The Ninth Homework

April 28, 2025

1. Consider the structure $\mathcal{R} = (\mathbb{R}, +, \times, <, 0, 1)$.Let $\mathcal{N}$ be the substructure of $\mathcal{R}$ given by the set of all $r \in \mathbb{R}$ such that the set $A_r = \{r\}$ is definable in $\mathcal{R}$ without parameters.Show that $\mathcal{N} \leqslant \mathcal{R}$.

   <u>SOLUTION</u>: $\forall \varphi (x_1 \cdots x_n) \in S, \quad \forall r_1 \cdots r_n \in \mathcal{R}$. Since $\{r_1\} \ldots [r_1]$ definable.$\exists \psi_1 \ldots \psi_n$ s.t $\forall b \in \mathbb{R}$ $\mathcal{R} \vDash \psi_i(b) \Leftrightarrow b = r_i \Leftrightarrow \mathcal{N} = \psi_i(b)$. so $\mathcal{N} \vDash \varphi (r_1 \ldots r_n) \Leftrightarrow \mathcal{N} \vDash \exists x_1 \ldots x_n \quad \varphi (x_1 \cdots x_1) \wedge \psi_1 (x_1) \wedge \psi_n (x_n) \Leftrightarrow \mathcal{R} \vDash \exists x_1 \ldots x_n \quad \varphi (x_1 \cdots x_1) \wedge \psi_1 (x_1) \wedge \psi_n (x_n) \Leftrightarrow \mathcal{R} \vDash \varphi (r_1 \ldots r_n)$ ■

2. Prove Theorem 5.3.

   Theorem 5.3 (Soundness, version II). Suppose that $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ and that $\Gamma \vdash \varphi$. Then for any $(\mathcal{M}, \nu)$, if $(\mathcal{M}, \nu) \vDash \Gamma$,[1] then $(\mathcal{M}, \nu) \vDash \varphi$.

   <u>SOLUTION</u>: See the textbook [P79-Theorem 5.14]. ■

3. Prove Corrollary 5.4.

   Theorem 5.4 (Soundness, version I). Suppose $\Gamma \subseteq \mathcal{L}$. If Γ is satisfiable, then Γ is consistent.

   <u>SOLUTION</u>: See the textbook [P81-Corollary 5.15]. ■

[1] $(\mathcal{M}, \nu) \vDash \Gamma$ abbreviates "$(\mathcal{M}, \nu) \vDash \gamma$, for every $\gamma \in \Gamma$".