MULTIBUMP SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH STEEP POTENTIAL WELL AND INDEFINITE POTENTIAL

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Dedicated to Jean Mawhin on the occasion of his 70th birthday.

ABSTRACT. We are concerned with the existence of single- and multi-bump solutions of the equation

$$-\Delta u + (\lambda a(x) + a_0(x))u = |u|^{p-2}u, \quad x \in \mathbb{R}^N;$$

where $p > 2$, and $p < \frac{2N}{N-2}$ if $N \geq 3$. We require that $a \geq 0$ is in $L_\text{loc}^\infty(\mathbb{R}^N)$ and has a bounded potential well $\Omega$, i.e. $a(x) = 0$ for $x \in \Omega$ and $a(x) > 0$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}$. Unlike most other papers on this problem we allow that $a_0 \in L_\infty(\mathbb{R}^N)$ changes sign. Using variational methods we prove the existence of multibump solutions $u_\lambda$ which localize, as $\lambda \to \infty$, near prescribed isolated open subsets $\Omega_1, \ldots, \Omega_k \subset \Omega$. The operator $L_0 := -\Delta + a_0$ may have negative eigenvalues in $\Omega_j$, each bump of $u_\lambda$ may be sign-changing.

1. Introduction and main result. We are concerned with the stationary nonlinear Schrödinger equation

$$\begin{cases}
-\Delta u + (\lambda a(x) + a_0(x))u = |u|^{p-2}u & x \in \mathbb{R}^N; \\
u(x) \to 0 & \text{as } |x| \to \infty;
\end{cases} \quad (S_\lambda)$$

here $p < 2^* = 2N/(N-2)^+$. We require that $a \geq 0$ and $\Omega := \text{int } a^{-1}(0) \neq \emptyset$. Thus for $\lambda > 0$ large the potential $\lambda a + a_0$ develops a steep potential well and one expects to find solutions which localize near its bottom $\Omega$. This problem has found much interest after being first considered in [2]–[4]; see the papers [10, 12] for recent results and references to the literature.

Fixing disjoint isolated open subsets $\Omega_1, \ldots, \Omega_k \subset \Omega$ we develop a method of constructing solutions $u_\lambda$ for $\lambda > 0$ large such that the restrictions $u_\lambda|_{\Omega_j}$ converge as $\lambda \to \infty$ towards a least energy solution of

$$-\Delta u + a_0(x)u = |u|^{p-2}u, \quad u \in H_0^1(\Omega_j), \quad (P_j)$$

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If $-\Delta + a_0$ is positive such a result has been proved in [5]. In that case, the trivial solution $u = 0$ is a nondegenerate local minimum of the variational functional associated to $(P_j)$, and the least energy solution is positive and of mountain pass type. More recently, Sato and Tanaka [10] considered the case where $a_0 \equiv 1$, so again $-\Delta + a_0$ is positive. It is well known that $(P_j)$ has an unbounded sequence $u^{(j)}_i$, $i \in \mathbb{N}$, of critical points. This uses the oddness of the nonlinearity in an essential way. Assuming $\Omega = \Omega_1 + \Omega_2$, Sato and Tanaka constructed for $j$ large solutions $u_\lambda \in H^1(\mathbb{R}^N)$ of $(S_\lambda)$ such that $u_\lambda|_{\Omega_1}$ converges towards $u^{(1)}_j$, the mountain solution of $(P_1)$, and $u_\lambda|_{\Omega_2}$ converges towards $u^{(2)}_j$, some $j \geq 1$.

In this paper we allow that $-\Delta + a_0$ is indefinite. As a consequence, the least energy solution of $(P_j)$ may change sign and will not be of mountain pass type in general. It is obtained via a higher dimensional linking argument, or via a minimization on a certain submanifold of $H^1_0(\Omega_j)$ of higher codimension. Our method is quite different from those of [5] and [10]. It does not use the oddness of the nonlinearity and can therefore be extended to deal with more general nonlinearities $f(u)$ instead of $|u|^{p-2}u$; see Remark 1.2.

Let us fix our hypotheses on $a$ and $a_0$:

$(V_1)$ $a \in L^\infty_\text{loc}(\mathbb{R}^N)$, $a \geq 0$, $\Omega := \text{int} a^{-1}(0) \neq \emptyset$ is bounded with $\partial \Omega$ smooth, $\liminf_{|x| \to \infty} a(x) > 0$;

$(V_2)$ $a_0 \in L^\infty(\mathbb{R}^N)$;

$(V_3)$ there exist nonempty disjoint open sets $\Omega_1, \ldots, \Omega_m \subset \Omega$ such that $\Omega = \bigcup_{1 \leq j \leq m} \Omega_j$. For each $j = 1, \ldots, m$ there holds $\Omega_j \cap \Omega \setminus \Omega_j = \emptyset$ and $-\Delta + a_0$ is nondegenerate in $H^1_0(\Omega_j)$.

It is well known that under assumptions $(V_2)$ and $(V_3)$ problem $(P_j)$ has a solution obtained via a linking argument applied to the energy functional

$$I_j(u) = \frac{1}{2} \int_{\Omega_j} (|\nabla u|^2 + a_0 u^2) - \frac{1}{p} \int_{\Omega_j} |u|^p.$$ 

In fact, the solution can also be obtained by minimizing $I_j$ on the Nehari-Pankov manifold; see Section 2. It is a least energy solution, i.e. it lies on the level

$$c_j := \inf \{ I_j(u) : u \in H^1_0(\Omega_j), u \neq 0 \text{ solves } (P) \},$$

and may be considered as ground state solution (see [11]). If 0 is a local minimum of $I_j$ then this solution is positive and of mountain pass type; otherwise it changes sign and has higher Morse index.

**Theorem 1.1.** Fix a subset $J \subset \{1, 2, \ldots, m\}$ and set $\Omega_J := \bigcup_{j \in J} \Omega_j$. Then for any $\varepsilon > 0$, there exists $\Lambda(\varepsilon) > 0$ such that for any $\lambda \geq \Lambda(\varepsilon)$, $(S_\lambda)$ has a solution $u_\lambda$ satisfying:

(i) For $j \in J$ there holds

$$\left| \int_{\Omega_j} \left( \frac{1}{2} (|\nabla u_\lambda|^2 + a_0 u_\lambda^2) - \frac{1}{p} |u_\lambda|^p \right) dx - c_j \right| \leq \varepsilon.$$

(ii) $\int_{\mathbb{R}^N \setminus \Omega_J} (|\nabla u_\lambda|^2 + (\lambda a + a_0) u_\lambda^2) \leq \varepsilon$
(iii) Every sequence \( \lambda_n \to \infty \) has a subsequence \( (\lambda_{n_i}) \) such that \( u_{\lambda_n} \to \bar{u} \) as \( i \to \infty \). The restriction \( \bar{u}|_{\Omega_j} \) is a least energy solution of \((P_j)\) for \( j \in J \). Moreover, \( \bar{u}(x) = 0 \) for \( x \in \mathbb{R}^N \setminus \Omega_j \).

This is a generalization of the result from [5] who considered the case where \(-\Delta + a_0\) is positive definite, so that \( I_j \) has mountain pass structure. A new feature in the proof of our result is a combination of a global linking applied in each \( H_0^1(\Omega_j) \), \( j \in J \), and a local linking near \( 0 \in H_0^1(\Omega_j) \), \( j \notin J \). These are extended to \( H^1(\mathbb{R}^N) \) and “added”. We believe that this technique can be used in a variety of other singular limit problems.

**Remark 1.2.** The results continue to hold for \(-\Delta + (\lambda a(x) + a_0(x))u = f(u)\) provided the nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) is continuous and satisfies the following conditions:

1. \( f(u) = o(u) \) as \( u \to 0 \).
2. \( |f(u)| \leq \gamma(1 + |u|^{p-1}) \) for some \( \gamma > 0 \).
3. \( F(u)/u^2 \to \infty \) as \( |u| \to \infty \) where \( F(u) = \int_0^u f \).
4. \( f(x) \) The map \( u \mapsto f(u)/|u| \) is strictly increasing in \( \mathbb{R} \setminus \{0\} \).

Also the hypotheses on the potential can be weakened. In \((V_1)\) the assumption \( \liminf_{|x| \to \infty} a(x) > 0 \) can be replaced by the following one: There exists \( M > 0 \) such that the measure of the set \( \{x \in \mathbb{R}^N : a(x) \leq M\} \) is finite; see [4]. In \((V_2)\) it suffices to assume that \( a_0 \in L^\infty_{loc}(\mathbb{R}^N) \) and \( \text{ess inf} \ a_0 > -\infty \). In order to keep the presentation readable we refrained from treating the most general situation.

**Remark 1.3.** If the least energy solutions \( \bar{u}_j \) of \((P_j)\) are isolated then Theorem 1.1 follows from [1]. In fact, one can show that they have nontrivial critical groups, hence [1, Theorem 1.4] applies. If they have nontrivial degree then according to [1, Theorem 1.2] there exists a connected set \( S \subset \{(\lambda, u) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N) : (\lambda, u) \text{ solves } (S_\lambda)\} \) of solutions such that for any sequence \((\lambda_n, u_n) \in S \) with \( \lambda_n \to \infty \) there holds \( u_n \to \sum_{j \in J} \bar{u}_j \) as \( n \to \infty \). If they are even nondegenerate, then [1, Theorem 1.3] yields a smooth function \( \lambda \to u_\lambda \) satisfying \( u_\lambda \to \sum_{j \in J} \bar{u}_j \) as \( \lambda \to \infty \).

Our paper is organized as follows: In section 2 we recall the Nehari-Pankov manifold and study the properties of the least energy solutions. Since the standard functional associated to \((S_\lambda)\) does not satisfy the Palais-Smale condition under our hypotheses, in Section 3 we construct and investigate a penalized functional \( J_\lambda \). This does satisfy the (PS)-condition for \( \lambda \) large and its critical points in a certain energy range are solutions of \((S_\lambda)\). In Section 4, we study the behavior of the eigenvalues and eigenspaces of \(-\Delta + \lambda a + a_0\) when \( \lambda \to \infty \). Based on this we construct a new linking and define a possible critical value for \( J_\lambda \), \( \lambda > 0 \) large, in Section 5. This is based on an intersection lemma which we prove in Section 6. Sections 5 and 6 are the new key ingredients of our work. Finally, Section 7 contains the proof of Theorem 1.1.

We will use \( C \) to denote various generic positive constants which are independent of \( \lambda \) and \( n \), and we will write \( o(1) \) and \( o_n(1) \) to denote quantities that tend to 0 as \( \lambda \to \infty \), resp. \( n \to \infty \).

2. The Nehari-Pankov manifold. We consider an open subset \( \mathcal{O} \subset \mathbb{R}^N \) and a potential \( b \in L^\infty_{loc}(\mathcal{O}) \) which is bounded below. The functional

\[
J(u) = \frac{1}{2} \int_{\mathcal{O}} (|\nabla u|^2 + b(x)u^2) - \frac{1}{p} \int_{\mathcal{O}} |u|^p
\]
is defined for \( u \in H^1(\Omega) \) satisfying \( \int_\Omega |b|^2 < \infty \). We write \( E \) for either of the energy spaces \( \{ u \in H^1(\Omega) : \int_\Omega |b|^2 < \infty \} \) or \( \{ u \in H^1_0(\Omega) : \int_\Omega |b|^2 < \infty \} \). In this paper the operator \(-\Delta + b(x)\) has finite Morse index and is nondegenerate on \( E \). Then \( E\) splits as an orthogonal sum \( E = E^- \oplus E^+ \) of the negative and positive eigenspace of \(-\Delta + b(x)\), and \( \dim E^- < \infty \). Let \( P^- : E \to E^- \) denote the orthogonal projection.

The Nehari-Pankov manifold is defined as

\[
\mathcal{N} := \{ u \in E \setminus \{ 0 \} : P^- \nabla J(u) = 0, DJ(u)[u] = 0 \} \subset E \setminus E^-.
\]

It has been introduced by Pankov [8] in a situation where \( \dim E^- = \infty \), and coincides with the Nehari manifold if \( E^- = \{ 0 \} \). In order to formulate certain geometric properties of \( \mathcal{N} \) we need some notation. For \( (s,t) \in \mathbb{R}^d \times \mathbb{R} \) and \( 0 < r < \| w \| < R \) we need some notation. For \( \sum_{i=1}^d s_i e_i + ((1-t)r + tR)w \). Then we have

\[
A_{w,r,R} := \{ v + tw : v \in E^-, \| v \| < R, \; t \in (r,R) \} \subset H_w.
\]

Then we have

\[
\mathcal{N} = \{ w \in E \setminus E^- : \nabla (J|H_w) = 0 \}.
\]

**Proposition 2.1.**

a) For every \( w \in E^+ \setminus \{ 0 \} \) there exist \( t_w > 0 \) and \( \varphi(w) \in E^- \) such that \( H_w \cap \mathcal{N} = \{ \varphi(w) + t_w \cdot w \} \).

b) For every \( w \in \mathcal{N} \) and every \( u \in H_w \setminus \{ w \} \) there holds \( J(u) < J(w) \).

c) \( c_0 := \inf_{u \in \mathcal{N}} J(u) > 0 \)

d) For every \( w \in \mathcal{N} \) there holds \( \| P^+ w \| > \max\{ \| P^- w \|, \sqrt{2c_0} \} \).

e) For \( w \in \mathcal{N} \) and \( 0 < r < \| w \| < R \) the map

\[
f : H_w \to E^- \times \mathbb{R}, \quad f(u) := (P^- \nabla J(u), DJ(u)[u]),
\]

has degree \( \deg(f, A_{w,r,R}, 0) = 1 \). Here we identify \( H_w \subset E^- \oplus \mathbb{R}w \) and \( E^- \times \mathbb{R}^+ \subset E^- \times \mathbb{R} \).

**Proof.** The proof of a) – d) can be found in [11]. For the proof of e) observe that \( f \) is homotopic to \( \nabla (J|H_w) : H_w \to E^- \oplus \mathbb{R}w \cong E^- \times \mathbb{R} \). By a) and b) the constrained functional \( J|H_w \) has a unique critical point, namely \( w \), which is the global maximum. Since the local degree of a global maximum is \( +1 \) we deduce

\[
\deg(f, A_{w,r,R}, 0) = \deg(\nabla (J|H_w), A_{w,r,R}, 0) = 1.
\]

\[ \square \]

**Remark 2.2.** Set \( d := \dim E^- \) and let \( e_1, \ldots, e_d \) be an orthonormal basis of \( E^- \). We also need the sets \( A := \{(s,t) \in \mathbb{R}^d \times \mathbb{R} : |s| \leq 1, \; 0 \leq t \leq 1 \} \) and \( B := \partial A \subset \mathbb{R}^{d+1} \). Given \( w \in \mathcal{N} \) and \( 0 < r < \| w \| < R \) the map

\[
h_{w,r,R} : (A,B) \to (E,E \setminus \mathcal{N}), \quad h_{w,r,R}(s,t) := R \sum_{i=1}^d s_i e_i + ((1-t)r + tR)w.
\]

is well defined. It is not difficult to see that all maps \( h_{w,r,R} \) are homotopic. As a consequence of Proposition 2.1 we have

\[
c_0 = \inf_{u \in \mathcal{N}} J(u) = \inf_{w \in \mathcal{N}} \max_{0 < c < \| w \|} J(u) = \inf_{u \in A_{w,r,R}} \max_{\gamma \in \Gamma(s,t) \in A} J \circ \gamma(s,t)
\]

where

\[
\Gamma = \{ \gamma : (A,B) \to (E,E \setminus \mathcal{N}) | \gamma|_B \text{ is homotopic to some } h_{w,r,R} \}.
\]

The proof of the following result is standard.
Proposition 2.3. If $J$ satisfies the Palais-Smale condition at the level $c_0 = \inf_{u \in \mathcal{N}} J(u)$ then $c_0$ is achieved by a least energy solution $u_0 \in \mathcal{N}$.

3. The penalized functional. We first construct a variational functional whose critical points (in a certain energy range) will be solutions of $(S_{\lambda})$ and which satisfies the Palais-Smale condition. By assumption $(V_3)$ there exist smoothly bounded open sets $\Omega_1', \ldots, \Omega_m' \subset \mathbb{R}^N$ such that

$$\Omega_j \subset \Omega_j', \quad \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset \quad \text{for } i \neq j, \quad \text{and} \quad \overline{\Omega_j'} \cap \partial \Omega_j = \emptyset.$$ 

Using $(V_1) - (V_3)$, we may choose $\Lambda_0 > 0$ such that

$$\Lambda_0 a(x) + a_0(x) \geq 1 \quad \text{if} \quad x \notin \Omega' := \bigcup_{j=1}^m \Omega_j'.$$

(3.1)

Setting $V_{\lambda} := \lambda a + a_0$ we look for solutions lying in the energy space

$$E := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_{\lambda u} u^2 < \infty \right\} \subset H^1(\mathbb{R}^N).$$

(3.2)

As a consequence of (3.1) the norms

$$\| u \|_{\lambda} := \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V_{\lambda u} u^2) \right)^{1/2}$$

are equivalent for $\lambda \geq \Lambda_0$, and satisfy $\| \cdot \|_{\lambda} \leq \| \cdot \|_{\lambda'}$ for $\lambda \leq \lambda'$. Occasionally we write $E_{\lambda}$ for $(E, \| \cdot \|_{\lambda'})$.

(3.3)

with embedding constant $C > 1$ independent of $\lambda$. The functional

$$I_{\lambda} : E \to \mathbb{R}, \quad I_{\lambda}(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_{\lambda u} u^2) - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p,$$

is of class $C^2$, and critical points of $I_{\lambda}$ are solutions of $(S_{\lambda})$. $I_{\lambda}$ is the standard functional associated to $(S_{\lambda})$.

Since $I_{\lambda}$ does not need to satisfy the Palais-Smale condition we shall now modify it. We first define for $t \in \mathbb{R}$ and $\delta > 0:

$$f_{\delta}(t) := \begin{cases} t^{p-2}t & \text{if } |t| \leq \delta \\ \delta^{p-2}t & \text{if } |t| > \delta \end{cases}$$

and set $F_{\delta}(t) := \int_0^t f_{\delta}(s)ds$. Let $\chi : \mathbb{R}^N \to [0, 1]$ denote the characteristic function of $\Omega'$. We consider the penalized nonlinearity

$$g_{\delta}(x, t) := \chi(|x|) t^{p-2}t + (1 - \chi(|x|)) f_{\delta}(t).$$

Setting $G_{\delta}(x, t) := \int_0^t g_{\delta}(x, s)ds$ we can now define the functional

$$J_{\lambda} : E \to \mathbb{R}, \quad J_{\lambda}(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_{\lambda}(x) u^2) - \int_{\mathbb{R}^N} G_{\delta}(x, u),$$

The constant $\delta$ is suppressed in the notation because it will be fixed. We only require that $3C\delta^{p-2} < 1$ with $C$ from (3.3). This implies in particular that $G_{\delta}(x, t) \leq t^2/2$ for $x \in \mathbb{R}^N \setminus \Omega'$. It is standard to check that $J_{\lambda}$ is of class $C^1$ and that its nontrivial critical points are solutions of

$$-\Delta u + (\lambda a(x) + a_0(x))u = g_{\delta}(x, u) \quad \text{in } \mathbb{R}^N.$$
If moreover $u$ satisfies $|u(x)| < \delta$ for all $x \in \mathbb{R}^N \setminus \Omega'$, then $u$ solves the original problem $(S_{\lambda})$.

**Proposition 3.1.** $J_{\lambda}$ satisfies the Palais-Smale condition for $\lambda \geq \Lambda_0$. More precisely, any sequence $(u_n)$ in $E$ with
\[
J_{\lambda}(u_n) \leq c, \quad \nabla J_{\lambda}(u_n) \rightharpoonup 0 \text{ strongly in } E_{\lambda},
\]
contains a strongly convergent subsequence in $E$.

For the proof we need the following

**Lemma 3.2.** Suppose that a sequence $(u_n)$ in $E$ satisfies (3.4). Then there exists a constant $M(c)$ which is independent of $\lambda$ such that
\[
\limsup_{n \to \infty} \|u_n\|_{\lambda}^2 \leq M(c).
\]

**Proof.** Setting $\varepsilon_n := \|\nabla J_{\lambda}(u_n)\|$ it follows from (3.4) that
\[
\int_{\Omega'} \left( \frac{1}{2} - \frac{1}{p} \right) |u_n|^p + \int_{\mathbb{R}^N \setminus \Omega'} \left( \frac{1}{2} f_\delta(u_n)u_n - F_\delta(u_n) \right)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} g_\delta(x, u_n)u_n - \int_{\mathbb{R}^N} G_\delta(x, u_n)
\]
\[
= J_{\lambda}(u_n) - \frac{1}{2} J'_{\lambda}(u_n)u_n \leq c + \varepsilon_n \|u_n\|_{\lambda}.
\]

Observe that for $|t| \in (\delta, \infty)$,
\[
\frac{1}{2} f_\delta(t) - F_\delta(t) = \frac{1}{2} \delta^{p-2}t^2 - \frac{1}{2} \delta^p - \frac{2}{2p} \delta^p + \frac{p}{2p} \delta^p \geq 0,
\]
and for $|t| \leq \delta$,
\[
\frac{1}{2} f(t) - F(t) = \left( \frac{1}{2} - \frac{1}{p} \right) |t|^p.
\]

Combining (3.6)-(3.8) we obtain
\[
\left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega'} |u_n|^p \leq c + o(1) + \varepsilon_n \|u_n\|_{\lambda}.
\]

Since $V_{\lambda}$ is non-increasing with respect to $\lambda$ and $\text{supp} \ V_{\lambda} \subset \Omega'$ for $\lambda \geq \Lambda_0$ we deduce for $\lambda \geq \Lambda_0$:
\[
\int_{\mathbb{R}^N} V_{\lambda}^- u_n^2 = \int_{\Omega'} V_{\lambda}^- u_n^2 \leq \int_{\Omega'} V_{\lambda}^- u_n^2 \leq C + \int_{\Omega'} |u_n|^p
\]
\[
\leq C \left( 1 + c + (\varepsilon_n) \|u_n\|_{\lambda} \right),
\]
where $C$ is a positive constant which is independent of $\lambda$ and $n$.

Using (3.4) once more, we obtain
\[
\left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_{\lambda}^+ u_n^2) - \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} V_{\lambda}^- u_n^2
\]
\[
+ \frac{1}{p} \int_{\mathbb{R}^N} g_\delta(x, u_n)u_n - \int_{\mathbb{R}^N} G(x, u_n) \leq J_{\lambda}(u_n) - \frac{1}{p} J'_{\lambda}(u_n)u_n \leq c + \varepsilon_n \|u_n\|_{\lambda}.
\]
A similar argument yields
\[
\frac{1}{p} \int_{\mathbb{R}^N} g_\delta(x, u_n) u_n - \int_{\mathbb{R}^N} G_\delta(x, u_n) \geq - \left( \frac{1}{2} - \frac{1}{p} \right) \delta^{p-2} \int_{\mathbb{R}^N \setminus \Omega'} u_n^2 \tag{3.11}
\]
Combining (3.10) and (3.11) gives
\[
(1 - \frac{1}{p}) \|u_n\|^2 \leq \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} \|\nabla u_n\|^2 + V_\lambda^+ u_n^2 \leq C \left( 1 + c + \varepsilon_n \|u_n\|_\lambda \right).
\]
Since \(\delta^{p-2} < 1\) it easily follows that there exists \(M(c)\) which is independent of \(\lambda \geq \Lambda_0\) such that (3.5) holds. This completes the proof of Lemma 3.2. \(\square\)

Now we can give the

Proof of Proposition 3.1. From Lemma 3.2, we know that \((u_n)\) is bounded in \(E_\lambda\), so after passing to a subsequence there holds

- \(u_n \rightharpoonup u\) weakly in \(E_\lambda\),
- \(u_n \rightharpoonup u\) strongly in \(L^q_{\text{loc}}(\mathbb{R}^N)\) for \(2 \leq q < 2^*\),
- \(u_n \to u\) a.e in \(\mathbb{R}^N\).

Now we prove that \(u_n \to u\) in \(E_\lambda\). First of all, it is easy to check that \(u\) is a critical point of \(J_\lambda(u)\), that is,

\[
\int_{\mathbb{R}^N} (\nabla u \nabla \psi + V_\lambda(x) u \psi) = \int_{\mathbb{R}^N} g_\delta(x, u) \psi \quad \text{for every } \psi \in E_\lambda.
\]

It follows from (3.4) that
\[
o_n(1) = (J_\lambda(u_n) - J_\lambda(u))(u_n - u)
\]
\[
= \int_{\mathbb{R}^N} (|\nabla (u_n - u)|^2 + V_\lambda(x)|u_n - u|^2) - \int_{\mathbb{R}^N} g_\delta(x, u_n)(u_n - u)
\]
\[
+ \int_{\mathbb{R}^N} g_\delta(x, u)(u_n - u)
\]
\[
= \|u_n - u\|_\lambda^2 - \int_{\Omega'} V_\lambda^-(x)|u_n - u|^2 - \int_{\Omega'} |u_n|^{p-2} u_n u_n - u
\]
\[
- \int_{\mathbb{R}^N \setminus \Omega'} f_\delta(u_n)(u_n - u) + \int_{\Omega'} |u|^{p-2} u_n(u_n - u) + \int_{\mathbb{R}^N \setminus \Omega'} f_\delta(u)(u_n - u)
\]
By the definition of \(f_\delta(t)\) we have
\[
\left| \int_{\mathbb{R}^N \setminus \Omega'} f_\delta(u_n)(u_n - u) \right|
\]
\[
\leq \left| \int_{\mathbb{R}^N \setminus \Omega'} (f_\delta(u_n) - \delta^{p-2} u_n)(u_n - u) \right| + \delta^{p-2} \left| \int_{\mathbb{R}^N \setminus \Omega'} u_n(u_n - u) \right|
\]
\[
\leq 3 \delta^{p-2} \|u_n - u\|_{L^2}^{p} + \delta^{p-2} \left| \int_{\mathbb{R}^N \setminus \Omega'} u(u_n - u) \right|
\]
Now \(u_n \to u\) in \(E_\lambda\) implies
\[
\int_{\mathbb{R}^N \setminus \Omega'} u(u_n - u) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N \setminus \Omega'} f_\delta(u)(u_n - u) \to 0.
\]
Finally, since \( u_n \to u \) strongly in \( L^p(\Omega') \), and since \( \| \cdot \|_{L^2} \leq C \| \cdot \|^2_{L^\infty} \), we deduce:

\[
(1 - 3C\delta^{p-2})\|u_n - u\|_{L^2}^2 \leq \|u_n - u\|_{L^2}^2 - 3\delta^{p-2}\|u_n - u\|_{L^2}^2 \leq 0.
\]

Thus \( (u_n) \to u \) in \( L^2(\Omega) \).

as \( n \to \infty \). Therefore \( u_n \to u \) in \( E_\lambda \) because \( 3C\delta^{p-2} < 1 \).

\[\square\]

**Proposition 3.3.** Suppose the sequences \( \lambda_n \to \infty \) and \( (u_n) \) in \( E \) satisfy

\[ J_{\lambda_n}(u_n) \leq c, \quad \|\nabla J_{\lambda_n}(u_n)\|_{L^\infty} \to 0. \tag{3.12} \]

Then, after passing to a subsequence, we have:

a) \( u_n \to u \) weakly in \( E \) for some \( u \in E \).

b) \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \), and \( u|_{\Omega_j} \) solves

\[
-\Delta u + a_0 u = |u|^{p-2} u \quad \text{in } \Omega_j
\]

for \( j = 1, \ldots, m \).

c) \( \|u_n - u\|_{\lambda_n} \to 0 \), consequently \( u_n \to u \) in \( H^1(\mathbb{R}^N) \).

\( (u_n) \) also satisfies for \( n \to \infty \):

\[
(i) \int_{\mathbb{R}^N} \lambda_n a(x)u_n^2 \to 0
\]

\[
(ii) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_{\lambda_n} u_n^2) \to 0
\]

\[
(iii) \int_{\Omega_j} (|\nabla u_n|^2 + V_{\lambda_n} u_n^2) \to \int_{\Omega_j} (|\nabla u|^2 + a_0(x)u^2) \quad \text{for } j = 1, \ldots, m.
\]

**Proof.** As in the proof of Lemma 3.2, one shows that \( \limsup_{n \to \infty} \|u_n\|_{L^\infty}^2 \leq M(c) \). Thus \( (u_n) \) stays bounded as \( n \to \infty \) in \( E \), so we may assume that for some \( u \in E \):

\[
u_n \to u \text{ weakly in } E,
u_n \to u \text{ a.e. in } \mathbb{R}^N,
u_n \to u \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^N) \]

for \( 2 \leq q < 2^* \).

Now we prove b). Setting \( C_k := \{ x \in \mathbb{R}^N : a(x) \geq \frac{1}{k} \} \), we have for \( n \) large:

\[
\int_{C_k} u_n^2 \leq \frac{k}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n a(x)u_n^2 = \frac{k}{\lambda_n} \int_{\mathbb{R}^N} (\lambda_n a(x) + a_0(x))u_n^2 - \frac{k}{\lambda_n} \int_{\mathbb{R}^N} a_0(x)u_n^2 \leq \frac{k}{\lambda_n} \|u_n\|_{L^\infty}^2 + \frac{k}{\lambda_n} \|a_0\|_{L^\infty} \|u_n\|_{L^2}^2 \to 0.
\]

It follows that \( u(x) = 0 \) in \( \bigcup_{k=1}^{\infty} C_k = \mathbb{R}^N \setminus \Omega \).

Next we have for any test function \( \varphi \in C_0^\infty(\Omega_j), j = 1, 2, \ldots, m \):

\[
|J'_{\lambda_n}(u_n)\varphi| \leq \|\nabla J_{\lambda_n}(u_n)\|_{L^\infty} \|\varphi\|_{\lambda_n} \to 0.
\]

Here we use the fact that \( \|\varphi\|_{\lambda_n} \) does not depend on \( \lambda_n \). It follows that

\[
\int_{\Omega_j} (\nabla u \nabla \varphi + a_0 u \varphi) = \int_{\Omega_j} g(x, u) \varphi.
\]

This implies b).
In order to prove c) we observe that
\[ J_\lambda'(u_n)(u_n - u) - J_\lambda'(u)(u_n - u) \]
\[ = \|u_n - u\|_{\lambda_n}^2 - \int_{\mathbb{R}^N \setminus \Omega'} f_\delta(u_n)(u_n - u) + \int_{\mathbb{R}^N \setminus \Omega'} f(u)(u_n - u) \]
\[ = - \int_{\Omega'} V_\lambda^-(u_n - u)^2 - \int_{\Omega'} |u_n|^{p-2} u_n (u_n - u) + \int_{\Omega'} |u|^{p-2} u (u_n - u). \]
Here we have used the fact that supp $V_\lambda^- \subset \Omega'$ for $n$ large. Since $u_n \to u$ in $L^p(\Omega')$, we have
\[ \int_{\Omega'} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) \to 0 \quad \text{and} \quad \int_{\Omega'} V_\lambda^-(u_n - u)^2 \to 0 \quad \text{as} \quad n \to \infty. \]
On the other hand
\[ |J_\lambda'(u_n)(u_n - u)| \leq \|\nabla J_\lambda(u_n)\|_{\lambda_n} \|u_n - u\|_{\lambda_n} \]
\[ \leq \|\nabla J_\lambda(u_n)\|_{\lambda_n} (\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) \to 0. \]
This implies
\[ \|u_n - u\|_{\lambda_n}^2 - \int_{\mathbb{R}^N \setminus \Omega'} (f_\delta(u_n) - f_\delta(u))(u_n - u) \to 0. \]
We obtain \( (1 - 3C\delta^{p-2})\|u_n - u\|_{\lambda_n}^2 \to 0 \) as in the proof of Proposition 3.1, hence c) holds.

It remains to prove d). Using c) we see that
\[ \frac{1}{2} \int_{\mathbb{R}^N} \lambda_n a(x) u_n^2 = \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega} \lambda_n a(x) u_n^2 = \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega} \lambda_n a(x) |u_n - u|^2 \]
\[ \leq \|u_n - u\|_{\lambda_n}^2 \to 0 \]
which proves (i); (ii) and (iii) also follow immediately from c).

Proposition 3.4. Given $c > 0$ there exists $\Lambda_c > \Lambda_0$ such that for $\lambda \geq \Lambda_c$ a critical point $u_\lambda$ of $J_\lambda$ with $|J_\lambda(u_\lambda)| \leq c$ satisfies $|u_\lambda| \leq \delta$ for $x \in \mathbb{R}^N \setminus \Omega'$.

Proof. Since $u_\lambda \in E_\lambda$ is a critical point of $J_\lambda(u)$ it satisfies the equation
\[ -\Delta u_\lambda + (\lambda \alpha(x) + a_0(x)) u_\lambda = g_\delta(x, u_\lambda), \quad \text{in} \quad \mathbb{R}^N. \]
Using that $u_\lambda$ is bounded in $E$ independent of $\lambda$, an argument as in the proof of [4, Lemma 5.1] shows that $\|u_\lambda\|_{L^\infty}$ is bounded independent of $\lambda$. On the other hand, by the definition of $g_\delta$, we know that $A_\delta(x) := g_\delta(x, u_\lambda(x))/u_\lambda(x)$ is bounded in $L^\infty(\mathbb{R}^N)$. Moreover, $(V_1)$ implies that the negative part of $W_\lambda := \lambda \alpha + a_0 - A_\delta$ is bounded uniformly in $\lambda$. It follows from [9, A.2.1] that the norm of $W_\lambda$ in the Kato class $K_N$ is bounded uniformly in $\lambda$. Thus by the subsolution estimate [9, Theorem C.1.2] there exists a constant $C$ which is independent of $\lambda$ such that
\[ |u_\lambda(x)| \leq C(r) \int_{B_r(x)} |u_\lambda|; \quad (3.13) \]
here $B_r(x) = \{ y \in \mathbb{R}^N : |x - y| < r \}$. Proposition 3.3 implies that for any sequence $\lambda_n \to \infty$, after passing to a subsequence there holds $u_{\lambda_n} \to u_0 \in H_0^1(\Omega)$ strongly in $E$, and therefore $u_{\lambda_n} \to 0$ strongly in $L^2(\mathbb{R}^N \setminus \overline{\Omega})$. Since $\lambda_n \to \infty$ was arbitrary, we have $u_\lambda \to 0$ strongly in $L^2(\mathbb{R}^N \setminus \overline{\Omega})$ as $\lambda \to \infty$. 

\[ \square \]
Thus, choosing \( r = \frac{1}{2} \text{dist}(\Omega, \mathbb{R}^N \setminus \Omega') \), we have uniformly in \( x \in \mathbb{R}^N \setminus \Omega' \) that
\[
|u_\lambda(x)| \leq C(r) \int_{B_r(x)} |u_\lambda(x)| \leq C(r)(\text{meas } B_r(x))^{1/2} \|u_\lambda\|_{L^2(B_r(x))}^{1/2} \\
\leq C(r)(\text{meas } B_r(x))^{1/2} \|u_\lambda\|_{L^2(\mathbb{R}^N \setminus \Omega)}^{1/2} \to 0.
\]
This completes the proof. \( \square \)

4. Behavior of eigenvalues and eigenspaces. Recall the smoothly bounded open neighborhoods \( \Omega'_j \) of \( \Omega_j \) from the definition of the penalized functional in Section 3, and denote \( X_j := H^1(\Omega'_j) \). Let \( \mu_{j,1}^\lambda < \mu_{j,2}^\lambda < \mu_{j,3}^\lambda < \ldots \) be the distinct eigenvalues of \( L_\lambda \) in \( X_j \) and let \( V_{j,n}, \ n \in \mathbb{N} \), be the corresponding eigenspaces. Similarly, let \( \mu_{j,1} < \mu_{j,2} < \mu_{j,3} < \ldots \) denote the distinct eigenvalues of \( L_0 = -\Delta + a_0 \) in \( E_j = H^1_0(\Omega_j) \) with eigenspaces \( V_{j,n} \). Then we have:

Lemma 4.1. \( \mu_{j,n}^\lambda \to \mu_{j,n} \) and \( V_{j,n}^\lambda \to V_{j,n} \) as \( \lambda \to \infty \).

Here \( V_{j,n}^\lambda \to V_{j,n} \) means that, given any sequence \( \lambda_i \to \infty \) and normalized eigenfunctions \( \psi_i \in V_{j,n}^\lambda \), there exists a normalized eigenfunction \( \psi \in V_{j,n} \) such that \( \psi_i \to \psi \) strongly in \( X_j \) along a subsequence.

Corollary 4.2. For \( \lambda \) large the operator \( -\Delta + \lambda a + a_0 \) on \( X_j = H^1(\Omega'_j) \) is nondegenerate and has finite Morse index \( d_j := \dim E_j^- \) uniformly in \( \lambda \).

Proof of Lemma 4.1. Since \( j \in \{1, \ldots, m\} \) is fixed, to simplify notation we denote \( \mu_{j,n}^\lambda \) by \( \mu_{n}^\lambda \), \( \mu_{j,n} \) by \( \mu_{n} \), \( V_{j,n}^\lambda \) by \( V_{n}^\lambda \), and \( V_{j,n} \) by \( V_{n} \). For \( n = 1 \) the result has been proved by Ding and Tanaka [5, Lemma 1.2]). Now suppose \( n \geq 2 \) and the result holds up to \( n - 1 \). Set
\[
d := \dim V_1 + \cdots + \dim V_{n-1} = \dim V_1^\lambda + \cdots + \dim V_{n-1}^\lambda.
\]
By the minmax description of the eigenvalues, see Reed and Simon [9, XIII.1], for instance, there holds:
\[
\mu_{n}^\lambda = \inf \left\{ (L_\lambda \psi, \psi) : \psi \in H^1(\Omega'_j), \|\psi\|_{L^2(\Omega'_j)} = 1, \psi \perp V_m^\lambda = 0 \text{ for } m = 1, \ldots, n-1 \right\} \\
= \max_{\phi_1, \ldots, \phi_d \in H^1(\Omega'_j)} \inf \left\{ (L_\lambda \psi, \psi) : \psi \in H^1(\Omega'_j), \|\psi\|_{L^2(\Omega'_j)} = 1, \psi \perp \phi_i = 0 \text{ for } i = 1, \ldots, d \right\}, \tag{4.1}
\]
and
\[
\mu_{n} = \inf \left\{ (L_0 \psi, \psi) : \psi \in H^1_0(\Omega_j), \|\psi\|_{L^2(\Omega_j)} = 1, \psi \perp V_m = 0 \text{ for } m = 1, \ldots, n-1 \right\} \\
= \max_{\phi_1, \ldots, \phi_{d-1} \in H^1_0(\Omega_j)} \inf \left\{ (L_0 \psi, \psi) : \psi \in H^1_0(\Omega_j), \|\psi\|_{L^2(\Omega_j)} = 1, \psi \perp \phi_i = 0 \text{ for } i = 1, \ldots, d - 1 \right\}. \tag{4.2}
\]
Since \( V_{n}^\lambda \to V_{n} \) for \( 1 \leq m \leq n - 1 \) as \( \lambda \to \infty \), and since \( (L_\lambda \psi, \psi) = (L_0 \psi, \psi) \), for every \( \psi \in H^1_0(\Omega_j) \), (4.1) and (4.2) imply:
\[
\limsup_{\lambda \to \infty} \mu_{n}^\lambda \leq \mu_{n}. \tag{4.3}
\]
In order to prove equality consider a sequence \( \lambda_i \to \infty \) and normalized eigenfunctions \( \psi_i \) corresponding to \( \mu_n^{\lambda_i} \). Then we have:

\[
\int_{\Omega_j'} \psi_i^2 = 1, \quad \int_{\Omega_j'} (|\nabla \psi_i|^2 + (\lambda_i a(x) + a_0(x))\psi_i^2) = \mu_n^{\lambda_i},
\]

and

\[
\psi_i \perp V_m^{\lambda_i} \quad \text{for } m = 1, 2, \ldots, n - 1.
\]

By (4.3), \( \psi_i \) is bounded in \( H^1(\Omega_j') \), so we may assume that \( \psi_i \to \psi \in H^1(\Omega_j') \) and \( \psi_i \to \psi \) in \( L^2(\Omega_j') \). It is easy to see that \( \psi = 0 \) in \( \Omega_j' \setminus \Omega_j \), because \( a(x) > 0 \) in \( \Omega_j' \setminus \Omega_j \). Since \( \partial \Omega_j \) is smooth it follows that \( \psi \in H^1_0(\Omega_j') \). Strong convergence in \( L^2(\Omega_j') \) implies \( \int_{\Omega_j} \psi^2 = \int_{\Omega_j'} \psi^2 = 1 \). Since by our induction assumption, \( V_m^{\lambda_i} \to V_m, \quad m = 1, \ldots, n - 1 \), we obtain

\[
\psi \perp V_m, \quad m = 1, \ldots, n - 1. \tag{4.4}
\]

By the minmax description of the \( n \)-th-eigenvalue there holds:

\[
\mu_n \leq \int_{\Omega_j} (|\nabla \psi|^2 + a_0(x)\psi^2) \leq \liminf_{i \to \infty} \int_{\Omega_j} (|\nabla \psi_i|^2 + (\lambda_i a(x) + a_0(x))\psi_i^2) = \liminf_{i \to \infty} \mu_n^{\lambda_i} \leq \mu_n. \tag{4.5}
\]

This and (4.3) show that \( \mu_n^{\lambda} \to \mu_n \) as \( \lambda \to \infty \). It also follows from (4.5) that \( \psi_i \to \psi \in V_n \) strongly in \( X_j \), hence \( V_n^{\lambda} \to V_n \).

5. **Definition of the critical value.** For \( j = 1, \ldots, m \), we set \( E_j := H^1_0(\Omega_j) \subset E \), where \( E \) is defined in (3.2), and consider the functional

\[
I_j : E_j \to \mathbb{R}, \quad I_j(u) = \frac{1}{2} \int_{\Omega_j} (|\nabla u|^2 + a_0 u^2) - \frac{1}{p} \int_{\Omega_j} |u|^p.
\]

By assumption \((V_3)\), \( E_j \) splits as the orthogonal sum \( E_j = E_j^- \oplus E_j^+ \) of the negative and positive eigenspace of \( -\Delta + a_0 \). As in Section 2 let \( P_j^- : E_j \to E_j^- \) denote the orthogonal projection. Since \( \Omega_j \) is bounded, \( p < 2N/(N - 2) \) if \( N > 2 \), \( I_j \) satisfies the Palais-Smale condition, hence the infimum of \( I_j \) on the Nehari-Pankov manifold

\[
\mathcal{N}_j = \{ u \in E_j \setminus \{0\} : P_j^- (\nabla I_j(u)) = 0, \; DI_j(u)[u] = 0 \}
\]

is achieved by some \( w_j \in \mathcal{N}_j \),

\[
c_j := \inf_{u \in \mathcal{N}_j} I_j(u) = I_j(w_j) > 0. \tag{5.1}
\]

We fix a subset \( J \subset \{1, 2, \ldots, m\} \), set \( d_j := \dim E_j^- \), and let \( c_{j,i} , \quad i = 1, \ldots, d_j \), be an orthonormal basis of \( E_j^- , \quad j = 1, \ldots, m \). We also need the sets

\[
A := \{(s_1, \ldots, s_m, t) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \times \mathbb{R}^J : \|s_i\|_\infty \leq 1, \quad i = 1, \ldots, m,
\]

\[
0 \leq t_j \leq 1, \quad j \in J\}
\]

and \( B := \partial A \). For \( R > \max_{j \in J} \|w_j\| \) large and \( 0 < r < \min_{j \in J} \|w_j\| \) small, to be determined below, we define the map \( \gamma_0 : A \to E \) by

\[
\gamma_0(s, t) := \sum_{j \in J} \left( R \sum_{i=1}^{d_j} s_{ji} e_{ji} + ((1 - t_j)r + t_j R)w_j \right) + \sum_{j \notin J} \left( r \sum_{i=1}^{d_j} s_{ji} e_{ji} \right).
\]
Observe that \( I_j(u) \leq 0 \) for \( u \in E_j^- \), and therefore
\[
\sum_{j \notin J} I_j \left( r \sum_{i=1}^{d_j} s_{ji} e_{ji} \right) \leq 0 \quad \text{for all } s_{ji}.
\]
Hence if some \( s_{ji} \neq 0 \) or some \( t_j \neq 0 \) then
\[
J_\lambda(\gamma_0(s,t)) = \sum_{j \in J} I_j \left( R \sum_{i=1}^{d_j} s_{ji} e_{ji} + ((1 - t_j)r + t_j R) w_j \right) + \sum_{j \notin J} I_j \left( r \sum_{i=1}^{d_j} s_{ji} e_{ji} \right)
\]
\[
\rightarrow -\infty
\]
as \( R \rightarrow \infty \). Also, if \( t_j = 0 \) for \( j \in J \) and \( r = 0 \) then \( J_\lambda(\gamma_0(s,t)) \leq 0 \). It follows that for \( R > 0 \) large and \( r > 0 \) small there holds
\[
J_\lambda(\gamma_0(s,t)) < \sum_{j \in J} c_j \quad \text{for all } (s, t) \in B, \lambda \geq 0.
\]
(5.2)

If \( r \) is small enough there exists \( \alpha > 0 \) such that
\[
I_j(u_j) \geq \alpha \| u_j \|_{E_j}^2, \quad u_j \in E_j^+, \quad \| u_j \|_{E_j} \leq r.
\]
(5.3)

We fix \( r, R \) satisfying (5.2) and (5.3). Now we define the sets
\[
\mathcal{H}_\lambda := \{ h : A \times [0, 1] \rightarrow E : h \in C^0, \ h(s, t, 0) = \gamma_0(s,t), \ J_\lambda(h(s, t, \tau)) \text{ is nonincreasing with respect to } \tau \}
\]
and
\[
\Gamma_\lambda := \{ \gamma : A \rightarrow E | \exists h \in \mathcal{H}_\lambda \forall (s, t) \in A : \gamma(s, t) = h(s, t, 1) \}.
\]
Finally we arrive at a minmax description of a possible critical value:
\[
c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{(s, t) \in A} J_\lambda(\gamma(s,t)).
\]
(5.4)

**Lemma 5.1.** \( c_\lambda \leq \sum_{j \in J} c_j \)

**Proof.** This follows from \( \gamma_0 \in \Gamma_\lambda \), the choice of the \( w_j \), and Proposition 2.1. \( \square \)

In order to obtain a lower bound for \( c_\lambda \) we need the smoothly bounded open neighborhoods \( \Omega'_j \) of \( \Omega_j \) from the definition of the penalized functional in Section 3. We consider the functional \( I_j^\lambda : X_j = H^1(\Omega'_j) \rightarrow \mathbb{R} \) defined by
\[
I_j^\lambda(u) := \frac{1}{2} \int_{\Omega'_j} (|\nabla u|^2 + (\lambda a + a_0) u^2) - \frac{1}{p} \int_{\Omega'_j} |u|^p,
\]
and its associated Nehari-Pankov manifold
\[
\mathcal{N}_j^\lambda := \{ u \in X_j \setminus \{0\} : Q_j^{\lambda-}(\nabla I_j^\lambda(u)) = 0, DI_j^\lambda(u)[u] = 0 \}.
\]
Here \( Q_j^{\lambda-} : X_j \rightarrow X_j^{\lambda-} \) is the orthogonal projection on the negative eigenspace associated to \( L_\lambda := -\Delta + \lambda a + a_0 \) in \( X_j \). As a consequence of Corollary 4.2 the results from Section 2 apply and the infimum
\[
c_j^\lambda := \inf_{u \in \mathcal{N}_j^\lambda} I_j^\lambda(u) > 0
\]
is achieved. We have the following asymptotic behavior for \( c_j^\lambda \) as \( \lambda \rightarrow \infty \).

**Lemma 5.2.** \( c_j^\lambda \rightarrow c_j \) as \( \lambda \rightarrow \infty \).
Proof. Clearly $N_j \subset N_j^\lambda$ because
\[
Q_j^\lambda - (\nabla I_j^\lambda(u_j)) = P_j^\lambda - (\nabla I_j(u_j)) \quad \text{and} \quad DI_j^\lambda(u_j)[u_j] = DI_j(u_j)[u_j]
\]
for every $u \in H_0^1(\Omega_j)$. It follows that
\[
c_j^\lambda \leq c_j. \tag{5.5}
\]
On the other hand, it is easy to see that $c_j^\lambda$ is nondecreasing with respect to $\lambda$. Thus (5.5) implies that the limit $\lim_{\lambda \to \infty} c_j^\lambda$ exists and
\[
\lim_{\lambda \to \infty} c_j^\lambda \leq c_j. \tag{5.6}
\]
Now we prove the inverse of (5.6). Indeed, since $I_j^\lambda$ satisfies the Palais-Smale condition, $c_j^\lambda$ is achieved by a critical point $w^\lambda$ of $I_j^\lambda$. Given a sequence $\lambda_i \to \infty$, we deduce from (5.6) that $w^{\lambda_i}$ is uniformly bounded in $H^1(\Omega_i)$, so we may assume $w^{\lambda_i} \to w$ in $H^1(\Omega_i)$. As in the proof of Proposition 3.3 one sees that $w^{\lambda_i} \to w$ strongly in $H^1(\Omega_i)$, $w \in H_0^1(\Omega_j)$, and $c_j^\lambda = I_j^\lambda(w^{\lambda_i}) \to I_j(w)$; in particular $w \neq 0$. Moreover,
\[
DI_{\lambda_i}(w^{\lambda_i})[w^{\lambda_i}] \to DI_j(w)[w]
\]
and
\[
Q_j^\lambda \nabla I_j^\lambda(w^{\lambda_i}) \to P_j \nabla I_j(w);
\]
here we also used Lemma 4.1. Thus $w \in N_j$ and
\[
c_j \leq I_j(w) = \lim_{\lambda \to \infty} c_j^\lambda. \tag{5.7}
\]
The lemma follows from (5.6) and (5.7).

Let $\Omega_0 := \bigcup_{j \notin J} \Omega_j$ and $\Omega'_0 := \bigcup_{j \notin J} \Omega'_j$. We denote $X_0 := H^1(\Omega_0') = \bigoplus_{j \notin J} X_j$ and $E_0 := H_0^1(\Omega_0) = \bigoplus_{j \notin J} E_j$. Let $X_0^\lambda^-$ be the negative eigenspace associated to $-\Delta + \lambda a + a_0$ in $X_0$, and let $E_0^-$ be the negative eigenspace associated to $-\Delta + a_0$ in $E_0$. Clearly $X_0^{\lambda^-} = \bigoplus_{j \notin J} X_j^{\lambda^-}$ and $E_0^- = \bigoplus_{j \notin J} E_j^-$. Finally, let $Q_0^{\lambda^-} : X_0 \to X_0^{\lambda^-}$ and $P_0^- : E_0 \to E_0^-$ be the orthogonal projections.

The following linking property for $\gamma \in \Gamma_\lambda$ is the key to the proof of the lower bound of $c_\lambda$. It will be proved in the next section.

Lemma 5.3. If $\lambda$ is sufficiently large, then for any $\gamma \in \Gamma_\lambda$, there exists $(s, t) \in A$ such that $u := \gamma(s, t)$ satisfies
\[
u_j := u|_{\Omega'_j} \in N_j^\lambda \quad \text{for} \quad j \in J,
\]
and
\[
u_0 \perp X_0^{\lambda^-}, \quad \|u_0\| < r. \tag{5.9}
\]

Lemma 5.4. $c_\lambda \geq \sum_{j \in J} c_j^\lambda$.

Proof. Lemma 5.3 yields that, given $\gamma \in \Gamma_\lambda$ there exists $(s, t) \in A$ such that $u := \gamma(s, t)$ satisfies (5.8) and (5.9). Using (5.3) this implies $I_0^\lambda(u_0) \geq 0$, hence
\[
\max_A J_\lambda \circ \gamma \geq J_\lambda(u) \geq \sum_{j \in J} I_j^\lambda(u_j) \geq \sum_{j \in J} c_j^\lambda.
\]

As a consequence of the lemmas 5.1, 5.4 and 5.2, we deduce:
Corollary 5.5. There holds \( \lim_{\lambda \to \infty} c_\lambda = \sum_{j \in J} c_j \) and for \( \lambda \) large, \( c_\lambda \) is achieved by a critical point \( u_\lambda \) of \( J_\lambda \).

Proof. In fact, for \( \lambda \) large enough (5.2) implies
\[ c_\lambda > \max_{(s,t) \in B} J_\lambda(\gamma_0(s,t)). \]

A standard argument now yields that \( c_\lambda \) is achieved by a critical point \( u_\lambda \) of \( J_\lambda \) provided \( \lambda \geq \Lambda_0 \) as in Proposition 3.1. As a consequence of Proposition 3.4, \( u_\lambda \) is a solution of \( (S_\lambda) \) for \( \lambda \) large. \( \square \)

6. Proof of Lemma 5.3. For \( u \in E \) we write \( u_j := u|_{\Omega_j}, j \in J_0 := J \cup \{0\} \). We need the map
\[ f_\lambda : E \to X_0^\lambda - \prod_{j \in J} (X_j^\lambda - \mathbb{R}) \]
defined by
\[ f_{\lambda,0} := Q_0^\lambda : E \to X_0^\lambda - \]
and for \( j \in J \):
\[ f_{\lambda,j} : E \to X_j^\lambda - \mathbb{R}, \quad f_{\lambda,j}(u) := (Q_j^\lambda(\nabla I_j^\lambda(u_j)), DI_j^\lambda(u_j)[u_j]). \]

Clearly we have:
\[ f_\lambda(u) = 0 \iff u_0 \perp X_0^\lambda - \text{ and } u_j \in \mathcal{N}_j^\lambda \text{ for } j \in J \quad (6.1) \]

Consider \( \gamma \in \Gamma_\lambda \) and let \( h \in \mathcal{H}_\lambda \) be a homotopy from \( \gamma_0 \) to \( \gamma \). We have to show that for \( \lambda \) large there exists \( (s,t) \in A \) such that \( u = \gamma(s,t) \) satisfies \( f_\lambda(u) = 0 \) and \( \|u_0\| < r \). This will be done with a degree argument.

First we claim that for \( (s,t,\tau) \in A \times [0,1], u := h(s,t,\tau) \), and \( \lambda \) large the following holds:
\[ f_\lambda(u) = 0 \implies \|u_0\|_{X_0} \neq r. \quad (6.2) \]

In order to see this we observe that Lemma 4.1 and (5.3) imply the existence of \( \beta > 0 \) such that
\[ I_0^\lambda(v) \geq \beta \text{ for all } v \in X_0^+, \|v\|_{X_0} = r, \]
and
\[ I_0^\lambda(v) \geq 0 \text{ for all } v \in X_0^+, \|v\|_{X_0} \leq r, \]
hold for \( \lambda \) large. Moreover, Lemma 5.2 shows that
\[ \sum_{j \in J} c_j < \sum_{j \in J} c_j^\lambda + \beta \]
for \( \lambda \) large. Now suppose that
\[ \|u_0\|_{X_0} = r. \quad (6.3) \]
Our choice of $\delta$ implies for $v \in E$ and $\lambda \geq \Lambda_0$ that
\begin{align*}
J_\lambda(v) &= \frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega'} \left( |\nabla v|^2 + (\lambda a + a_0)v^2 \right) - \int_{\mathbb{R}^N \setminus \Omega'} G_\delta(x, v) \\
&\quad + \sum_{j \in J_0} \left( \frac{1}{2} \int_{\Omega_j'} \left( |\nabla v|^2 + (\lambda a + a_0)v^2 \right) - \int_{\Omega_j'} G_\delta(x, v) \right) \\
&\geq \sum_{j \in J_0} \left( \frac{1}{2} \int_{\Omega_j'} \left( |\nabla u|^2 + (\lambda a + a_0)v^2 \right) - \frac{1}{p} \int_{\Omega_j'} |v|^p \right) \\
&= \sum_{j \in J_0} I_j^*(v|\Omega_j').
\end{align*}
Thus we get for $u = h(s, t, r)$
\begin{equation}
J_\lambda(u) \geq \sum_{j \in J_0} I_j^*(u_j) \geq \beta + \sum_{j \in J} c_j^\lambda > \sum_{j \in J} c_j.
\tag{6.4}
\end{equation}
On the other hand, using that $J_\lambda(h(s, t, \tau))$ is nonincreasing with respect to $\tau \in [0, 1]$ we have
\begin{equation*}
J_\lambda(u) = J_\lambda(h(s, t, \tau)) \leq J_\lambda(h(s, t, 0)) = J_\lambda(\gamma_0(s, t)) \leq \sum_{j \in J} c_j
\end{equation*}
which contradicts with (6.4). This contradiction implies that (6.3) is impossible, which proves (6.2).

Now we consider the sets
\begin{equation*}
\mathcal{G}_\lambda := \{(s, t, \tau) \in A \times [0, 1] : f_\lambda(h(s, t, \tau)) = 0\}
\end{equation*}
and
\begin{equation*}
\mathcal{G}_\lambda^0 := \{(s, t, \tau) \in \mathcal{G}_\lambda : u = h(s, t, \tau) \text{ satisfies } \|u_0\|_{X_0} < r\}.
\end{equation*}
By (6.2), for $\lambda$ large there exists a neighborhood $U_\lambda$ of $\mathcal{G}_\lambda^0$ in $A \times [0, 1]$ such that $U_\lambda \cap (\mathcal{G}_\lambda \setminus \mathcal{G}_\lambda^0) = \emptyset$. We define $U_\lambda^* := \{(s, t) \in A : (s, t, \tau) \in U_\lambda\}$. The lemma is proved if we can find $(s, t) \in U_\lambda^*$ such that $f_\lambda(\gamma(s, t)) = 0$. By the homotopy invariance of the degree we have
\begin{equation}
\deg(f_\lambda \circ \gamma, U_\lambda^1, 0) = \deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0).
\tag{6.5}
\end{equation}
Setting
\begin{equation}
s^* = (0, \ldots, 0) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m} \quad \text{and} \quad t^* = \left( \frac{1-r}{R-r}, \ldots, \frac{1-r}{R-r} \right) \in \mathbb{R}^J
\tag{6.6}
\end{equation}
we have
\begin{equation*}
\mathcal{G} \cap (A \times \{0\}) = \{(s^*, t^*, 0)\},
\end{equation*}
and therefore
\begin{equation}
\deg(f_\lambda \circ \gamma_0, U_\lambda^0, 0) = \deg(f_\lambda \circ \gamma_0, A, 0)
\tag{6.7}
\end{equation}
Clearly $\gamma_0$ is linear in $(s, t)$ and defines a homeomorphism
\begin{equation*}
\gamma_0 : A \to A' := B_{0,r} \times \prod_{j \in J} A_{w_j, r, R} \subset E_0^- \times \prod_{j \in J} H_{w_j} \subset H^1_0(\Omega).
\end{equation*}
Here \( A_{w_j, r, R} \subset H_{w_j} \subset E_j^- \oplus \mathbb{R}w_j \) is defined as in (2.1) and (2.2), and
\[
B_{0, r} := \left\{ u \in E_0^- : u = r \sum_{j \notin J} d_j \sum_{i=1}^{d_j} s_{ji} e_{ji}, |s_{ji}| \leq 1 \right\}.
\]
It follows that
\[
\deg(f_\lambda \circ \gamma_0, A, 0) = \pm \deg(f_\lambda, A', 0). \tag{6.8}
\]
Moreover, since \( A' \subset H_0^1(\Omega) \) we have for \( u \in A' \) that \( u_j = u|_{\Omega_j} \in H_0^1(\Omega_j) \). This implies
\[
Q_0^-(u_0) = P_0^-(u_0),
\]
and for \( j \in J \):
\[
Q_j^-(\nabla I_j^\lambda(u_j)) = P_j^-(\nabla I_j(u_j)), \quad DI_j^\lambda(u_j)[u_j] = DI_j(u_j)[u_j].
\]
Thus for \( u \in A \) we have \( f_\lambda(u) = (g_j(u_j))_{j \in J_0} \) with \( g_0(u) = P_0^-(u) \) and
\[
g_j(u_j) = \left(P_j^-(\nabla I_j(u_j)), DI_j(u_j)[u_j]\right), \quad j \in J.
\]
Now Proposition 2.1 e) yields
\[
\deg(f_\lambda, A', 0) = \deg(g_0, B_{0, r}, 0) \cdot \prod_{j \in J} \deg(g_j, A_{w_j, r, R}, 0) = 1. \tag{6.9}
\]
The equations (6.5)-(6.9) imply the existence of \((s, t) \in U_\lambda^1\) with \( f_\lambda(\gamma(s, t)) = 0 \). It follows that \( u = \gamma(s, t) \) satisfies \( \|u_0\|_{X_0} < r \), in addition to \( f_\lambda(u) = 0 \). This proves Lemma 5.3.

7. Proof of Theorem 1.1. For \( u \in E \) and \( M \subset \mathbb{R}^N \) measurable we use the notation
\[
\|u\|_{\lambda, M, \beta} := \left( \int_M (|\nabla u|^2 + (\lambda a(x) + a_0(x))u^2) \right)^{1/2}.
\]
We choose \( \varepsilon > 0 \) small so that \( B_\varepsilon(0, E_j) \) contains only \( 0 \in E_j \) as critical point of \( I_j \), for all \( j \notin J \). We also require that \( \varepsilon < \sqrt{2pc_j/(p-2)} \) for \( j \in J \). Now we define
\[
D_\lambda^\varepsilon = \left\{ u \in E_\lambda : \|u\|_{\lambda, M, \beta} \leq \varepsilon/3, \quad \|u\|_{\lambda, \Omega_j} - \sqrt{2pc_j/(p-2)} \leq \varepsilon/3 \text{ for all } j \in J \right\}.
\]
Setting \( c^* := \sum_{j \in J} c_j \), it is easy to check that \( D_\lambda^\varepsilon \cap J^\varepsilon_\lambda \) contains all functions of the form
\[
w(x) = \begin{cases} v_j(x) & x \in \Omega_j, j \in J, \\ 0 & x \in \mathbb{R}^N \setminus \Omega_j; \end{cases}
\]
where \( v_j \) minimizes \( I_j \) in \( N_j \); see Section 5.

**Lemma 7.1.** There exists \( \sigma_0 > 0 \) and \( \Lambda_1 \geq \Lambda_0 \) such that
\[
\|\nabla J_\lambda (u)\| \geq \sigma_0 \quad \text{for } \lambda \geq \Lambda_1 \text{ and } u \in (D_\lambda^\varepsilon \setminus D_\lambda^\varepsilon) \cap J^\varepsilon_\lambda 
\]
We consider the associated flow is well defined, Lipschitz continuous and satisfies
\begin{equation}
\lim_{n \to \infty} \| u_n \|_{\lambda_n, S^I} = \int_{\Omega_1} (|\nabla u|^2 + a_0(x)u^2) \quad \text{for} \ 1 \leq j \leq m, \quad (7.2)
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \| u_n \|_{\lambda_n, \mathbb{R}^N \setminus \Omega} = 0. \quad (7.3)
\end{equation}
This implies that \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \). Since \( |u|_{\Omega} \| < \varepsilon \) for \( j \notin J \) we also have \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega_j \). On the other hand, (7.2) and our choice of \( \varepsilon \) imply \( u_{|\Omega_j} \neq 0 \) for \( j \in J \), hence \( I_j(u_{|\Omega_j}) \geq c_j \) for \( j \in J \). Then \( J_{\lambda_n}(u_n) \leq c^* \) yields \( I_j(u_{|\Omega_j}) = c_j \) for \( j \in J \).

From this we deduce
\begin{equation}
\int_{\Omega_j} (|\nabla u|^2 + a_0u^2) = \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} c_j = 2pc_j/(p-2) \quad \text{for} \ j \in J,
\end{equation}
hence \( u_n \in D_{\lambda_n}^2 \) for large \( n \) by (7.2) and (7.3), contradicting \( u_n \in D_{\lambda_n}^2 \setminus D_{\lambda_n}^\varepsilon \).

The following proposition is the key of the proof of our main result.

**Proposition 7.2.** Let \( \Lambda_1 \) be the constant given in Lemma 7.1 and \( \Lambda_\varepsilon \) the constant from Proposition 3.4. Then for \( \lambda \geq \max\{\Lambda_1, \Lambda_\varepsilon \} \) there exists a solution \( u_{\lambda} \) of (S\( \lambda \)) satisfying \( u_{\lambda} \in D_{\lambda}^\varepsilon \setminus J_{\lambda}^\varepsilon \).

**Proof.** We argue indirectly and assume that \( J_{\lambda} \) has no critical points in \( D_{\lambda}^\varepsilon \setminus J_{\lambda}^\varepsilon \).

Since \( J_{\lambda} \) satisfies the Palais-Smale condition, there exists a constant \( d_{\lambda} > 0 \) such that
\begin{equation}
\| \nabla J_{\lambda}(u) \\|_{\lambda} \geq d_{\lambda} \quad \text{for all} \ u \in D_{\lambda}^\varepsilon \setminus J_{\lambda}^\varepsilon. \quad (7.4)
\end{equation}

By Lemma 7.1 there holds
\begin{equation}
\| \nabla J_{\lambda}(u) \\|_{\lambda} \geq \sigma_0 \quad \text{for all} \ u \in (D_{\lambda}^\varepsilon \setminus D_{\lambda}^\varepsilon) \cap J_{\lambda}^\varepsilon.
\end{equation}

Let \( \varphi : E \to \mathbb{R} \) be a Lipschitz continuous function such that
\begin{equation}
\varphi(u) = \begin{cases} 1 & \text{for} \ u \in D_{\lambda}^{3\varepsilon/2}, \\ 0 & \text{for} \ u \notin D_{\lambda}^{3\varepsilon} 
\end{cases}
\end{equation}
and \( 0 \leq \varphi(u) \leq 1 \) for every \( u \in E \). Then the vector field
\begin{equation}
V: J_{\lambda}^\varepsilon \to E, \quad V(u) = -\varphi(u) \frac{\nabla J_{\lambda}(u)}{\| \nabla J_{\lambda}(u) \\|_{\lambda}},
\end{equation}
is well defined, Lipschitz continuous and satisfies
\begin{equation}
\| V(u) \|_{\lambda} \leq 1 \quad \text{for all} \ u.
\end{equation}

We consider the associated flow \( \eta : [0, \infty) \times J_{\lambda}^\varepsilon \to J_{\lambda}^\varepsilon \) defined by
\begin{equation}
\dot{\eta}(\tau, u) = \frac{d\eta}{d\tau}(\tau, u) = V(\eta(\tau, u)), \quad \eta(0, u) = u.
\end{equation}

Obviously \( \eta \) satisfies
\begin{equation}
\frac{d}{d\tau} J_{\lambda}(\eta(\tau, u)) = -\varphi(u) \| \nabla J_{\lambda}(u) \\|_{\lambda} \leq 0, \quad (7.6)
\end{equation}
and
\begin{equation}
\eta(\tau, u) = u \quad \text{for all} \ \tau \geq 0, \ u \in J_{\lambda}^\varepsilon \setminus D_{\lambda}^\varepsilon. \quad (7.7)
\end{equation}
We consider $\eta(\tau, \gamma_0)$ for large $\tau$. Since $\gamma_0(s, t) \not\in D^{j, \lambda}_x$ for $(s, t) \in B$, (7.7) implies

\[ \eta(\tau, \gamma_0(s, t)) = \gamma_0(s, t) \quad \text{for } (s, t) \in B, \quad \tau \geq 0. \quad (7.8) \]

Recall that $\text{supp} \gamma_0(s, t) \subseteq \bigcup_{j \in J} \overline{D_j}$ for every $(s, t) \in A$, hence $J_\lambda(\gamma_0(s, t))$ and $\|\gamma_0(s, t)\|_{\lambda, \Omega^c}$ etc. do not depend on $\lambda \geq 0$. On the other hand

\[ J_\lambda(\gamma_0(s, t)) \leq c^* \quad \text{for } (s, t) \in A \]

and there exists a unique $(s^*, t^*) \in A$, see (6.6), with $J_\lambda(\gamma_0(s^*, t^*)) = c^*$, that is, $\gamma_0(s^*, t^*))|_{\Omega_j} = w_j$ for $j \in J$ and $\gamma_0(s^*, t^*)|(x)|_{\Omega_j} = 0$ for $j \notin J$. Thus we have

\[ m_0 := \max \{J_\lambda (u) : u \in \gamma_0(A) \setminus D^{j, \lambda}_x\} < c^* \quad (7.9) \]

is independent of $\lambda$.

Now we claim that for large $\bar{\tau}$,

\[ \max_{(s, t) \in A} J_\lambda(\eta(\bar{\tau}, \gamma_0(s, t))) \leq \max \{m_0, c^* - \sigma_0 \varepsilon / 6\} \quad (7.10) \]

with $\sigma_0$, $m_0$ from (7.1), (7.9), respectively. In fact, (7.9) yields $J_\lambda(\eta(\tau, \gamma_0(s, t))) \leq m_0$ if $\gamma_0(s, t) \not\in D^{j, \lambda}_x$, $\tau \geq 0$. In the case $\gamma_0(s, t) \in D^{j, \lambda}_x$ we consider the behavior of $\bar{\eta}(\tau) := \eta(\tau, \gamma_0(s, t))$. We set $\bar{d}_\lambda := \min \{d_\lambda, \sigma_0\}$ and $\bar{\tau} = \sigma_0 \mu / 6 \bar{d}_\lambda$, where $d_\lambda$ is from (7.4). We consider two cases:

1) $\bar{\eta}(\tau) \in D^{j, \lambda}_x \bar{\tau}$ for all $\tau \in [0, \bar{\tau}]$.
2) $\bar{\eta}(\tau_0) \in \partial D^{j, \lambda}_x$ for some $\tau_0 \in [0, \bar{\tau}]$.

In case 1) we have $\varphi(\bar{\eta}(\tau)) \equiv 1$ and $\|\nabla J_\lambda(\bar{\eta}(\tau))\|_{\lambda, \bar{\tau}} \geq \bar{d}_\lambda$ for all $\tau \in [0, \bar{\tau}]$. Then (7.1) implies

\[ J_\lambda(\bar{\eta}(\tau)) = J_\lambda(\gamma_0(s, t)) + \int_0^\tau \frac{d}{ds} J_\lambda(\bar{\eta}(s)) \]

\[ = J_\lambda(\gamma_0(s, t)) - \int_0^\tau \varphi(\bar{\eta}(s)))\|\nabla J_\lambda(\bar{\eta}(s))\|_{\lambda} ds \leq c^* - \int_0^\tau \bar{d}_\lambda ds = c^* - \bar{d}_\lambda \bar{\tau} = c^* - \sigma_0 \varepsilon / 6. \]

In case 2) there exist $0 \leq \tau_1 < \tau_2 \leq \bar{\tau}$ such that

\[ \bar{\eta}(\tau_1) \in \partial D^{j, \lambda}_x, \quad \bar{\eta}(\tau_2) \in \partial D^{j, \lambda}_x, \quad (7.11) \]

and

\[ \bar{\eta}(\tau) \in D^{j, \lambda}_x \setminus D^{j, \lambda}_x \quad \text{for all } \tau \in [\tau_1, \tau_2]. \quad (7.12) \]

It follows from (7.11) that

\[ \|\bar{\eta}(\tau_1)\|_{\lambda, \partial \Omega^c_j} \leq \varepsilon / 3 \quad \text{and} \quad \|\bar{\eta}(\tau_1)\|_{\lambda, \Omega_j^c} - \sqrt{2pc_j/(p - 2)} \leq \varepsilon / 3 \quad \text{for all } j \in J \]

and

\[ \|\bar{\eta}(\tau_2)\|_{\lambda, \partial \Omega^c_j} = \varepsilon / 2 \quad \text{or} \quad \|\bar{\eta}(\tau_2)\|_{\lambda, \Omega_j^c} - \sqrt{2pc_j/(p - 2)} = \varepsilon / 2 \quad \text{for some } j \in J. \]

This immediately implies

\[ \|\bar{\eta}(\tau_1) - \bar{\eta}(\tau_2)\|_{\lambda} \geq \varepsilon / 6. \quad (7.13) \]
Now (7.5), (7.13) and the mean value theorem imply $\tau_2 - \tau_1 \geq \varepsilon/6$. Using (7.1) we deduce

$$J_\lambda(\tilde{\eta}(\bar{\tau})) = J_\lambda(\gamma_0(s,t)) - \int_{\tau_1}^{\tau_2} \sigma_0 ds = c^* - \sigma_0(\tau_2 - \tau_1) \leq c^* - \sigma_0\mu/6$$

and thus (7.10) is proved.

Now we define $\tilde{h}(s,t,r) := \eta(r\bar{\tau}, \gamma_0(s,t))$ and $\tilde{\gamma}(s,t) := \tilde{h}(s,t,1) = \eta(\bar{\tau}, \gamma_0(s,t))$. Observe that $\tilde{h} \in H_\lambda$ due to (7.6), (7.8), hence $\gamma \in \Gamma_\lambda$. Thus we have $c_\lambda \leq J_\lambda(\tilde{\gamma}(s,t)) \leq \max\{m_0, c^* - \sigma_0\mu/6\}$ (7.14)

However by Corollary 5.5 we have $c_\lambda \to c^*$ as $\lambda \to \infty$. This contradicts (7.10), and thus $J_\lambda$ has a critical point $u_\lambda \in D_{\varepsilon,\lambda}$. By Proposition 3.4, $u_\lambda$ is a solution of the original problem ($S_\lambda$).

Finally we easily prove the main result.

Proof of Theorem 1.1. Let $u_\lambda$ be a solution of ($S_\lambda$) obtained in Proposition 7.2. Applying Proposition 3.3, for any given sequence $\lambda_n \to \infty$ we can extract a subsequence, which satisfies the conclusion of Proposition 3.3. With the same argument as in the proof of Lemma 7.1, we can extract a subsequence of $u_{\lambda_n}$ such that $u_{\lambda_n} \to u$ in $E$ along this subsequence, and $u|_{\mathbb{R}^N \setminus \Omega_j} \equiv 0$. Furthermore

$$\lim_{n \to \infty} \int_{\Omega_j} \left( \frac{1}{2}(|\nabla u_{\lambda_n}|^2 + a_0(x)u_{\lambda_n}^2) - \frac{1}{p}|u_{\lambda_n}|^p \right) = c_j \quad \text{for } j \in J$$

(7.15)

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega_j} (|\nabla u_{\lambda_n}|^2 + (\lambda_n a(x) + a_0(x))u_{\lambda_n}^2) = 0.$$ (7.16)

Since the limits in (7.15) and (7.16) do not depend on the choice of the sequence $\lambda_n \to \infty$ Theorem 1.1 is proved. \hfill \Box

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