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On the least energy solutions for semilinear Schrödinger equation with electromagnetic fields involving critical growth and indefinite potentials

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\textbf{ABSTRACT}
In this paper, we are concerned with the following semilinear Schrödinger equation with electromagnetic fields and critical growth

\[-(\nabla + iA(x))^2 u + (\lambda a(x) - \delta)u = |u|^{2^* - 2}u, \ x \in \mathbb{R}^N\]

for sufficiently large $\lambda$, where $N \geq 4$, $a(x) \geq 0$ and its zero set is not empty, $2^*$ is the critical Sobolev exponent, $\delta > 0$ is a constant such that the operator $-(\nabla + iA(x))^2 + \lambda a(x) - \delta$ might be indefinite but is non-degenerate. Using variational method and modified Nehari–Pankov method, we prove the equation admits a least energy solution which localizes near the potential well $\int \{a^{-1}(0)\}$. The results we obtain here extend the corresponding results for the Schrödinger equation which involves critical growth but does not involve electromagnetic fields.

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\section{Introduction and main results}

Let us consider the following semilinear Schrödinger equation

\[-(\nabla + iA(x))^2 u + (\lambda a(x) - \delta)u = |u|^{2^* - 2}u, \ x \in \mathbb{R}^N\]  \hspace{1cm} (1.1)

for sufficiently large $\lambda$, where $i$ is the imaginary unit, $\delta > 0$ is a constant, $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = +\infty$ for $N = 1, 2$ is the critical Sobolev exponent, $A(x) = (A_1(x), A_2(x), \ldots, A_N(x))$ is the real-valued magnetic vector potential, $A_j(x)$ is a real-valued on $\mathbb{R}^N$ for $j = 1, 2, \ldots, N$, $a(x)$ is the real-valued electric potential.

Solutions to this type are related to the existence of the standing wave solutions $\psi(x, t) = e^{-i\delta t}u(x)$ of the following equation

\[i\frac{\partial \psi}{\partial t} = -(\nabla + iA(x))^2 \psi + \lambda a(x)\psi - |\psi|^{p-2}\psi, \]

and thus $u(x)$ satisfies

\[-(\nabla + iA(x))^2 u(x) + (\lambda a(x) - \delta)u(x) = |u(x)|^{p-2}u(x), \ x \in \mathbb{R}^N. \]  \hspace{1cm} (1.2)
We compare our problem with the related problem

\[-(\hbar \nabla + iA(x))^2 u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^N. \tag{1.3}\]

In past three decades, much attention has been devoted to the study of the existence of one-bump or multi-bump bound states for the nonlinear Schrödinger equation with electromagnetic fields (1.3). See, for example, the existence of solutions of (1.3) has been proved by Esteban and Lions [1] for \( \hbar > 0 \) fixed and for special classes of magnetic fields. They proved the existence by solving an appropriate minimization problem for the corresponding energy functional in the case of \( N = 2 \) and \( N = 3 \). Kurata [2] proved the existence of the least energy solution of (1.3) for \( \hbar > 0 \) under the suitable conditions associating with \( A(x) \) and \( V(x) \). Cingolani [3] proved the multiple results of solutions of (1.3) which concentrate at a single point for small \( \hbar > 0 \) by using topological argument, and she also proved that the magnetic fields \( A(x) \) only contribute to the phase factor of the solitary solutions of (1.3) as \( \hbar \) small enough. Cingolani and Secchi [4] proved the existence of one-bump bound states of (1.3) which concentrates at a non-degenerate critical point of \( V(x) \) as \( \hbar \) goes to zero. Cao and Tang [5] verified the existence and uniqueness of multi-bump bound states of (1.3) which concentrate simultaneously near several different non-degenerate critical points of \( V(x) \) as \( \hbar \) goes to zero. Liang and Zhang [6] proved the existence and multiplicity of standing wave solutions of (1.3) with critical nonlinearity. Tang [7,8] considered the existence of multiple solutions of the semilinear Schrödinger equations with electromagnetic fields which localize near the potential well for the subcritical growth. Fu et al. [9] considered the existence of multi-bump bound states of the nonlinear Schrödinger system with electromagnetic fields which localize near the potential well. Tang and Wang [10] proved the existence of the least energy solution for semilinear Schrödinger equation with electromagnetic fields and critical growth (1.1) by using variational method. Moreover, they showed that the solution was localized near the potential well \( int \{ a^{-1}(0) \} \) for \( \lambda \) large enough.

It is worth pointing out that the existence of solutions of (1.3) for \( A(x) \equiv 0 \) has been extensively studied. Using Lyapunov–Schmidt reduction, Floer and Weinstein [11] established the existence of standing wave solution of (1.3) when \( N = 1, \ p = 3 \) and \( V(x) \) is a bounded function with a non-degenerate critical point for sufficiently small \( \hbar > 0 \). Moreover, they showed that the solution concentrates near the given non-degenerate critical point of \( V(x) \) as \( \hbar \) tends to 0. For more results, we can see Ambrosetti et al. [12]; Ambrosetti et al. [13]; Cingolani and Lazzo [14]; Cingolani and Nolasco [15]; Del Pino and Felmer [16,17]; Oh [18,19] and references therein.

In this paper, we consider Schrödinger equation with electromagnetic fields involving critical growth and indefinite potential (1.1). We focus on the existence of the least energy solution which localizes near the potential well \( int \{ a^{-1}(0) \} \). For the similar investigation involving critical growth and indefinite potential without electromagnetic fields, one can refer to the paper by Tang [20].

For the study of Schrödinger equations with indefinite potentials without electromagnetic fields, we firstly refer to Ding and Wei [21]. In that paper, the authors considered the following problems

\[
\begin{cases}
-\Delta u(x) + \lambda V(x)u(x) = \lambda |u(x)|^{p-2}u(x) + \lambda g(x, u), \quad x \in \mathbb{R}^N \\
u(x) \rightarrow 0 \quad \text{as} \ |x| \rightarrow \infty,
\end{cases} \tag{1.4}
\]

where \( V(x) \) can be negative in some domains in \( \mathbb{R}^N \), \( g(x, u) \) is some perturbation term. Both for subcritical case \((2 < p < 2^*)\) and critical case \((p = 2^*)\), using variational methods the authors proved that there exists \( \Lambda > 0 \) such that for \( \lambda > \Lambda \), (1.4) admits at least one nontrivial solution. For the indefinite potentials, we also refer the work by Szulkin and Weth [22]. In that paper, the authors gave a new minimax characterization of the corresponding critical value and hence reduced the indefinite problem to a definite one. They also presented a precise description to the Nehari–Pankov manifold which is useful even for other problems. For the study of indefinite potential Schrödinger equations, we also refer the reader to Bartsch and Tang [23] and Guo and Tang [24] and references therein.
To state our main results, we firstly give some notations and remarks. Suppose $A_j \in C^\gamma_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$ for some $0 < \gamma < 1$, we write $\nabla A u = (\nabla + iA)u$. Let us denote

$$H^1_A(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N) \},$$

which is a Hilbert space under the scalar product

$$(u, v) = \Re \int_{\mathbb{R}^N} \left( (\nabla u + iAu) \cdot (\nabla v + iAv) + u\overline{v} \right) \, dx,$$

and the norm induced by the product $(\cdot, \cdot)$ is

$$\|u\|_{H^1_A(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \left( |\nabla_A u|^2 + |u|^2 \right) \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^N} \left( |\nabla u|^2 + (|A(x)|^2 + 1)|u|^2 \right) \, dx - 2\Re \int_{\mathbb{R}^N} iA(x)u\overline{\nabla u} \, dx \right)^{\frac{1}{2}}.$$ 

Let $H^{0,1}_A(\Omega)$ be the Hilbert space defined by the closure of $C_0^\infty(\Omega, \mathbb{C})$ under the scalar product $(\cdot, \cdot)$, thus

$$\|u\|_{H^{0,1}_A(\Omega)}^2 = \int_\Omega (|\nabla_A u|^2 + |u|^2) \, dx.$$ 

By the diamagnetic inequality (see Esteban and Lions [1]), we know that for $u \in H^1_A(\mathbb{R}^N)$, $|u| \in H^1(\mathbb{R}^N)$ and

$$|\nabla_A u(x)| \geq |\nabla u(x)| \quad \text{for a.e. } x \in \mathbb{R}^N.$$ 

On the other hand, we have the following further remark.

**Remark 1.1:** In general, $H^1_A(\mathbb{R}^N) \nsubseteq H^1(\mathbb{R}^N)$, and $H^1(\mathbb{R}^N) \nsubseteq H^1_A(\mathbb{R}^N)$. However, it is proved by Arioli and Szulkin [25] that if the domain is restricted to a bounded domain $\Omega$, there exist $c_1, c_2 > 0$ only depending on $\Omega$ such that

$$c_1 \|u\|_{H^1(\Omega)} \leq \|u\|_{H^1_A(\Omega)} \leq c_2 \|u\|_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$. It implies that the two spaces are equivalent.

Now we give our assumptions on $A(x)$ and $a(x)$ as follows.

(A1) $A_j(x) \in C^\gamma_{\text{loc}}(\mathbb{R}^N, \mathbb{R}) (j = 1, 2, \ldots, N)$ for some $0 < \gamma < 1$.  

(A2) $a(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $a(x) \geq 0$ and $
\Omega := \text{int} \{ a^{-1}(0) \}$ is nonempty and has smooth boundary and $\Omega = a^{-1}(0)$.  

(A3) $\lim \inf_{|x| \to \infty} a(x) > 0$.

**Remark 1.2:** From our assumptions (A2), (A3) on $a(x)$, we can see that the zero set $\Omega$ of $a(x)$ is a bounded domain in $\mathbb{R}^N$. Indeed, we can replace the assumption (A3) by the following weaker one: There exists $M > 0$ such that the measure of the set $\{ x \in \mathbb{R}^N : a(x) \leq M \}$ is finite. Obviously assumption (A3) is stronger than this assumption. To see this, we only need to take $M = \frac{1}{2} \lim \inf_{|x| \to \infty} a(x)$. This weaker assumption implies that the measure of the zero set $\Omega$ of $a(x)$ is finite.

**Remark 1.3:** From our assumptions (A1)–(A3) on $A(x)$ and $a(x)$, we can find that the operator $-(\nabla + iA(x))^2$ has discrete spectrum in $H^{0,1}_A(\Omega)$, and we denote its eigenvalues as $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$.

We assume further that

(A4) The operator $-(\nabla + iA(x))^2 - \delta$ defined on $H^{0,1}_A(\Omega)$ is non-degenerate. Namely, $\delta > 0, \delta \neq \mu_k$ for any $k \geq 1$. 

Remark 1.4: One can easily see that if $0 < \delta < \mu_1$, the operator $-(\nabla + iA(x))^2 + \lambda a(x) - \delta$ is positively definite in $H_A^{0,1}(\Omega)$ for any $\lambda > 0$, and thus the least energy solution of (1.1) is positive and may associate with the mountain pass argument. However, if $\delta > \mu_1$, the operator $-(\nabla + iA(x))^2 + \lambda a(x) - \delta$ might be indefinite in $H_A^{0,1}(\Omega)$ which implies the least energy solution of (1.1) may change sign and will not be of mountain pass type in general.

We denote $V_\lambda := \lambda a(x) - \delta$ and take a Hilbert space

$$E_\lambda = \left\{ u \in H_A^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)|u|^2 \, dx < \infty \right\}$$

with the induced norm

$$\|u\|^2_{E_\lambda} = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\lambda^+ |u|^2) \, dx,$$

where $V_\lambda^+ := \max\{V_\lambda, 0\}$. By the assumptions $(A_1) - (A_3)$, it is easy to see that $E_\lambda$ is continuously embedded in $H_A^1(\mathbb{R}^N)$ for $\lambda$ large enough.

We denote the operator $L_\lambda := -(\nabla + iA(x))^2 + \lambda a(x) - \delta$ and $L_0 := -(\nabla + iA(x))^2 - \delta$. From our assumption on $(A_4)$, $H_A^{0,1}(\Omega)$ splits as an orthogonal sum $E_0^- \oplus E_0^+$ according to the eigenspaces of $L_0$. By Remark 1.2, we know that $\inf_{E_\lambda} \sigma_c(L_\lambda) \geq \lambda M$ and $L_\lambda$ has finite Morse index on $E_\lambda$, where $\sigma_c(L_\lambda)$ denotes the essential spectrum of operator $L_\lambda$ in $E_\lambda$. Thus $E_\lambda$ splits as an orthogonal sum $E_\lambda = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+$ according to the negative, zero and positive eigenspace of $L_\lambda$ and $\dim E_\lambda^- \cup E_\lambda^0 < \infty$. Indeed, by the Corollary 2.2 in next section, we will see that for $\lambda$ large, $E_\lambda^0$ is indeed the zero space $\{0\}$, which implies that for $\lambda$ large, we have $E_\lambda = E_\lambda^- \oplus E_\lambda^+$.

In order to study the asymptotic behavior of the least energy solutions for (1.1), we need to define the modified Nehari–Pankov manifold.

Let $P^-_\lambda : E_\lambda \to E_\lambda^-$ and $P^-_0 : H_A^{0,1}(\Omega) \to E_0^-$ denote the orthogonal projections.

The variational functional of (1.1) is defined as follows:

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u + iA(x)u|^2 + (\lambda a(x) - \delta)|u|^2) \, dx - \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \, dx, \quad (1.5)$$

and its critical points correspond to the solutions of (1.1).

We define the corresponding modified Nehari–Pankov manifold as follows:

$$N_\lambda = \{ u \in E_\lambda \setminus \{0\} : P^-_\lambda \nabla J_\lambda(u) = 0, J'_\lambda(u)u = 0 \} \subset E_\lambda \setminus E_0^-$$

with the corresponding level

$$c_\lambda := \inf_{u \in N_\lambda} J_\lambda(u). \quad (1.6)$$

In present paper, we want to prove that (1.1) admits a least energy solution $u_\lambda$ which achieves $c_\lambda$ for large $\lambda$, and such that $u_\lambda$ converge as $\lambda \to \infty$ towards a least energy solution of

$$-(\nabla + iA(x))^2 u - \delta u = |u|^{2^*-2} u, \quad u \in H_A^{0,1}(\Omega), \quad (D)$$

it lies on the level

$$c_0 := \inf_{N_0} I(u), \quad (1.7)$$

where

$$N_0 = \{ u \in H_A^{0,1}(\Omega) \setminus \{0\} : P^-_0 \nabla I(u) = 0, I'(u)u = 0 \}$$

is the Nehari–Pankov manifold, and

$$I(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 - \delta |u|^2) \, dx - \frac{1}{2}\int_{\Omega} |u|^2 \, dx$$
is the corresponding variational functional of \((D)\).

**Remark 1.5:** The Nehari–Pankov manifold was firstly introduced by Pankov [26]. This type of manifold coincides with the Nehari manifold if all eigenvalues are positive, i.e., the negative eigenfunction space is \([0]\). Moreover, the minimizer for \(c_0\) is a least energy solution of \((1.1)\). By the definition of the modified Nehari–Pankov manifold \(\mathcal{N}_\lambda\), it is easy to see that \(P^-_\lambda u = P^0_\lambda u\) for any \(u \in H^{0,1}_A(\Omega)\). As proved in Lemma 4.2, we can see that the minimizer for \(c_\lambda\) is indeed a least energy solution of \((1.1)\), the solution with smallest energy among all nontrivial solutions.

Our main results are the following.

**Theorem 1.6:** Assume that \(N \geq 4\) and \((A_1) - (A_4)\) are satisfied. Then there exists \(\Lambda_0 > 0\) such that for \(\lambda \geq \Lambda_0\), \((1.1)\) admits a least energy solution \(u_\lambda\) which achieves \(c_\lambda\). Furthermore, for any sequences \(\lambda_n \to +\infty\), \(\{u_{\lambda_n}\}\) has a subsequence converging to \(u\) such that \(u\) is a least energy solution of \((D)\).

Namely \(u\) solves \((D)\) and \(I(u) = c_0\).

### 2. Preliminary lemma

In this section, we present a preliminary lemma which is needed for the proof of our main results.

Firstly, we give an asymptotic behavior result for operator \(L_\lambda := -(\nabla + iA(x))^2 + \lambda a(x) - \delta\). By Remark 1.2, we know that \(\inf \sigma_c(L_\lambda) \geq \lambda M\), where \(\sigma_c(L_\lambda)\) denotes the essential spectrum of operator \(L_\lambda\), and \(M\) is the same constant appeared in Remark 1.2. Thus we may assume that \(\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_{k_\lambda} < \sigma_c(L_\lambda)\) are the distinct eigenvalues of \(L_\lambda\) in \(E_\lambda\), and \(k_\lambda \in \mathbb{N}\) goes to \(\infty\) as \(\lambda\) goes to \(\infty\). Let \(V^\lambda_j (j \leq k_\lambda)\) be the corresponding eigenspaces of \(\mu^\lambda_j\). As Remark 1.3, the operator \(L_0 = -(\nabla + iA(x))^2 - \delta\) has discrete spectrum in \(H^{0,1}_A(\Omega)\), and we denote them as \(\mu_1 < \mu_2 < \mu_3 < \cdots\), which are the distinct eigenvalues of \(L_0\) in \(H^{0,1}_A(\Omega)\). Let \(V_j\) denotes the corresponding eigenspace of \(\mu_j\).

The following lemma can be proved by the similar arguments to that in Tang [20], we only give the result and the proof is omitted.

**Lemma 2.1:** For any \(k \in \mathbb{R}^N\) fixed, \(\mu^\lambda_k \to \mu_k\) and \(V^\lambda_k \to V_k\) as \(\lambda \to \infty\).

Here \(V^\lambda_k \to V_k\) means that, given any sequence \(\lambda_i \to \infty\) and normalized eigenfunctions \(\psi_i \in V^\lambda_k\), there exists a normalized eigenfunction \(\psi \in V_k\) such that \(\psi_i \to \psi\) strongly in \(H^{1,0}_A(\mathbb{R}^N)\) along a subsequence.

**Corollary 2.2:** For \(\lambda\) large the operator \(- (\nabla + iA(x))^2 + \lambda a(x) - \delta\) on \(E_\lambda\) is non-degenerate and has finite Morse index \(d_j := \dim E^-_\lambda\) uniformly in \(\lambda\).

### 3. Modified Nehari–Pankov manifold

In this section, we give some properties of the modified Nehari–Pankov manifold.

Recall that the modified Nehari–Pankov manifold

\[
\mathcal{N}_\lambda = \{ u \in E_\lambda \setminus \{0\} : P^-_\lambda \nabla I_\lambda (u) = 0, J'_\lambda (u)u = 0 \} \subset E_\lambda \setminus E^0_\lambda,
\]

and the Nehari–Pankov manifold

\[
\mathcal{N}_0 = \{ u \in H^{0,1}_A(\Omega) \setminus \{0\} : P^-_0 \nabla I(u) = 0, I'(u)u = 0 \} \subset H^{0,1}_A(\Omega) \setminus E^0_0.
\]

The corresponding levels are

\[
c_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda (u) \quad \text{and} \quad c_0 := \inf_{u \in \mathcal{N}_0} I(u).
\]

For the Nehari–Pankov manifold \(\mathcal{N}_0\), we have the following lemma due to Szulkin and Weth [22].
Lemma 3.1: For any \( w \in H^{0,1}_A(\Omega) \setminus E_0^- \), set
\[
H_w := \{ v + tw : v \in E_0^-, t > 0 \}.
\]

The following properties hold:
(a) \( \mathcal{N}_0 = \{ w \in H^{0,1}_A(\Omega) \setminus E_0^- : \nabla(I|H_w) = 0 \} \).
(b) For every \( w \in E_0^- \setminus \{0\} \), there exist \( t_w > 0 \) and \( \varphi(w) \in E_0^- \) such that \( H_w \cap \mathcal{N}_0 = \{ \varphi(w) + t_w \cdot w \} \).
(c) For every \( w \in \mathcal{N}_0 \) and every \( u \in H_w \setminus \{w\} \), there holds \( I(u) < I(w) \).
(d) \( c_0 = \inf_{u \in \mathcal{N}_0} I(u) > 0 \).

For the modified Nehari–Pankov manifold, we have the following Lemma.

Lemma 3.2: For any \( w \in E_\lambda \setminus E_0^- \), set
\[
\hat{H}_w := \{ v + tw : v \in E_0^-, t > 0 \}.
\]

The following properties hold:
(i) \( \hat{\mathcal{N}}_\lambda = \{ w \in E_\lambda \setminus E_0^- : \nabla(J_\lambda|\hat{H}_w) = 0 \} \).
(ii) Let
\[
\hat{E}_\lambda^+ := \left\{ w \in E_\lambda^+ : \int_{\mathbb{R}^N} w_e dx = 0, i = 1, 2, \ldots, k \right\}.
\]
where \( e_i \) is the eigenfunction of the operator \( L_0 \). Then for any \( w \in \hat{E}_\lambda^+ \setminus \{0\} \), there exist \( t_w > 0 \) and \( \varphi(w) \in E_0^- \) such that
\[
\hat{H}_w \cap \hat{\mathcal{N}}_\lambda = \{ \varphi(w) + t_w \cdot w \}.
\]

(iii) For any \( w \in \hat{\mathcal{N}}_\lambda \) and \( u \in \hat{H}_w \setminus \{w\} \) there holds \( J_\lambda(u) < J_\lambda(w) \).
(iv) \( c_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) \geq \tau > 0 \) for some small \( \tau > 0 \) independent of large \( \lambda \).

Proof: The proof of this lemma is similar to the proof of Lemma 2.8 in Tang and Wang [27], we omit the details.

Remark 3.3: By Lemma 3.2, we conclude that for each \( w \in E_\lambda \setminus E_0^- \), the set \( \mathcal{N}_\lambda \) intersects \( \hat{H}_w \) in exactly one point \( \tau(w) := \varphi(w) + t_w \cdot w \) which is the unique global maximum point of \( J_\lambda|\hat{H}_w \). Moreover, the map \( \tau : w \mapsto \varphi(w) + t_w \cdot w \) is continuous and the restriction of \( \tau \) to the unit sphere \( S_\lambda^+ \) in \( \hat{E}_\lambda^+ \) is a homeomorphism between \( S_\lambda^+ \) to \( \mathcal{N}_\lambda \). Thus the least energy level \( c_\lambda \) has a minimax characterization given by
\[
c_\lambda = \inf_{w \in \hat{E}_\lambda^+ \setminus \{0\}} \max_{u \in \hat{H}_w} J_\lambda(u). \tag{3.1}
\]

Remark 3.4: By the above Remark 3.3, we know that the codimension of \( \mathcal{N}_\lambda \) equals to the codimension of \( S_\lambda^+ \) which is \( \dim E_\lambda^+ + 1 \) and is finite by Corollary 2.2 for \( \lambda \) large.

4. Compactness of Palais–Smale sequence

In this section, we are focused on the compactness of Palais–Smale sequence.

By Lemma 3.2, we know that \( c_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) \). Thus for any \( \lambda > 0 \) fixed, from Remark 3.4, there exists, by the Ekeland’s variational principle, a Palais–Smale (P. S. for shortage) sequence \( \{u_n\} \) with
\[
J_\lambda(u_n) \to c_\lambda, \quad J'_\lambda(u_n) \to 0. \tag{4.1}
\]
In fact, for the detail of the proof to the existence of the Palais–Smale sequence \( \{u_n\} \), one can refer to the proof in Guo and Tang [24] and Pankov [26].
Taking a notation
\[ S_A := \inf_{u \in H^1_0(\Omega)} \| \nabla u \|_2^2. \]
Then, we have \( S_A = S \) (see Lemma 3.2 in Tang and Wang [10]), where \( S \) is the best Sobolev constant
\[ S = \inf_{u \in H^1_0(\Omega)} \| \nabla u \|_2^2. \]

Firstly, we give that any Palais–Smale sequence is bounded for \( \lambda \) large enough.

**Lemma 4.1:** There exists a positive constant \( \Lambda_0 > 0 \) such that if \( \lambda \geq \Lambda_0 \), and \( \{ u_n \} \) is a Palais–Smale sequence satisfying (4.1), then there exists a constant \( C > 0 \) which is independent of \( \lambda \) and \( n \) such that
\[ \lim sup_{n \to \infty} \| u_n \|_{\lambda} \leq C \| u_n \|_{\lambda}. \] (4.2)

**Proof:** Setting \( \varepsilon_n := \| \nabla J_\lambda(u_n) \| \), it follows from (4.1) that
\[ \int_{\mathbb{R}^N} \left( \frac{1}{2} - \frac{1}{2^*} \right) |u_n|^2 dx = J_\lambda(u_n) - \frac{1}{2} J'_\lambda(u_n) u_n \leq c_\lambda + \varepsilon_n \| u_n \|_{\lambda}. \] (4.3)

On the other hand, we take \( R > 0 \) such that \( \overline{\Omega} \subset B_R(0) \), and let \( \eta := \inf_{|x| \geq R} V_\lambda(x) > 0 \). Taking \( \Lambda_0 > 0 \) such that \( \Lambda_0 \eta \geq \delta \). Since \( V^-_\lambda \) is non-increasing with respect to \( \lambda \) and \( \text{supp } V^-_\lambda \subset B_R(0) \) for \( \lambda \geq \Lambda_0 \), we deduce for \( \lambda \geq \Lambda_0 \):
\[ \int_{\mathbb{R}^N} V^-_\lambda |u_n|^2 dx = \int_{B_R(0)} V^-_\lambda |u_n|^2 dx \leq \int_{B_R(0)} V^-_{\Lambda_0} |u_n|^2 dx \]
\[ \leq C \int_{B_R(0)} |u_n|^2 dx \leq C(c_\lambda + \varepsilon_n \| u_n \|_{\lambda}), \] (4.4)
where \( C \) is a positive constant which is independent of \( \lambda \) and \( n \).

Using (4.1) once more, we obtain
\[ \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + V_\lambda |u_n|^2) dx \]
\[ = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + V^+_\lambda |u_n|^2) dx - \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} V^-_\lambda |u_n|^2 dx \]
\[ = J_\lambda(u_n) - \frac{1}{2^*} J'_\lambda(u_n) u_n \]
\[ \leq c_\lambda + \varepsilon_n \| u_n \|_{\lambda}. \] (4.5)

Combining (4.4) and (4.5), we obtain
\[ \left( \frac{1}{2} - \frac{1}{2^*} \right) \| u_n \|_{\lambda}^2 = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + V^+_\lambda |u_n|^2) dx \]
\[ \leq C(c_\lambda + \varepsilon_n \| u_n \|_{\lambda}). \]

Thus it is easy to see that there exists a constant \( C \) which is independent of \( \lambda \geq \Lambda_0 \) such that (4.2) holds. This completes the proof of Lemma 4.1.

Now, we give the main compactness result.
Lemma 4.2: Suppose \( \{u_n\} \) is Palais–Smale sequence of \( J_\lambda \) satisfying (4.1) with
\[
c_\lambda < \frac{1}{N} S_N^2,
\]
where \( S \) is the best Sobolev constant. Then there exists a subsequence of \( \{u_n\} \) which converges strongly in \( E_\lambda \) to a solution \( u_\lambda \) of (1.1) such that \( J_\lambda(u_\lambda) = c_\lambda \).

Proof: By Lemma 4.1, we know that \( \{u_n\} \) is bounded in \( E_\lambda \). Hence there is a function \( u_\lambda \in E_\lambda \) such that up to a subsequence,
\[
\begin{align*}
 u_n &\to u_\lambda \quad \text{in } L^{2^*}(\mathbb{R}^N), \\
 u_n &\to u_\lambda \quad \text{in } H^{1}_A(\mathbb{R}^N), \\
 u_n &\to u_\lambda \quad \text{in } L^{2}_{\text{loc}}(\mathbb{R}^N), \\
 u_n &\to u_\lambda \quad \text{a.e. in } \mathbb{R}^N.
\end{align*}
\]

With a standard argument, it is easy to check that \( J'_\lambda(u_\lambda) = 0 \) and \( J_\lambda(u_\lambda) \geq 0 \). Let \( v_n = u_n - u_\lambda \), then we have
\[
\begin{align*}
 v_n &\to 0 \quad \text{in } L^{2^*}(\mathbb{R}^N), \\
 v_n &\to 0 \quad \text{in } H^{1}_A(\mathbb{R}^N), \\
 v_n &\to 0 \quad \text{in } L^{2}_{\text{loc}}(\mathbb{R}^N), \\
 v_n &\to 0 \quad \text{a.e. in } \mathbb{R}^N.
\end{align*}
\]

Using Brezis–Kato’s Lemma, we can prove that \( \{v_n\} \) is also a Palais–Smale sequence of \( J_\lambda \) satisfying
\[
J'_\lambda(v_n) \to 0
\]
and
\[
\lim_{n \to \infty} J_\lambda(v_n) = c_\lambda - J_\lambda(u_\lambda) \leq c_\lambda < \frac{1}{N} S_N^2.
\]

Now we prove that \( v_n \to 0 \) strongly in \( E_\lambda \). Indeed, since \( \{v_n\} \) is a Palais–Smale sequence of \( J_\lambda \), we only need to show that \( v_n \to 0 \) in \( L^{2^*}(\mathbb{R}^N) \). We prove it by a contradiction argument. Without loss of generality, up to a subsequence, we assume on the contrary that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^{2^*} dx = b > 0.
\]

Since \( \{v_n\} \) is also bounded in \( E_\lambda \), we have
\[
o(1) = J'_\lambda(v_n) \cdot v_n = \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + (\lambda a(x) - \delta)|v_n|^2) dx - \int_{\mathbb{R}^N} |v_n|^{2^*} dx,
\]
which implies that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + (\lambda a(x) - \delta)|v_n|^2) dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^{2^*} dx = b > 0.
\]

Thus
\[
\lim_{n \to \infty} J_\lambda(v_n) = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2^{*}} \right) \int_{\mathbb{R}^N} |v_n|^{2^*} dx = \frac{1}{N} b,
\]
which implies that
\[
b < S_N^{2^*}.
\]
On the other hand, since \( v_n \to 0 \) in \( L^2(B_R(0)) \) and \( \Lambda_0 a_0 \geq \delta \), where \( a_0 := \inf_{|x| \geq R} a(x) > 0 \). Thus we have for \( \lambda \geq \Lambda_0 \)

\[
b = \lim_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla_A v_n|^2 + (\lambda a(x) - \delta)|v_n|^2)dx
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla_A v_n|^2 dx + \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (\lambda a(x) - \delta)|v_n|^2 dx
\]

\[
+ \lim_{n \to \infty} \int_{B_R(0)} (\lambda a(x) - \delta)|v_n|^2 dx
\]

\[
\geq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla_A v_n|^2 dx.
\] (4.10)

Combining (4.7), (4.10) and the diamagnetic inequality, we obtain that

\[
S = \inf_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 dx}{(\int_{\mathbb{R}^N} |v|^2 dx)^\frac{2}{N}} \leq \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2 dx}{(\int_{\mathbb{R}^N} |v_n|^2 dx)^\frac{2}{N}} \leq \frac{\int_{\mathbb{R}^N} |\nabla_A v_n|^2 dx}{(\int_{\mathbb{R}^N} |v_n|^2 dx)^\frac{2}{N}} \leq b^2 \frac{2}{N},
\]

i.e., \( b \geq S^2 \), which contradicts with (4.9) and thus we proved that \( v_n \to 0 \) strongly in \( L^{2^*}(\mathbb{R}^N) \) and the proof of Lemma 4.2 is completed.

The following lemma can be proved by the similar arguments to that in Capozzi et al. [28] and Tang and Wang [10].

**Lemma 4.3:** For \( N \geq 4 \) and \( \lambda \geq \Lambda_0 \), we have \( c_\lambda < \frac{1}{N} S^2 \), where \( c_\lambda \) is defined as in (1.6) and \( \Lambda_0 \) is taken as in Lemma 4.2.

**Proof:** By the definition of \( c_\lambda \) we know that \( c_\lambda \leq c_0 \), where \( c_0 \) is defined as in (1.7), thus in order to complete the proof, we only need to show that \( c_0 < \frac{1}{N} S^2 \).

Assume that \( 0 \in \Omega \), let \( u_\varepsilon(x) := \psi(x) U_\varepsilon(x) \cdot e^{-i\chi(x)} \), where

\[
U_\varepsilon(x) = \frac{[N(N - 2)]^{\frac{N-2}{4}}}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}}, \quad \psi(x) \in C_0^\infty(\Omega, [0, 1])
\]

is a cut-off function, \( \psi(x) \equiv 1 \) in \( B_\varepsilon(0) \) and \( \psi(x) \equiv 0 \) when \( |x| \geq r, \chi(x) = \sum_{j=1}^n A_j(x)j_1 \). It is easy to see \( (A - \nabla \chi)(0) = 0 \). By the continuity of \( A(x) \), for \( \forall \xi > 0 \), there exists \( \delta_0 > 0 \) such that it holds \( |(A - \nabla \chi)(x)| < \xi \) for \( |x| < \delta_0 \). Let \( w_\varepsilon = \frac{u_\varepsilon}{||u_\varepsilon||_{2^*}} \) such that \( \|w_\varepsilon\|_{2^*} = 1 \). Set

\[
Y_\varepsilon = \sup_{t \geq 0} I(t w_\varepsilon) = \sup_{t \geq 0} \left[ \frac{1}{2} t^2 \int_{\Omega} (|\nabla_A w_\varepsilon|^2 - \delta |w_\varepsilon|^2) dx - \frac{1}{2^*} t^{2^*} \int_{\Omega} |w_\varepsilon|^{2^*} dx \right].
\]

It is easy to see \( \lim_{t \to \infty} I(t w_\varepsilon) = -\infty \). Therefore, \( \sup_{t \geq 0} I(t w_\varepsilon) \) is achieved at some \( t_\varepsilon > 0 \). On the other hand, \( I'(t_\varepsilon w_\varepsilon) = t_\varepsilon \|\nabla_A w_\varepsilon\|_2^2 - \delta t_\varepsilon \|w_\varepsilon\|_2^2 - t_\varepsilon^{2^*-1} = 0 \). Thus,

\[
t_\varepsilon^{2^*-1} = t_\varepsilon \|\nabla_A w_\varepsilon\|_2^2 - \delta t_\varepsilon \|w_\varepsilon\|_2^2 \leq t_\varepsilon \|\nabla_A w_\varepsilon\|_2^2.
\] (4.11)

it follows that \( t_\varepsilon \leq \|\nabla_A w_\varepsilon\|_2 \). Since the function \( g : \mapsto \frac{1}{2} t^2 \|\nabla_A w_\varepsilon\|_2^2 - \frac{1}{2^*} t^{2^*} \) is strictly monotone increasing in the interval \( [0, \|\nabla_A w_\varepsilon\|_2] \), we obtain

\[
Y_\varepsilon \leq \frac{1}{N} \|\nabla_A w_\varepsilon\|_2^{\frac{2^*}{2^*-2}} - \frac{1}{2^*} \delta t_\varepsilon \|w_\varepsilon\|_2^2 < \frac{1}{N} \|\nabla_A w_\varepsilon\|_2^{\frac{2^*}{2^*-2}}.
\] (4.12)
By a direct calculation (see also Brezis and Nirenberg [29]), for $N \geq 4$, we have
\begin{align}
\|\nabla (\psi U_\varepsilon(x))\|_2^2 &= S_0^N + O(\varepsilon^{N-2}), \\
\|\psi U_\varepsilon(x)\|_2^2 &= O(\varepsilon), \\
\|\psi U_\varepsilon(x)\|_{2^*}^2 &= S_0^N + O(\varepsilon^N).
\end{align}

Using (4.13)–(4.15), we can obtain that
\[
\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega \left[ |\nabla (\psi U_\varepsilon)|^2 + \psi^2 U_\varepsilon^2 |\nabla \chi - A|^2 \right] \, dx \\
\leq S_0^N + \varepsilon^2 O(\varepsilon) + O(\varepsilon^{N-2})
\]
and
\[
\|u_\varepsilon\|_{2^*}^2 = S_0^{N-2} + O(\varepsilon^{N-2}).
\]
Combining (4.12) and $w_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_2}$, we have $Y_\varepsilon < \frac{1}{N} S_0^N$. As a consequence, we conclude that $c_0 < \frac{1}{N} S_0^N$.

**Proposition 4.4:** For $N \geq 4$ and $\lambda \geq \Lambda_0$, there is a least energy solution $u_\lambda$ of (1.1) which achieves $c_\lambda$.

**Proof:** This is a direct result of Lemmas 4.2 and 4.3.

Up to now, we indeed have proved that for $\lambda$ large, there exists a least energy solution $u_\lambda$ for (1.1), which is the part for the existence of the least energy solution of Theorem 1.6. In the following, we come to give the asymptotic behavior of the least energy solutions of (1.1) as $\lambda$ goes to infinity. Namely, we prove another part of Theorem 1.6.

**5. Asymptotic behavior of the least energy solution**

In this section, we study the asymptotic behavior of the least energy solutions of (1.1) as $\lambda$ goes to infinity.

Suppose $u_\lambda \in E_\lambda$ is the least energy solution of (1.1) such that
\[ J_\lambda(u_\lambda) = c_\lambda = \inf_{u \in N_\lambda} J_\lambda(u), \]
we have the following asymptotic property for $u_\lambda$ and $c_\lambda$ which is the main ingredient of this section.

**Proposition 5.1:** $\lim_{\lambda \to +\infty} c_\lambda = c_0$ and for any sequence $\{\lambda_n\}$ ($\lambda_n \to +\infty$), up to a subsequence $u_{\lambda_n} \to u$ strongly in $H^1_A(\mathbb{R}^N)$. Here $u$ is a least energy solution of (D) which achieves $c_0$.

**Proof:** Clearly, $N_0 \subset N_\lambda$ because for $u \in H^{0,1}_A(\Omega)$,
\[ P^-_\lambda (\nabla J_\lambda(u)) = P^0_\lambda (\nabla I(u)) \quad \text{and} \quad J'_\lambda(u)u = I'(u)u. \]
Thus by the definition of $c_\lambda$ and $c_0$, it is easy to see that $c_\lambda \leq c_0$ for $\lambda \geq \Lambda_0$. On the other hand it is not difficult to check that $c_\lambda$ is monotonically increasing as $\lambda$ growth. Thus we may assume that $\lim_{\lambda \to +\infty} c_\lambda = k \leq c_0$ which implies for any sequence $\{\lambda_n\}$ ($\lambda_n \to +\infty$), $c_{\lambda_n} \to k \leq c_0$. We assume that $u_n$ is such that $c_{\lambda_n}$ is achieved, by Lemma 4.1, $\{u_n\}$ is bounded in $E_{\lambda_n}$, and thus it is also bounded in $H^1_A(\mathbb{R}^N)$. As a consequence, there exists $u \in H^1_A(\mathbb{R}^N)$ such that
\[ u_n \to u \quad \text{in} \quad L^2(\mathbb{R}^N), \]
\[ u_n \to u \quad \text{in } H^1_A(\mathbb{R}^N), \]
\[ u_n \to u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \]
\[ u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \]

We claim that \( u_{|\Omega^c} = 0 \), where \( \Omega^c := \{ x \in \mathbb{R}^N \setminus \Omega \} \). Indeed, if not, there exists a compact subset \( F \subset \Omega^c \) with \( \text{dist} \{ F, \partial \Omega \} > 0 \) such that \( u|_F \neq 0 \) and
\[
\int_F u_n^2 dx \to \int_F u^2 dx > 0.
\]

Moreover, there exists \( \varepsilon_0 > 0 \) such that \( a(x) \geq \varepsilon_0 \) for any \( x \in F \).

By the choice of \( \{ u_n \} \), we have
\[
0 = J'_{\lambda_n}(u_n) \cdot u_n
= \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + (\lambda_n a(x) - \delta) |u_n|^2) dx - \int_{\mathbb{R}^N} |u_n|^{2^*_s} dx,
\]

hence for \( n \) large
\[
J_{\lambda_n}(u_n) = \left( \frac{1}{2} - \frac{1}{2^*_s} \right) \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + (\lambda_n a(x) - \delta) |u_n|^2) dx
\geq \left( \frac{1}{2} - \frac{1}{2^*_s} \right) \int_F (\lambda_n \varepsilon_0 - \delta) |u_n|^2 dx
\to +\infty \quad \text{as } n \to \infty.
\]

This leads to a contradiction and thus \( u_{|\Omega^c} = 0 \). By the smoothness assumption on the boundary \( \partial \Omega \), we have \( u \in H^{0,1}_A(\Omega) \).

Now we show that
\[
u_n \to u \quad \text{strongly in } L^{2^*_s}(\mathbb{R}^N).
\]

Note that \( c_{\lambda_n} \leq c_0 < \frac{1}{N} S^N_2 \). Now, we take \( v_n = u_n - u \) and suppose on the contrary that (5.1) is not true, then up to a subsequence, we may assume that \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^{2^*_s} dx = b > 0 \). By a similar argument as the proof of Lemma 4.2, we can show that \( b \geq S^N_2 \) which implies that \( k = \lim_{n \to \infty} c_{\lambda_n} \geq \frac{1}{2^*} S^N_2 \). This also leads contradiction. Namely, we proved that (5.1) holds. Moreover,
\[
\lim_{n \to \infty} \inf \int_{\mathbb{R}^N} |\nabla A u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla A u|^2 dx.
\]

Since \( u_n \) is the solution of (1.1), where \( \lambda \) is instead of \( \lambda_n \), we have
\[
J'_{\lambda_n}(u_n) \cdot u_n = \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + (\lambda_n a(x) - \delta) |u_n|^2) dx - \int_{\mathbb{R}^N} |u_n|^{2^*_s} dx \to 0,
\]

which implies that for \( n \) large enough
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + (\lambda_n a(x) - \delta) |u_n|^2) dx
= \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_s} dx = \int_{\mathbb{R}^N} |u|^{2^*_s} dx
\geq \lim_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla A u_n|^2 + |u_n|^2) dx
\geq \int_{\mathbb{R}^N} (|\nabla A u|^2 + |u|^2) dx.
This implies
\[ \int_{\mathbb{R}^N} |u|^2\,dx \geq \int_{\mathbb{R}^N} (|\nabla A u|^2 + |u|^2)\,dx. \]
As a consequence of \( u|_{\Omega^c} = 0 \), we have
\[ \int_{\Omega} (|\nabla A u|^2 - \delta |u|^2)\,dx \leq \int_{\Omega} (|\nabla A u|^2 + |u|^2)\,dx \leq \int_{\Omega} |u|^2\,dx. \]
Then there exists \( \alpha \in (0, 1] \) such that
\[ \int_{\Omega} (|\nabla A \alpha u|^2 + |\alpha u|^2)\,dx = \int_{\Omega} |\alpha u|^{2^*}\,dx. \]
Furthermore,
\[ I(\alpha u) = \left( \frac{1}{2} - \frac{1}{2^{*}} \right) \int_{\Omega} (|\nabla A \alpha u|^2 - \delta |\alpha u|^2)\,dx \leq \frac{1}{N} \liminf_{n \to \infty} \int_{\Omega} \left( |\nabla A u_n|^2 + (\lambda_n a(x) - \delta) |u_n|^2 \right)\,dx = k. \]
This implies \( k \geq c_0 \), which leads to a contradiction. Hence, we have proved that \( \lim_{\lambda \to +\infty} c_\lambda = c_0 \).

By the definition of \( c_0 \) we have \( I(u) \geq c_0 \). On the other hand, by the strong convergence of \( \{u_n\} \), we have
\[ I(u) = \left( \frac{1}{2} - \frac{1}{2^{*}} \right) \int_{\Omega} |u|^{2^*}\,dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*}\,dx = \lim_{n \to \infty} J_{\lambda_n}(u_n) \]
\[ = \lim_{n \to \infty} c_{\lambda_n} = k \leq c_0. \]
Thus we proved that \( I(u) = k = c_0 \). Namely, \( u \) is indeed a least energy solution of \( (D) \) which achieves \( c_0 \), and thus the proof of Proposition 5.1 is completed.

\[ \square \]

**The Proof of Theorem 1.6:** They are the direct results of Propositions 4.4 and 5.1.

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