Mathematical Logic

Xianghui Shi

School of Mathematical Science
Beijing Normal University

Spring, 2025

Next

1. Propositional Logic
   - $\mathcal{L}_0$-formulas
   - Truth Assignments

Truth assignment

Definition 2.1

A set of truth values consists of two distinct elements:[a]

$$F, \quad \text{called } \textit{falsity}; \qquad\qquad T, \quad \text{called } \textit{truth}.$$

[a] Some people use 0 and 1 instead of F, T.

Definition 2.2

A truth assignment over $\{A_1, \ldots, A_n\}$ is a function

$$\nu : \{A_1, \ldots, A_n\} \to \{T, F\}.$$

Truth assignment

Question

How many different truth assignments over $\{A_1, \ldots, A_n\}$ are there?

Definition 2.3

A truth assignment for $\mathcal{L}_0$ is a function

$$\nu : \{A_n \mid n \in \mathbb{N}\} \to \{T, F\}.$$

Theorem 2.1

Suppose that ν is a truth assignment for $\mathcal{L}_0$. Then there is a unique function $\bar{\nu}$ defined on $\mathcal{L}_0$ such that

1. *For all n, $\bar{\nu}(A_n) = \nu(A_n)$.*
2. *For all $\psi \in \mathcal{L}_0$,*
$$\bar{\nu}(\neg\psi) = \begin{cases} T, & \text{if } \bar{\nu}(\psi) = F; \\ F, & \text{otherwise.} \end{cases}$$
3. *For all ψ_1 and ψ_2 in $\mathcal{L}_0$,*
$$\bar{\nu}(\psi_1 \to \psi_2) = \begin{cases} F, & \text{if } \bar{\nu}(\psi_1) = T \text{ and } \bar{\nu}(\psi_2) = F; \\ T, & \text{otherwise.} \end{cases}$$

Rank of formulas in $\mathcal{L}_0$

By the Unique Readability of formulas in $\mathcal{L}_0$, one can define a rank function $\rho : \mathcal{L}_0 \to \mathbb{N}$ as follows:

- $\rho(\langle A_n \rangle) = 0$, for $A_n \in S_0$;
- if $\varphi \equiv (\neg\psi)$, then $\rho(\varphi) = \rho(\psi) + 1$;
- if $\varphi \equiv (\psi_1 \to \psi_2)$, then $\rho(\varphi) = \max\{\rho(\psi_1), \rho(\psi_2)\} + 1$

It is not difficult to see that

Proposition 2.2

For $\varphi \in \mathcal{L}_0$, $\rho(\varphi) = \sup\{\rho(\psi) + 1 \mid \psi$ is a subformula of $\varphi\}$.

Question

What can you say about the range of ρ, $\mathrm{ran}(\rho)$?

Proof of Theorem 2.1

Construction of $\bar{\nu}$:

Base step. For each $n \in \mathbb{N}$, define $\bar{\nu}(A_n) = \nu(A_n)$.
Inductive step. Suppose that $\rho(\varphi) = s + 1$ and $s \geqslant 0$, that $\bar{\nu}$ is defined on all $\mathcal{L}_0$-formulas of rank $\leqslant s$.

- If $\varphi = (\neg\psi)$, then define
$$\bar{\nu}(\varphi) = \begin{cases} T, & \text{if } \bar{\nu}(\psi) = F; \\ F, & \text{otherwise.} \end{cases}$$
- If $\varphi = (\psi_1 \to \psi_2)$, then define
$$\bar{\nu}(\varphi) = \begin{cases} F, & \text{if } \bar{\nu}(\psi_1) = T \text{ and } \bar{\nu}(\psi_2) = F; \\ T, & \text{otherwise.} \end{cases}$$

Uniqueness:

show $\bar{\nu}(\varphi) = \bar{\nu}'(\varphi)$, by induction on the rank of $\varphi \in$ wff.

Uniqueness (local version)

Theorem 2.3

Suppose that $\varphi \in \mathcal{L}_0$ *and that* ν *and* μ *are truth assignments that agree on the propositional symbols which occur in* φ. *Written as* $\mu \sim_\varphi \nu$. *Then* $\bar{\nu}(\varphi) = \bar{\mu}(\varphi)$.

Corollary 2.4 (Uniqueness)

Suppose that ν *and* μ *are truth assignments that agree on all the propositional symbols* $\{A_n \mid n \in \mathbb{N}\}$. *Written as* $\mu \sim \nu$. *Then* $\bar{\nu}(\varphi) = \bar{\mu}(\varphi)$, *for all* $\varphi \in \mathcal{L}_0$.

True tables for connectives

p	$\neg p$
1	0
0	1

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
1	1	1	1	1	1
1	0	1	0	0	0
0	1	1	0	1	0
0	0	0	0	1	1

Question

Why do we assign "1" to $p \to q$ whenever p is "0"?

Want

p	q	$p \wedge q$	$p \to^* q$	$p \wedge q \to^* p$
1	1	1	1	1
1	0	0	0	a
0	1	0	$a = 1$	b
0	0	0	$b = 1$	b

Satisfaction

Definition 2.4

1. A truth assignment ν satisfies a formula φ, $\nu \models \varphi$, iff $\bar{\nu}(\varphi) = T$. Similarly, ν satisfies a set of formulas Γ, $\nu \models \Gamma$, iff $\nu \models \varphi$ for all $\varphi \in \Gamma$.
2. φ is a tautology（恒真式）iff every truth assignment ν satisfies φ.
3. $\varphi \in \mathcal{L}_0$ (or $\Gamma \subset \mathcal{L}_0$) are satisfiable iff there is a truth assignment which satisfies φ (or Γ, respectively).
4. φ is a contradiction if and only if there is no truth assignment which satisfies φ.

Examples

Example 2.1

1. $\neg(\neg A_1 \to A_2)$.
2. $\neg A_1 \to (A_1 \to A_2)$.

If you, by luck, try $\nu(A_1) = \nu(A_2) = F$, you can see that both formulas are satisfiable. Truth table provides a systematic way to check if they are tautologies or contradictions.

Example 2.2

Truth table for $\neg(\neg A_1 \to A_2)$:

A_1	A_2	$\neg A_1$	$\neg A_1 \to A_2$	$\neg(\neg A_1 \to A_2)$
1	1	0	1	0
1	0	0	1	0
0	1	1	1	0
0	0	1	0	1

Example 2.3

Compute the true table for $\varphi = p \to (\neg p \vee q) \wedge q$ and $\psi = p \to (q \to r)$.

p	q	r	$\neg p \vee q$	$(\neg p \vee q) \wedge q$	$q \to r$	φ	ψ
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	0
1	0	1	0	0	1	0	1
1	0	0	0	0	1	0	1
0	1	1	1	1	1	1	1
0	1	0	1	1	0	1	1
0	0	1	1	0	1	1	1
0	0	0	1	0	1	1	1

Theorem 2.5

There is an algorithm to determine whether a propositional formula φ is a tautology, satisfiable, or a contradiction.

Proof

Suppose φ involves n propositional symbols. The idea is to use the truth table for φ.

- List the construction sequence of φ, and set them the first row.
- List all the possible 2^n possible truth assignment as the first n columns.
- Compute the truth values for each row (i.e. each truth assignment). Complete the truth table.
- Decide if its category according to the number of T's in the (rightmost) column for φ. □

Reading assignment

Read about algorithm efficiency and P = NP problem.

Example 2.4

Prove that the followings are tautologies:

1. $p \to p$
2. $p \to (q \to p)$
3. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

Exercise 2.1

Compute the truth tables for

1. $p \to (\neg p \wedge q) \vee q$.
2. $(\neg p \to q) \wedge q \wedge r$.

Exercise 2.2

Show that the following are tautologies:

1. $\neg\neg p \to p$.
2. $(p \to q) \to (\neg q \to \neg p)$.
3. $(p \to q) \leftrightarrow (\neg p \vee q)$.

A puzzle

Exercise (课外思考题)

You are in a land inhabited by people who either always tell the truth or always tell falsehoods. You come to a fork in the road and you need to know which path leads to the capital. There is a local resident there, but he has time only to reply to one yes-or-no question. What one question should you ask so as to learn which path to take?

Liar-Truthteller puzzle

Consider "Is φ true?", where for example $\varphi \equiv$ "$0 < 1$".

φ	T-teller	Liar
1	yes	no
0	no	yes

Problem: Lair and Truthteller do NOT agree on the truth of φ.

KEY: Replace φ by a statement whose truth they don't agree on,
$\varphi^* \equiv$ "if I ask whether φ, you would say 'yes'"

φ	T-teller	Liar
1	yes	yes
0	no	no

Outcome: Lair and Truthteller agree on the truth of φ^*.

Tautologically equivalent

Definition 2.5

We say that two propositional formulas α and β are tautologically equivalent（恒真等价）, written as

$$\alpha \Leftrightarrow \beta,$$

if they have the same truth table, or equivalently,

$$\alpha \leftrightarrow \beta$$

is a tautology.

Logic laws from tautological equivalences

Theorem 2.6

Let p, q, r be any three propositions. Then

1. (commutative 交换律)
   - $p \wedge q \Leftrightarrow q \wedge p$.
   - $p \vee q \Leftrightarrow q \vee p$.
2. (associative 结合律)
   - $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$.
   - $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$.
3. (distributive 分配律)
   - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
   - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Theorem 2.6 (Cont'd)

4. (idempotent 幂等律)

$$p \vee p \Leftrightarrow p \wedge p \Leftrightarrow p$$

5. (double negation 双重否定)

$$\neg\neg p \Leftrightarrow p$$

6. (De Morgan's law)
   - $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
   - $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Theorem 2.6 (Cont'd)

7. (absorbing 吸收律)

$$p \vee (p \wedge q) \Leftrightarrow p \wedge (p \vee q) \Leftrightarrow p$$

8. (zero-one 零一律)
   - $p \vee \neg p \Leftrightarrow 1$
   - $p \wedge \neg p \Leftrightarrow 0$
   - $p \vee 1 \Leftrightarrow 1$
   - $p \wedge 1 \Leftrightarrow p$
   - $p \vee 0 \Leftrightarrow p$
   - $p \wedge 0 \Leftrightarrow 0$

Theorem 2.7

Let p, q, r be any three propositions. Then

1. $p \to q \Leftrightarrow \neg p \vee q$
2. $p \to q \Leftrightarrow \neg q \to \neg p$
3. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
4. $\neg(p \to q) \Leftrightarrow p \wedge \neg q$
5. $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$.

Examples

Equivalent formulas can be used to simplify formulas or to convert a formula to a certain form.

Example 2.5

1. Show that $\neg(p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$.
2. Simplify $p \wedge \neg q \rightarrow (\neg p \vee q) \wedge q$.

Exercise 2.3

Show that $(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$.

Examples (solution)

1. Apply Distribution:

$$
\begin{aligned}
& (p \vee q) \wedge \neg(p \wedge q) \\
\Leftrightarrow\ & (p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q) \\
\Leftrightarrow\ & (p \wedge \neg q) \vee (\neg p \wedge q) \\
\Leftrightarrow\ & \neg(p \leftrightarrow q), && \text{(Theorem 2.7-5)}
\end{aligned}
$$

2. Apply the equivalent form of $a \to b$,

$$
\begin{aligned}
& p \wedge \neg q \to (\neg p \vee q) \wedge q \\
\Leftrightarrow\ & \neg(p \wedge \neg q) \vee ((\neg p \vee q) \wedge q)) \\
\Leftrightarrow\ & (\neg p \vee q) \vee ((\neg p \vee q) \wedge q)) && \text{(De Morgan's Law)} \\
\Leftrightarrow\ & (\neg p \vee q) && \text{(吸收律)}
\end{aligned}
$$

Normal disjunction formula

Definition 2.6

Let $A_1, \ldots, A_n$ be propositional symbols. The basic conjunction terms（基本合取项）of $A_1, \ldots, A_n$ are formulas of the form:

$$Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n,$$

where each Q_i is either A_i or $\neg A_i$.

Normal disjunction formulas（析取范式）are formulas that are disjunctions of basic conjunction terms.[a]

[a]Also called Disjunctive Normal Form, abbreviated as **DNF**.

Proposition 2.8

Every satisfiable propositional formula is tautologically equivalent to a unique normal disjunction formula.

First get the basic conjunction terms via the truth table:

m_i	basic conjunction terms	p	q	r	$p \to (q \to r)$
m_{000}	$\neg p \wedge \neg q \wedge \neg r$	0	0	0	1
m_{001}	$\neg p \wedge \neg q \wedge r$	0	0	1	1
m_{010}	$\neg p \wedge q \wedge \neg r$	0	1	0	1
m_{011}	$\neg p \wedge q \wedge r$	0	1	1	1
m_{100}	$p \wedge \neg q \wedge \neg r$	1	0	0	1
m_{101}	$p \wedge \neg q \wedge r$	1	0	1	1
m_{110}	$p \wedge q \wedge \neg r$	1	1	0	0
m_{111}	$p \wedge q \wedge r$	1	1	1	1

Then take disjunction of the terms corresponding to "1" to get the equivalent normal form:

$$p \to (q \to r) \Leftrightarrow m_{000} \vee m_{001} \vee m_{010} \vee m_{011} \vee m_{100} \vee m_{101} \vee m_{111}.$$

Normal conjunction formula

Definition 2.7

Let $A_1, \ldots, A_n$ be propositional variables. The basic disjunction terms（基本析取项）of $A_1, \ldots, A_n$ are formulas of the form:

$$Q_1 \vee Q_2 \vee \cdots \vee Q_n,$$

where each Q_i is either A_i or $\neg A_i$.

Normal conjunction formulas（合取范式）are formulas that are conjunctions of basic disjunction terms.

Proposition 2.9

Every satisfiable propositional formula is tautologically equivalent to a unique normal conjunction formula.

Exercise 2.4

Find equivalent normal disjunction formulas for:

1. $(p \wedge q) \vee r$;
2. $(\neg p \rightarrow q) \wedge q \wedge r$.

Exercise (课外思考题)

Find equivalent normal conjunction formulas for the above formulas.

Truth functions

Definition 2.8

An n-ary truth function is a function whose domain is the set of sequences of T's and F's of length n, written $\{T, F\}^n$ and whose range is contained in $\{T, F\}$, i.e. a function

$$f : \{T, F\}^n \to \{T, F\}.$$

Question

How many distinct n-ary truth functions are there?

Each $\varphi \in \mathcal{L}_0$ defines a truth function/table.

Suppose $\varphi \in \mathcal{L}_0$ and the propositional symbols in φ are contained in $\{A_1, \ldots, A_n\}$. For each $\sigma \in \{T, F\}^n$, let ν_σ be the truth assignment on $\{A_1, \ldots, A_n\}$ such that

$$\nu_\sigma(A_i) = \text{ the } i^{\text{th}} \text{ element of } \sigma.$$

Define

$$f_\varphi(\sigma) = \bar{\nu}_\sigma(\varphi).$$

Reversely, each truth function can be realized by a formula in $\mathcal{L}_0$.

Theorem 2.10

Suppose that $f : \{T, F\}^n \to \{T, F\}$ is a truth function. Then there is a formula φ such that $f_\varphi = f$.

In this case, we say f is realized by φ.

Functional complete sets

Definition 2.9

A set of propositional connectives C is complete if every truth function $f : \{F, T\}^n \to \{F, T\}$ for $n \geqslant 1$ can be realized by a formula using only the connective symbols in C.

The following sets of connectives are complete:

- $\{\neg, \wedge, \vee\}$ and $\{\neg, \wedge, \vee, \to, \leftrightarrow\}$, by NDF
- $\{\neg, \to\}$,
- $\{\neg, \wedge\}$ and $\{\neg, \vee\}$.

Below is a complete list of all the binary connectives.

Symbol	Equivalent	Remarks
1	1	0-ary
0	0	0-ary
Γ_1	p	1-ary
Γ_2	q	1-ary
$\neg_1$	$\neg p$	1-ary
$\neg_2$	$\neg q$	1-ary

The rest 10 are truly binary:

Symbol	Equivalent	Remarks
$\wedge$	$p \wedge q$	且
$\vee$	$p \vee q$	或
$\rightarrow$	$p \rightarrow q$	如果 p 就 q
$\leftarrow$	$p \leftarrow q$	如果 q 就 p
$\leftrightarrow$	$p \leftrightarrow q$	p 当且仅当 q
$\oplus$	$(p \vee q) \wedge \neg(p \wedge q)$	异或，相当于对称差
$\uparrow$	$\neg(p \wedge q)$	与非
$\downarrow$	$\neg(p \vee q)$	或非
$<$	$\neg p \wedge q$	相当于 $q \backslash p$
$>$	$p \wedge \neg q$	相当于 $p \backslash q$

Exercise 2.5

Show that $\{\uparrow\}$ and $\{\downarrow\}$ are complete.

Exercise (课外思考题)

Prove that $\{\vee\}$, $\{\wedge\}$, $\{\rightarrow\}$, $\{\neg, \leftrightarrow\}$ are not complete.