Mathematical Logic

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Recall

Two aspects of a formal language.

- Syntax
  - formulas
  - connectives
- Semantics
  - truth value/truth assignment
  - truth table/truth function

Next

1. Propositional Logic
   - $\mathcal{L}_0$-formulas
   - Truth Assignments
   - Proof System for $\mathcal{L}_0$

A proof system for $\mathcal{L}_0$

Suppose that φ_1, φ_2 and φ_3 are $\mathcal{L}_0$-formulas. Then each of the following $\mathcal{L}_0$-formulas is a logical axiom:

(Group I axioms)

- $(\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3))$
- $\varphi_1 \to \varphi_1$
- $\varphi_1 \to (\varphi_2 \to \varphi_1)$

(Group II axioms)

- $\varphi_1 \to (\neg\varphi_1 \to \varphi_2)$

(Group III axioms)

- $(\neg\varphi_1 \to \varphi_1) \to \varphi_1$

(Group IV axioms)

- $\neg\varphi_1 \to (\varphi_1 \to \varphi_2)$
- $\varphi_1 \to (\neg\varphi_2 \to \neg(\varphi_1 \to \varphi_2))$

Δ_0 denote the set of all logical axioms.

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Δ_0 denote the set of all logical axioms.

Proposition 3.1

Every logical axioms above is a tautology.

Γ-proof

Suppose that $\Gamma \subseteq \mathcal{L}_0$.

Definition 3.1

Suppose that $s = \langle \varphi_i : i \leqslant n \rangle$ is a finite sequence of propositional formulas. s is a Γ-proof if for each $i \leqslant n$ at least one of the following happens:

- $\varphi_i \in \Gamma$;
- φ_i is a logical axiom;
- there exists $j_1, j_2 < i$ such that $\varphi_{j_2} = \varphi_{j_1} \to \varphi_i$. This rule is called Modus Ponens.

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Definition 3.2

$\Gamma \vdash \varphi$ (Γ proves φ) iff there exists a finite sequence $s = \langle \varphi_i : i \leqslant n \rangle$ such that s is a Γ-proof and such that $\varphi_n = \varphi$.

Such sequence s is called a proof from Γ to φ, and φ is called a consequence of Γ.

When $\Gamma = \varnothing$, write $\vdash \varphi$.

Some properties of Γ-proofs

1. If s is a Γ-proof, and t is an initial segment of s, then t is also a Γ-proof.
2. If $s = \langle \varphi_i : i \leqslant n \rangle$ and $t = \langle \psi_i : i \leqslant m \rangle$ are two Γ-proofs, then so is

$$s + t = \langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \rangle.$$

Definition 3.3

Suppose that $\Gamma \subseteq \mathcal{L}_0$.

1. Γ is inconsistent if for some formula φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
2. Γ is consistent if Γ is not inconsistent.
3. Γ is maximally consistent if and only if for each formula ψ if $\Gamma \cup \{\psi\}$ is consistent then $\psi \in \Gamma$.

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Compare with Γ being **satisfiable**. Our ultimate goal is to show that

$$\Gamma \text{ is consistent if and only if } \Gamma \text{ is satisfiable.}$$

We first show the ‘if” direction.

Soundness

Theorem 3.2 (Soundness, version I)

If $\Gamma \subseteq \mathcal{L}_0$ is satisfiable. Then Γ is consistent.

Definition 3.4 (Logical implication)

Suppose $\Gamma \subseteq \mathcal{L}_0$ and $\varphi \in \mathcal{L}_0$. Then Γ logically implies φ, write $\Gamma \models \varphi$, if and only if for every truth assignment ν, $\nu \models \Gamma$ implies $\nu \models \varphi$.

Theorem 3.3 (Soundness, version II)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$ *and that* $\Gamma \vdash \varphi$. *Then* $\Gamma \models \varphi$.

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Proof.

Suppose $s = \langle \varphi_i : i = 1, \cdots, n \rangle$ is a Γ-proof. Prove by induction on $|s|$, the length of s.

BASE STEP $n = 1$: $\varphi_1 \in \Gamma \cup \Delta_0$.

INDUCTIVE STEP $n = k + 1$: Suppose the theorem is true for all proofs of length $\leqslant k$, in particular, true for φ's in $s|i = \langle \varphi_1, \cdots, \varphi_i \rangle$ for $i \leqslant k$. Consider φ_{k+1}. there are three cases:

- $\varphi_{k+1} \in \Gamma$
- $\varphi_{k+1} \in \Delta_0$
- φ_{k+1} is obtained by MP

□

Lemmas for the completeness of $\mathcal{L}_0$

Lemma 1 (Inference)

Suppose $\Gamma \subseteq \mathcal{L}_0$*,* $\varphi, \psi \in \mathcal{L}_0$*. Suppose* $\Gamma \vdash \psi$ *and* $\Gamma \vdash (\psi \to \varphi)$*. Then* $\Gamma \vdash \varphi$*.*

Lemmas for the completeness of $\mathcal{L}_0$

Lemma 1 (Inference)

Suppose $\Gamma \subseteq \mathcal{L}_0$, $\varphi, \psi \in \mathcal{L}_0$. *Suppose* $\Gamma \vdash \psi$ *and* $\Gamma \vdash (\psi \to \varphi)$. *Then* $\Gamma \vdash \varphi$.

Proof

No logical axioms required.

Let s be a Γ-proof for $\Gamma \vdash \psi$, (denoted as $\Gamma \vdash_s \psi$)
t be a Γ-proof for $\Gamma \vdash (\psi \to \varphi)$.
Then $s + t + \langle \varphi \rangle$ is a Γ-proof for $\Gamma \vdash \varphi$. □

Lemma 2 (Deduction)

Suppose $\Gamma \subseteq \mathcal{L}_0$, $\varphi, \psi \in \mathcal{L}_0$ *and* $\Gamma \cup \{\varphi\} \vdash \psi$. *Then* $\Gamma \vdash (\varphi \to \psi)$.

Lemma 2 (Deduction)

Suppose $\Gamma \subseteq \mathcal{L}_0$, $\varphi, \psi \in \mathcal{L}_0$ *and* $\Gamma \cup \{\varphi\} \vdash \psi$. *Then* $\Gamma \vdash (\varphi \to \psi)$.

Proof (Group I axioms are needed).

Let $s = \langle \varphi_1, \cdots, \varphi_n = \psi \rangle$ be a $\Gamma \cup \{\varphi\}$-proof for $\Gamma \cup \{\varphi\} \vdash \psi$. We prove by induction on $|s|$ that there is a Γ-proof s^* for $\Gamma \vdash (\varphi \to \varphi_n)$.

BASE STEP $|s| = 1$: then $\varphi_1 \in \Gamma \cup \{\varphi\}$

If $\varphi_1 = \varphi$, *let* $s^* = \langle (\varphi_1 \to \varphi_1) \rangle$.

If $\varphi_1 \neq \varphi$, *let* $s^* = \langle \varphi_1, \varphi_1 \to (\varphi \to \varphi_1), \varphi \to \varphi_1 \rangle$.

INDUCTIVE STEP $|s| = k + 1$:

$\varphi_{k+1} \in \Gamma \cup \{\varphi\} \cup \Delta_0$: *same as the case* $|s| = 1$.

φ_{k+1} *is obtained by MP:* $\varphi_{j_1}, \varphi_{j_2} = \varphi_{j_1} \to \varphi_{k+1}$.[a] □

[a]Students fill in the rest.

Lemma 3

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\psi \in \mathcal{L}_0$. *Suppose that* Γ *is inconsistent. Then* $\Gamma \vdash \psi$.

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Proof

Uses the Deduction lemma and Group II axioms.

Let φ, s, t be such that $\Gamma \vdash_s \varphi$ and $\Gamma \vdash_t \neg\varphi$. Below is a Γ-proof for $\Gamma \vdash \psi$:

$$s + t + \langle \varphi \to (\neg\varphi \to \psi), \neg\varphi \to \psi, \psi \rangle.$$

□

Lemma 4

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi \in \mathcal{L}_0$*. Suppose that* Γ *is consistent. Then at least one of* $\Gamma \cup \{\varphi\}$ *or* $\Gamma \cup \{\neg\varphi\}$ *is consistent, possibly both.*

Lemma 4

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi \in \mathcal{L}_0$. *Suppose that* Γ *is consistent. Then at least one of* $\Gamma \cup \{\varphi\}$ *or* $\Gamma \cup \{\neg\varphi\}$ *is consistent, possibly both.*

Proof

Needs Group III axioms.

Towards a contradiction, suppose both are inconsistent.

- Apply Lemma 3 to $\Gamma = \Gamma \cup \{\neg\varphi\}$ and $\psi = \varphi$, we get $\Gamma \cup \{\neg\varphi\} \vdash \varphi$.
- By deduction, $\Gamma \vdash (\neg\varphi \to \varphi)$.
- Use Δ_0-III axiom, $(\neg\varphi \to \varphi) \to \varphi$.
- By Inference, $\Gamma \vdash \varphi$.
- $\Gamma \cup \{\varphi\}$ is inconsistent, by Lemma 3, $\Gamma \cup \{\varphi\} \vdash \neg\varphi$.
- Use Deduction and Inference, we have $\Gamma \vdash (\varphi \to \neg\varphi)$ and hence $\Gamma \vdash \neg\varphi$. Therefore Γ is inconsistent. Contradiction! □

Corollary 3.4

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi \in \mathcal{L}_0$. *Suppose that* Γ *is maximally consistent. Then*

1. *Either* $\varphi \in \Gamma$ *or* $(\neg\varphi) \in \Gamma$.
2. *If* $\Gamma \vdash \varphi$ *then* $\varphi \in \Gamma$.

Corollary 3.4

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi \in \mathcal{L}_0$. *Suppose that* Γ *is maximally consistent. Then*

1. *Either* $\varphi \in \Gamma$ *or* $(\neg\varphi) \in \Gamma$.
2. *If* $\Gamma \vdash \varphi$ *then* $\varphi \in \Gamma$.

Proof.

1. By Lemma 4, either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{\neg\varphi\}$ is consistent, by maximality, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
2. If $\varphi \notin \Gamma$, then $\neg\varphi \in \Gamma$. Therefore $\Gamma \vdash \neg\varphi$, plus the assumption $\Gamma \vdash \varphi$, Γ must be inconsistent. Contradiction! □

Lemma 5

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi_1, \varphi_2 \in \mathcal{L}_0$. *Suppose that* Γ *is maximally consistent. Then* $(\varphi_1 \to \varphi_2) \in \Gamma$ *iff either* $\varphi_1 \notin \Gamma$ *or* $\varphi_2 \in \Gamma$.

Lemma 5

Suppose $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi_1, \varphi_2 \in \mathcal{L}_0$. *Suppose that* Γ *is maximally consistent. Then* $(\varphi_1 \to \varphi_2) \in \Gamma$ *iff either* $\varphi_1 \notin \Gamma$ *or* $\varphi_2 \in \Gamma$.

Proof

Uses Group IV axioms.

"$\Leftarrow$":

- Suppose $\varphi_1 \notin \Gamma$. By maximality, $\neg\varphi_1 \in \Gamma$. Use Axiom Δ_0-IV-2: $\neg\varphi_1 \to (\varphi_1 \to \varphi_2)$. By MP and maximality, $(\varphi_1 \to \varphi_2) \in \Gamma$.
- Suppose $\varphi_2 \in \Gamma$. Use Axiom Δ_0-I-3: $\varphi_2 \to (\varphi_1 \to \varphi_2)$. Thus using Deduction and maximality, one has $(\varphi_1 \to \varphi_2) \in \Gamma$.

"$\Rightarrow$": Suppose $\varphi_1 \in \Gamma$ and $\varphi_2 \notin \Gamma$. Use Axiom Δ_0-IV-2,[a]

$$\varphi_1 \to (\neg\varphi_2 \to \neg(\varphi_1 \to \varphi_2)).$$

Apply Deduction twice, one has $\neg(\varphi_1 \to \varphi_2) \in \Gamma$. As Γ is consistent, it must be that $(\varphi_1 \to \varphi_2) \notin \Gamma$. □

[a] One can prove "$\Rightarrow$" without using Δ_0-IV-2.

Lemma 6

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is maximally consistent. Then* Γ *is satisfiable.*

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Proof.

Define a truth assignment ν as follows. For $i \in \mathbb{N}$,

$$\nu(A_n) = \begin{cases} T, & \text{if } \langle A_n \rangle \in \Gamma \\ F, & \text{if } \langle A_n \rangle \notin \Gamma \end{cases}$$

CLAIM. *This* $\bar{\nu}$ *works. In fact, for every* $\varphi \in \mathcal{L}_0$, $\bar{\nu}(\varphi) = T$ *iff* $\varphi \in \Gamma$.

Prove by induction on the construction of φ.

- Corollary 3.4 takes care of the case $\varphi = (\neg\psi)$
- Lemma 5 is for the case $\varphi = (\varphi_1 \to \varphi_2)$. □

Lemma 7

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is consistent. Then there exists a set* $\Gamma^* \subset \mathcal{L}_0$ *such that* $\Gamma \subseteq \Gamma^*$ *and such that* Γ^* *is maximally consistent.*

Lemma 7

Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Then there exists a set $\Gamma^* \subset \mathcal{L}_0$ such that $\Gamma \subseteq \Gamma^*$ and such that Γ^* is maximally consistent.

Proof.

- Enumerate all propositional formulas $\{\varphi_n : n \in \mathbb{N}\}$: At step n, enumerate all (finitely many) $\mathcal{L}_0$ formulas of length $\leqslant n$ and using only proportional symbols in $A_1, \cdots, A_n$.
- Extending Γ in ω many steps: $\Gamma_0 \subset \cdots \subset \Gamma_n \subset \cdots$, so that each Γ_n is consistent. Let $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ if $\Gamma_n \cup \{\varphi_n\}$ is consistent; otherwise let $\Gamma_{n+1} = \Gamma_n$.
- At last, let $\Gamma^* = \bigcup_n \Gamma_n$. Verify that Γ^* is maximally consistent. □

Completeness Theorem

Theorem 3.5 (Completeness, version I)

Suppose that $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is consistent. Then* Γ *is satisfiable.*

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This follows from Lemma 7 and then Lemma 6.

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Theorem 3.6 (Completeness, version II)

Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$ and that $\Gamma \models \varphi$. Then $\Gamma \vdash \varphi$.

Exercise 3.1

1. Find out which of the following formulas is a tautology without using Truth table.
   - $(((A_1 \to A_1) \to A_2) \to A_2)$
   - $((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$
2. For $\Gamma \subseteq \mathcal{L}_0$ and ψ in $\mathcal{L}_0$, show that

$$\Gamma \cup \{\varphi\} \vDash \psi \quad \text{if and only if} \quad \Gamma \vDash (\varphi \to \psi).$$

3. Two physicists, A and B, and a logician C, are wearing hats, which they know are either black or white but not all white. A can see the hats of B and C; B can see the hats of A and C; C is blind. Each is asked in turn if they know the color of their own hat. The answers are: A:"No". B:"No". C:"Yes". What color is C's hat and how does C know.

Exercise (思考题)

Show that

1. $\vdash \neg\neg\alpha \to \alpha$
2. $\vdash (\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
3. $\vdash \alpha \to \neg\neg\alpha$
4. $\vdash (\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$
5. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \beta$, then $\Gamma \vdash \beta$.