Equivalent versions of three theorems in propositional logic

Soundness Theorem（可靠性定理）

Theorem 0.1. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

(2) Any satisfiable set of formulas is consistent.

Proof. (1) $\Rightarrow$ (2). Let Γ be a set of formulas satisfied by a truth assignment ν. We want to show that Γ is consistent. Suppose NOT. Then there is a formula φ such that

$$\Gamma \vdash \varphi \quad \text{and} \quad \Gamma \vdash \neg\varphi.$$

By (1), we have

$$\Gamma \models \varphi \quad \text{and} \quad \Gamma \models \neg\varphi.$$

Hence,

$$\bar{\nu}(\varphi) = 1 \quad \text{and} \quad \bar{\nu}(\neg\varphi) = 1.$$

But this is impossible! Hence Γ must be consistent.

(2) $\Rightarrow$ (1). Suppose $\Gamma \vdash \varphi$, we want to show that $\Gamma \models \varphi$. We prove by contradiction. Suppose there exist a truth assignment ν such that

- $\bar{\nu}(\gamma) = 1$, for every $\gamma \in \Gamma$; and
- $\bar{\nu}(\varphi) = 0$.

This means that ν satisfies $\Gamma \cup \{\neg\varphi\}$. By (2), $\Gamma \cup \{\neg\varphi\}$ is consistent. But from the assumption $\Gamma \vdash \varphi$, we have $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ and $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$. Contradiction! So it must be that $\Gamma \models \varphi$. ☐

Completeness Theorem（完全性定理）

Theorem 0.2. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

(2) Any consistent set of formulas is satisfiable.

Proof. (1) $\Rightarrow$ (2). Let Γ be a consistent set of formulas. If Γ is unsatisfiable, since no truth assignment satisfies Γ, $\Gamma \models \varphi$ for any formula φ, in particular $A_1 \wedge \neg A_1$. But then by (1), $\Gamma \vdash A_1 \wedge \neg A_1$, contradicting the assumption that Γ is consistent.

(2) $\Rightarrow$ (1). Suppose $\Gamma \vDash \varphi$ and $\Gamma \nvdash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent.[1] (2), $\Gamma \cup \{\neg\varphi\}$ is satisfied by some truth assignment ν. In particular, $\bar{\nu}(\neg\varphi) = 1$. But from $\Gamma \models \varphi$, we have $\bar{\nu}(\varphi) = 1$. Contradiction! So if $\Gamma \models \varphi$, it must be that $\Gamma \vdash \varphi$. ☐

[1]This is in fact an "if and only if". If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by definition. By Deduction, $\Gamma \vdash \neg\varphi \to \varphi$. By Group III axiom, $\Gamma \vdash \varphi$. The direction that $\Gamma \cup \{\neg\varphi\}$ is consistent implies $\Gamma \nvdash \varphi$ is proved in the second part of the proof for Soundness Theorem.

Compactness Theorem（紧致性定理）

Theorem 0.3. The following two statements are equivalent:

(1) For any set of formula Γ, if $\Gamma \models \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \models \varphi$.

(2) For any set of formula Γ, if every finite subset Γ_0 of Γ is satisfiable, then Γ is satisfiable.

Proof. (1) $\Rightarrow$ (2). Suppose that every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable. Consider the falsity $\varphi \equiv (A_1 \wedge \neg A_1)$. If Γ is unsatisfiable, then $\Gamma \models \varphi$, since no truth assignment satisfies Γ. By (1), for for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \models \varphi$. As φ is false, there is no truth assignments satisfies Γ_0, contradicting the assumption that every finite subset of Γ is satisfiable.

(2) $\Rightarrow$ (1). Suppose $\Gamma \models \varphi$. Assume that for any finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \models \varphi$ fails, i.e. $\Gamma_0 \cup \{\neg\varphi\}$ is satisfiable. It follows that every finite subsets of $\Gamma \cup \{\neg\varphi\}$ is satisfiable. By (2), $\Gamma \cup \{\neg\varphi\}$ is satisfiable, contradicting that $\Gamma \models \varphi$. This proves (2) $\Rightarrow$ (1). $\square$