Mathematical Logic

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Part I. Propositional Logic

Next

1. Propositional Logic
   - $\mathcal{L}_0$-formulas

Formal Language

Examples:

1. (Digital sequence understood by computer)

   $$0010101010000010111101000$$

2. (Programme Language, eg. C)

   ```
   while (s>10) do {s-=1};
   ```

3. (Propositional/Sentential Logic)

   $$(\neg((p \vee q) \rightarrow p))$$

4. (First-Order Logic)

   $$\forall \varepsilon \exists \delta \forall x(|x-a|<\delta \rightarrow |f(x)-c|<\varepsilon)$$

5. (Modal Logic) ($\Box \sim$ 必然, $\Diamond \sim$ 可能)

   $$\neg(\Diamond p) \leftrightarrow \Box(\neg p)$$

Formal languages usually

1. translate a restricted class of natural language.
2. have a fix set of atomic symbols and formation rules.
3. are precise and unambiguous.

Propositional logic formalizes certain type of assertions in natural language.

Definition 1.1

An **assertion** is a sentence that is either true or false.

Language for Propositional Logic

Example 1.1

Which are assertions?

1. $2 + 3 = 4$.
2. 你吃了饭吗?
3. The earth is round.
4. 明天会下雨。
5. 这个活动太精彩了!
6. 2 is a prime number.

More complicated ones:

Example 1.2

1. 明天不会下雨。
2. 明天下雨或者刮大风。
3. 明天下雨而且刮大风。
4. 如果明天下雨，明天就会刮大风。
5. 明天下雨当且仅当明天刮大风。

Let $p \equiv$ “明天下雨”，$q \equiv$ “明天刮大风”：

$\neg p$	明天不会下雨。
$p \vee q$	明天下雨或者刮大风。
$p \wedge q$	明天下雨而且刮大风。
$p \rightarrow q$	如果明天下雨，明天就会刮大风。
$p \leftrightarrow q$	明天下雨当且仅当明天刮大风。

Symbols

The following 3 types of elements are extracted from our natural language:

1. parenthesis（括号）: (,).
2. propositional connectives（命题连接词）:

$\neg$	not
$\to$	if $\cdots$, then $\cdots$

3. proposition symbols（命题符号）:

$$A_1, A_2, \cdots, A_n, \cdots$$

$\mathcal{L}_0$-Formulas

Definition 1.2

The **propositional language** $\mathcal{L}_0$ is the smallest set L such that L is a set of finite sequences of symbols in

$$S_0 = \{(,), \neg, \rightarrow\} \cup \{A_n \mid n \in \mathbb{N}\}.$$

and such that

1. $\langle A_n \rangle \in L$, for each $n \in \mathbb{N}$. [a]
2. If $s \in L$, then $(\neg s) \in L$.
3. If $s, t \in L$, then $(s \rightarrow t) \in L$.

[a] $\langle A \rangle$ denotes the length-1 sequence that consists of only one symbol A.

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The existence of the "smallest" such L needs some explanation. To see that $\mathcal{L}_0$ is well defined, we give an equivalent definition of $\mathcal{L}_0$.

$\mathcal{L}_0$ is well defined

Let $(*)$ denote the three conditions in the previous definition. Let $(S_0)^{<\omega} =_{\text{def}} \bigcup_{n \in \mathbb{Z}^+} (S_0)^n$, the set of all finite sequences of symbols in S_0.

Theorem 1.1

Let $\mathcal{L}_0^* = \bigcap \{L \subseteq (S_0)^{<\omega} \mid L$ satisfies $(*)\}$. Then $\mathcal{L}_0 = \mathcal{L}_0^*$.

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Theorem 1.1

Let $\mathcal{L}_0^* = \bigcap \{L \subseteq (S_0)^{<\omega} \mid L \text{ satisfies } (*)\}$. Then $\mathcal{L}_0 = \mathcal{L}_0^*$.

Proof.

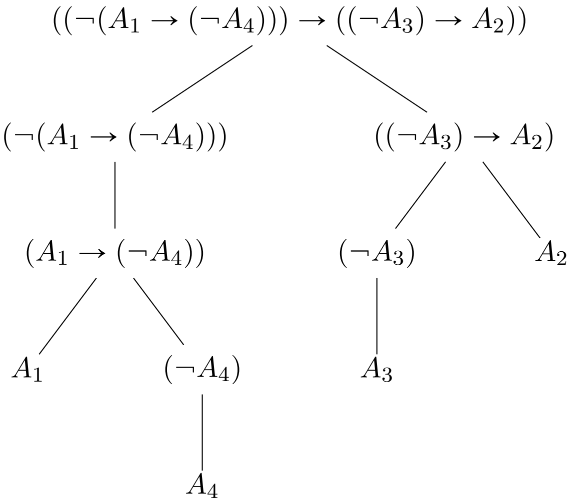
- Let $\Lambda = \{L \subseteq (S_0)^{<\omega} \mid L \text{ satisfies } (*)\}$.
- Then $\Lambda \neq \varnothing$, as $(S_0)^{<\omega} \in \Lambda$. Thus $\mathcal{L}_0^*$ is well defined.
- $\mathcal{L}_0^*$ satisfies $(*)$: Check $1, 2, 3$. Therefore $\mathcal{L}_0^* \supseteq \mathcal{L}_0$.
- By definition, $\mathcal{L}_0^* \subseteq L$, for all $L \in \Lambda$, in particular $\mathcal{L}_0^* \subseteq \mathcal{L}_0$. Therefore $\mathcal{L}_0^* = \mathcal{L}_0$. □

Well-formed formula

Definition 1.3

A finite sequence of elements in S_0 is called **well-formed formulas** (or simply **formula** or **wff**) if it can be built-up from $\{A_n \mid n \in \mathbb{N}\}$ by applying the following *formula-building operations* **finitely** many times:

$$\mathcal{E}_{\neg}(s) = (\neg s),$$
$$\mathcal{E}_{\rightarrow}(s, t) = (s \rightarrow t).$$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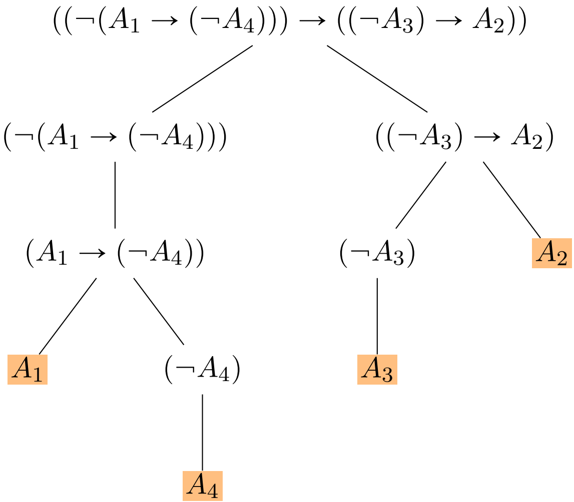
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Readability

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Proof.

- For " $\Rightarrow$ ", verify that wff satisfies $(*)$. So $\mathcal{L}_0 \subseteq$ wff.
- For " $\Leftarrow$ ", prove that (by induction on the least number of construction steps of $\varphi \in$ wff)
  - $\mathcal{L}_0$ contains all $\langle A_n \rangle$'s, and
  - $\mathcal{L}_0$ is closed under the two wff-operators: $\mathcal{E}_\neg$, $\mathcal{E}_\rightarrow$. □

Corollary 1.3 (Readability)

Suppose $\varphi \in \mathcal{L}_0$. *Then exactly one of the following applies.*

1. *There is an* n *such that* $\varphi = \langle A_n \rangle$.
2. *There is a* $\psi \in \mathcal{L}_0$ *such that* $\varphi = (\neg\psi)$.
3. *There are* ψ_1 *and* ψ_2 *in* $\mathcal{L}_0$ *such that* $\varphi = (\psi_1 \to \psi_2)$.

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Proof.

As $\mathcal{L}_0 =$ wff, we have the readability of $\varphi \in \mathcal{L}_0$.

For the "exact"-ness, it suffices to verify that the three cases are mutually exclusive.

- Case 1 consists of only one symbol
- Case 2 starts with "$(\neg$"
- Case 3 starts with "$(($" or "$(A_n$" for some A_n. □

Subformulas

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Subformulas

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The following definition of *subformula* is natural and often used in practice, however, it's not well defined unless the Uniqueness of Readability (to be discussed later) is proved.

Definition 1.4 (Subformula, an inductive definition)

The set $S(\varphi)$ of all subformulas of a given $\varphi \in \mathcal{L}_0$ is defined inductively as follows:

$$S(\langle A_n \rangle) = \{\langle A_n \rangle\}, \quad \text{for } n \in \mathbb{N}$$
$$S((\neg\alpha)) = S(\alpha) \cup \{(\neg\alpha)\}$$
$$S((\alpha \to \beta)) = S(\alpha) \cup S(\beta) \cup \{(\alpha \to \beta)\} \qquad (\star)$$

For the proof of Unique Readability, we use a bit-by-bit definition of subformulas.

Definition 1.5

Suppose s, t are finite sequences, φ, ψ are formulas.

1. t is a block-subsequence of s $\cdots[\cdots]\cdots$
2. t is a (proper) initial segment of s $[\cdots]\cdots\cdots$
3. an occurrence of s in φ $\cdots[\text{-}s\text{-}]\cdots$
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Question

Let s be a finite sequence of length n. How many block-subsequence of s are there?

Unique Readability

Theorem 1.4 (Unique Readability)

Suppose $\varphi \in \mathcal{L}_0$. Then exactly one of the following applies.

1. *There is an n such that $\varphi = A_n$.*
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3. *There are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$.*

Further, in cases (2) and (3), the subformulas ψ, ψ_1 and ψ_2 are unique, respectively.

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The Unique Readability enables us to prove by induction on the construction of formulas.

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The uniqueness of case 1 and 2 are self-clear.
To prove the uniqueness of ψ_1 and ψ_2 in case 3, we need

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- $|\varphi| = 1$. Then $s = \varnothing$. Vacuously true: as $\varnothing \notin \mathcal{L}_0$.

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    - $\varphi \equiv (\neg\psi)$, if $s \in \mathcal{L}_0$, then it must be $s \equiv (\neg\theta)$, some $\theta \in \mathcal{L}_0$. But then $\theta \subsetneq_{\text{init}} \psi$ and $|\psi| < |\varphi|$. Contradiction!

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  - $\varphi \equiv (\psi_1 \to \psi_2)$, if $s \in \mathcal{L}_0$, it must be that $s \equiv (\theta_1 \to \theta_2)$, for some $\theta_1, \theta_2 \in \mathcal{L}_0$.

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  - $\varphi \equiv (\psi_1 \to \psi_2)$, if $s \in \mathcal{L}_0$, it must be that $s \equiv (\theta_1 \to \theta_2)$, for some $\theta_1, \theta_2 \in \mathcal{L}_0$.
    - $\psi_1 \neq \theta_1$, one of $\{\psi_1, \theta_1\}$ is a proper initial segment of the other. Contradiction!
    - $\psi_1 = \theta_1$, one of $\{\psi_2, \theta_2\}$ is a proper initial segment of the other. Contradiction! □

Question

Fix a 1-1 enumeration of S_0. Give an algorithm to enumerate

1. $(S_0)^{<\omega} := \bigcup_{n\in\mathbb{N}}(S_0)^n$, the set of finite sequences of members in S_0
2. $\mathcal{L}_0$.

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Exercise 1.1

1. For which natural numbers n are there elements of $\mathcal{L}_0$ of length n? Provide detailed argument. (Hint: $n \neq 2, 3, 6$)
2. Show that a sequence φ is an element of $\mathcal{L}_0$ if and only if there is a finite sequence of sequences $\langle\varphi_1, ..., \varphi_n\rangle$ such that $\varphi_n = \varphi$, and for each $i \leqslant n$,
   - either there is an m such that $\varphi_i = \langle A_m\rangle$,
   - or there is a $j < i$ such that $\varphi_i = (\neg\varphi_j)$,
   - or there are $j_1, j_2 < i$ such that $\varphi_i = (\varphi_{j_1}\to\varphi_{j_2})$.

Polish Notation

Though parentheses are helpful for human eyes, it is possible to drop parentheses without loss of clarity. Let $S_0^* = S_0 - \{(,)\}$.

Definition 1.6

Let $\mathcal{P}_0$ be the smallest set $P \subseteq (S_0^*)^{<\omega}$ such that

1. For each n, $A_n \in P$.
2. If ψ_1 and ψ_2 belong to P, then so do $\neg\psi_1$ and $\rightarrow \psi_1\psi_2$.

Theorem 1.5

For any $s \in (S_0^*)^{<\omega}$, $s \in \mathcal{P}_0 \Leftrightarrow s \in \mathcal{P}_0$-wff.

$\mathcal{P}$-wff

Definition 1.7

A finite sequence of elements in S_0 is called $\mathcal{P}_0$-**wff** if it can be built-up from $\{A_n \mid n \in \mathbb{N}\}$ by applying the following formula-building operations finitely many times:

$$\mathcal{D}_\neg(s) = \neg s,$$
$$\mathcal{D}_\to(s, t) = \to s\, t.$$

Example 1.3

$$\to \neg \to A_1 \neg A_4 \to \neg A_3 A_2.$$

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It is our early example: $((\neg(A_1 \to (\neg A_4))) \to ((\neg A_3) \to A_2))$.

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Exercise 1.2

State and prove the unique readability theorem for $\mathcal{P}_0$.

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$$\to \neg \to A_1 \neg A_4 \to \neg A_3\, A_2.$$

It is our early example: $((\neg(A_1 \to (\neg A_4))) \to ((\neg A_3) \to A_2))$.

Exercise 1.2

State and prove the unique readability theorem for $\mathcal{P}_0$.

Question

Try to write the Reverse Polish version of the above formula.

Polish Notation and Reverse Polish Notation

Disadvantage: hard to decode by human
Advantage: processed faster by computer

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Reading assignment

Find out more about Polish and reverse Polish notations, as well as SVO, SOV, VSO, etc. S = Subject, V = Verb, O = Object

Word order

Word order	English equivalent	Proportion of languages		Example languages
SOV	"Cows grass eat."	45%		Ancient Greek, Bengali, Burmese, Hindi/Urdu, Japanese, Korean, Latin, Persian, Sanskrit, Tamil, Telugu, Turkish, etc
SVO	"Cows eat grass."	42%		Chinese, Dutch, English, French, German, Hausa, Italian, Malay, Portuguese, Russian, Spanish, Swahili, Thai, Vietnamese, etc
VSO	"Eat cows grass."	9%		Biblical Hebrew, Classical Arabic, Filipino, Irish, Māori, Tuareg–Berber, Welsh
VOS	"Eat grass cows."	3%		Car, Fijian, Malagasy, Q'eqchi', Terêna
OVS	"Grass eat cows."	1%		Hixkaryana, Urarina
OSV	"Grass cows eat."	0%		Tobati, Warao

Frequency distribution of word order in languages surveyed by Russell S. Tomlin in the 1980s

Priority of operators

To establish a more compact notation,

1. The outermost parentheses are omitted.
2. The priority of operators are ordered as: $\neg$ is higher than $\rightarrow$. [1] e.g.

$$B \rightarrow \neg A \quad \text{is} \quad (B \rightarrow (\neg A))$$

3. When connectives of the same priority are repeated, grouping is to the right:

$$A \rightarrow B \rightarrow C \quad \text{is} \quad (A \rightarrow (B \rightarrow C))$$

[1] When $\vee$, $\wedge$ and $\leftrightarrow$ are considered:

$$\neg \qquad (\vee, \wedge) \qquad (\rightarrow, \leftrightarrow).$$

Other connectives

Other connectives $\vee$, $\wedge$, $\leftrightarrow$ are treated as abbreviations of formulas (involving $\{\neg, \to\}$ only) as follows:

$$
\begin{array}{rcl}
p \vee q & \text{iff} & \neg p \to q \\
p \wedge q & \text{iff} & \neg(p \to \neg q) \\
p \leftrightarrow q & \text{iff} & (p \to q) \wedge (q \to p)
\end{array}
$$

This treatment will be justified by the next section.