The Eighth Homework

April 28, 2025

1. Show that for every pair of $\mathcal{L}$ formulas φ and ψ, $\{\varphi, (\neg\varphi)\} \vdash \psi$.

   SOLUTION: Since $(\varphi \to (\neg\varphi \to \psi))$ is a logical axiom,

   $$\langle(\varphi \to (\neg\varphi \to \psi)), \varphi, (\neg\varphi \to \psi), \neg\varphi, \psi\rangle$$

   witnesses that $\{\varphi, (\neg\varphi)\} \vdash \psi$. ■

2. Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Show that $\Gamma \vdash \varphi$.

   SOLUTION:

   Method 1: With Completeness. If there is a model of Γ also satisfies $(\neg\varphi)$, then $\Gamma \cup \{(\neg\varphi)\}$ is consistent. So every model of Γ satisfies φ, and thus $\Gamma \vdash \varphi$.

   Method 2: Without Completeness. By the definition of consistent, there is a formula ψ such that $\Gamma \cup \{(\neg\varphi)\} \vdash \{\psi, (\neg\psi)\}$, by Question 1, we can derive that $\Gamma \cup \{(\neg\varphi)\} \vdash \varphi$. By deduction, $\Gamma \vdash ((\neg\varphi) \to \varphi)$. Assume that s is a Γ-proof of $((\neg\varphi) \to \varphi)$, then $s^\frown\langle(((\neg\varphi) \to \varphi) \to \varphi), \varphi\rangle$ witnesses that $\Gamma \vdash \varphi$. ■