# Mathematical Logic

## – Final Review –

Spring, 2025

# Propositional Logic

**Need to know:**

- *Concepts*:
  - formulas, Polish notation
  - Truth table
  - Logical implication
  - Normal disjunction/conjunction form
  - Complete set of connectives, etc.
- *Techniques*: convert formulas, present a deduction
- *Reminder:* write explicit reasons for each step of deduction.

# Binary Connectives

| Symbol | Equivalent | Remarks |
|---|---|---|
| $1$ | $1$ | 0-ary |
| $0$ | $0$ | 0-ary |
| $\Gamma_1$ | $p$ | 1-ary |
| $\Gamma_2$ | $q$ | 1-ary |
| $\neg_1$ | $\neg p$ | 1-ary |
| $\neg_2$ | $\neg q$ | 1-ary |

说明：这里列出所有 2-元的命题连接词。建议自选组合，练习验证连接词集合的完全性。例如：$\{\downarrow\}$, $\{0, \to\}$ 等。

The rest $10$ are truly binary:

| Symbol | Equivalent | Remarks |
|---|---|---|
| $\wedge$ | $p \wedge q$ | 且 |
| $\vee$ | $p \vee q$ | 或 |
| $\rightarrow$ | $p \rightarrow q$ | 如果 $p$ 就 $q$ |
| $\leftarrow$ | $p \leftarrow q$ | 如果 $q$ 就 $p$ |
| $\leftrightarrow$ | $p \leftrightarrow q$ | $p$ 当且仅当 $q$ |
| $\oplus$ | $(p \vee q) \wedge \neg(p \wedge q)$ | 异或，相当于对称差 |
| $\uparrow$ | $\neg(p \wedge q)$ | 与非 |
| $\downarrow$ | $\neg(p \vee q)$ | 或非 |
| $<$ | $\neg p \wedge q$ | 相当于 $q \setminus p$ |
| $>$ | $p \wedge \neg q$ | 相当于 $p \setminus q$ |

Show that the following sets of connectives are complete.

1. $\{0, \to\}$, where $0$ is the constant connective.
2. $\{\uparrow\}$, where $p \uparrow q \Leftrightarrow \neg(p \wedge q)$.

1. Use the fact that $\{\neg, \to\}$ is complete:

$$\neg p \Leftrightarrow \neg p \vee 0 \Leftrightarrow p \to 0$$

2. Use the fact that $\{\neg, \wedge\}$ is complete:

$$\neg p \Leftrightarrow \neg(p \wedge p) \Leftrightarrow p \uparrow p$$
$$p \wedge q \Leftrightarrow \neg(p \uparrow q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q).$$

Alternative solutions: One can also derive $\vee$ from $\uparrow$ as follows.

$$p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q) \Leftrightarrow (\neg p) \uparrow (\neg q) \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$$

# More examples

- $\{\uparrow\}$
- $\{\downarrow\}$
- $\{\neg, >\}$
- $\{\neg, \leftarrow\}$
- $\{0, \rightarrow\}$
- $\{\wedge, 0\}$
- $\{\vee, 1\}$
- $\{\oplus, 1\}$
- $\{\leftrightarrow, \oplus, >\}$

# Polish Notation

Translate the following formula (in *Polish Notation*) into an $\mathcal{L}_0$-formula, and find its equivalent normal disjunctive form.

$$p \neg q \; r \; s \neg \rightarrow \rightarrow \rightarrow \; \neg q \neg r \; s \neg \rightarrow \neg \rightarrow \neg \rightarrow \neg$$

SOLUTION: The corresponding $\mathcal{L}_0$-formula is

$$\neg(\neg(\neg p \rightarrow (q \rightarrow (r \rightarrow \neg s))) \rightarrow \neg(\neg q \rightarrow \neg(r \rightarrow \neg s)))$$

Denote it as $\varphi \equiv \neg(\neg A \rightarrow \neg B)$. If $\mu(\varphi) = T$, it must be that $\mu(A) = F$ and $\mu(B) = T$. Hence $\mu(p) = F$, $\mu(q \rightarrow (r \rightarrow \neg s)) = F$. Then $\mu(q) = T$ and $\mu(r \rightarrow \neg s) = F$. At last $\mu(r) = T$ and $\mu(s) = T$. [1] So it is equivalent to

$$\neg p \wedge q \wedge r \wedge s.$$

[1] Demonstrate the solution using tautological equivalence on board.

# First-Order Logic

**Need to know:**

- *Concepts*:
  - language, formula/sentence,
  - deduction, consistency,
  - structure, satisfaction,
  - (homo/iso/auto)-morphism of models, definable sets,
  - EC/$\mathrm{EC}_\Delta$-class, etc.
  - Soundness, Completeness and Compactness.

# First-Order Logic

**Need to know:**

- *Techniques*:
  - translate sentences in natural language into first-order formulas,
  - convert formulas,
  - present a deduction,
  - application of Compactness.

# Example

## Completeness for the first order logic

Show that the following two statements are equivalent:

1. For any set of formula $\Gamma$, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.
2. Any consistent set of formulas is satisfiable.

# Solution

## $1 \Rightarrow 2.$

- Let $\Gamma$ be a consistent set of formulas.
- Note that $\Gamma \models \varphi$ means that for any model $\mathcal{M}$ and any assignment $\nu : V \to |\mathcal{M}|$, if $(\mathcal{M}, \nu)$ satisfies every formula in $\Gamma$, then $(\mathcal{M}, \nu)$ satisfies $\varphi$ as well.
- If $\Gamma$ is unsatisfiable, since $\Gamma$ is satisfied by no models and assignments, $\Gamma \models \varphi$ holds vacuously for any formula $\varphi$, in particular for $\varphi \equiv \neg(x \hat{=} x)$.
- But then by 1, $\Gamma \vdash \neg(x \hat{=} x)$, contradicting the assumption that $\Gamma$ is consistent.

# Solution

## $2 \Rightarrow 1.$

- Assume towards a contradiction that $\Gamma \models \varphi$ and $\Gamma \nvdash \varphi$.
- Then $\Gamma \cup \{\neg\varphi\}$ is consistent. By 2, $\Gamma \cup \{\neg\varphi\}$ is satisfied by some model $\mathcal{M}$ and some assignment $\nu : V \to |\mathcal{M}|$.
- In particular, $(\mathcal{M}, \nu) \models \neg\varphi$.
- But from $\Gamma \models \varphi$, we have $(\mathcal{M}, \nu) \models \varphi$. Contradiction!
- So if $\Gamma \models \varphi$, it must be that $\Gamma \vdash \varphi$.

# Example

Give a proof for the following

$$\{\forall x\,(\gamma(x) \to \neg\varphi(x)), \forall x\,(\psi(x) \to \gamma(x))\} \vdash \forall x\,(\psi(x) \to \neg\varphi(x)).$$

## Solution.

| | | |
|---|---|---|
| (1) | $\forall x\,(\gamma(x) \to \neg\varphi(x))$ | Hypothesis |
| (2) | $\gamma(z) \to \neg\varphi(z)$ | $\Delta$-2, $z$ is new for $\varphi$, $\psi$, $\gamma$ |
| (3) | $\forall x\,(\psi(x) \to \gamma(x))$ | Hypothesis |
| (4) | $\psi(z) \to \gamma(z)$ | $z$ is substitutable for $x$ in (3) |
| (5) | $\psi(z) \to \neg\varphi(z)$ | MP[(4,2),$\Delta$-1] |
| (6) | $\forall z\,(\psi(z) \to \neg\varphi(z))$ | Generalization, $z$ is not free in hypothesis |
| (7) | $\forall x\,(\psi(x) \to \neg\varphi(x))$ | Change $z$ to $x$, $x$ is not free in (6). |

$\square$

# More Exercises on proofs

Suppose $x$ does not occur in $\psi$. Show

- $\vdash \forall x\,(\varphi(x) \to \psi) \to (\exists x\,\varphi(x) \to \psi)$.
- $\vdash (\exists x\,\varphi(x) \to \psi) \to \forall x\,(\varphi(x) \to \psi)$.
- $\vdash \forall x\,(\psi \to \varphi(x)) \to (\psi \to \forall x\,\varphi(x))$.
- $\vdash (\psi \to \forall x\,\varphi(x)) \to \forall x\,(\psi \to \varphi(x))$.

Here $\exists x \varphi(x)$ is treated as the abbreviation of $\neg\forall x\neg\varphi(x)$.

# More Exercises on proofs

In fact, let $A \Leftrightarrow B$ abbreviates that

$$\vdash A \to B \quad \text{and} \quad \vdash B \to A.$$

Then we have

**Theorem**

*Let $\varphi$, $\psi$ be any two formulas. Then*

1. $\forall x\,(\varphi \wedge \psi) \Leftrightarrow \forall x\,\varphi \wedge \forall x\,\psi$.
2. $\exists x\,(\varphi \vee \psi) \Leftrightarrow \exists x\,\varphi \vee \exists x\,\psi$.

If $x$ does not occur in $\varphi$, then

3. $\forall x\,(\varphi \vee \psi) \Leftrightarrow \varphi \vee \forall x\,\psi$.
4. $\exists x\,(\varphi \wedge \psi) \Leftrightarrow \varphi \wedge \exists x\,\psi$.

If $x$ does not occur in $\varphi$, then

5. $\forall x\,(\varphi \to \psi) \Leftrightarrow \varphi \to \forall x\,\psi$.
6. $\exists x\,(\varphi \to \psi) \Leftrightarrow \varphi \to \exists x\,\psi$.

If $x$ does not occur in $\psi$, then

7. $\forall x\,(\varphi \to \psi) \Leftrightarrow \exists x\,\varphi \to \psi$.
8. $\exists x\,(\varphi \to \psi) \Leftrightarrow \forall x\,\varphi \to \psi$.

# Example

Consider the language $A = \{E\}$, where $E$ is a two-place predicate symbol. Consider the two structures $(\mathbb{N}, <)$ and $(\mathbb{R}, <)$ for the language. Find a sentence true in one structure and false in the other.

**Solution**.

- $\exists x \forall y (\neg (y \hat{=} x) \to E(x, y))$.

  $(\mathbb{N}, <)$ has the $<$-least element, but $(\mathbb{R}, <)$ doesn't.

- $\forall x \forall y \, (E(x, y) \to \exists z \, (E(x, z) \wedge E(z, y)))$.

  $(\mathbb{R}, <)$ is dense, and $(\mathbb{N}, <)$ is not.

# Example

Let $P$ be a $2$-place predicate, $\mathcal{M} = (M, I)$ be the $\mathcal{L}_{\{P\}}$-structure where $M = \{a, b, c, d\}$ and

$$I(P) = \{(a,b),(b,a),(a,c),(b,d),\\ (a,d),(b,c),(c,e),(d,e)\}$$

Classify the definability of every subset of $M$.

# Solution

- $\varnothing, \{c,d\}, \{a,b\}, \{e\}, \{a,b,e\}, \{c,d,e\}, \{a,b,c,d\}, M$ are definable.

  $\varphi_M(x) \equiv (x \doteq x), \quad \varphi_\varnothing \equiv \neg\varphi_M, \quad \varphi_{\{c,d\}}(x) \equiv \exists y \exists z (E(x,z) \wedge E(y,x) \wedge \neg E(x,y)),$

  $\varphi_{\{e\}}(x) \equiv \neg\exists y (E(x,y)), \quad \varphi_{\{a,b\}}(x) \equiv \exists y (\neg(x \doteq y) \wedge E(x,y) \wedge E(y,x)),$

  $\varphi_{\{a,b,e\}} \equiv \varphi_{\{a,b\}} \vee \varphi_{\{e\}}, \quad \varphi_{\{c,d,e\}} \equiv \varphi_{\{c,d\}} \vee \varphi_{\{e\}}, \quad \varphi_{\{a,b,c,d\}} \equiv \varphi_{\{a,b\}} \vee \varphi_{\{c,d\}}.$

- The rest are not definable:
  - $\{a\}, \{a,c\}, \{a,d\}, \{a,c,d\}, \{a,e\}, \{a,c,e\}, \{a,d,e\}, \{a,c,d,e\},$

    $\{b\}, \{b,c\}, \{b,d\}, \{b,c,d\}, \{b,e\}, \{b,c,e\}, \{b,d,e\}, \{b,c,d,e\}$

    are not definable – use the bijection $\pi_1$ that swaps $a, b$ and pointwise fixes $c, d, e$;
  - $\{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{c,e\}, \{a,c,e\}, \{b,c,e\}, \{a,b,c,e\},$

    $\{d\}, \{a,d\}, \{b,d\}, \{a,b,d\}, \{d,e\}, \{a,d,e\}, \{b,d,e\}, \{a,b,d,e\}$

    are not definable – use the bijection $\pi_2$ that swaps $c, d$ and pointwise fixes $a, b, e$.

# Example

We say $S \subseteq |\mathfrak{A}|^n$ is *definable* in a structure $\mathfrak{A}$ if there is a formula $\varphi(\bar{x})$ in the language for $\mathfrak{A}$ such that

$$S = \{(\bar{a}) \in |\mathfrak{A}|^n \mid \mathfrak{A} \models \varphi[\bar{a}]\}.$$

1. For each of the following sets, give a formula which defines it in $(\mathbb{N}; <)$.
   - $O = \{0\}$.
   - $A = \{1\}$.
   - $E = \{(m, n) \mid m + 3 < n\}$.

# Example

2. For each of the following sets, give a formula which defines it in $(\mathbb{N}; +, \cdot)$.
    - $O = \{0\}$
    - $A = \{1\}$
    - $B = \{2\}$
    - $E = \{(m, n) \mid n$ is the successor of $m$ in $\omega\}$.
    - $F = \{(m, n) \mid m < n$ in $\omega\}$.

# Solution

1. In $(\mathbb{N}; <)$,
   - $0$ is the only $<$-least element. Let $\varphi_0(x) \equiv \forall y\, (y \neq x \to x < y)$. Then
   $$\{0\} = \{x \mid (\omega, <) \models \varphi_0(x)\}.$$
   - $1$ is the $<$-least element of $\omega \setminus \{0\}$. Let $\varphi_1(x) \equiv$
   $$0 < x \wedge \forall y\, (0 < y \wedge \neg(y \mathrel{\hat{=}} x) \to x < y),$$
   where $0 < y$ abbreviates $\forall z\, (\varphi_0(z) \to z < y)$. Then $\varphi_1(x)$ defines $A$.
   - $E$ is defined by $\varphi_2(x, y) \equiv$
   $$\exists u\, \exists v\, \exists w\, (x < u \wedge u < v \wedge v < w \wedge w < y).$$

2. In $(\mathbb{N}; +, \cdot)$,
   - $0$ is the unit of $+$. So $\{0\}$ is defined by
   $$\psi_0(x) \equiv \forall y \, (y + x \hat{=} y).$$
   - $1$ is the unit of $\cdot$, hence $\{1\}$ is defined by
   $$\psi_1(x) \equiv \forall y \, (y \cdot x \hat{=} y).$$
   - $2$ is obtained by $1 + 1$, so $\{2\}$ is defined by
   $$\psi_2(x) \equiv \exists y \, (\psi_1(y) \wedge x \hat{=} y + y).$$

2. (Cont'd)

- In $E$, every $(m, n)$ has the relation $n = m + 1$, so $E$ is simply defined by

$$\psi_3(x, y) \equiv \exists z \, (\psi_1(z) \wedge x + z \hat{=} y).$$

- $m < n$ if and only if $m + k = n$ for some $k > 0$, so a definition for $F$ is as follows,

$$\psi_4(x, y) \equiv \exists z \, (\neg \psi_0(z) \wedge x + z \hat{=} y).$$

# Example

Consider the structure $\mathfrak{R} = (\mathbb{R}, +, \cdot)$.

1. Give a formula which defines in $\mathfrak{R}$ the set $[0, \infty)$
2. Show that for every two rational numbers $a < b$, the interval $[a, b)$ is definable in $\mathfrak{R}$.

# Solution

1. Every non-negative real can be written as the square of some real, so

$$x \in [0, \infty) \quad \text{iff} \quad \exists y\, (x \hat{=} y \cdot y).$$

2. First, we claim that

   If $\{a\}$ and $\{b\}$ are definable, then so is $[a, b)$.

   Suppose $\varphi(x)$ defines $\{a\}$ and $\psi(x)$ defines $\{b\}$, then $[a, b)$ is defined by

$$\theta(x) \equiv \exists u \exists v\, (\varphi(u) \wedge x \hat{=} u + v \cdot v) \\ \wedge \neg(\exists u \exists v\, (\psi(u) \wedge x \hat{=} u + v \cdot v)).$$

# Solution

Next, we show that

Every rational number is definable.

Since every rational number is the quotient of two integers, thus definable by formulas of the form $q \cdot x \hat{=} p$, where $p$, $q$ are integers, it suffices to show that

Every integer is definable.

# Solution

Note that $\{0\}$ and $\{1\}$ are definable, as we did before. Then every positive integer $n$ is defined by

$$x \hat{=} \underbrace{(\cdots(1+1)+\cdots)+1}_{n}$$

and each negative integer $-n$ is defined by the formula $x + n \hat{=} 0$.

# Example

1. Show that the function

$$\begin{aligned} \pi(0) &= 0; \\ \pi(2^n \cdot 3^m \cdot k) &= 2^m \cdot 3^n \cdot k, \end{aligned}$$

   where $m, n \geq 0$, $k > 0$, $k$ is divisible by neither $2$ and $3$, is an automorphism of $(\mathbb{N}; \cdot)$.

2. Use (1) to show that the addition relation,

$$R = \{(m, n, p) \mid p = m + n\},$$

   is not definable in $(\mathbb{N}; \cdot)$.

# Solution

1. Clearly, $\pi$ is surjective. $\pi$ being one-to-one follows from the factorization theorem in number theory

$$\begin{aligned}
&2^{u_1} \cdot 3^{v_1} \cdot k_1 \neq 2^{u_2} \cdot 3^{v_2} \cdot k_2 \\
\Leftrightarrow\ &u_1 \neq u_2 \vee v_1 \neq v_2 \vee k_1 \neq k_2 \\
\Leftrightarrow\ &3^{u_1} \cdot 2^{v_1} \cdot k_1 \neq 3^{u_2} \cdot 2^{v_2} \cdot k_2.
\end{aligned}$$

Next, we show that $\pi$ preservers multiplication:

$$\begin{aligned}
&(2^{u_1} \cdot 3^{v_1} \cdot k_1) \cdot (2^{u_2} \cdot 3^{v_2} \cdot k_2) = 2^{u_3} \cdot 3^{v_3} \cdot k_3 \\
\Leftrightarrow\ &u_3 = u_1 + u_2 \wedge v_3 = v_1 + v_2 \wedge k_3 = k_1 \cdot k_2 \\
\Leftrightarrow\ &(3^{u_1} \cdot 2^{v_1} \cdot k_1) \cdot (3^{u_2} \cdot 2^{v_2} \cdot k_2) = 3^{u_3} \cdot 2^{v_3} \cdot k_3
\end{aligned}$$

# Solution

2. We need the following basic property of definable set: if $\pi$ is an automorphism on $\mathfrak{A}$, then $\pi[D] = D$.

   Now to show that the set $R$ is not definable, it suffices to find an $(m, n, p) \in R$ such that $\pi(m, n, p) \notin R$. Consider $(3, 4, 7) \in R$, we have $\pi(3, 4, 7) = (2, 9, 7) \notin R$. Hence the addition function is not definable in $(\omega, \cdot)$.

# Example

考虑如下的一阶语言 $\mathcal{L}_A$，$A = \{+, \cdot, 0, 1\}$，其中，$+$ 和 $\cdot$ 为两个二元运算符号，0 和 1 是两个常数符号，

1. 用 $\mathcal{L}_A$ 公式写出如下命题的形式表达式
   - $+$ 和 $\cdot$ 是可交换的，可结合的；
   - $\cdot$ 对 $+$ 满足分配律；
   - 0 是 $+$ 的单位元，1 是 $\cdot$ 的单位元，它们不相同；
   - 每一个元素都有唯一的一个 $+$ 逆元素（即它们经过 $+$ 运算后结果为 0）；

# Example

*(Cont'd)*

- 每一个非 0 元素都有唯一的一个 $\cdot$ 逆元素（即它们经过 $\cdot$ 运算后结果为 1）；
- 若两个元素经过 $\cdot$ 运算后结果为 0，则其中之一必为 0。

将前述命题取为公理之后得到一个公理系统，记为 $F$。

2. 证明 $F$ 是一个相容（即无矛盾）的系统。

# Example

3. 用 $A$ 记表达式 $\exists x\,(x \cdot x + 1 = 0)$。
   - 证明新系统 $F \cup \{A\}$ 仍相容。
   - 证明新系统 $F \cup \{\neg A\}$ 仍相容。
4. 为 $F$ 加进一组新的公理 $H$ 以满足如下的要求：
   令 $E = F \cup H$
   - $E$ 是一个相容的系统；
   - $E$ 的任何一个模型中都有无穷多个元素。

# Solution

1. - For addition, $\forall x \forall y\, (x + y = y + x)$, and

$$\forall x \forall y \forall z\, ((x + y) + z = x + (y + z))$$

   For multiplication, $\forall x \forall y\, (x \cdot y = y \cdot x)$, and

$$\forall x \forall y \forall z\, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

   - $\forall x \forall y \forall z\, (z \cdot (x + y) = z \cdot x + z \cdot y)$.
   - For $0$, $\forall x\, (x = 0 \leftrightarrow \forall y\, (x + y = y))$.
     For $1$, $\forall x\, (x = 1 \leftrightarrow \forall y\, (x \cdot y = y))$.
     $0$ and $1$ are different: $\neg(0 = 1)$.

# Solution

(Cont'd)

- $\forall x \exists y\,(x + y = 0 \wedge \forall z\,(x + z = 0 \to z = y))$.
- $\forall x\,(\neg(x = 0) \to \exists y\,(x \cdot y = 1 \wedge \forall z\,(x \cdot z = 1 \to z = y)))$.
- $\forall x \forall y\,(x \cdot y = 0 \to x = 0 \vee y = 0)$.

1. $F$ is in fact the axiom system for fields. It is satisfied by the field of reals $(\mathbb{R}, +, \cdot, 0, 1)$. By Soundness, $F$ is consistent.

# Solution

3. $A$ says there is an imaginary number (as $i = \sqrt{-1}$ in the field of complex number). $(\mathbb{C}, +, \cdot, 0, 1)$ is a model for $F \cup \{A\}$, $(\mathbb{R}, +, \cdot, 0, 1)$ is a model for $F \cup \{\neg A\}$. So by Soundness, $F \cup \{A\}$ and $F \cup \{\neg A\}$ are consistent.

4. $F$ can also be satisfied by models of the form $(\mathbb{Z}_p, +, \cdot, 0, 1)$, $p$ is a prime. To exclude models like these, we add the following sentences: for each number $n \in \mathbb{N}$, let

$$\psi_n(x) \equiv \neg(\underbrace{x + \cdots + x}_{n} = 0).$$

# Solution

4. (Cont'd)

   Let $H = \{\forall x\, \psi_n(x) \mid n \in \mathbb{N}\}$.[2] Every model of $F \cup H$, if any, must be infinite, since for any $n < m \in \mathbb{N}$,

   $$\underbrace{1 + \cdots + 1}_{m} \neq \underbrace{1 + \cdots + 1}_{n}.$$

   Note that $(\mathbb{Q}, +, \cdot, 0, 1)$ and the two fields mentioned before all satisfy $F \cup H$. By Soundness again, $F \cup H$ is consistent.

REMARK. As a corollary of the argument, there is no $H$ such that the models of $E$ are exactly all the finite models of $F$.

[2] It suffices to use $H = \{\neg\psi_p \mid p \text{ is prime}\}$.

Key items to be tested in the final exam:

- Express statements in formulas
- Complete set of connectives
- RPN, DNF, Truth table
- Proofs in propositional logic and 1st-order logic
- Six theorems
- Applications of compactness
- definable sets
- Elementary classes

- Final: May 22 (Thursday), 10:00-11:40 @ 教七 106

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# GOOD LUCK!