Mathematical Logic

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1. Propositional Logic
   - $\mathcal{L}_0$-formulas
   - Truth Assignments
   - Proof System for $\mathcal{L}_0$
   - Compactness of $\mathcal{L}_0$

Compactness

Definition 4.1

We say $\Gamma \subseteq \mathcal{L}_0$ is finitely satisfiable if every finite $\Gamma_0 \subset \Gamma$ is satisfiable, i.e. there is a truth assignment ν such that for all $\psi \in \Gamma_0$, $\bar{\nu}(\psi) = T$.

Theorem 4.1 (Compactness for $\mathcal{L}_0$)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$ *and* $\Gamma \models \varphi$. *Then there is a finite* $\Gamma_0 \subset \Gamma$ *such that* $\Gamma_0 \models \varphi$.

Theorem 4.2 (Compactness, Version II)

A set of formulas is satisfiable if and only if every finite subset is satisfiable.[a]

[a] This version can be proved using ultraproduct of models.

Topological Compactness

Underlying set: the set of all truth assignments.

$$V = \{\nu \mid \nu : S_0 \to \{T, F\}\}.$$

Topology: $\tau = \{\mathcal{O}_\Gamma \mid \Gamma \subset \mathcal{L}_0\}$, where for each Γ,

$$\mathcal{O}_\Gamma = \{\nu \mid \bar{\nu}(\varphi) = F,\ \exists\, \varphi \in \Gamma\}$$

The closed sets are exactly of the form

$$\mathcal{B}_\Gamma = \{\nu \mid \bar{\nu}(\varphi) = T,\ \forall\, \varphi \in \Gamma\}.$$

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Theorem 4.3

The space (V, τ) is Hausdorff and Compact.

<u>CLAIM</u>. τ is a topology.

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CLAIM. τ is a topology.

1. $\varnothing = \mathcal{O}_{\{A_1 \to A_1\}}$, and $V = \mathcal{O}_{\{\neg A_1 \to A_1\}}$, thus $\varnothing, V \in \tau$.
2. $\bigcup\{\mathcal{O}_\Gamma \mid \Gamma \in I\} = \mathcal{O}_{\cup\{\Gamma \mid \Gamma \in I\}}$.
3. $\mathcal{O}_{\Gamma_1} \cap \mathcal{O}_{\Gamma_2} = \mathcal{O}_{\{\varphi_1 \vee \varphi_2 \mid \varphi_1 \in \Gamma_1, \varphi_2 \in \Gamma_2\}}$.

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CLAIM. (V, τ) is Hausdorff.

Suppose $\nu_1, \nu_2 \in V$ and $\nu_1 \neq \nu_2$. Let n be s.t.

$$\nu_1(A_n) \neq \nu_2(A_n).$$

We may assume that $\nu_1(A_n) = 0$ and $\nu_2(A_n) = 1$. Then $\nu_1 \in \mathcal{O}_{\{A_n\}}$ and $\nu_2 \in \mathcal{O}_{\{\neg A_n\}}$, and

$$\mathcal{O}_{\{A_n\}} \cap \mathcal{O}_{\{\neg A_n\}} = \varnothing.$$

Topological Compactness

Let (X, τ) be a topological space. Suppose (X, τ) is a Hausdorff.

Definition

- Let $\mathcal{F}$ be a collection of closed subsets of X. If every nonempty finite subcollection $\mathcal{F}_0 \subset \mathcal{F}$ has nonempty intersection, i.e. $\bigcap \mathcal{F}_0 \neq \varnothing$, we say $\mathcal{F}$ has finite intersection property (FIP).
- We say (X, τ) is compact, if any nonempty collection $\mathcal{F}$ of closed subsets of X with FIP has nonempty intersection, i.e. $\bigcap \mathcal{F} \neq \varnothing$.

- The Compactness theorem in fact is a consequence of Tychonoff's theorem (which says the product of compact spaces is compact) applied to compact Stone spaces (Hausdorff + totally disconnected).

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Top. $\Rightarrow$ Log. Finite satisfiability says that $\{\mathcal{B}_{\{\varphi\}} \mid \varphi \in \Gamma\}$ has the finite intersection property (FIP). By Top. Comp., $\bigcap_{\varphi\in\Gamma}\mathcal{B}_{\{\varphi\}} = \mathcal{B}_\Gamma \neq \varnothing$.

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Log. $\Rightarrow$ Top. Suppose $\{B_\Gamma \mid \Gamma \in \mathcal{G}\}$ has the FIP. We show $\bigcup_{\Gamma\in\mathcal{G}}\Gamma$ is finitely satisfiable. Let $\Lambda \subseteq \bigcup\mathcal{G}$ be finite, choose finite $\mathcal{G}_0 \subseteq \mathcal{G}$, such that $\Lambda \subseteq \bigcup\mathcal{G}_0$. By FIP,

$$\mathcal{B}_\Lambda \supseteq \mathcal{B}_{\cup\mathcal{G}_0} = \bigcap_{\Gamma\in\mathcal{G}_0}\mathcal{B}_\Gamma \neq \varnothing.$$

Applications of Compactness

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Every set M can be (totally) ordered.

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Assign p_{ab} for each $(a, b) \in M \times M$. Consider a Σ_M s.t.

1. $p_{ab} \to \neg p_{ba}$, for $a \in M$.
2. $p_{ab} \wedge p_{bc} \to p_{ac}$, for $a, b, c \in M$.
3. $p_{ab} \vee p_{ba}$, for $a, b \in M, a \neq b$.

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If an assignment $\nu \models \Sigma_M$, then the set

$$\{(a, b) \mid \nu(p_{ab}) = T\} \text{ is a total order over } M.$$

To show every finite $\Sigma \subset \Sigma_M$ is satisfiable, it suffices to show that

CLAIM. *For every finite* $K \subset M$*, the corresponding* $\Sigma_K = \Sigma_M | K$ *is satisfiable, i.e. every finite* K *is totally orderable.*

[1] Such u exists, as otherwise K would be infinite.

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PROOF. Prove by induction on $|K|$.

- The case $|K| = 1$ is trivial. Suppose the claim holds for all K' of size $< |K|$, and now consider K.
- Select a $u \in K$ such that there is no $v \in K$ satisfies that $\Sigma_K \vdash p_{vu}$.[1] Let $K' = K \backslash \{u\}$. $|K'| < |K|$, by the inductive hypothesis, $\Sigma_{K'}$ is satisfiable, say via ν'.
- Extend ν' by setting $\nu(p_{uv}) = 1$ for any $v \in K'$ and $\nu(p_{vu}) = 0$ for all $v \in K$, and $\nu \upharpoonright K' = \nu'$. It is routine to check that ν gives a total order on K, i.e. $\nu \models \Sigma_K$.

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4-coloring infinite planar graphs

Theorem 4.5

A graph (V, E) is k-colorable iff every finite subgraph (V_0, E_0) is k-colorable.

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So the famous 4-**color theorem** (every finite planar graph is 4-colorable) implies that all planar graph is 4-colorable. Assign $p_{v,i}$ for each $v \in V$ and $1 \leqslant i \leqslant k$. Consider a set Σ such that

1. $p_{v1} \vee \cdots \vee p_{vk}$, for $v \in V$;
2. $\neg(p_{vi} \wedge p_{vj})$, for $v \in V$, $1 \leqslant i < j \leqslant k$;
3. $\neg(p_{ui} \wedge p_{vi})$, for $(u, v) \in E$, $1 \leqslant i \leqslant k$.

The ultrafilter theorem

Theorem 4.6

Every subset $\mathcal{F} \subset \mathscr{P}(X)$ with the FIP property, i.e.:

$$\text{for all finite } F \subset \mathcal{F}, \bigcap F \neq \varnothing$$

can be extended to an ultrafilter on X.

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Assign p_u for each $u \in \mathscr{P}(X)$. Consider a Σ such that for $u, v \in \mathscr{P}(X)$,

- p_X, $\neg p_\varnothing$;
- $p_u \to p_v$, if $u \subseteq v$;
- $p_u \wedge p_v \to p_{u \cap v}$;
- $p_{u^c} \leftrightarrow \neg p_u$, where $u^c := u^c$;
- p_f, for $f \in \mathcal{F}$.

- Without loss of generality, we may assume that $\mathcal{F}$ is a filter.
- For $A \subseteq \mathscr{P}(X)$, let $\langle A \rangle$ be the smallest subset of $\mathscr{P}(X)$ containing A and closed under complement and finite $\cap$, $\cup$.
- $(\langle A \rangle, \cup, \cap, *^c, \subset)$ is the subalgebra of $(\mathscr{P}(X), \cup, \cap, *^c, \subseteq)$ generated by A.
- Let $\Sigma_{\langle A \rangle} = \Sigma | \langle A \rangle$, i.e. the part of Σ with only $u, v \in \langle A \rangle$ and $f \in \mathcal{F} \cap \langle A \rangle$.
- If $\Sigma_0 \subset \Sigma$ is finite, then there is a finite $A \subset \mathscr{P}(X)$ such that $\Sigma_0 \subset \Sigma_{\langle A \rangle}$, thus it sufffices to show that

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CLAIM. *For every finite $A \subseteq \mathscr{P}(X)$, $\Sigma_{\langle A \rangle}$ is satisfiable.*

- For $\langle A \rangle$ there is a $\subset$-minimal $A^* \subseteq A$ such that $\langle A \rangle = \langle A^* \rangle$. We may assume that $A =$ such A^*.

Prove by induction on $|A|$. The case $n = 1$ is trivial. Now suppose the claim holds for all A' with $|A'| < |A|$, work with $|A|$. There are two cases:

- for every $u \in \langle A \rangle$, $\Sigma_{\langle A \rangle} \vdash p_u \vee p_{u^c}$.
  - As $\varnothing \notin \mathcal{F}$, for every $u \in \langle A \rangle$, exactly one of $\{u, u^c\}$ is in $\mathcal{F}$.
  - Thus $\Sigma_{\langle A \rangle} \vdash \mathcal{F} \cap \langle A \rangle$ is an ultrafilter over $\langle A \rangle$.
  - For $u \in \langle A \rangle$, set ν such that

$$\nu(p_u) = 1 \quad \text{iff} \quad u \in \mathcal{F}.$$

Otherwise, ...

- there is a $u \in \langle A \rangle$ such that $\Sigma_{\langle A \rangle} \not\models p_u \vee p_{u^c}$.
  - Let A' be maximal such that $A' \subset A$ and $\langle A' \rangle \subseteq \langle A \rangle \setminus \{u\}$.
  - By the minimality assumption on A, we may assume that $|A'| + 1 = |A|$. By the ind. hyp., there is some $\nu' \models \Sigma_{\langle A' \rangle}$.
  - Let $\mathcal{U}_{A'} = \{w \in \langle A' \rangle \mid \nu'(p_w) = 1\}$.
  - $\mathcal{U}_{A'}$ is finite, so $\bigcap \mathcal{U}_{A'} \neq \varnothing$.
  - One of $\{u, u^c\}$ has non-empty intersection with $\bigcap \mathcal{U}_{A'}$, say u.
  - For $w \in \langle A \rangle$, set ν such that

$$\nu(p_w) = 1 \quad \text{iff} \quad w \supset u \cap (\textstyle\bigcap \mathcal{U}_{A'})$$

Verify that $\nu \models \Sigma_{\langle A \rangle}$.

Exercise 4.1

Use Compactness to show that every partial order $<_0$ on a set X can be extended to a total order $<$ on X.

Exercise (思考题)

1. Suppose $\varnothing \notin \mathcal{F} \neq \varnothing$. Prove that $\mathcal{F}$ is a filter on X iff $u \cap v \in \mathcal{F} \Leftrightarrow u \in \mathcal{F} \wedge v \in \mathcal{F}$ for every $u, v \subseteq X$.
2. Suppose $\varnothing \notin \mathcal{F} \neq \varnothing$ is a filter on X. Show that $\mathcal{F}$ is an ultrafilter iff $u \cup v \in \mathcal{F} \Leftrightarrow u \in \mathcal{F} \vee v \in \mathcal{F}$.
3. An ultrafilter U on X is **principal** iff there is an $x \in X$ such that $U = \{u \subset X \mid x \in u\}$. Show that every ultrafilter on a finite X is principal.

König's tree lemma

Theorem 4.7 (König's Lemma)

If T is a tree such that every node has finitely many successors, and contains arbitrarily long finite path, then T has an infinite path.

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Assign p_t for each $t \in T$. Let Γ consist of

1. $\bigvee_{t \in T_k} p_t$, for $k \in \mathbb{N}$;
2. $\neg(p_s \wedge p_t)$ for $s \neq t \in T_k$, $k \in \mathbb{N}$.
3. $p_t \to p_s$ for $s, t \in T$ such that $s <_T t$.

Here T_k is the k-th level of T.

The marriage problem (in Linguistic)

Theorem 4.8

*Suppose each word has finitely many meanings and every k words have $\geqslant k$ different meanings. Then there is an injective f : {**W**ords} $\to$ {**M**eanings}.*

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Finite cases are easy. For infinite W, for $(v, n) \in W \times M$, assign p_{vn}. Consider a set Σ such that

1. $p_{vn_1} \vee \cdots \vee p_{vn_l}$, for $v \in W$, n_i's are the meanings of v.
2. $\neg(p_{vm} \wedge p_{vn})$, for $v \in W$ and $m \neq n \in M$.
3. $\neg(p_{vn} \wedge p_{wn})$, for $v \neq w \in W$ and $n \in M$.