} \text{Re} \int |u|^{p-2} u \overline{\psi} - (p - 2) \frac{||u||^{2}_h}{(\int |u|^p)^{2}} \int |u|^{p-4} \text{Re}(u \overline{\varphi}) \text{Re}(u \overline{\psi}) - \frac{||u||^{2}_h}{(\int |u|^p)^{2}} \text{Re} \int |u|^{p-2} \varphi \overline{\psi} \right\}. \]

(5.10)

We will show that for \( j = 1, 2 \)

\[ D^2 K_h(\phi_h + \omega_h) \left( (1 + \alpha)^{j-1} \frac{\partial A_{h}^{m}}{\partial y_s^j}, (1 + \alpha)^{j-1} \frac{\partial A_{h}^{j}}{\partial y_\ell^j} \right) \]

is the main term in (5.9), while the others are of higher-order \( O(h) \). Indeed,

\[ D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A_{h}^{m}}{\partial y_s^j}, \frac{\partial A_{h}^{j}}{\partial y_\ell^j} \right) = \frac{2}{(\int |\phi_h + \omega_h|^p)^{2/p}} \left\{ \left\langle \frac{\partial A_{h}^{m}}{\partial y_s^j}, \frac{\partial A_{h}^{j}}{\partial y_\ell^j} \right\rangle_h \right\} \]

\[ - \frac{2||\phi_h, \frac{\partial A_{h}^{j}}{\partial y_\ell^j}||_h}{(\int |\phi_h + \omega_h|^p)^{2/p}} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A_{h}^{m}}{\partial y_s^j} \]

\[ - \frac{2||\phi_h, \frac{\partial A_{h}^{m}}{\partial y_s^j}||_h}{(\int |\phi_h + \omega_h|^p)^{2/p}} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A_{h}^{j}}{\partial y_\ell^j} \]
As in Section 4, we find

\begin{align*}
+p + 2 \frac{\| \phi_h + \omega_h \|^2}{(\int |\phi_h + \omega_h|^p)^2} & \Re \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A_h^j}{\partial y_\ell^j} \\
\times \Re \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A_h^m}{\partial y_\ell^m} & \\
-(p - 2) \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} & \Re \int |\phi_h + \omega_h|^{p-4} \\
\times \Re \left( \phi_h + \omega_h \right) \frac{\partial A_h^j}{\partial y_\ell^j} & \Re \left( \phi_h + \omega_h \right) \frac{\partial A_h^m}{\partial y_\ell^m} \\
- \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} & \Re \int |\phi_h + \omega_h|^{p-2} \frac{\partial A_h^j}{\partial y_\ell^j} \frac{\partial A_h^m}{\partial y_\ell^m} \right).
\end{align*}

(5.11)

As in Section 4, we find

\begin{equation}
\Re \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A_h^j}{\partial y_\ell^j} = O(h), \quad j = 1, 2, \quad \ell = 1, \ldots, N,
\end{equation}

\begin{equation}
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A_h^j}{\partial y_\ell^j}, \frac{\partial A_h^m}{\partial y_\ell^m} \right)
\end{equation}

\begin{align*}
= & \frac{2}{(\int |\phi_h + \omega_h|^p)^{2/p}} \left\{ \left( \frac{\partial A_h^j}{\partial y_\ell^j}, \frac{\partial A_h^m}{\partial y_\ell^m} \right) \right\}_h - (p - 2) \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} \\
\times \Re \int |\phi_h + \omega_h|^{p-4} & \Re \left( \phi_h + \omega_h \right) \frac{\partial A_h^j}{\partial y_\ell^j} \Re \left( \phi_h + \omega_h \right) \frac{\partial A_h^m}{\partial y_\ell^m} \\
- \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} & \Re \int |\phi_h + \omega_h|^{p-2} \frac{\partial A_h^j}{\partial y_\ell^j} \frac{\partial A_h^m}{\partial y_\ell^m} \right).
\end{align*}

(5.12)

By (5.12) we get

\begin{equation}
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A_h^j}{\partial y_\ell^j}, \frac{\partial A_h^m}{\partial y_\ell^m} \right) = O(h) \quad \text{if } m, j = 1, 2, \quad m \neq j.
\end{equation}

(5.13)
Next, we consider the case $m = j$. Applying the identity

$$\left(\phi_h + \omega_h, \frac{\partial^2 A_h^m}{\partial y^m_s \partial y^m_\ell} \right)_h$$

$$= \ell_h(\phi_h + \omega_h) \Re \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial^2 A_h^m}{\partial y^m_s \partial y^m_\ell}$$

to (5.12) we obtain

$$D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A_h^2}{\partial y^2_s}, \frac{\partial A_h^2}{\partial y^2_\ell} \right)$$

$$= \frac{2}{(\int |\phi_h + \omega_h|^{p})^{2/p}} \left\{ \left( \frac{\partial A_h^2}{\partial y^2_s}, \frac{\partial A_h^2}{\partial y^2_\ell} \right)_h + \left( \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell} \right)_h \right\}$$

$$+ \left( \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell} \right)_h + \left( \omega_h, \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell} \right)_h - \ell_h(\phi_h + \omega_h) \Re \int |\phi_h + \omega_h|^{p-2}$$

$$\times (\phi_h + \omega_h) \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell} - (p - 2) \ell_h(\phi_h + \omega_h) \int |\phi_h + \omega_h|^{p-4}$$

$$\times \Re \left[ (\phi_h + \omega_h) \frac{\partial A_h^2}{\partial y^2_s} \right] \Re \left[ (\phi_h + \omega_h) \frac{\partial A_h^2}{\partial y^2_\ell} \right]$$

$$- \ell_h(\phi_h + \omega_h) \Re \int |\phi_h + \omega_h|^{p-2} \frac{\partial A_h^2}{\partial y^2_\ell} \frac{\partial A_h^2}{\partial y^2_s} + O(h) \right\},$$

(5.14)

$$\Re \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell}$$

$$= \Re \int |A_h^1|^p \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell} + \Re \int |A_h^2|^p \frac{\partial^2 A_h^2}{\partial y^2_\ell \partial y^2_s}$$

$$+ \left( \frac{p - 2}{2} + 1 \right) \Re \int |A_h^1|^p \omega_h \frac{\partial^2 A_h^2}{\partial y^2_s \partial y^2_\ell}$$

$$+ \frac{p - 2}{2} \Re \int |A_h^2|^p \omega_h \frac{\partial^2 A_h^2}{\partial y^2_\ell \partial y^2_s} + o(h),$$

(5.15)
Substituting (5.15) and (5.16) into (5.14) we obtain

\[
Re \int |\phi_h + \omega_h|^{p-2} \frac{\partial A_h^2}{\partial y_{1}^2} \frac{\partial A_h^2}{\partial y_{2}^2}
= \Re \int |A_h|^p \frac{\partial A_h^2}{\partial y_{1}^2} \frac{\partial A_h^2}{\partial y_{2}^2} + \Re \int |A_h|^p \frac{\partial A_h^2}{\partial y_{1}^2} \frac{\partial A_h^2}{\partial y_{2}^2}
+ \left( \frac{p-2}{2} + 1 \right) \Re \int |A_h|^p \omega_h \frac{\partial A_h^2}{\partial y_{1}^2} \frac{\partial A_h^2}{\partial y_{2}^2}
+ \frac{p-2}{2} \Re \int |A_h|^p \omega_h e^{2y_2} \frac{\partial A_h^2}{\partial y_{1}^2} \frac{\partial A_h^2}{\partial y_{2}^2} + o(h).
\]

(5.16)

Similarly, we have the following estimate:

\[
\int |\phi_h + \omega_h|^{p-4} \Re \left[ (\phi_h + \omega_h) \frac{\partial A_h^2}{\partial y_{1}^2} \right] \Re \left[ (\phi_h + \omega_h) \frac{\partial A_h^2}{\partial y_{2}^2} \right]
= \int |A_h|^p \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{1}^2} \right] \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{2}^2} \right]
+ \left( \frac{p-4}{2} + 1 \right) \Re \int |A_h|^p \omega_h \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{1}^2} \right] \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{2}^2} \right]
+ \frac{p-4}{2} \Re \int |A_h|^p \omega_h e^{2y_2} \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{1}^2} \right] \Re \left[ A_h^2 \frac{\partial A_h^2}{\partial y_{2}^2} \right] + o(h).
\]

Since \((x_h, y_h^1, y_h^2)\) is a critical point of \(L_h(x, y^1, y^2)\), and hence \(\phi_h + \omega_h\) is a critical point of \(K_h\), we have

\[
\ell_h(\phi_h + \omega_h) = 1 + O(h).
\]

Substituting (5.15) and (5.16) into (5.14) we obtain

\[
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A_h^2}{\partial y_{1}^2}, \frac{\partial A_h^2}{\partial y_{2}^2} \right)
= \frac{2}{(\int |\phi_h + \omega_h|^p)^{2/p}} \left\{ \left( \frac{\partial A_h^2}{\partial y_{1}^2}, \frac{\partial A_h^2}{\partial y_{2}^2} \right) + \left( A_h^2, \frac{\partial A_h^2}{\partial y_{1}^2} \right) \right\}_h.
\]
\[ + \text{Re} \int |A_h^2|^{p-2} A_h^2 \frac{\partial^2 A_h^2}{\partial y^2 \partial y_s^2} + (p - 2) \int |A_h^2|^{p-4} \text{Re} \left[ \frac{\partial A_h^2}{\partial y^2} \right] \text{Re} \left[ \frac{\partial A_h^2}{\partial y_s^2} \right] \]

\[ + \text{Re} \int |A_h^2|^{p-2} \frac{\partial A_h^2}{\partial y^2} \frac{\partial A_h^2}{\partial y_s^2} + O(h \| \omega_h \|_h) \}

\[ = \frac{2}{(f |\phi_h + \omega_h|^p)^{2/p}} \left\{ \frac{1}{2} \frac{\partial^2}{\partial y^2 \partial y_s^2} (A_h^2, A_h^2)_h + \frac{1}{p} \frac{\partial^2}{\partial y^2 \partial y_s^2} \int |A_h^2|^{p} + O(h \| \omega_h \|_h) \right\} \]

\[ = \frac{2}{(f |\phi_h + \omega_h|^p)^{2/p}} \left\{ \frac{1}{2} \frac{\partial^2}{\partial y^2 \partial y_s^2} (A_h^2, A_h^2)_h + O(h \| \omega_h \|_h) \right\}. \quad (5.17) \]

It is easy to see that

\[ \frac{1}{2} \frac{\partial^2}{\partial y^2 \partial y_s^2} (A_h^2, A_h^2)_h \]

\[ = \frac{\partial^2}{\partial y^2 \partial y_s^2} \int |A(hx + a^2 + y_h^2) - A(a^2 + y_h^2)|^2 U_a^2(x) \]

\[ + \frac{\partial^2}{\partial y^2 \partial y_s^2} V(hx + a^2 + y_h^2) \int U_a^2(x) + O(h \| \omega_h \|_h) \]

\[ = \frac{\partial^2 V(a^2)}{\partial a^2 \partial a^2} \int U_a^2(x) + O(h + h \| \omega_h \|_h). \quad (5.18) \]

From the estimates of \( \omega_{j,h,s}, \beta_{j,t,\ell}, \gamma_{j,t} \), for \( j = 1, 2, \ t = 1, 2, \ \ell = 1, 2, \ldots, N \) in Appendix C, it is easy to check that

\[ D^2 K_h(\phi_h + \omega_h) \left( \omega_{s,h}^2, \frac{\partial A_h^2}{\partial y_s^2} \right) = O(\| \omega \|_h) = O(h), \quad (5.19) \]

\[ D^2 K_h(\phi_h + \omega_h) \left( \omega_{s,h}^2, \sum_{t=1}^{2} \left\{ \beta_{t,h}^2 A_h + A_h^t \sum_{n=1}^{N} \gamma_{n,t,h}^2 \frac{\partial A_h^t}{\partial y_n^2} \right\} \right) \]

\[ = O(\| \omega \|_h) = O(h), \quad (5.20) \]

\[ D^2 K_h(\phi_h + \omega_h) \left( \sum_{t=1}^{2} \left\{ \beta_{t,h}^2 A_h + \sum_{n=1}^{N} \gamma_{n,t,h}^2 \frac{\partial A_h^t}{\partial y_n^2} \right\}, \right) \]
\[
\frac{\partial A_h^2}{\partial y_y^2} + \sum_{t=1}^2 \left\{ \beta_{t,h}^2 A_h^t + \sum_{n=1}^N \gamma_{n,t,h} \frac{\partial A_h^t}{\partial y_n} \right\}
= O(\|\omega\|) = O(h).
\] (5.21)

Thus, combining (5.18)–(5.21) we have

\[
\frac{\partial^2 L_h(x_h, y_{h1}, y_{h2})}{\partial y_{y1} \partial y_{y2}} = \lambda_3^* \frac{\partial^2 V(a^2)}{\partial a_{e} \partial a_{e}} + O(h),
\] (5.22)

where

\[
\lambda_3^* = \frac{(1 + \alpha)^2 \int U_{a^2}^2(x)}{\left( \int U_{a^1}^p(x) + (1 + \alpha)^p \int U_{a^2}^p(x) \right)^{2/p}}.
\]

Similarly, we obtain

\[
\frac{\partial^2 L_h(x_h, y_{h1}, y_{h2})}{\partial y_{y1} \partial y_{y1}} = \lambda_2^* \frac{\partial^2 V(a^1)}{\partial a_{e} \partial a_{e}} + O(h),
\] (5.23)

where

\[
\lambda_2^* = \frac{\int U_{a^1}^2(x)}{\left( \int U_{a^1}^p(x) + (1 + \alpha)^p \int U_{a^2}^p(x) \right)^{2/p}}.
\]

Combining (5.13), (5.19) and (5.20) we deduce

\[
\frac{\partial^2 L_h(x_h, y_{h1}, y_{h2})}{\partial y_{y1} \partial y_{y2}} = O(h).
\] (5.24)

Similarly, we get

\[
\frac{\partial^2 L_h(x_h, y_{h1}, y_{h2})}{\partial \alpha \partial y_{y1}} = \lambda_{1,\alpha} + O(1),
\] (5.25)

\[
\frac{\partial^2 L_h(x_h, y_{h1}, y_{h2})}{\partial \alpha \partial y_{y2}} = \lambda_{4,\alpha} \sum_{s=1}^N \frac{\partial^2 V(a^1)}{\partial a_{e} \partial a_{s}} y_{1s} + O(h).
\] (5.26)
\[
\frac{\partial^2 L_h(\mathbf{x}_h, \mathbf{y}_1^1, \mathbf{y}_2^2)}{\partial \mathbf{x} \partial \mathbf{y}_2^2} = \lambda_{5, \alpha} \sum_{s=1}^{N} \frac{\partial^2 V(\mathbf{a}_s^2)}{\partial \mathbf{a}_s \partial \mathbf{a}_s} y_s^1 + O(h), \tag{5.27}
\]

where

\[
\lambda_{1, \alpha} = -\frac{2 \int U^p_{a_1} \int U^p_{a_2}}{(\int U^p_{a_1} + (1 + \alpha)p \int U^p_{a_2})^{\frac{2}{p} + 2}} \cdot \left\{ \left[ (p - 1)(1 + \alpha)^{p-2} - 1 \right] \int U^p_{a_1} + \left[ (p + 1)(1 + \alpha)^p - 3(1 + \alpha)^{2p-2} \right] \int U^p_{a_2} \right\},
\]

\[
\lambda_{4, \alpha} = -\frac{2(1 + \alpha)^{p-1} \int U^2_{a_1} \int U^p_{a_2}}{(\int U^p_{a_1} + (1 + \alpha)p \int U^p_{a_2})^{\frac{2}{p} + 1}},
\]

\[
\lambda_{5, \alpha} = \frac{2(1 + \alpha) \int U^2_{a_2} \int U^p_{a_1}}{(\int U^p_{a_1} + (1 + \alpha)p \int U^p_{a_2})^{\frac{2}{p} + 1}}.
\]

By (5.22)–(5.27) it follows that

\[
sign(\text{Jac} \left( \nabla L_h(\mathbf{x}, \mathbf{y}_1^1, \mathbf{y}_2^2) \right) ) = \text{sign} \left( \det B_1 \det B_2 \lambda_{1, \alpha} \lambda_{2, \alpha}^2 \lambda_{3, \alpha}^3 + O(|y_1^1| + |y_2^2| + |\alpha|) + O(h) \right) = (-1)^{n_1 + n_2 + 1}
\]

provided \( \delta > 0 \) is sufficiently small, \( h > 0 \) is sufficiently small, where \( B_1 = D^2 V(\mathbf{a}_1) \), \( B_2 = D^2 V(\mathbf{a}_2) \), so Proposition 5.1 is proved.

Appendix A. Technical estimates

In this appendix, we will give a number of equalities or inequalities used in the previous sections.

Let \( a^1, \ldots, a^k \) be given distinct points in \( \mathbb{R}^N \) and \( \delta > 0 \) is small that \( B_{5\delta}(a^m) \cap B_{5\delta}(a^j) \) is empty if \( m \neq j, m, j = 1, \ldots, k \). \( y_1^1, y_2^2 \) denote points in \( B_\delta \). \( F^k \) is as in Section 3. \( \alpha \in (0, 1) \) is a fixed number, and we will assume \( A(x) = (A_1(x), \ldots, A_k(x)) \in C^1(\mathbb{R}^{N \times k}) \) and \( V \in C^2(\mathbb{R}^N) \). Estimates to those can be found in Cao and Heinz [8],...
but we state them for completeness.

\[
\text{Re} \int |U_{am}|^{q_1-1} |U_{aj}|^{q_2-1} U_{am} e^{i \sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) \cdot A(a^m + y^m)} U_{aj} e^{i \sigma_j + i \left( x - \frac{a^j + y^j}{h} \right) \cdot A(a^j + y^j)}
\]

\[
= O \left( \exp \left( - \frac{1}{2h} \min(q_1, q_2) \sqrt{V_0} |a^m + y^m - a^j - y^j| \right) \right),
\]

(A.1)

\[
\int \nabla U_{am} \left( x - \frac{a^m + y^m}{h} \right) \nabla U_{aj} \left( x - \frac{a^j + y^j}{h} \right)
\]

\[
= O \left( \exp \left( - \frac{1}{2h} \sqrt{V_0} |a^m + y^m - a^j - y^j| \right) \right),
\]

(A.2)

where in (B.1), (B.2), \( m, j = 1, \ldots, k, m \neq j, q_1, q_2 > 0 \).

\[
\int V(hx) U_{am}^2 \left( x - \frac{a^m + y^m}{h} \right)
\]

\[
= \begin{cases} 
V(a^m) \int U_{am}^2 + \left( V(y^m + a^m) - V(a^m) \right) \int U_{am}^2 + O(h) & \text{if } V \in C^1(\mathbb{R}^N), \\
V(a^m) \int U_{am}^2 + O(h^2 + |y|^2) & \text{if } V \in C^2(\mathbb{R}^N), \ DV(a^i) = 0.
\end{cases}
\]

(A.3)

By (B.3) and the definition of \( U_{am} \) we have

\[
\left\| U_{am} \left( x - \frac{a^i + y^i}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) \cdot A(a^m + y^m)} \right\|_h^2
\]

\[
= \begin{cases} 
\int U_{am}^p + O(\|y^m\|) \int U_{am}^2 + O(h) & \text{if } V \in C^1(\mathbb{R}^N), \\
\int U_{am}^p + O(h^2 + |y|^2) & \text{if } V \in C^2(\mathbb{R}^N), \ DV(a^i) = 0.
\end{cases}
\]

(A.4)

In the remaining of this appendix, we will always assume \( \omega \in \bigcap_{m=1}^k F_{h,a^m,y^m} \).

\[
\text{Re} \int U_{am}^{p-2} \left( x - \frac{a^m + y^m}{h} \right) e^{i \sigma_m + i \left( x - \frac{a^m + y^m}{h} \right) \cdot A(a^m + y^m)} \tilde{\omega}(x)
\]

\[
= \begin{cases} 
O(\|y^m\| + h) \|\omega\|_h & \text{if } V \in C^1(\mathbb{R}^N), \\
O(h^2 + |y|^2) \|\omega\|_h & \text{if } V \in C^2(\mathbb{R}^N), \ DV(a^i) = 0.
\end{cases}
\]

(A.5)
Let $p^* = \min\{p - 1, 2\}$. We have

$$
\text{Re} \int \left| \sum_{m=1}^{k} \mathcal{Z}_m U_{am} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} + \omega(x) \right|^{p-2} \left( \sum_{m=1}^{k} U_{am} \left( x - \frac{a^i + y^i}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} + \omega(x) \right) \tilde{\phi}(x)
$$

$$
= \sum_{m=1}^{k} \mathcal{Z}_m^{p-1} \left\{ \text{Re} \int U_{a^m}^{p-2} \left( x - \frac{a^m + y^m}{h} \right) U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) \right. \\
\times e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} \tilde{\phi}(x) \\
+ (p - 1) \mathcal{Z}_m^{p-2} \text{Re} \int U_{a^m}^{p-2} \left( x - \frac{a^m + y^m}{h} \right) \omega(x) \varphi(x) \}
$$

$$
+ O(\|\omega\|^{p^*}_{h} \|\varphi\| + o(h^2) \|\varphi\|), \quad (A.6)
$$

where $\varphi \in E_h$

$$
\int \left| \sum_{m=1}^{k} \mathcal{Z}_m U_{am} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m + \omega(x))} \right|^{p} \\
= \sum_{m=1}^{k} \left\{ \mathcal{Z}_m^{p} \int U_{a^m}^{p} + p \mathcal{Z}_m^{p-1} \text{Re} \int U_{a^m}^{p-2} \left( x - \frac{a^m + y^m}{h} \right) U_{a^m} \left( x - \frac{a^m + y^m}{h} \right) \\
\times e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} \tilde{\phi}(x) + \frac{p(p - 1)}{2} \mathcal{Z}_m^{p-2} \int U_{a^m}^{p-2} \\
\times \left( x - \frac{a^m + y^m}{h} \right) |\omega(x)|^2 \right\} + O(\|\omega\|^{p^*+1}_{h}) + o(h^2).
$$

$$
\int \left| \sum_{m=1}^{k} \mathcal{Z}_m U_{am} \left( x - \frac{a^m + y^m}{h} \right) e^{i\sigma_m + i\left(x - \frac{a^m + y^m}{h}\right)A(a^m + y^m)} + \omega(x) \right|^{p-2} |\varphi(x)|^2 \\
= \sum_{m=1}^{k} \int \mathcal{Z}_m^{p-2} U_{a^m}^{p-2} \left( x - \frac{a^m + y^m}{h} \right) |\varphi(x)|^2 + o(h^2) \|\varphi\|^2_{h} \\
+ O(\|\omega\|^{p^*-1}_{h} \|\varphi\|^2_{h}).
$$

(A.8)
Appendix B. An estimate on \( \omega \in \bigcap_{m=1}^{k} F_{h,a^m,\sigma_m,y^m} \)

In this appendix, we will give an estimate for elements in \( \bigcap_{m=1}^{k} F_{h,a^m,\sigma_m,y^m} \). We have

**Proposition B.1.** There exists \( \rho > 0 \) such that for all \( \omega \in \bigcap_{m=1}^{k} F_{h,a^m,\sigma_m,y^m} \)

\[
\| \omega \|_h^2 - (p - 2) \sum_{m=1}^{k} \int |A^m_h|^{p-4} \text{Re} A^m_h \bar{\bar{\omega}}} \text{Re} A^m_h \bar{\bar{\omega}}} - \sum_{m=1}^{k} \int |A^m_h|^{p-2} |\omega(x)|^2 \\
\geq \rho \| \omega \|_h^2,
\]

(B.1)

for all \( y^m \in B_\delta \) (\( m = 1, \ldots, k \)), provided \( \delta > 0 \) is sufficiently small.

**Proof.** We follow argument of Lemma 3.2 in Cingolani and Secchi [13], with some minor modifications due to the multiple-peaks. We define

\[
T_{h,a^m,\sigma_m,y^m} = \left\{ A^m_h, \frac{\partial}{\partial y_j} A^m_h, j = 1, \ldots, N \right\}.
\]

Then \( F_{h,a^m,\sigma_m,y^m} = (T_{h,a^m,\sigma_m,y^m})^\perp \) and \( \bigcap_{m=1}^{k} F_{h,a^m,\sigma_m,y^m} = \left( \bigcup_{m=1}^{k} T_{h,a^m,\sigma_m,y^m} \right)^\perp \). We define

\[
T_{h,a^m,\sigma_m,y^m} = \left\{ iA^m_h, A^m_h, \frac{\partial}{\partial y_j} A^m_h, j = 1, \ldots, N \right\}.
\]

Thus as in [13], it suffices to prove (B.1) for all \( v \in \text{span}\{iA^1_h, iA^2_h, \ldots, iA^k_h, \phi\} \), where \( \phi \perp \bigcup_{m=1}^{k} T_{h,a^m,\sigma_m,y^m} \). Let

\[
L_h(\phi_1, \phi_2) = \langle \phi_1, \phi_2 \rangle_h - (p - 2) \sum_{m=1}^{k} \int |A^m_h|^{p-4} \text{Re} A^m_h \bar{\bar{\phi_1}}} \text{Re} A^m_h \bar{\bar{\phi_2}}} - \sum_{m=1}^{k} \int |A^m_h|^{p-2} |\phi_1 \phi_2|.
\]

To complete the proof of Proposition B.1, by the same argument in [13], it is enough to prove that for some constants \( C_1 > 0, C_2 > 0 \) and \( m = 1, \ldots, k \),

\[
L_h(iA^m_h, iA^m_h) \leq - C_1 < 0,
\]

\[
L_h(\phi, \phi) \geq C_2 \| \phi \|_h, \forall \phi \perp \bigcup_{m=1}^{k} T_{h,a^m,\sigma_m,y^m}.
\]
The proof of the above two inequality is similar with the proof of Lemma 3.2 in [13] and we omitted here. □

Appendix C. Analysis of $\omega_{h, x, y^1, y^2}$

Lemma C.1. Suppose $\omega_{h, x, y^1, y^2} \in F_{h, a^1, \sigma_1, y^1} \cap F_{h, a^2, \sigma_2, y^2}$. If we decompose $\omega_{h, x, y^1, y^2}$ in the following way:

$$\frac{\partial \omega_{h, x, y^1, y^2}}{\partial y^j} = \omega_{\ell, h}^m + \sum_{t=1}^2 \left\{ \xi_{\ell, h}^2 t \mathcal{A}_h^t + \sum_{n=1}^N \gamma_{n, \ell, h}^2 t \right\},$$

(C.1)

where $\omega_{\ell, h}^m \in F_{h, a^1, \sigma_1, y^1} \cap F_{h, a^2, \sigma_2, y^2}$, $m = 1, 2$, $\ell = 1, \ldots, N$, and $\mathcal{A}_h^t$ is as defined in Section 5, then

$$\| \omega_{\ell, h}^m \| = O(\| \omega_{h, x, y^1, y^2} \|_h), \quad m = 1, 2,$$

(C.2)

$$\xi_{\ell, h}^m, j = O(h)\| \omega_{h, x, y^1, y^2} \|_h, \quad j, m = 1, 2, \ell = 1, \ldots, N,$$

(C.3)

$$\gamma_{s, \ell, h}^m = O(h + \| \omega_{h, x, y^1, y^2} \|_h), \quad j, m = 1, 2, \ell, s = 1, \ldots, N,$$

(C.4)

$$\left\| \frac{\partial \omega_{h, x, y^1, y^2}}{\partial \mathcal{X}} \right\| = O(h + \| \omega_{h, x, y^1, y^2} \|_h).$$

(C.5)

Proof. Since proofs of estimates similar to (C.3), (C.4), and (C.5) can be found in Appendix C in Cao and Heinz [8], here, we only give the proof of (C.2).

Generality, we just consider the term $\omega_{\ell, h}^1$. For any $(\tilde{x}, \tilde{y}, \tilde{z}) \in N_\delta$, we denote by $\pi_{\tilde{x}, \tilde{y}, \tilde{z}}$ the projection of $\omega_{\ell, h}^1$ onto $F_{h, a^1, \sigma_1, y^1} \cap F_{h, a^2, \sigma_2, y^2}$. It is easy to see that $\pi$ is $C^1$ with respect to $(\tilde{x}, \tilde{y}, \tilde{z}) \in N_\delta$ (see Glangetas [19]). Since $\pi_{\tilde{x}, \tilde{y}, \tilde{z}} \in F_{h, a^1, \sigma_1, \tilde{y}, \tilde{z}} \cap F_{h, a^2, \sigma_2, \tilde{y}, \tilde{z}}$ for $(\tilde{x}, \tilde{y}, \tilde{z}) \in N_\delta$, we have

$$DK_h(\tilde{\phi}_h + \tilde{\omega}_h)(\pi_{\tilde{x}, \tilde{y}, \tilde{z}}) = 0,$$

where $\tilde{\phi}_h = \tilde{\mathcal{A}}_h^1 + (1 + \tilde{x})\tilde{\mathcal{A}}_h^2$ is defined as in Section 5, replacing $x, y^1, y^2$ by $\tilde{x}, \tilde{y}^1, \tilde{y}^2$, respectively. Differentiating with respect to $\tilde{y}^1$, we have

$$D^2K_h(\tilde{\phi}_h + \tilde{\omega}_h) \left( \frac{\partial \tilde{\mathcal{A}}_h^1}{\partial \tilde{y}^1} + \frac{\partial \tilde{\omega}_h}{\partial \tilde{y}^1}, \pi_{\tilde{x}, \tilde{y}, \tilde{z}} \right) + DK_h(\tilde{\phi}_h + \tilde{\omega}_h) \left( \frac{\partial \pi_{\tilde{x}, \tilde{y}, \tilde{z}}}{\partial \tilde{y}^1} \right) = 0.$$

(C.6)
Let \((\tilde{x}, \tilde{y}^1, \tilde{y}^2) = (x, y^1, y^2)\), Thus \(\pi_{x, y^1, y^2} = \omega^1_{\ell, h}\), and since \(\phi_h + \omega_h\) is a critical point of \(K_h(u)\), we have that (C.6) is equivalent to

\[
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A^1_{\ell}}{\partial y^1_{\ell}}, \frac{\partial \omega_h}{\partial y^1_{\ell}}, \omega^1_{\ell, h} \right) = 0.
\]

This implies

\[
D^2 K_h(\phi_h + \omega_h)(\omega^1_{\ell, h}, \omega^1_{\ell, h}) + D^2 K_h(\phi_h + \omega_h)
\times \left( \frac{\partial A^1_{\ell}}{\partial y^1_{\ell}}, \sum_{t=1}^{2} \left\{ \beta_{t, \ell, h}^A A^t_{\ell} + A^t_{\ell} \sum_{n=1}^{N} \gamma_{n, \ell, h}^A \frac{\partial A^t_{\ell}}{\partial y^1_n} \right\}, \omega^1_{\ell, h} \right) = 0.
\]

(C.7)

First of all, we prove that for \(h\) small enough,

\[
D^2 K_h(\phi_h + \omega_h)(\omega^1_{\ell, h}, \omega^1_{\ell, h}) \geq \rho' \| \omega^1_{\ell, h} \|_h,
\]

(C.8)

where \(\rho' > 0\) is independent of \(h\). We find

\[
D^2 K_h(\phi_h + \omega_h)(\omega^1_{\ell, h}, \omega^1_{\ell, h})
\]

\[
= \frac{2}{(\int |\phi_h + \omega_h|^p)^2} \left\{ \left( \omega^1_{\ell, h}, \omega^1_{\ell, h} \right) \right\}_h
\]

\[
- \frac{4}{\int |\phi_h + \omega_h|^p} \text{Re} \int |\phi_h + \omega_h|^{p-2}(\phi_h + \omega_h)\overline{\omega^1_{\ell, h}}
\]

\[
+ (p + 2) \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} \left( \text{Re} \int |\phi_h + \omega_h|^{p-2}(\phi_h + \omega_h)\overline{\omega^1_{\ell, h}} \right)^2
\]

\[
- (p - 2) \text{Re} \int |\phi_h + \omega_h|^2 \int |\phi_h + \omega_h|^{p-4} \text{Re}[(\phi_h + \omega_h)\overline{\omega^1_{\ell, h}}]
\]

\[
\times \text{Re}[(\phi_h + \omega_h)\overline{\omega^1_{\ell, h}}] - \text{Re} \frac{\| \phi_h + \omega_h \|^2}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-2} |\omega^1_{\ell, h}|^2 \right\}.
\]

From Proposition A.1, we obtain (C.8). Secondly, we prove that

\[
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A^1_{\ell}}{\partial y^1_{\ell}}, \sum_{t=1}^{2} \left\{ \beta_{t, \ell, h}^A A^t_{\ell} + A^t_{\ell} \sum_{n=1}^{N} \gamma_{n, \ell, h}^A \frac{\partial A^t_{\ell}}{\partial y^1_n} \right\}, \omega^1_{\ell, h} \right)
\]

\[
= O(h)\| \omega^1_{\ell, h} \|_h.
\]

(C.9)
In fact

\[
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A^1_{\ell,h}}{\partial y^1}, \omega^1_{\ell,h} \right) = \frac{2}{(\int |\phi_h + \omega_h|^p)^{2/p}} \left\{ \left( \frac{\partial A^1_{\ell,h}}{\partial y^1} \right) \right\}_h
\]

\[
= -\frac{2\left(\phi_h + \omega_h, \omega^1_{\ell,h}\right)}{\int |\phi_h + \omega_h|^p} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
= -\frac{2\left(\phi_h + \omega_h, \frac{\partial A^1_{\ell,h}}{\partial y^1}\right)}{\int |\phi_h + \omega_h|^p} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
+ (p + 2) \frac{||\phi_h + \omega_h||^2_h}{(\int |\phi_h + \omega_h|^p)^{2/p}} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
\times \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
- (p - 2) \frac{||\phi_h + \omega_h||^2_h}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-4} \text{Re} \left\{ (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1} \right\} \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
- \frac{||\phi_h + \omega_h||^2_h}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-2} \frac{\partial A^1_{\ell,h}}{\partial y^1}\left( \omega^1_{\ell,h} \right).
\]

Since \( \omega^1_{\ell,h} \in F_{h,a^1,\sigma_1,y^1} \cap F_{h,a^2,\sigma_2,y^2} \), there holds

\[
D^2 K_h(\phi_h + \omega_h) \left( \frac{\partial A^1_{\ell,h}}{\partial y^1}, \omega^1_{\ell,h} \right) = \frac{2}{(\int |\phi_h + \omega_h|^p)^{2/p}} \left\{ \left( \frac{\partial A^1_{\ell,h}}{\partial y^1} \right) \right\}_h
\]

\[
= -\frac{2\left(\phi_h + \omega_h, \frac{\partial A^1_{\ell,h}}{\partial y^1}\right)}{\int |\phi_h + \omega_h|^p} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
+ (p + 2) \frac{||\phi_h + \omega_h||^2_h}{(\int |\phi_h + \omega_h|^p)^{2/p}} \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
\times \text{Re} \int |\phi_h + \omega_h|^{p-2} (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
- (p - 2) \frac{||\phi_h + \omega_h||^2_h}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-4} \text{Re} \left\{ (\phi_h + \omega_h) \frac{\partial A^1_{\ell,h}}{\partial y^1} \right\} \frac{\partial A^1_{\ell,h}}{\partial y^1}
\]

\[
- \frac{||\phi_h + \omega_h||^2_h}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-2} \frac{\partial A^1_{\ell,h}}{\partial y^1}\left( \omega^1_{\ell,h} \right).
\]
\[+(p+2)\frac{\|\phi_h + \omega_h\|_h^2}{\left(\int |\phi_h + \omega_h|^p\right)^2} Re \int |\phi_h + \omega_h|^{p-2}(\phi_h + \omega_h)\overline{\omega_{1,h}^{\ell}}\]

\[\times Re \int |\phi_h + \omega_h|^{p-2}(\phi_h + \omega_h) \frac{\partial A_1^h}{\partial y_1^\ell}\]

\[-(p-2)\frac{\|\phi_h + \omega_h\|_h^2}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-4} Re \left[ (\phi_h + \omega_h) \overline{\omega_{1,h}^{\ell}} \right] \frac{\partial A_1^h}{\partial y_1^\ell}\]

\[-\frac{\|\phi_h + \omega_h\|_h^2}{\int |\phi_h + \omega_h|^p} \int |\phi_h + \omega_h|^{p-2} \frac{\partial A_1^h}{\partial y_1^\ell} \overline{\omega_{1,h}^{\ell}}\].

From Section 4, we know that \(\|\omega_h\|_h = O(h)\) and \(\left\langle A_1^h, \frac{\partial A_1^h}{\partial y_1^\ell} \right\rangle_h = O(h)\), thus it is easy to check that (C.9) holds.

Combining (C.7), (C.8) and (C.9) we obtain the proof of (C.2). \(\square\)

**Remark.** By a similar process, if one replace \(A_1^h\) by \(A\) in (C.1), then the corresponding estimates (C.2)–(C.5) still holds.

**References**


