The 9^{th} Homework

April 29, 2025

1. Suppose $\mathcal{M}$ is an $\mathcal{L}_A$ -structure. Let $A^* = A \cup \{c_m \mid m \in M\}$. Identify $\mathcal{M}$ as an $\mathcal{L}_{A^*}$-structure. Show that for any $\mathcal{N}$, if $\mathcal{N} \models \mathrm{Th}_{\mathcal{L}_{A^*}(\mathcal{M})}$, then $\mathcal{M} \preccurlyeq \mathcal{N}$.

   <u>Solution</u>: Suppose $\mathcal{N} \models \mathrm{Th}_{\mathcal{L}_{A^*}(\mathcal{M})}$, consider the map $f : \mathcal{M} \longrightarrow \mathcal{N}$ s.t. $f(m) = (c_m)^{\mathcal{N}}$. Then for all $\mathcal{N} \models \mathrm{Th}_{\mathcal{L}_{A^*}(\mathcal{M})}$, for all $\varphi \in \mathcal{L}_A$ and for all $m_1 \dots m_k \in \mathcal{M}$

$$\begin{aligned}\mathcal{M} \vDash \varphi[m_1 \dots m_k] &\Leftrightarrow \mathcal{M} \vDash \varphi\,(c_{m_1} \dots c_{m_k}) \\ &\Leftrightarrow \mathcal{N} \vDash \varphi\,(c_{m_1} \dots c_{m_k}) \\ &\Leftrightarrow \mathcal{N} \vDash \varphi[f(m_1) \dots f(m_k)]\end{aligned}$$

   Theorefore $\mathcal{M} \preccurlyeq \mathcal{N}$ ■

2. Suppose that $\mathcal{M}$ is an infinite $\mathcal{L}_{\mathcal{A}}$-structure. Show that there is an $\mathcal{M}_1$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are elementarily equivalent and $\mathcal{M}_1$ has an element which is not the interpretation of any constant symbol. (Hint: Apply Compactness to $\Gamma = \mathrm{Th}_{\mathcal{L}_{\mathcal{A}}}(\mathcal{M}) \cup \{\neg(x_1 \hat{=} c) \mid c \in \mathcal{C} \cap A\}$)

   <u>Solution</u>: Let $\Sigma = \{\neg(x \hat{=} c) \mid c \in \mathcal{C} \cap A\}$ and $\Gamma = \mathrm{Th}_{\mathcal{L}_{\mathcal{A}}}(\mathcal{M}) \cup \Sigma$. For arbitrary finite subset $\Gamma_0 \subseteq \Gamma$, suppose $\Gamma_0 \cap \Sigma \subseteq \{\neg(x \hat{=} c_i) \mid i \leq n\}$ for some $n \in \mathbb{N}$. Since $\mathcal{M}$ is infinite, there is an $\mathcal{M}$-assignment ν sending x to some element in $\mathcal{M}$ other than any $c_i^{\mathcal{M}}$ $(i \leq n)$, and $(\mathcal{M}, \nu) \vDash \Gamma_0$. So Γ_0 is satisfiable, and by compactness, Γ is also satisfiable. Therefore any model $\mathcal{M}_1$ satisfies Γ is elementarily equivalent to $\mathcal{M}$ and has an element which is not the interpretation of any constant symbol. ■